# MODIFIED BLOCK REPLACEMENT MODELS IN DISCRETE AND CONTINUOUS TIME

A THESIS
SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

BY
FELİN ARUN
APRİL, 2000

TS
181.6
.A78
2000

# MODIFIED BLOCK REPLACEMENT MODELS IN DISCRETE AND CONTINUOUS TIME

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL

ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Pelin Arun

April, 2000

TS
181.6
.A78
2000
B052405

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Vladimir V. Anisimov(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Ülkü Gürler

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Emre Berk

Approved for the Institute of Engineering and Sciences:

Prof. Mehmet Baray
Director of Institute of Engineering and Sciences

# ABSTRACT

## MODIFIED BLOCK REPLACEMENT MODELS IN DISCRETE AND CONTINUOUS TIME

Pelin Arun
M.S. in Industrial Engineering
Supervisor: Prof. Vladimir V. Anisimov
April, 2000

In this study, we present modified multi-component block replacement policies. Units (items) are replaced only at prescribed times $jT$, $j = 1, 2, ...$ A failed unit is changed with a good one with probability $\alpha$. Replacement time is negligible. Three replacement policies for models that are not represented as renewal processes are provided under this setup. Some reliability characteristics are discussed.

In the first model, total control is considered. All units are controlled at time $jT$, $j = 1, 2, ....$ In the second model, a partial (group) control is studied in which a sample of size $n$, $(0 < n < N)$ is taken from all units to inspect. And the last model deals with cyclic control: units are divided into $r$ parties. Party $k$ is controlled at time $jT$, $j = 1, 2, ...$ where $j \equiv k$ (modulus $r$), $k = 1, 2, ..., r-1$ and if $k$ is equal to zero then party $r$ is controlled. A comparison between the partial (group) control and cyclic control is provided. We also introduced cyclic partial control which combines the partial and cyclic control policies. The cyclic partial control and cyclic control is compared as well. Cost type of functionals are considered and optimal replacement interval $T^*$ is studied as well.

*Key Words*: Replacement Policies, Block Replacement, Total Control, Partial Control, Cyclic Control.

# ÖZET

## KESİKLİ VE SÜREKLİ ZAMANDA FARKLILAŞTIRILMIŞ ÖBEK DEĞİŞTİRME MODELLERİ

Pelin Arun
Endüstri Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Prof. Vladimir V. Anisimov
Nisan, 2000

Bu çalışmada, kesikli ve sürekli zamanda çok bileşenli farklılaştırılmış öbek değiştirme modelleri sunuluyor. Toplamda $N$ tane olan birimlerden herbiri rassal bozulmalara maruz kalıyor. Sistemdeki bozulmuş birimler önceden belirlenmiş $jT$, $j = 1, 2, ...$ zamanlarında $\alpha$ olasılığıyla değiştiriliyor. Yenilenme süreci şeklinde gösterilemeyen modellerde değiştirme zamanları göz önünde bulundurulmuyor. Bu doğrultuda ortaya konulan üç modelin bazı güvenilirlik özellikleri tartışılıyor.

İlk model olan toplam kontrolda sistemdeki tüm birimler $jT$, $j = 1, 2, ...$ zamanlarında kontrol ediliyor. İkinci model olarak kısmı kontrol ortaya konuluyor. Bu modelde sistemden alınan $n$, $(0 < n < N)$, büyüklüğündeki örneklemin kontrol edilmesi varsayılıyor. Son model olan çevrimsel kontrolda sistem sabit $r$ gruba bölünüyor ve $jT$, $j = 1, 2, ...$ kontrol zamanlarında, $j \equiv k \pmod r$ $k = 1, 2, ..., r-1$ durumunda grup $k$, eğer $k$ sıfıra eşitse grup $r$ kontrol ediliyor. Kısmi ve çevrimsel kontrolun karşılaştırılması yapılıyor. Ayrıca kısmi ve çevrimsel kontrolun birleşiminden meydana gelen çevrimsel-kısmi kontrol tanıtılıyor. Bu çalışmada son olarak maliyet tipi işlevleri ve en iyi değiştirme aralığı $T^*$ çalışılıyor.

*Anahtar sözcükler*: Değiştirme Kuralları, Öbek Degiştirme, Toplam Kontrol, Kısmi Kontrol, Çevrimsel Kontrol.

