



# On the $(Q, r)$ policy for perishables with positive lead times and multiple outstanding orders

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## Abstract

We consider an inventory system for perishables with fixed lifetimes, positive replenishment lead times and lost sales in the presence of non-negligible fixed ordering costs. The system is studied under the lotsize reorder level  $(Q, r)$  policy. An exact analysis of this system based on the stationary distribution of the remaining lifetime process is provided by Berk and Gürler (Oper Res 56(5):1238–1246, 2008) under the restriction that there is at most one outstanding order at any time ( $r < Q$ ). In this work, we generalize their results to allow for more than one outstanding orders ( $r \geq Q$ ). We provide the operating characteristics of the inventory system and construct the exact expected cost rate expression using a renewal theoretic approach. An illustrative numerical study indicates that allowing for multiple outstanding orders ( $r \geq Q$ ) may result in significant savings in the expected cost rate, compared to the case with  $r < Q$ . In particular, when the fixed lifetimes are short and the ordering costs are low, expected costs can be reduced by more than half.

**Keywords** Perishable inventory · Lot size-reorder point policy · Lost sales · Effective lifetime · Multiple outstanding orders

## 1 Introduction

Perishable inventories are commonly encountered in practice as fashion goods, foodstuffs, pharmaceuticals, blood products or composite materials. In the related literature, Weiss (1980) may be cited as the earliest work with fixed lifetime. With positive lead times, Schmidt and Nahmias (1985) provide the first exact analytical treatment of a system with fixed lifetimes and under continuous review, extended by Perry and Posner (1998). Nahmias and Wang

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(1979) provide the first approximate treatment of perishable inventory with fixed ordering costs and a constant lead time. Chiu (1995) and Olsson (2014) provide other heuristic solutions. Tekin et al. (2001) propose a time-based control policy for fixed lifetimes under a special aging process. Studies considering random lifetime models include Nahmias (1975), Liu (1990), Kalpakam and Shanthi (2001), Kalpakam and Shanthi (2006), and Gürler and Özkaya (2008).

Most of the available inventory literature on perishables under continuous review with positive lead times employ the lot size-reorder point policy class. (See Kouki et al. 2015 for an extensive taxonomy of the available models and policies.) It is known that this policy class is not necessarily optimal. However, the structure of the optimal policy for perishables in the presence of non-negligible lead times is still an open question. As stated in Schmidt and Nahmias (1985), it is unlikely that anyone interested in a real problem would be able to find or to use an *optimal policy*. A reasonable alternative to *optimal policies* which is commonly employed is to find *the best policy from a prespecified class...*" (emphasis added). Berk and Gürler (2008) focused on this policy class and provided a  $(Q, r)$  inventory model with lost sales under the restriction of  $r < Q$ . Similarly, we consider a model where each item in the same batch has the same fixed lifetime, under continuous review setting with a fixed positive lead time and non-negligible ordering cost. Our model herein differs from theirs in that we allow for  $r \geq Q$ , that is, there may be more than one outstanding orders at any time. This extension makes the analysis more complicated as the remaining shelf life process is now expressed as a vector, the elements of which are related to each other in a special structure. However, our numerical examples show that this generalization albeit at a cost of increased technicality pays for itself. As illustrated in our numerical section, allowing for more than one outstanding order would be most beneficial where the desired service levels (fraction of stockout times) would be high with highly variable demand and/or relatively small (but non-negligible) fixed ordering costs and in some cases expected cost rate is reduced by more than half. An important by-product of our model is the following. As the constant lifetimes of items go to infinity in the limit, our model subsumes the non-perishable inventories. The exact analysis of the lost sales model is not available in the literature for non-perishables with Markovian demands, constant lead times and positive fixed ordering costs. The authors are familiar with only Hill (1992, 1994) where a non-perishable inventory system is analyzed under the restriction of at most two outstanding orders at any time. Hence, our model may be viewed as a unifying framework for a class of lost sales inventory systems. Furthermore, our generalized model could serve as a building block to address joint pricing (Liu et al. 2015; Avinadav et al. 2017; Chintapalli 2015) and more novel customer behaviors during stockouts (Amirthakodi et al. 2015; Ioannidis et al. 2013; Al Hamadi et al. 2015).

The rest of the paper is organized as follows. Section 2 introduces the basic assumptions of the perishable inventory system under consideration and main results existing for the case  $r > Q$  are presented. Some novel characteristics of the sequence of effective lifetime distributions are introduced in Sect. 3. In Sect. 4, the operating characteristics of the model are derived and the objective function is constructed for the case where  $r$  is not an integer multiple of  $Q$ . In Sect. 5, we address the special case where  $r$  is an integer multiple  $m$  of  $Q$ . Section 6 presents some numerical results on the benefits of alleviating the restriction  $r < Q$  and some sensitivity results. Finally, Sect. 7 provides concluding remarks.

## 2 The model assumptions and preliminaries

Unit external demands are generated according to a Poisson process with rate  $\lambda$ . Replenishment is done in batches and there is a fixed, positive procurement lead time,  $L$ . All of the items in a batch have identical lifetimes. After joining stock, a batch has a fixed, finite lifetime of  $\tau$  time units, beyond which it is no longer usable. The items are withdrawn from stock to satisfy the demand according to the *FIFO* policy. Each unit held in the stock incurs a holding cost  $h$  per unit time. Each unit that perishes incurs a perishing cost of  $p$ . All unmet demand is lost at a unit lost sales cost of  $\pi$ . There is a fixed non-zero ordering cost  $K$ . The inventory system is monitored continuously and the following modified  $(Q, r)$  control policy based on the inventory position (on hand inventory plus outstanding orders) is employed.

**Policy** A replenishment order of size  $Q$  is placed whenever the inventory position hits or crosses  $r$ , whichever occurs first.

To see how the policy operates, let  $m$  be a positive integer defined as  $m := \lceil \frac{r}{Q} \rceil$  for a  $(Q, r)$  pair, where  $\lceil x \rceil$  denotes the smallest integer greater than  $x$ . We have  $(m - 1)Q \leq r < mQ$ . Under the above policy, items join stock in batches of  $Q$  units and all items in a batch have the same lifetime. So long as inventory depletion occurs through demand, inventory position decreases incrementally by one unit and a reorder decision is triggered when the inventory position hits  $r$ . However, it is also possible that items in the oldest batch simultaneously perish; then, the reorder decision is triggered either when the inventory position hits  $r$  or crosses  $r$  momentarily depending on whether or not  $r$  is an integer multiple of  $Q$ . If the reorder point is an integer multiple of the lot size ( $r = (m - 1)Q$ ), the inventory position hits  $r$  at instances of order placement by perishing; otherwise, reordering is triggered as the inventory position crosses  $r$ . In both cases, the inventory position drops exactly to the level  $(m - 1)Q$ . Furthermore, we make the following observations: (i) At steady state, the inventory position takes on values in the interval  $[r + 1, Q + r]$  in both cases  $(m - 1)Q \leq r < mQ$  and  $r = (m - 1)Q$ . (ii) There may be at most  $m + 1$  different batches if  $(m - 1)Q \leq r < mQ$  and  $m$  different batches if  $r = (m - 1)Q$  in the inventory system. (iii) Between two consecutive order placements, there is always an instance when the inventory position hits  $mQ$  for the first time. This note will become important when we discuss the embedded cycles in the subsequent sections. (iv) The inventory system can be fully characterized by the inventory position at any point in time and the times elapsed since each of the batches in the system was ordered.

To illustrate the above particulars of the policy, we provide the following examples. Suppose  $Q = 5$  and  $r = 3$ . Then,  $m = \lceil \frac{3}{5} \rceil = 1$ . At any instance at most one order may be outstanding. The policy orders 5 units when the inventory position drops to 3 by demand or to 0 ( $= (m - 1)Q$ ) by perishing, whichever occurs first. At steady state, the inventory position takes values in the interval  $[4, 8]$ . Suppose now,  $Q = 5$  and  $r = 5$  so that  $m = 2$  which implies that there may be at most two outstanding orders at any time. The policy orders 5 units whenever the inventory position drops to 5 by demand or perishing. At steady state, the inventory position takes on values between  $[6, 10]$ . Lastly, suppose that  $Q = 5$  and  $r = 7$  so that  $m = 2$  again. There may be at most two outstanding orders at any time. An order of 5 units is ordered when the inventory drops to 7 by demand or crosses 7 dropping momentarily to 5 by perishing of the oldest batch, whichever occurs first. At steady state, the inventory position takes on values between  $[8, 12]$ .

In our analysis, we impose no restrictions on the possible values that  $Q$  and  $r$  may assume other than that they are non-negative. However, as will be illustrated below, when the reorder point  $r$  is an integer multiple of the order quantity  $Q$ , some event realizations possible for

$r \neq Q$  are not observed which consequently results in some simplifications. Hence, we address the cases for  $r > (m - 1)Q$  and  $r = (m - 1)Q$  separately. We first discuss the former and defer the discussion of the latter to Sect. 5.

Our modeling approach is similar to that in Berk and Gürlér (2008) and rests on the concept of an *embedded cycle* defined, herein, as the time between two consecutive instances at which the *inventory position* hits  $mQ$ . Such instances provide extreme ease in the analysis: At the beginning of an embedded cycle, a new batch is ‘issued’ for use; the batch ‘issued’ is the oldest among the most recent  $m$  batches due to the *FIFO* rule; during the embedded cycle, only this batch is being used to satisfy demand; at the end of the embedded cycle, this batch has been depleted completely either by demand or perishing. Note that, in the model herein, a stockout period occurs (when it does) at the beginning of an embedded cycle whereas in Berk and Gürlér (2008) such periods are by definition at the end. For brevity, we shall refer to the batch issued at the beginning of an embedded cycle as the *current* batch, for the entirety of the embedded cycle. We also define the *effective lifetime* of a batch as the lifetime  $\tau$  plus the lead time  $L$  minus the time elapsed since that particular batch was ordered. That is, the effective lifetime is the remaining lifetime at the beginning of an embedded cycle if the batch is already in stock; otherwise, it is the remaining lead time at that moment plus the lifetime  $\tau$ . The effective lifetime of a batch may, in general, take on values over  $(0, \tau + L]$ . At embedded cycle beginnings (when the inventory position is  $mQ$  by definition), the inventory system can be fully characterized by an  $m$ -dimensional array of the effective lifetimes of the most recent  $m$  batches. If one observes these arrays over consecutive embedded cycle beginnings, they would appear as a sequence of random vectors, to which we shall refer as the sequence of *effective lifetime vector*. Next, we obtain the probability distribution and certain properties of this sequence.

### 3 Effective lifetime distribution

In this section, we show that the effective lifetime (vector) sequence has the Markov property, establish certain properties leading to the ergodicity of the process, and derive the stationary effective lifetime (vector) distribution.

Suppose we start observing the inventory system operating under the proposed policy at time  $t = 0$  with  $m$  batches of  $Q$  items in the system. Let  $\{T_n, n \geq 1\}$  be the sequence of time epochs at which the inventory position hits  $mQ$  for the  $n$ th time; with  $T_1 = 0$ . Then,  $IP(T_n) = mQ$  for all  $n \geq 1$ , where  $IP(t)$  is the inventory position at time  $t$ . Let  $\{\mathbf{Z}_n, n \geq 1\}$  be the sequence of effective lifetimes of the  $m$  batches in the system at  $T_n$ , where  $\mathbf{Z}_n = \{Z_{n,1}, \dots, Z_{n,m}\}$ ; and  $\mathbf{z}_n = \{z_{n,1}, \dots, z_{n,m}\}$  denote a particular realization of  $\mathbf{Z}_n$ , where  $0 < z_{n,1} \leq z_{n,2} \leq \dots \leq z_{n,m} \leq \tau + L$ . Considering the system between two consecutive instances when the inventory position hits  $mQ$ , we develop the expressions for the limiting probability distribution of the effective lifetime sequence  $\{\mathbf{Z}_n, n \geq 1\}$ .

We first demonstrate that the effective lifetime process  $\{\mathbf{Z}_n, n \geq 1\}$  possesses the Markov property, which is a crucial assumption for the validity of our modeling approach. Let  $\{Y_j, j \geq 1\}$  be the sequence of Poisson demand arrival times in chronological order and let  $N(t)$  be the counting process of the arrivals in  $(0, t]$ . Then,  $\{T_n, n \geq 1\}$  is a sequence of stopping times for  $N(t)$ . Hence, the time between the  $(N(T_n) + j)$ th demand arrival and the last stopping time, i.e.,  $Y_{N(T_n)+j} - T_n$  has an Erlang  $j$  distribution with rate  $\lambda$ , independent of the events prior to  $T_n$ . The time between  $T_{n+1}$  and  $T_n$  corresponds to the  $n$ th embedded cycle for  $n \geq 1$  as defined above.

As an example, consider the following sample path: Set the time origin,  $T_1 = t = 0$ , as the beginning of Embedded Cycle 1, where the oldest batch in the system has an effective lifetime  $z_{1,1} = \tau$  (hence all the other  $z_{i,1} \leq \tau$ ). Suppose the inventory position drops to  $r$  after  $mQ - r$  demands have arrived and a replenishment order is given at  $t = Y_{mQ-r}$ . During the lead time period of length  $L$ , the remaining  $r - (m - 1)Q$  units are also depleted by demand and Embedded Cycle 1 is completed before the lead time period ends, starting Embedded Cycle 2 ( $T_2 = Y_Q$ ). This realization is characterized by the events  $Y_Q < z_{1,1} = \tau$  and  $Y_Q - Y_{Q-r} < L$ . Embedded Cycle 1 has lasted for  $T_2 - T_1 = Y_Q$  time units; hence, the second embedded cycle begins with a batch of  $Q$  items with an effective lifetime of  $z_{2,1} = z_{1,2} - [T_2 - T_1] = z_{1,2} - Y_Q$ . Similarly, we have  $z_{2,i} = z_{1,i+1} - [T_2 - T_1]$  for  $i = 2, \dots, m - 1$ . The youngest batch (the most recently placed order) in the system has an effective lifetime of  $z_{2,m} = \tau + L - [Y_Q - Y_{mQ-r}]$ , where the bracketed term corresponds to the time elapsed since the corresponding order was placed. Suppose in Embedded Cycle 2, the inventory position drops to  $r$  at time  $t = Y_{N(T_2)+mQ-r}$  and an order of size  $Q$  is placed and at the end of the lead time, there are still some unsold items left over. Those items are then depleted by demand at time  $t = Y_{N(T_2)+Q}$  without perishing and Embedded Cycle 3 starts ( $T_3 = Y_{N(T_2)+Q}$ ). This realization is characterized by the events  $Y_{N(T_2)+mQ-r} - T_2 < z_{2,1}$ ,  $Y_{N(T_2)+Q} - Y_{N(T_2)+mQ-r} > L$  and  $Y_{N(T_2)+Q} - T_2 < z_{2,1}$ . At the beginning of Embedded Cycle 3, the effective lifetimes for the batches in the system are as follows:  $z_{3,i} = z_{2,i+1} - [T_3 - T_2]$  for  $i = 1, \dots, m - 1$  and  $z_{3,m} = \tau + L - [T_3 - Y_{N(T_2)+mQ-r}]$ . Assume that  $z_{3,1} > \tau$ , implying that this batch corresponds to an outstanding order, and that the system loses the demand that occur, if any, over the time segment  $w(z_{3,1})$ , where  $w(x) = (x - \tau)^+ = \max(0, x - \tau)$ . A positive value of  $w(x)$  implies a stockout period with duration of that length, which occurs at the beginning of the corresponding embedded cycle under our definition herein of an embedded cycle. Suppose in this embedded cycle, the inventory level drops to  $r$  at  $t = Y_{N(T_3)+N(w(z_{3,1}))+mQ-r}$  and an order of size  $Q$  is placed. Note that the demands arriving over the time segment  $(T_3, T_3 + w(z_{3,1}))$  are lost to the system since there is no stock on hand during this time. At the end of the lead time, there are still unsold items and unlike the previous case, however, suppose some of these left over items perish at time  $t = T_4$  before they are depleted by demand. This perishing event completes the third embedded cycle and starts the fourth one ( $T_4 = T_3 + z_{3,1}$ ). This realization is characterized by the events  $Y_{N(T_3)+N(w(z_{3,1}))+Q} - T_3 > z_{3,1}$ ,  $Y_{N(T_3)+N(w(z_{3,1}))+mQ-r} - T_3 < z_{3,1}$  and  $Y_{N(T_3)+N(w(z_{3,1}))+Q} - Y_{N(T_3)+N(w(z_{3,1}))+mQ-r} > L$ . At the beginning of Embedded Cycle 4, we have  $z_{4,i} = z_{3,i+1} - [T_4 - T_3] = z_{3,i+1} - z_{3,1}$  for  $i = 1, 2, \dots, m - 1$  and  $z_{4,m} = \tau + L - [T_4 - Y_{N(T_3)+N(w(z_{3,1}))+mQ-r}]$ . Assume that  $z_{4,1} > \tau$  and an order is placed at  $t = T_4 + z_{4,1}$  when all items on hand perish before a total of  $mQ - r$  units can be sold. This perishing event which triggers a reordering decision also completes the fourth embedded cycle and starts Embedded Cycle 5 ( $T_5 = T_4 + z_{4,1}$ ). This realization is characterized by the event  $Y_{N(T_4)+N(w(z_{4,1}))+mQ-r} - T_4 < z_{4,1}$ . At the beginning of the fifth embedded cycle, we have  $z_{5,i} = z_{4,i+1} - [T_5 - T_4] = z_{4,i+1} - z_{4,1}$  for  $i = 1, 2, \dots, m - 1$  and  $z_{5,m} = \tau + L$ . The process continues in this fashion. As illustrated by the foregoing discussion, the effective lifetime vector  $\mathbf{Z}_{n+1}$  at the beginning of the  $(n + 1)$ th embedded cycle is completely determined by (i)  $\mathbf{Z}_n$  and, (ii) the Poisson demand arrival process after the stopping time  $T_n$ . Therefore, the embedded process  $\{\mathbf{Z}_n, n \geq 1\}$  has the Markov property. Due to possible stockouts, the demand process within an embedded cycle is not identical to the sales process within it. For brevity, we, henceforth define  $X_{r-(m-1)Q} = Y_{N(T_n)+N(w(z_{n,1}))+Q} - Y_{N(T_n)+N(w(z_{n,1}))+mQ-r}$ ,  $X_Q = Y_{N(T_n)+N(w(z_{n,1}))+Q} - [T_n + w(z_{n,1})]$ , and  $X_{mQ-r} = Y_{N(T_n)+N(w(z_{n,1}))+mQ-r} - [T_n + w(z_{n,1})]$ . They denote, respectively, the times that elapse until  $r - (m - 1)Q$ ,  $Q$ , and  $mQ - r$  units are sold since the beginning of