To Arun Family...

# ACKNOWLEDGEMENT

I would like to express my sincere gratitude to Prof. Vladimir V. Anisimov for his supervision and encouragement during my graduate study. He has been so kind and patient in all my desperate times. His trust and understanding motivated me and let this thesis come to an end.

I am indebted to Assoc. Prof. Ülkü Gürler and Asst. Prof. Emre Berk for accepting to read and review this thesis.

I would like to take this opportunity to thank Gonca Yıldırım for being such a good friend in all Bilkent life. Without her friendship and support, I would not be able to bear with all this time. I would also like to thank my officemate Banu Yüksel for all for her morale support and encouragement in all my hard times. I am grateful to Ayten Türkcan, Eşmen Işın Toyman and Filiz Gürtuna for all for their helps and encouragement. I would like to thank also Hande Yaman, Evrim Didem Güneş and Aysegül Toptal for their friendship, support and all the nice times we spent together.

I would like to express my gratitude to Ahmet Toktaş for his love and kindness. I owe so much to him for happiness he brought to my life.

# Contents


1 INTRODUCTION 1

1.1 Some Maintenance Policies and Literature Review . . . . . 4

2 DISCRETE TIME MODELS 10

2.1 Analysis of Behaviour of Items . . . . . 10

2.2 Total Control with Block Replacement . . . . . 15

2.3 Partial (Group) Control with Block Replacement 24

2.4 Cyclic Control with Block Replacement . . . . . 27

2.5 Comparison of Partial (Group) Control and Cyclic Control . . . 30

3 CONTINUOUS TIME MODELS 33

3.1 A Continuous Time Model without Replacement . . . . . 33

3.2 Total Control with Block Replacement . . . . . 36

3.3 Partial (Group) Control with Block Replacement 40

3.4 Cyclic Control with Block Replacement . . . . . 42

3.5 Comparison of Partial (Group) Control and Cyclic Control . . . 44

3.6 Correspondence Between Discrete and Continuous Time 46

3.7 Cyclic Partial Control with Block Replacement 47

**4 COST CONSIDERATIONS 50**

4.1 Formulation and Analysis 50

**5 CONCLUSION 71**

**Bibliography 73**

# List of Figures


2.1 The graph of simulated and expected number of failed units for $N = 1000$ and $q = 0.05$ . . . . . 13

2.2 The partition of $N$ units. . . . . 27

2.3 Control times of parties, $jT$, $j = 1, 2, ....$ 27

3.1 Summary of Total Control. . . . . 36

3.2 Summary of Partial (Group) Control. . . . . 40

4.1 The graph of $C(T)$ when $C_0 = 1$, $C_1 = 0$, $C_2 = 0.0125$, $N = 100$ and $\lambda = 1$. . . . . 61

# List of Tables

4.1 Long-run Expected Costs of Total Control, Partial Control and Cyclic Control for Different Values of Parameters. . . . . 67

4.2 Simulated $T^*$ values when $\alpha = 1$, $C_0 = 1$, $C_1 = 0$, $C_2 = 0.0124867$, $N = 100$ and $\lambda = 1$. 67

4.3 Signs of $f(x)$. (If $f(x) > 0$, partial control is better than total control.) . . . . . 67

# Chapter 1

# INTRODUCTION

Reliability [1] is design engineering discipline which applies scientific knowledge to assure a product will perform its intended function for the required duration within a given environment. This includes designing in the ability to maintain, test, and support the product throughout its total life cycle. Reliability is best described as product performance over time. This is accomplished concurrently with other design disciplines by contributing to the selection of the system architecture, materials, processes, and components -both software and hardware; followed by verifying the selections made by thorough analysis and test.

The skills and knowledge required to achieve reliable products are:

- statistical analysis
- product reliability modeling for selection of redundancy versus component reliability
- trade study analysis
- reliability predictions

[1] Received from http://www.ewch.ieee.org/soc/rs

- worst case stress/tolerance/sigma design performance analysis
- engineering based physics of failure
- failure modes/effects/criticality analysis
- reliability test planning and testing – product stress screening/accelareted life/demonstration
- failure analysis
- maintenance concept definition
- maintainability analysis
- maintainability test planning and demonstration
- supportability analysis
- derating analysis
- human engineering analysis
- product safety analysis
- reliability/maintainability/system safety/quality/logistics support/human factors/software performance monitoring
- product effectiveness.