an embedded cycle or the first instance within an embedded cycle at which there is positive stock, whichever occurs first. Note that  $X_{r-(m-1)Q}$ ,  $X_Q$  and  $X_{mQ-r}$  are independent Erlang  $r - (m - 1)Q$ , Erlang  $Q$  and Erlang  $mQ - r$  variables with rate  $\lambda$ , respectively. For an Erlang  $j$  variable, its probability density function (p.d.f.) is denoted by  $h_j(\cdot)$ , its cumulative distribution function (c.d.f) by  $H_j(\cdot)$ , and its complementary c.d.f. by  $\bar{H}_j(\cdot)$ .

The effective lifetimes for the  $(n + 1)$ th embedded cycle are obtained by considering the following events. (i)  $E_1 : \{X_Q < \min(Z_{n,1}, \tau) = Z_{n,1} - w(Z_{n,1})\}$  where all  $Q$  items of the oldest batch of embedded cycle  $n$  are sold without perishing; (ii)  $E_2 : \{X_{mQ-r} < Z_{n,1} - w(Z_{n,1}) < X_Q\}$  where some of the items in the oldest batch of embedded cycle  $n$  perish after reordering; and (iii)  $E_3 : \{X_{mQ-r} > Z_{n,1} - w(Z_{n,1})\}$  where some of the items in the oldest batch of embedded cycle  $n$  perish before the reorder point  $r$  is reached through sales. We have

$$Z_{n+1,i} = \begin{cases} Z_{n,i+1} - w(Z_{n,1}) - X_Q & i = 1, 2, \dots, m - 1 \text{ and } E_1 \\ \tau + L - X_{r-(m-1)Q} & i = m \text{ and } E_1, \\ Z_{n,i+1} - Z_{n,1} & i = 1, 2, \dots, m - 1 \text{ and } E_2 \\ \tau + L - [Z_{n,1} - w(Z_{n,1}) - X_{mQ-r}] & i = m \text{ and } E_2, \\ = \tau + L + X_{mQ-r} - \min(Z_{n,1}, \tau) & \\ Z_{n,i+1} - Z_{n,1} & i = 1, \dots, m - 1 \text{ and } E_3 \\ \tau + L & i = m \text{ and } E_3. \end{cases} \quad (1)$$

**Remark** In event  $E_1$  above, we need  $Z_{(n+1),m} > L$ , since, otherwise,  $\tau + L - X_{r-(m-1)Q} < L$ , which will imply  $\tau < X_{r-(m-1)Q}$ . However, in this realization,  $X_Q = X_{r-(m-1)Q} + X_{mQ-r} < \min(Z_{n,1}, \tau) < \tau$  which implies  $X_{r-(m-1)Q} < \tau$  and this conflicts with the previous finding. Similarly, in event  $E_2$ , we conclude that  $Z_{n+1,m} > L$ , since otherwise  $\tau + L - \min(Z_{n,1}, \tau) + X_{mQ-r} < L$ , which implies that  $X_{mQ-r} < \min(Z_{n,1}, \tau) < \tau$ , which is not possible. Therefore,  $L < Z_{n,m} \leq \tau + L$  for  $n \geq 1$ . But, we have  $0 \leq Z_{n,1} \leq Z_{n,2} \leq \dots \leq Z_{n,m}$  for all  $n$ . The state space of the system is  $S = \{(x_1, x_2, \dots, x_m) : 0 \leq x_i \leq \tau + L, i = 1, \dots, m - 1; L \leq x_m \leq \tau + L\}$ . Let  $\mathcal{B}^m$  be the Borel  $\sigma$ -algebra generated by the subsets of  $S$ . Without loss of generality, we consider the sets  $A \in \mathcal{B}^m$  which are in the form  $A = (0, z_1] \times (0, z_2] \times \dots \times (L, z_m]$ , where  $z_i \leq \tau + L, i = 1, \dots, m - 1$  and  $L \leq z_m \leq \tau + L$ . Let  $\mathbf{x} = (x_1, x_2, \dots, x_m)$ ,  $\mathbf{Z}_n = (Z_{n,1}, Z_{n,2}, \dots, Z_{n,m})$ . Then, we have the following result.

**Theorem 1** (Transition probability function of  $\mathbf{Z}_n$ ).

For  $r \neq (m - 1)Q$ , let  $P(A|\mathbf{x}) \equiv P(Z_{n+1,i} \leq z_i, i = 1, \dots, m | \mathbf{Z}_n = \mathbf{x})$ . Then,

$$P(A|\mathbf{x}) = \begin{cases} H_{mQ-r}[\min(x_1, \tau) - (\tau + L - z_m)] \bar{H}_{r-(m-1)Q}(\tau + L - z_m) \\ - \int_{\tau+L-z_m}^{\min(x_1, \tau)} H_{mQ-r}(m_x - u) dH_{r-(m-1)Q}(u) \\ + I(z_m = \tau + L) \bar{H}_{mQ-r}(\min(x_1, \tau)) & \text{if } x_1 \geq m_1, \\ 0 & \text{if } x_1 < m_1, \end{cases} \quad (2)$$

where  $m_x = m_1 - (x_1 - \tau)^+$ ,  $m_1 = \max_{1 \leq i \leq m-1} \{x_{i+1} - z_i\}$  and  $I(\cdot)$  is the indicator function.

**Proof** See ‘‘Appendix’’. □

**Proposition 1** (Boundedness)

(i) Given  $\mathbf{Z}_n = \mathbf{x}$ , we have

$$E[Z_{n+1,i} | \mathbf{x}] = x_{i+1} - x_1 + \int_0^{\min(x_1, \tau)} dH_Q(u), \text{ for } i \leq m - 1, \quad (3)$$

$$E[Z_{n+1,m}|\mathbf{x}] = \tau + L - \int_0^{\min(x_1, \tau)} H_{mQ-r}(u) \bar{H}_{r-(m-1)Q}(\min(x_1, \tau) - u) du. \tag{4}$$

(ii) For  $0 < a < \tau$  and  $x_1 > \max(\tau + L - a, \tau)$ ,

$$\sum_{i=1}^m E[Z_{n+1,i}|Z_n = \mathbf{x}] \leq \sum_{i=1}^m x_i - \epsilon,$$

where

$$\epsilon = \int_0^a H_{mQ-r}(u) \bar{H}_{r-(m-1)Q}(\tau - u) du.$$

**Proof** See ‘‘Appendix’’. □

**Theorem 2** (Ergodicity) *The process  $\{Z_n, n \geq 1\}$  is ergodic.*

**Proof** Recall that if there are no closed classes in a Markov chain, except for the entire class of states, the chain is irreducible (Cinlar (2013), p. 127). Inspecting (2), we see that for any  $A \in \mathcal{B}^m$ , there exists some  $\mathbf{x} \in A$  such that  $P(A|x) = 0$ . In particular, for any  $\mathbf{x}$  that satisfies  $x_i \leq z_i, i = 1, \dots, m$  and  $x_1 < m_1, P(A|x) = 0$ . That is,  $p(Z_{n+1} \in A|Z_n = x \in A) = 0$  and  $Z_{n+1}$  gets out of the subset  $A$ . This implies that a set of the form  $A$  given above is not closed. Similarly, it is easy to check that  $p(A|x) = 1$  if  $z_i = \tau + L, i = 1, \dots, m$ , in which case,  $A$  is the entire state space. Hence, irreducibility follows. We now invoke Theorem 2.1 of Laslett et al. (1978) for a multi-dimensional Euclidean space and take  $g(\mathbf{x}) = \sum_{i=1}^m x_i$ . Then, Proposition 1 implies the boundedness of the mean hitting times of the process, which implies the ergodicity. Using Theorem 1, the transient and stationary distribution of the effective lifetime vector can be obtained by linking the  $n$ th and  $(n + 1)$ th cycles,

$$F_{n+1}(z_1, z_2, \dots, z_m) = \int \dots \int P(A|x_1, \dots, x_m) dF_n(x_1, \dots, x_m). \tag{5}$$