Many researches have been conducted on maintenance models for systems with stochastic failure in the last forty years. Maintenance models can be applied to a variety of areas such as military, industry, health, and the environment. Due to technological advances in recent decades, systems become more complicated and require new technologies, control policies and methodologies. Whichever the environment is under consideration, an important point is the determination of when to replace the system to guarantee that the system is used effectively, efficiently and less costly and it is available whenever need arises.

In the thesis, we presented partial control policy with a multi-unit system. This model assumes that a random sample of $n$ units is inspected at pre-scheduled times $jT$, $j = 1, 2, ...$ and each failed item is replaced by new one with probability $\alpha$. Consequently, the points of inspection are in general not renewal points and it is not possible to apply directly renewal theory. In literature, there are no known results about partial control of multi-unit models.

Since multi-unit maintenance models are much more difficult to analyze, the results of the single-unit models are used frequently (see examples in Section 1.1). In the usual block replacement policy (with single or multi-unit system), all units in the system are replaced simultaneously by new ones at prescribed times $jT$, $j = 1, 2, ...$ Therefore, all known results about usual block replacement policy (to the best of our knowledge) are based on renewal theory.

In this study, we present three modified multi-unit block replacement policies. In all models, there are $N$ units (items) which are subject to random failures. Each failed item is changed with probability $\alpha$. Replacements are allowed only at times $jT$, $j = 1, 2, ...$ and $T > 0$ is fixed. Some reliability characteristics are discussed under this setup.

The first model is a total control policy in which all units are controlled at times $jT$, $j = 1, 2, ...$ Because of shortage of spare units, lack of money or workers, controlling all units may not be possible every time. So, we proposed a model called partial (group) control which depends on controlling a sample taken from all units of size $n$, $(0 < n < N)$. The last model is a cyclic control in which all units are divided into $r$ parties. Then, all parties are numbered from 1 to $r$. Party $k$ is controlled at time $jT$, $j = 1, 2, ...$ where $j \equiv k$ (modulus $r$) $k = 1, 2, ..., r-1$ and if $k$ is equal to zero then party $r$ is controlled. We also introduced cyclic partial control which combines the partial and cyclic control policies. Under cyclic control policy, a random sample of $m < n$ units are taken from party $k$ at time $jT$, $j = 1, 2, ...$ $j \equiv k$ (modulus $r$) $k = 1, 2, ..., r-1$ and if $k$ is equal to zero then a random sample taken from party $r$ is controlled. Then,

the failed units in the sample taken from the party is replaced by probability $\beta$.

The problem of multi-unit systems with general lifetimes practically is not analytically solvable. Thus, we study the exponential case. Since we replaced the failed items with good ones with a certain probability, the processes in our models are not renewal and regenerative processes. Hence, when we are studying the long-run behaviour of the number of failed items in the system and the cost functionals, we use the asymptotic results for Markov processes with discrete interference of chance.

## 1.1 Some Maintenance Policies and Literature Review

Various maintenance policies are studied with a view toward information concerning their basic stochastic characteristics, such as the distribution of the number of failures, the expected time to in-service failure etc. Some decisions concerning replacement, repair and inspection are made in the study of maintenance policies.

Replacement decision making involves the problem of specifying a replacement policy which balances the cost of failures of a unit during operation against the cost of planned replacements. Two widely used replacement policies policies are age replacement and block replacement. Under *age replacement*, the system is replaced upon failure or upon reaching a specified age $T$ whichever occurs first where $T$ is considered as constant. Under *block replacement*, the system is replaced upon at failure and at times $jT$, $j = 1, 2, ...$ (see Barlow and Proschan [6]).

There are some studies in which some replacement policies are compared.In

1979, Berg and Epstein [8] compared age, block and failure replacement (in which no preventive replacements are made at all) and provided a simple rule for choosing the least costly of the tree policies for any possible values of the planned replacement costs. Langberg [18] made the stochastic comparisons of the number of failures and removals in an interval $[0, s]$ under age and block replacement policies.

There is a large amount of literature on several maintenance models. In sequel, studies related to other maintenance actions and maintenance models are provided briefly.