Specifically, we have the following result. □

**Corollary 1** *Define the  $m$ -dimensional arrays  $\mathbf{z} = (z_1, z_2, \dots, z_m), \underline{\mathbf{z}} = (0, z_1, z_2, \dots, z_{(m-1)}), \mathbf{u} = (t, \omega(t) + x + y, \dots, \omega(t) + x + y)$  and  $\mathbf{v} = (t, t, \dots, t)$ ; and let  $\partial/\partial z_i F_n(\mathbf{z})$  denote the partial derivative of  $F_n(\cdot)$  w.r.t.  $z_i$ .*

(a) For  $0 < z_1 \leq z_2 \leq \dots \leq z_m$  and  $L \leq z_m < \tau + L$ ,

$$\begin{aligned} F_{n+1}(\mathbf{z}) &= \int_{t=0}^{\tau+L} \int_{y=\tau+L-z_m}^{\min(t, \tau)} \int_{x=0}^{\min(t, \tau)-y} \frac{\partial}{\partial z_1} F_n(\mathbf{z} + \mathbf{u}) dH_{mQ-r}(x) dH_{r-(m-1)Q}(y) dt \\ &+ \int_{t=0}^{\tau+L} \int_{x=0}^{\min((t-\omega(t)), (z_m+\omega(t)-\tau-L))} \frac{\partial}{\partial z_1} F_n(\mathbf{z} + \mathbf{v}) \\ &\times \bar{H}_{r-(m-1)Q}(t - \omega(t) - x) dH_{mQ-r}(x) dt \\ &+ \int_{t=0}^{\tau+L} \frac{\partial}{\partial z_1} F_n(\mathbf{z} + \mathbf{v}) \bar{H}_{mQ-r}(t - \omega(t)) dt. \end{aligned}$$

(b) For  $0 < z_1 \leq z_2 \leq \dots \leq z_m = \tau + L$ ,

$$F_{n+1}(\mathbf{z}) = \int_{t=0}^{\tau+L} \frac{\partial}{\partial z_1} F_n(\mathbf{z} + \mathbf{v}) \bar{H}_{mQ-r}(t - \omega(t)) dt.$$

(c) As  $n \rightarrow \infty, F_{n+1}(\mathbf{z}) = F_n(\mathbf{z}) = F(\mathbf{z})$ , where  $F(\mathbf{z})$  denotes the stationary distribution of the effective lifetime process.

The last part of the above corollary follows from the ergodicity of the process which implies that the limiting distribution of the effective lifetime process exists.  $F(\mathbf{z})$  is found via the set of Volterra-type equations in the above corollary.

## 4 Operating characteristics and objective function

In this section, we obtain the expressions for the operating characteristics of the inventory system at hand and construct the objective function of the decision model.

### 4.1 Operating characteristics

We derive the operating characteristics of the system for a given value of the effective lifetime,  $\mathbf{Z} = \mathbf{z} = (z_1, \dots, z_m)$ . Hence, the entities derived may be viewed as conditional; later, we shall uncondition them over the steady state effective lifetime vector. We first write the (conditional) embedded cycle length,

$$[CL|\mathbf{z}] = \begin{cases} X_Q + w(z_1) & \text{if } X_Q < z_1 - w(z_1), \\ z_1 & \text{if } X_Q \geq z_1 - w(z_1). \end{cases} \quad (6)$$

Recall that  $w(z_i)$  is the time that will pass in an embedded cycle before the  $i$ th oldest batch joins stock (the stockout period), which may be zero. The first event above corresponds to depleting the batch of items through demand, and the second corresponds to not being able to sell the items in the batch prior to the expiry of their lifetimes. Carrying out the expectations, we have the following conditional expectations:

$$E[CL|\mathbf{z}] = z_1 - \int_{x=0}^{z_1 - w(z_1)} dH_Q(x). \quad (7)$$

After standard calculus, we arrive at the following expression:

$$E[CL|\mathbf{z}] = \int_0^{z_1 - w(z_1)} x dH_Q(x) - \int_{z_1 - w(z_1)}^{\infty} z_1 dH_Q(x) = z_1 - \int_0^{\min(z_1, \tau)} dH_Q(x). \quad (8)$$

Next, we consider the (conditional) total stock-years (i.e., area under the inventory curve in an embedded cycle),  $[OH|\mathbf{z}]$ . For convenience, we define the following events:

$$\begin{aligned} E_{1a} &: \{X_Q < X_{mQ-r} + L; X_Q < z_1 - w(z_1)\}, \\ E_{1b} &: \{X_{mQ-r} + L < X_Q < z_1 - w(z_1)\}, \\ E_{2a} &: \{z_1 - w(z_1) - L < X_{mQ-r} < z_1 - w(z_1); X_Q > z_1 - w(z_1)\}, \\ E_{2b} &: \{X_{mQ-r} < z_1 - w(z_1) - L; X_Q > z_1 - w(z_1)\}, \\ E_3 &: \{X_{mQ-r} > z_1 - w(z_1)\}. \end{aligned}$$

Event  $E_{1a}$  corresponds to the case where the oldest batch is depleted by demand without perishing, an order is given during the embedded cycle but this order does not arrive before the embedded cycle terminates. Event  $E_{1b}$  corresponds to the case where the oldest batch is depleted with demand and the order given during the embedded cycle arrives before the embedded cycle is completed. Event  $E_{2a}$  describes the case where the oldest batch perishes after reordering and the last ordered batch does not join the stock before the embedded cycle ends; event  $E_{2b}$  corresponds to the case where the oldest batch perishes after reordering and

the ordered batch arrives before perishing; and, finally, event  $E_3$  corresponds to the case where the oldest batch perishes before the reorder point is reached. Then,

$$[OH|\mathbf{z}] = \begin{cases} \left\{ \sum_{i=1}^Q X_i \right\} + \left\{ \sum_{i=2}^m Q[X_Q - (w(z_i) - w(z_1))]^+ \right\} + \{0\} & \text{if } E_{1a}, \\ \left\{ \sum_{i=1}^{N(z_1-w(z_1))} X_i + (z_1 - w(z_1))[Q - N(z_1 - w(z_1))] \right\} \\ + \left\{ \sum_{i=2}^m Q[z_1 - w(z_i)]^+ \right\} + \{0\} & \text{if } E_{2a}, \\ \left\{ \sum_{i=1}^Q X_i \right\} + \left\{ \sum_{i=2}^m Q[X_Q - (w(z_i) - w(z_1))]^+ \right\} \\ + \left\{ Q[X_Q - X_{mQ-r} - L] \right\} & \text{if } E_{1b}, \\ \left\{ \sum_{i=1}^{N(z_1-w(z_1))} X_i + (z_1 - w(z_1))[Q - N(z_1 - w(z_1))] \right\} \\ + \left\{ \sum_{i=2}^m Q[z_1 - w(z_i)]^+ \right\} + \left\{ Q[z_1 - w(z_1) - X_{mQ-r} - L] \right\} & \text{if } E_{2b}, \\ \left\{ \sum_{i=1}^{N(z_1-w(z_1))} X_i + (z_1 - w(z_1))[Q - N(z_1 - w(z_1))] \right\} \\ + \left\{ \sum_{i=2}^m Q[z_1 - w(z_i)]^+ \right\} + \{0\} & \text{if } E_3. \end{cases}$$

The expression for  $[OH|\mathbf{z}]$  for each event consists of three components (within curly brackets): The stock-years computed for the items in the batch that is currently being used, for those in the previously placed  $m - 1$  orders which are already in stock or may join stock during the current cycle, and finally, for the most recently placed order. For example, for Event  $E_{1a}$ , the oldest batch will be carried in stock within an embedded cycle for the time until  $Q$  units are sold (one by one); the remaining batches ( $i = 2$  through  $m$ ) of size  $Q$  each will be carried in inventory for the time segment during which they are on hand; the most recently placed will result in zero positive stock since the embedded cycle ends before the lead time does. The stock-years are computed similarly for the other events.

The stock dynamics under the model herein is such that the stock-years computed for the oldest batch and the most recently placed order are identical to those obtained for the model with  $m = 1$  and the same effective lifetimes for the oldest batch in Berk and Gürlér (2008). Then, the expected stock-years expression for multiple outstanding orders can be written as follows.

$$E[OH|\mathbf{z}] = E[\tilde{O}H|z_1 - w(z_1)] + Q \sum_{i=2}^m \int_{w(z_i)-w(z_1)}^{z_1-w(z_1)} [u + w(z_1) - w(z_i)] dH_Q(u) + Q \sum_{i=2}^m [z_1 - w(z_i)]^+ \bar{H}_Q(z_1 - w(z_1)), \tag{9}$$

where  $E[\tilde{O}H|x]$  is the corresponding expression in Berk and Gürlér (2008) for  $m = 1$  (modified to allow for  $m > 1$ ) with the oldest batch having an effective lifetime of  $z_1 = x$ ,

$$E[\tilde{O}H|x] = Q[\eta(Q, r, x) + x\bar{H}_Q(x) - \frac{mQ-r}{\lambda}H_{mQ-r+1}(x-L) + \frac{(Q+1)}{2\lambda}H_{Q+1}(x) + \gamma(Q, r, x)] - \frac{\lambda x^2}{2}\bar{H}_{Q-1}(x), \tag{10}$$

and

$$\begin{aligned} \gamma(Q, r, x) &= H_{mQ-r}(x-L)[x - L\bar{H}_{r-(m-1)Q}(L) - \frac{r}{\lambda}H_{r-(m-1)Q+1}(L)], \\ \eta(Q, r, x) &= \int_0^{x-L} \left[ \frac{r - (m-1)Q}{\lambda} H_{r-(m-1)Q+1}(x-u) - (x-u)H_{r-(m-1)Q}(x-u) \right] dH_{mQ-r}(u). \end{aligned} \tag{11}$$

The conditional number of perishing units in an embedded cycle is  $[P|z] = Q - N(z_1 - w(z_1))$  if  $X_Q > z_1 - w(z_1)$ , and zero, otherwise. Then, the corresponding conditional expectation is given by

$$E[P|z] = \sum_{i=0}^{Q-1} (Q-i)P\{N(z_1 - w(z_1)) = i\} = \sum_{i=0}^{Q-1} (Q-i) \frac{e^{-\lambda(z_1 - w(z_1))} [\lambda(z_1 - w(z_1))]^i}{i!}. \quad (12)$$

Stockout occurs within a cycle if  $z_1 > \tau$ . Then, the conditional expectation of the number of lost sales is

$$E[LS|z] = \lambda w(z_1). \quad (13)$$

## 4.2 Objective function

The objective for our problem is the minimization of the expected cost rate,  $TC(Q, r)$  which is a function of  $Q$  and  $r$ . The approach taken to construct the expected cost rate is similar to that of Berk and Gürlér (2008) and builds on the results of Ross (1970) and Tijms (1994), with some modifications needed due to the multi-dimensional nature of the effective lifetime process.

We show that, for the inventory system at hand, the expected average cost based on regenerative cycles can be written via the embedded cycles defined above. We begin with stating the equivalence result of Ross (1970) for a general semi-Markov decision process (SMDP). In our model, the stochastic behavior of the inventory system at any time  $t$  is characterized by the multidimensional process  $W(t) = \{\mathbf{Z}(t), IP(t), t \geq 0\}$ , where  $\mathbf{Z}(\mathbf{t}) = (Z_1(t), Z_2(t), \dots, Z_m(t))$  is the effective lifetime process corresponding to the number of outstanding orders with state space  $S = \{(s_1, s_2, \dots, s_m) : 0 \leq s_i \leq \tau + L, i = 1, \dots, m-1; L \leq s_m \leq \tau + L\}$  and  $IP(t)$  is the inventory position at time  $t$ . Recall that  $\mathbf{Z}_n$  is the remaining lifetime vector of outstanding orders when the inventory position ( $IP$ ) hits  $mQ$  for the  $n$ th time. Without loss of generality, let the initial state of the system be  $\mathbf{w} = \mathbf{w}(0) = (\mathbf{z}, Q)$ ,  $\mathbf{z} \in S$ ,  $T = \inf\{t > 0 : \mathbf{W}(t) = \mathbf{w}, \mathbf{W}(t^-) \neq \mathbf{w}\}$  and  $N = \min\{n > 0 : \mathbf{Z}_{n+1} = \mathbf{z}\}$ . Hence,  $T$  is the first return time to state  $w$ , and, since we fix the initial inventory position at  $Q$ ,  $N$  determines the number of transitions it takes to return to the initial state. Considering the inventory system in continuous time, let  $C(t)$  denote the cost incurred over the interval  $(0, t]$ . Also, let  $C_i \equiv C_i(\mathbf{Z}_i, \mathbf{X})$  and  $L_i \equiv L_i(\mathbf{Z}_i, \mathbf{X})$  be the cost and the length of the  $i$ th embedded cycle ( $i = 1, 2, \dots$ ), where  $\mathbf{X}$  denotes the array of inter-arrival times of Poisson demands within the  $i$ th embedded cycle, independent of  $(\mathbf{Z}_1, \dots, \mathbf{Z}_i)$ . Following Ross, we define the two average cost criteria as follows:

$$\phi_1(\mathbf{w}) = \overline{\lim}_{t \rightarrow \infty} E[C(t)/t | \mathbf{W}(0) = \mathbf{w}], \quad (14)$$

$$\phi_2(\mathbf{z}) = \overline{\lim}_{n \rightarrow \infty} \frac{E\left[\sum_{i=1}^n C_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{z}\right]}{E\left[\sum_{i=1}^n L_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{z}\right]}, \quad (15)$$

where  $\phi_1(\mathbf{w})$  is the limiting value of the expected cost per time as time tends to infinity given that the initial system state was  $\mathbf{w}$ , and  $\phi_2(\mathbf{z})$  is the limiting value of the expected cost per time across a number of completed embedded cycles as that number of embedded cycles tends to infinity. The two expected cost constructs are equivalent if the stochastic system at hand satisfies the condition that transitions of  $\mathbf{Z}_n$  (embedded cycle completions) do not take place 'too quickly'. In the case of  $m = 1$ , this is established easily by the fact that each embedded cycle contains one full lead time period of finite length. However, for  $m > 1$ , this condition does not hold; instead, we need the following result.

**Lemma 1** *If  $E[T|\mathbf{w}]$  is finite, then  $E[N|\mathbf{w}] < \infty$  and  $T = \sum_{n=1}^N L_n$ .*

**Proof** Let  $N(\equiv N|\mathbf{w})$  be the number of *embedded* cycles within a *regenerative* cycle, i.e., the number of times the inventory position hits  $mQ$  between two consecutive visits to state  $\mathbf{w}$ . These  $N$  embedded cycles can be partitioned into consecutive *blocks* each consisting of  $(m + 1)$  consecutive embedded cycles. The first block will be composed of the embedded cycles 1 through  $m + 1$ , the second block will be composed of the embedded cycles  $m + 2$  through  $2(m + 1)$ , and so on. If  $N$  is not an integer multiple of  $(m + 1)$ , there will be one incomplete block (at the end) consisting of less than  $m + 1$  embedded cycles; otherwise, there will be no such incomplete cycle. For  $j \geq 1$ , let  $\Gamma_j = \sum_{i=(j-1)(m+1)+1}^{j(m+1)} L_i$ . Then,  $\Gamma_1$  is the length of the first block,  $\Gamma_2$  is the length of the second block, and so on. We can write, in general,  $T = \sum_{j=1}^K \Gamma_j + R_e$  where  $R_e$ , if non-zero, corresponds to the last segment in  $(0, T]$  which is an incomplete block, and  $K$  is the (random) number of complete blocks within a regenerative cycle, starting at initial state  $\mathbf{w}$ . Then,  $\sum_{j=1}^K \Gamma_j < T$  and  $E[\sum_{j=1}^K \Gamma_j] < E[T]$ . Next, we show that the length of any block cannot be smaller than the lead time;  $\Gamma_j \geq L$  for  $j \geq 1$ . Consider the  $k$ th block; the first embedded cycle of this block is the  $(k(m + 1) + 1)$ th embedded cycle starting at time  $T_{k(m+1)+1}$ . Suppose that  $t$  is the time of the reordering decision during this embedded cycle, which ends at  $T_{k(m+1)+2}$ . Clearly,  $T_{k(m+1)+1} < t \leq T_{k(m+1)+2}$ . By  $T_{k(m+1)+2}$ , exactly  $Q$  units of stock have been depleted since the beginning of the embedded cycle  $k(m + 1) + 1$ . Similarly, by the end of the last embedded cycle in the  $k$ th block,  $T_{k(m+1)+m+1}$ , exactly  $mQ$  units of stock will have been depleted either by demand or perishing. The batch ordered at  $t$  arrives at  $t + L$ . If  $T_{k(m+1)+m+1} \geq t + L$ , then there is positive stock in the system at the beginning of the next  $k + 1$ th block. This implies  $\Gamma_k = [T_{(k+1)(m+1)+1} - T_{k(m+1)+1}] \geq L$ . Otherwise, the system experiences a stockout only after which sales can occur; then, by definition of an embedded cycle, we must have  $T_{(k+1)(m+1)+1} > t + L$ . This also implies  $\Gamma_k = [T_{(k+1)(m+1)+1} - T_{k(m+1)+1}] \geq L$ . The lower bound on the duration of a block, in turn, implies  $E[K]L < E[\sum_{j=1}^K \Gamma_j] < E[T] < \infty$ . Furthermore,  $N < (m + 1)(K + 1)$  gives  $E[N] < (m + 1)(E[K] + 1) \leq (m + 1)(E[T]/L + 1)$ , and it follows from the finiteness of  $E[T]$  that  $E[N] < \infty$ . Hence, as in Berk and Gürlér (2008) for  $m = 1$ , Theorem 1 of Ross (1970) establishes the equivalence of  $\phi_1(\mathbf{w}) = \phi_2(\mathbf{z})$  for  $m > 1$ . □