Kaio and Osaki [16] studied the probabilistic characteristics of discrete and continuous lifetime distributions. Then, they applied the specific distributions to typical age and block replacements. The results are summarized in tables for the quick reference in real systems.

Block replacement policy derives its name from the commonly employed practice of replacing a block or group of units in the system at prescribed times $jT$, $j = 1, 2, ...$ independent of the failure history of the system. However, the studies on block replacement policy are usually single-unit, see e.g., Nakagawa [22]. He proposed new block replacement policies which convert the usual age, block, periodic and inspection models to discrete time replacement models. Other examples for single-unit block replacement policy have appeared in [4] [5] [6] [7] [11] [13] [14] [19] [20] [21] [27] and [28].

Some studies [4] [13] [17] have been written for optimization of replacement policies. Aven and Dekker [4] presented a general framework which covers many age and block replacement for this purpose.

Although the simple block replacement policy is simple to understand and easy to implement, the important disadvantage of this policy is that sometimes almost-new systems are replaced at prescribed replacement times $jT$, $j = 1, 2, ...$ Many modifications have been introduced to avoid this unnecessary

waste: Allowing the system to remain inactive, replacing the failed item by a used one or less reliable one are some modifications. also for repairable systems, two types of repair have been considered; minimal and imperfect repair. Barlow and Hunter [5] introduced the block replacement policy with minimal repair at failure. Under *minimal repair*, the failed unit is repaired so that it functions again, but has the same failure rate and the same effective age as at the time of failure. Under *imperfect repair*, the failed unit is repaired with probability $p$ it is returned to the "good-as-new" state (perfect repair), with probability $q = 1 - p$ it is returned to the functioning state, but it is only as good as a device of age equal to its age at failure (imperfect repair).

In 1979, Nakagawa [19] [20] studied block and age replacement with imperfect repair. After introducing imperfect repair concept in [19], he presented optimum policies when preventive maintenance is imperfect. A year later, he gave a summary of imperfect preventive maintenance policies with minimal repair in [21].

Beichelt [7] proposed a generalized block replacement policy for a system with two failure modes. If the system is in mode 1, it is removed by minimal repair. If the system failure is in mode 2, it is removed by replacement.

Other studies in imperfect repair has appeared in [11] and [14]. Fontenot and Proschan in [14] presented two models. In model I, they stated a modified age replacement model. At the beginning, a perfect repair is scheduled to take place at time $T$ (a constant) a cost $c_2$. If the device fails at a time $t_1$ prior to $T$, it is repaired at a cost $c_1 < c_2$. This repair is perfect with probability $p$ and imperfect (minimal) with probability $q = 1 - p$. Model II is the same as model I except in one concept: the unit is replaced (perfectly repaired) on the next failure after $k - 1$ successive imperfect repairs.

Sheu [27] [28] studied the modified block replacement policies in detail. In his models, an operating system is periodically exchanged at times $kT$, $k = 1, 2, ...$ independently of its failure history. After a generalization of the policy, he

proposed a model in which an operating system is either placed by a new or used one or minimally repaired or remains inactive until the next planned replacement at failure. In another policy, he introduced a modified block replacement with two variables and general random minimal cost. The cost of $i$th minimal repair at age $y$ consists of two parts: $C(y)$ is the age dependent random part, $c_i(y)$ is the deterministic part which depends on the age and the number of minimal repair. Thus, minimal repair cost id random. If the system fails in $[(k-1)T, (k-1)T+T_0)$ it is either replaced by a new one or minimally repaired, and if $[(k-1)T+T_0, kT)$ it is either minimally repaired or remains inactive until the next planned replacement.

The multi-unit maintenance models can be divided into the two main categories: "Preventive" and "preparedness" maintenance models (see in [12]). These models based on the knowledge of the state of systems. *Preventive* maintenance models assume that the state of equipment subject to stochastic failure is always known with certainty. *Preparedness* maintenance models also deal with equipment which fails stochastically; however, the state of equipment is assumed to be unknown unless either inspection or replacement is carried out. An additional decision related to an optimal inspection schedule must be made in these models.

Replacement models are included in the preventive maintenance models. In sequel, studies related replacement models with multi-unit system are provided. Tango [30] investigated a multi-unit system. He proposed a modified block replacement policy using less reliable items. If operating items fail in $[(k-1)T, kT-\nu)$, they are replaced by new items, and in $[(k-1)T-\nu, kT)$, they are replaced by less reliable items than new items. (Less reliable items should be cheaper and thus less durable than new items.) Haurie and L'ecuyer [15] formulated a group maintenance replacement problem in continuous time for multi-component system having identical elements. Also, the numerical computation of optimal and suboptimal strategies of group preventive replacement are done for discrete time.

Due to the fact that items in a system function under the same environmental conditions like temperature, humidity and vibrations, component lifetimes are generally stochastically dependent. Thus, they are also economically dependent because of doing preventive maintenance to functioning units. In [23], Özekici discusses the effects of these dependencies on periodic replacement policies and provides useful characterizations of the optimal replacement policy.

In 1990, Ritchken and Wilson [24] introduced $(m, T)$ group replacement policy which combines the m-failure and T-age policies. All items are replaced by new ones when the system is of age $T$, or when $m$ failures have occurred whichever comes first.

Optimal group maintenance policies for a set of $M$ identical machines subject to stochastic failures are considered in [3] [31]. Assaf and Shanthikumar [3] proposed a control limit policy which minimizes the expected cost per unit time over an infinite horizon when costs are incurred due to loss of production and repair only. In this model it is assumed that the number of failed machines in the system is unknown unless an inspection is carried out. Therefore, this model is an example for preparedness maintenance policy. In a part of their study, the following policy is considered: Inspection takes place every $\tau$ units of time and all failed machines, if any, are repaired at each inspection epoch. Let $W(\tau)$ be the average cost of this policy per unit of time. Then,

$$W(\tau) = \frac{C_0^I + C_0(1 - e^{-N\lambda\tau}) + N(C_1 - (C_2/\lambda))(1 - e^{-\lambda\tau})}{\tau} + NC_2$$

where $C_0$ is the overhead cost of repair, $C_1$ is a cost of repair per machine and $C_2$ is the cost of production loss for each failed machine during operation. Also, a positive cost $C_0^I$ is paid every time such an inspection is performed.

In [31], Van Der Dyn Schouten and Vanneste introduced four possible states for each component: good, doubtful, preventive maintenance is due and failed. They considered two types of control policies which based on the number of doubtful components at component failure epochs. After introducing a general model with identical lifetime distributions for individual components, they

propose an approximate model in which the four possible states are identified with certain age intervals for each individual component.

Some other studies about multi-unit maintenance policies are the following: In 1992, Sheu and Griffith [29] studied multi-unit systems with dependent life-lenghts having certain multivariate distributions. Components are repaired upon failure (perfect or imperfect repair) depending on the different sources of failure. In the same year, several new results which connect the properties of block replacement policies with properties of corresponding renewal function and excess lifetimes are obtained in [26]. In [10] the concept of repair replacement is introduced by Block, Langberg and Savits. In repair replacement, items are repaired if they fail and replaced only if they survive beyond a certain fixed time from the last repair or replacement.

In [9], it is assumed that performance of $M$ identical machines deteriorates with the operating time since the last maintenance and no random failures occur. Any machine can be taken off line for maintenance at any time, (that is, it is not a periodic inspection or a periodic replacement.) The maintenance times are independently and identically distributed. Under these assumptions, many maintenance policies are performed when the expected profit rate is maximized.

The rest of the thesis is organized as follows: In Chapter 2, after analyzing the behaviour of items in the system, we present total control, partial (group) control and cyclic control in discrete time. Some reliability characteristics are discussed. We compare partial (group) control and cyclic control when we consider the number of failed units in the system. In Chapter 3, all analysis for discrete time is done in continuous time. In Chapter 4, some cost functionals are studied and the optimal replacement interval $T^*$ is obtained. A simulation of total control policy is provided and the comparison with analytical result is given. Finally, we present our conclusion in Chapter 5.

# Chapter 2

# DISCRETE TIME MODELS

In this chapter, we present modified multi-component block replacement policies in which units (items) are replaced only at prescribed discrete times $T, 2T, \ldots$. In the first section, we will analyze the behaviour of items in the system and determine the number of failed units at time $k$, $k = 0, 1, 2, \ldots$ . In sequel sections, total control, partial (group) control and cyclic control policies for discrete time will be introduced.