Let  $\mathcal{C}(z) = E[C_i(Z_i, \mathbf{X})|Z_i = z]$  and  $\mathcal{L}(z) = E[L_i(Z_i, \mathbf{X})|Z_i = z]$  for  $i \geq 1$ . The expectations are independent of the index  $i$  when  $Z_i = z$  is given and are calculated with respect to the interarrival times of Poisson demands  $\mathbf{X}$ , as provided above. Generalizing the results of Tijms (1994) to continuous state spaces, we have the following.

**Theorem 3** *Let  $F(\cdot)$  be the limiting distribution function of  $\{Z_i, i \geq 1\}$ . Then,*

$$\phi_2(\mathbf{z}) = \frac{\int_{\mathbf{z}} \mathcal{C}(\mathbf{z}) dF(\mathbf{z})}{\int_{\mathbf{z}} \mathcal{L}(\mathbf{z}) dF(\mathbf{z})}. \tag{16}$$

We can now construct the expected cost rate,  $TC(Q, r)$ , as follows:

$$TC(Q, r) = \frac{K + \int_{\mathbf{z}} (hE[OH|\mathbf{Z} = \mathbf{z}] + pE[P|\mathbf{Z} = \mathbf{z}] + \pi E[LS|\mathbf{Z} = \mathbf{z}]) dF(\mathbf{z})}{\int_{\mathbf{z}} E[CL|\mathbf{Z} = \mathbf{z}] dF(\mathbf{z})}. \tag{17}$$

## 5 Special case: reorder point is an integer multiple of order quantity

Suppose  $r = (m - 1)Q$ ,  $m \geq 1$ . In this case, the order placement instances and the embedded cycle beginning/ending points coincide, unlike the cases encountered when  $r \neq (m - 1)Q$ . One of the implications of this is that, the event  $E_2 : \{X_{mQ-r} < Z_{n,1} - w(Z_{n,1}) < X_Q\}$  (described in Sect. 3) corresponding to the realization where some of the items in the oldest batch of the embedded cycle  $n$  perish after reordering, is never realized. This results in some simplifications in the expressions. Below, we reproduce some of the main results for the special case  $r = (m - 1)Q$ .

For notational convenience, the state space of the system is described slightly differently as  $SS = \{(x_1, x_2, \dots, x_m) : 0 \leq x_i \leq \tau + L, i = 1, \dots, m - 1; x_m = \tau + L\}$ . As before, let  $\mathcal{B}^m$  be the Borel  $\sigma$ -algebra generated by the subsets of  $SS$ . Without loss of generality, we consider the sets  $A \in \mathcal{B}^m$  which are in the form  $A = (0, z_1] \times (0, z_2] \times \dots \times (0, z_{m-1}] \times [\tau + L]$ , where  $z_i \leq \tau + L, i = 1, \dots, m - 1$ . Let  $\mathbf{x} = (x_1, x_2, \dots, x_{m-1}, \tau + L)$ ,  $\mathbf{Z}_n = (Z_{n,1}, Z_{n,2}, \dots, Z_{n,m-1}, \tau + L)$ . Then, we have the following result analogous to Theorem 1.

**Theorem 15** *If  $r = (m - 1)Q$ , then the transition probability matrix  $P(A|\mathbf{x}) \equiv P(Z_{n+1,i} \leq z_i, i = 1, \dots, m - 1, Z_{n+1,m} = \tau + L | \mathbf{Z}_n = \mathbf{x})$  is given by*

$$P(A|\mathbf{x}) = \begin{cases} \tilde{H}_Q([m_1 - (x_1 - \tau)^+]^+) & \text{if } x_1 \geq m_1, \\ 0 & \text{if } x_1 < m_1. \end{cases} \quad (18)$$

**Proof** See ‘‘Appendix’’. □

Also, by directly applying the characteristics of the special case to Proposition 1, we have the following result.

**Proposition 15** (i) *Given  $\mathbf{Z}_n = \mathbf{x}$ , we have*

$$\begin{aligned} E[Z_{n+1,i}|\mathbf{x}] &= x_{i+1} - x_1 + \int_0^{\min(x_1, \tau)} dH_Q(u), \quad i \leq m - 1, \\ E[Z_{n+1,m}|\mathbf{x}] &= \tau + L. \end{aligned} \quad (19)$$

(ii) *For  $0 < a < \tau$  and  $x_1 > \max(\tau + L - a, \tau)$ ,*

$$\sum_{i=1}^m E[Z_{n+1,i} | \mathbf{Z}_n = \mathbf{x}] \leq \sum_{i=1}^m x_i - \epsilon, \text{ where } \epsilon = \int_0^a dH_{mQ-r}(u). \quad (20)$$

The ergodicity of the process for the special case also follows as stated below.

**Theorem 25** *The process  $\{\mathbf{Z}_n, n \geq 1\}$  is ergodic.*

**Proof** Similar to Theorem 2, omitted. □

For  $r = (m - 1)Q$ , the expressions for expected cycle length, expected number of units that perish, and expected lost sales do not change but there is a slight change in the expressions for expected on hand. Note that the events  $E_{1b}$ ,  $E_{2a}$ ,  $E_{2b}$  (as discussed in Sect. 4.1 in relation to the on-hand expression) are not realized for  $r = (m - 1)Q$ , since  $X_{mQ-r} = X_Q$ . Hence, only the remaining events  $E_{1a} : \{X_Q < z_1 - w(z_1)\}$  and  $E_3 : \{X_{mQ-r} > z_1 - w(z_1)\}$  are to be considered. The expected on hand expression given by (9) is still valid and the expressions given in (10) and (11) change slightly as follows. For  $r = (m - 1)Q$ , we have the two cases below: (i) If  $m = 1$ , then  $\gamma(Q, r, x) = (x - L)H_Q(x - L)$ ,  $\eta(Q, r, x) = 0$  and

$$E[\tilde{O}H|x] = Q[x\bar{H}_Q(x) - \frac{Q}{\lambda}H_{Q+1}(x-L) + \frac{(Q+1)}{2\lambda}H_{Q+1}(x) + (x-L)H_Q(x-L)] - \frac{\lambda x^2}{2}\bar{H}_{Q-1}(x). \tag{21}$$

(i) If  $m \geq 2$ , then  $\gamma(Q, r, x) = \eta(Q, r, x) = 0$  and

$$E[\tilde{O}H|x] = Q[x\bar{H}_Q(x) + \frac{(Q+1)}{2\lambda}H_{Q+1}(x)] - \frac{\lambda x^2}{2}\bar{H}_{Q-1}(x). \tag{22}$$

Note that  $m = 1$  corresponds to the case  $r = 0$ , no safety stock is held in inventory and orders are placed only when the on-hand inventory drops to zero. In this case, the remaining lifetime of the batch is a degenerate random variable which always takes the value  $\tau + L$ . When  $m = 2$ , the remaining lifetime of the younger batch,  $Z_2$  is always  $\tau + L$  and hence the lifetime vector reduces effectively to a single dimension. In general, for any  $m$ , the effective dimension of the remaining lifetime vector is  $m - 1$ . For the cases of  $m = 2$  and  $m = 3$ , respectively, we provide below the distribution functions of the effective lifetime distributions: (i) For  $m = 2$ , the univariate lifetime distribution is

$$F_{n+1, Z_1}(z_1) = \int_{x=\tau+L-z_1}^{\tau+L} \bar{H}_Q(\tau+L-z_1-(x-\tau)^+)dF_{n, Z_1}(x). \tag{23}$$

(ii) For  $m = 3$ , the bivariate lifetime distribution is

$$\begin{aligned} &F_{n+1, Z_1, Z_2}(z_1, z_2) \\ &= \int_{x_2 > z_1}^{\tau+L} \int_{x_1 > \max(x_2 - z_1, \tau + L - z_2)}^{\tau+L} \bar{H}_Q(m_1 - (x_1 - \tau)^+)dF_{n, Z_1, Z_2}(x_1, x_2). \\ &F_{n+1, Z_1, Z_2}(z_1, z_2) \\ &= \int_{x_2 > \tau + L - z_1 - z_2}^{\tau+L} \int_{x_1 > x_2 - z_1}^{\min(x_2, \tau + L)} \bar{H}_Q(m_1 - (x_1 - \tau)^+)dF_{n, Z_1, Z_2}(x_1, x_2) \\ &+ \int_{x_2 > \tau + L - z_2}^{\tau + L + z_1 + z_2} \int_{x_1 > \tau + L - z_2}^{\min(x_2, \tau + L)} \bar{H}_Q(m_1 - (x_1 - \tau)^+)dF_{n, Z_1, Z_2}(x_1, x_2). \end{aligned} \tag{24}$$

Note that the arguments leading to the proof of Lemma 1 and Theorem 3 also follow similarly except that the modified definition of the state space  $S$  above is used. Hence, these results are also valid for the special case of  $r = (m - 1)Q$  and the expected cost rate expression given in (17) is valid with the modified expected on-hand related expressions given in (21) and (22).

### 6 Illustrative examples

In this section, we provide some examples to illustrate the sensitivity of the optimal values of the policy parameters w.r.t. operating environment parameters and, the restrictiveness of the  $r < Q$  assumption and benefits ensuing from alleviating this restriction. For the numerical examples, we fix  $L = 1, h = 1, \lambda = 10$  and  $\pi = 40$  but vary the lifetime ( $\tau = 2, 2.5, \text{ and } 3$ ), the fixed ordering cost ( $K = 5, 10, 50, \text{ and } 100$ ) and the unit perishing cost ( $p = 10 \text{ and } 50$ ). We retain the notation already introduced and use  $FR$  to denote the expected fraction of stock-out time within an embedded cycle defined as  $FR = \int_{\mathbf{z}} w(z_1)dF(\mathbf{z}) / \int_{\mathbf{z}} E[CL|\mathbf{Z} = \mathbf{z}]dF(\mathbf{z})$ . The expressions given in Theorems 1 and 1S for the steady state distribution function of shelf

lives do not have closed forms. Therefore, we resorted to numerical methods to obtain the effective lifetime distributions and used a discretization of size  $\Delta = (\tau + L)/k$ . Note that the domain for  $Z_m$  is  $[L, \tau + L]$  whereas the domains for  $Z_i$ 's are  $[0, \tau + L]$  for  $i = 1, \dots, m - 1$ . To ensure consistent discretization for  $Z_m$  and the other  $Z_i$ 's, we chose  $k$  around 100 when the dimension of the shelf life distribution is two or less and, around 40 for when it is 3 in our numerical study. For optimization, we used exhaustive search over a broad range of policy parameter values, and observed unimodality in all of the cases considered.

We present our results in Tables 1 and 2. The optimal lotsize-reorder point pairs, the corresponding expected total cost rate and stockout fractions are shown, respectively, by  $(Q^*, r^*)$ ,  $E(TC^*)$  and  $FR^*$ ; the corresponding maximum number of outstanding orders is  $m^*$ . Their counterparts under the  $r < Q$  restriction ( $m = 1$ ) are shown by  $(Q, r)_1$ ,  $E(TC)_1$ , and  $FR_1$ . The percentage deviation of the  $E(TC)_1$  from the optimal is denoted by GAP%.

The sensitivity of the optimal values of policy parameters  $(Q^*, r^*)$  w.r.t. the parameters of the operating environment is as follows. Both  $Q^*$  and  $r^*$  are non-decreasing in shelf life  $\tau$ , but they are non-increasing in unit perishing cost  $p$ . But the maximum number of outstanding orders,  $m$  is non-increasing in  $\tau$  whereas it is also non-decreasing in  $p$ , which implies that for systems with higher unit perishing costs such as pharmaceuticals the  $r < Q$  restriction is more costly. As the fixed ordering cost  $K$  increases,  $Q^*$  is non-decreasing and  $r^*$  is non-increasing such that  $m$  is overall non-increasing.

**Table 1** Comparison of the optimal and one-outstanding-order-restricted lotsize-reorder policies;  $p = 10$

Parameters	$K$	$(Q, r)_1$	$E(TC)_1$	$FR_1$	$(Q^*, r^*)$	$E(TC^*)$	$FR^*$	$m^*$	GAP%
$\tau = 2$	5	(15, 14)	28.53	98.55	(8, 16)	21.72	99.40	3	31.35
	10	(15, 14)	32.07	98.55	(9, 16)	27.91	99.37	2	14.91
	50	(15, 14)	60.58	98.52	(15, 14)	60.58	98.52	1	0.00
	100	(17, 13)	94.63	97.01	(17, 13)	94.63	97.01	1	0.00
$\tau = 2.5$	5	(15, 14)	22.50	98.73	(9, 16)	19.37	99.45	2	16.16
	10	(15, 14)	25.86	98.73	(11, 16)	24.42	99.48	2	5.90
	50	(17, 14)	51.46	98.76	(17, 14)	51.46	98.76	1	0.00
	100	(19, 14)	79.54	98.62	(19, 14)	79.54	98.62	1	0.00
$\tau = 3$	5	(16, 15)	20.10	99.35	(10, 16)	18.63	99.50	2	7.89
	10	(16, 15)	23.23	99.35	(13, 16)	22.84	99.58	2	1.71
	50	(20, 14)	45.95	98.95	(20, 14)	45.96	98.95	1	0.00
	100	(23, 14)	69.48	98.77	(23, 14)	69.48	98.77	1	0.00

**Table 2** Comparison of the optimal and one-outstanding-order-restricted lotsize-reorder policies;  $p = 50$

Parameters	$K$	$(Q, r)_1$	$E(TC)_1$	$FR_1$	$(Q^*, r^*)$	$E(TC^*)$	$FR^*$	$m^*$	GAP%
$\tau = 2$	5	(13, 12)	42.75	95.90	(7, 15)	26.49	98.81	3	61.38
	10	(13, 12)	46.55	95.90	(8, 14)	34.57	98.06	2	34.65
$\tau = 2.5$	5	(14, 13)	28.69	97.71	(7, 16)	21.16	99.35	3	35.59
	10	(14, 13)	32.22	97.71	(9, 15)	27.09	99.00	2	18.94
$\tau = 3$	5	(15, 14)	22.38	98.76	(9, 16)	19.13	99.45	2	16.99
	10	(15, 14)	25.68	98.76	(11, 16)	24.11	99.52	2	6.51

We observe that the savings can be significant when the restriction is alleviated. The deviations are increasing as the optimal number of outstanding orders ( $m^*$ ) increases, as the fixed ordering cost decreases, as the unit perishing cost increases, and as shelf life decreases. Although not reported herein, similar effects are observed as unit lost sales cost increases.

### 7 Concluding remarks

In this paper, we provide the generalization of the  $(Q, r)$  model for perishables with constant lifetimes and lead times for multiple outstanding orders. Expressions are given for the transient and steady state distribution of remaining lifetimes of all batches at hand; and operating characteristics are derived using an embedded Markov chain approach. From our numerical study, we observed that the contribution of the generalization to allow more than one outstanding orders becomes more pronounced when ordering costs and lifetimes are small, perishing and lost sales costs are high. Even in systems holding items with the long lifetime, small fixed ordering cost enforces the system to have multiple outstanding orders. Among all parameters, lifetime and fixed ordering cost are the most important ones playing very important rule in sensitivity of other parameters and the number of outstanding orders. The batch size is affected directly by unit perishing cost and reorder point by unit lost sales cost. The number of orders outstanding is very sensitive to any increase in lead time.

### Appendix

**Proof of Theorem 1** Suppose  $Z_{n,i} = x_i, i = 1, \dots, m$  be given and, for brevity of notation let  $k = r - (m - 1)Q$  so that  $Q - k = mQ - r$ . Then, referring to the expressions in (1), we have, for  $z_m < \tau + L$ ,

$$\begin{aligned}
 P(A|\mathbf{x}) &= P(x_{i+1} - (x_1 - \tau)^+ - X_Q \leq z_i, i = 1, \dots, m - 1; \\
 &\tau + L - X_k \leq z_m; X_Q \leq \min(x_1, \tau)) \\
 &\quad + P(x_1 \geq m_1; \tau + L + X_{Q-k} - \min(x_1, \tau) \leq z_m; X_{Q-k} \leq \min(x_1, \tau) \leq X_Q) \\
 &= P(m_x - X_k < X_{Q-k} \leq \min(x_1, \tau) - X_k; \tau + L - z_m \leq X_k < \min(x_1, \tau)) \\
 &\quad + P(m_1 < x_1; \min(x_1, \tau) - X_k \leq X_{Q-k} \leq \min(x_1, \tau) \\
 &\quad - (\tau + L - z_m); \tau + L - z_m \leq X_k) \\
 &= \int_{\tau+L-z_m}^{\min(x_1, \tau)} [H_{Q-k}(\min(x_1, \tau) - u) - H_{Q-k}(m_x - u)]dH_k(u) \\
 &\quad + \int_{\tau+L-z_m}^{\infty} [H_{Q-k}(\min(x_1, \tau) - (\tau + L - z_m)) - H_{Q-k}(\min(x_1, \tau) - u)]dH_k(u)
 \end{aligned}$$

$$\begin{aligned}
&= \int_{\tau+L-z_m}^{\min(x_1, \tau)} [H_{Q-k}(\min(x_1, \tau) - (\tau + L - z_m) - H_{Q-k}(m_x - u))] dH_k(u) \\
&\quad + H_{Q-k}(\min(x_1, \tau) - (\tau + L - z_m)) \bar{H}_k(\min(x_1, \tau)) \\
&= H_{Q-k}(\min(x_1, \tau) - (\tau + L - z_m)) \bar{H}_k(\tau + L - z_m) \\
&\quad - \int_{\tau+L-z_m}^{\min(x_1, \tau)} H_{Q-k}(m_x - u) dH_k(u)
\end{aligned}$$

and, for  $z_m = \tau + L$ ,

$$\begin{aligned}
P(A|\mathbf{x}) &= P(x_{i+1} - x_1 \leq z_i, i = 1, \dots, m-1; X_{Q-k} \geq \min(x_1, \tau)) \\
&= P(x_1 \geq m_1, X_{Q-k} \geq \min(x_1, \tau)).
\end{aligned}$$

Combining the cases, we obtain the result.

**Proof of Theorem 15** Suppose  $r = (m-1)Q$ . Then,  $Z_{n,m} = \tau + L$ . Let  $Z_{n,i} = z_i$ ,  $i = 1, \dots, m-1$ , and  $z_{n,m} = \tau + L$ . First, assume  $z_1 > \tau$ ;  $X_Q \leq \min(z_1, \tau)$ . According to the discussion in Sect. 5, only the realizations  $E_1$  and  $E_3$  of Sect. 3 are valid for this special case. Hence, for given  $Z_{n,i} = x_i$ ,  $i = 1, \dots, m$  we have:

If  $X_Q < \min(z_1, \tau)$ ,

$$\begin{aligned}
z_{n+1,i} &= x_{i+1} - (x_1 - \tau)^+ - X_Q; \quad i = 1, \dots, m-2, \\
z_{n+1,m-1} &= \tau + L - (x_1 - \tau)^+ - X_Q, \\
z_{n+1,m} &= \tau + L,
\end{aligned}$$

and if  $X_Q > \min(z_1, \tau)$ ,

$$\begin{aligned}
z_{n+1,i} &= x_{i+1} - x_1; \quad i = 1, \dots, m-2, \\
z_{n+1,m-1} &= \tau + L - x_1, \\
z_{n+1,m} &= \tau + L.
\end{aligned}$$

Then, we have

$$\begin{aligned}
P(A|\mathbf{x}) &= P(Z_{n+1,i} \leq z_i, i = 1, \dots, m | Z_{n,i} = x_i, i = 1, \dots, m) \\
&= P(x_{i+1} - (x_1 - \tau)^+ - X_Q \leq z_i, i = 1, \dots, m-2; \tau + L - \min(x_1, \tau) - X_Q \leq z_{m-1}; \\
&\quad X_Q \leq \min(x_1, \tau)) \\
&\quad + P(x_{i+1} - z_i \leq x_1, i = 1, \dots, m-2; \tau + L - z_{m-1} \leq x_1, \min(x_1, \tau) < X_Q) \\
&= P(x_{i+1} - z_i - (x_1 - \tau)^+ < X_Q \leq \min(x_1, \tau); i = 1, \dots, m-1) \\
&\quad + I\left(x_1 \geq \max_{i=1, \dots, m-1} \{x_{i+1} - z_i\}\right) \bar{H}_Q(\min(x_1, \tau)).
\end{aligned}$$

Recalling  $m_1 = \max_{i=1, \dots, m-1} \{x_{i+1} - z_i\}$  the above expression is written as

$$P(A|\mathbf{x}) = P(m_1 - (x_1 - \tau)^+ < X_Q \leq \min(x_1, \tau) + I(x_1 \geq m_1) \bar{H}_Q(\min(x_1, \tau)).$$

Observe that  $P(A|\mathbf{x}) = 1 - H_Q(m_1)$  if  $x_1 < \tau$ ,  $x_1 \geq m_1$ , and zero otherwise. Also, if  $x_1 \geq \tau$ ,  $x_1 \geq m_1$ ,  $P(A|\mathbf{x}) = 1 - H_Q(m_1 - x_1 + \tau)$ . This yields the following compact expression  $P(A|\mathbf{x}) = \bar{H}_Q(m_1 - (x_1, \tau)^+) I(x_1 \geq m_1)$ .

**Proof of Proposition 1** Note that  $x_1 - (x_1 - \tau)^+ = \min(x_1, \tau)$ . Then, rewriting the expressions in (1), we have, for  $i = 1, \dots, m-1$

$$Z_{n+1,i} = x_{i+1} - x_1 + (\min(x_1, \tau) - X_Q)^+, \quad (25)$$

and

$$Z_{n+1,m} = \tau + L - X_k I(X_Q < \min(x_1, \tau)) + (X_{Q-k} - \min(x_1, \tau)) I(X_{Q-k} < \min(x_1, \tau) < X_Q). \tag{26}$$

(i) (19) follows by taking the expectation with respect to  $X_Q$ . Also, referring to (26), we have

$$E[X_k I(X_Q < \min(x_1, \tau))] = \int_0^{\min(x_1, \tau)} u H_{Q-k}(\min(x_1, \tau) - u) dH_k(u),$$

$$P(X_{Q-k} < \min(x_1, \tau) < X_Q) = H_{Q-k}(\min(x_1, \tau)) - H_k(\min(x_1, \tau)),$$

and

$$E[X_{Q-k} I(X_{Q-k} < \min(x_1, \tau) < X_Q)]$$

$$= \int_0^{\min(x_1, \tau)} \int_{\min(x_1, \tau) - y}^{\min(x_1, \tau)} y dH_k(u) dH_{Q-k}(y) + \int_0^{\min(x_1, \tau)} \int_{\min(x_1, \tau)}^{\infty} y dH_k(u) dH_{Q-k}(y)$$

$$= \int_0^{\min(x_1, \tau)} u \bar{H}_k(\min(x_1, \tau) - u) dH_{Q-k}(u)$$

$$= \min(x_1, \tau) \bar{H}_{Q-k}(\min(x_1, \tau)) - \int_0^{\min(x_1, \tau)} H_{Q-k}(u) \bar{H}_k(\min(x_1, \tau) - u) du$$

$$- \int_0^{\min(x_1, \tau)} (\min(x_1, \tau) - u) H_{Q-k}(u) dH_k(u),$$

where the last equality follows from integration by parts. Now, (4) follows from taking the expectation of (26).

(ii) According to the part (i) above,

$$\sum_{i=1}^m E[Z_{n+1,i} | Z_n = \mathbf{x}] \leq \sum_{i=1}^{m-1} [x_{i+1} - (x_1 - \tau)^+] + \tau + L$$

$$- \int_0^{\min(x_1, \tau)} H_{Q-k}(u) \bar{H}_k(\min(x_1, \tau) - u) du$$

$$= \sum_{i=2}^m x_i - (m - 1)(x_1 - \tau)^+ + \tau + L - \int_0^{\min(x_1, \tau)} H_{Q-k}(u) \bar{H}_k(\min(x_1, \tau) - u) du$$

$$= \sum_{i=1}^m x_i - [(m - 1)(x_1 - \tau)^+ + x_1$$

$$+ \int_0^{\min(x_1, \tau)} H_{Q-k}(u) \bar{H}_k(\min(x_1, \tau) - u) du - (\tau + L)].$$

Let  $C$  be the term in square brackets. If  $x_1 > \tau + L - a$ ,

$$C \geq x_1 + \int_0^\tau H_{Q-k}(u) \bar{H}_k(\tau - u) du - \tau - L$$

$$> \tau + L - \int_a^\tau H_{Q-k}(u) \bar{H}_k(\tau - u) du + \int_0^\tau H_{Q-k}(u) \bar{H}_k(\tau - u) du - \tau - L$$

$$= \int_a^\tau H_{Q-k}(u) \bar{H}_k(\tau - u) du.$$

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