## 2.1 Analysis of Behaviour of Items

Consider a discrete time control policy. Suppose that there are $N$ independent items each of which is in good condition at the beginning in the system. Let $Q_k$ be the number of failed items at time $k$, $k = 1, 2, \ldots$. Assume that every unit may fail independently of the failures of the other units with the probability $q$. Thus, the distribution of failed items is a binomial distribution. Let $Bin(N, q)$ denote the Binomial random variable with parameters $N$ and $q$, that is, $P(\xi = i) = \begin{pmatrix} N \\ i \end{pmatrix} p^i q^{N-i}$ where $p = 1 - q$ and $i = 0, 1, 2, \ldots, N$. In future we will use also the notion of double stochastic binomial random variable.

**Definition 2.1** *If $Z$ is some integer random variable with values $\{1, 2, ...\}$. We will denote a double stochastic binomial random variable $Bin(Z, \alpha)$ such that*

$$P(Bin(Z, \alpha) = k) = \sum_{n \geq k} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} P(Z = n).$$

*Expectation of the double stochastic binomial random variable is*

$$E[Bin(Z, \alpha)] = \alpha E(Z).$$

Then we can write the following stochastic relation between number of failed items at step $k$ and $k+1$.

$$Q_{k+1} = Q_k + Bin(N - Q_k, q), \quad k = 0, 1, 2, ... \tag{2.1.1}$$

In order to find the expected number of failed items at each step, we can use the following recurrent relation. If $m_k = E(Q_k)$ is the expected number of failed items at step $k$, $k = 0, 1, 2, ....$ For step $k+1$,

$$m_{k+1} = m_k + (N - m_k)q$$

or

$$m_{k+1} = Nq + m_k p \quad \text{where} \quad p = 1 - q. \tag{2.1.2}$$

Since all units are in the good condition at the beginning, the expected number of failed items at initial step is zero, that is, $m_0 = 0$. Then by Equation (2.1.2), we have the following,

$$\begin{aligned}
m_1 &= Nq. \\
m_2 &= Nq + m_1 p = Nq(1+p). \\
m_3 &= Nq + m_2 p = Nq(1+p+p^2). \\
&\vdots \\
m_k &= Nq + m_{k-1} p = Nq(1+p+p^2+\ldots+p^{k-1}) \\
&= N(1-p)\frac{1-p^k}{1-p}.
\end{aligned}$$

Therefore, we get

$$m_k = N(1-p^k), \quad k = 0,1,2,... \tag{2.1.3}$$

As a result, we can say that number of failed items at step $k$, $k = 0,1,2,...$ has binomial distribution with parameters $N$ and $1-p^k$ for $k = 0,1,2,....$
If the number of failed units on the initial step is greater than zero, that is, $Q_0 > 0$, with mean $m_0$ then the recurrent relation (2.1.2) gives the following:

$$\begin{aligned}
m_1 &= Nq + m_0 p. \\
m_2 &= Nq + m_1 p = Nq(1+p) + p^2 m_0. \\
m_3 &= Nq + m_2 p = Nq(1+p+p^2) + p^3 m_0.
\end{aligned}$$

$$\begin{aligned}
m_k &= Nq + m_{k-1}p = Nq(1+p+p^2+\ldots+p^{k-1}) + p^k m_0. \\
&= N(1-p)\frac{1-p^k}{1-p} + p^k m_0.
\end{aligned}$$

Finally,

$$m_k = N(1-p^k) + p^k m_0, \quad k = 0,1,2,... \tag{2.1.4}$$

When we compare Equation (2.1.3) and (2.1.4), there is an additional term containing expected number of failed units at the beginning in Equation (2.1.4).

**Example:**

In order to demonstrate the use of the model, the case $N = 1000$ and $q = 0.05$ is analyzed. The number of failed units were obtained by simulation and the expected number of failed units for each step was calculated analytically. Since $N = 1000$ and $q = 0.05$ the function of expected number of failed items is given by

$$m_k = 1000(1-0.95^k) \quad \text{where} \quad k = 0,1,2,... \quad \text{is any step.}$$

At the end of 136 steps all units are failed. The graph of output is shown in Figure 2.1.

In Figure 2.1, the graph of number of failed items, which was obtained by

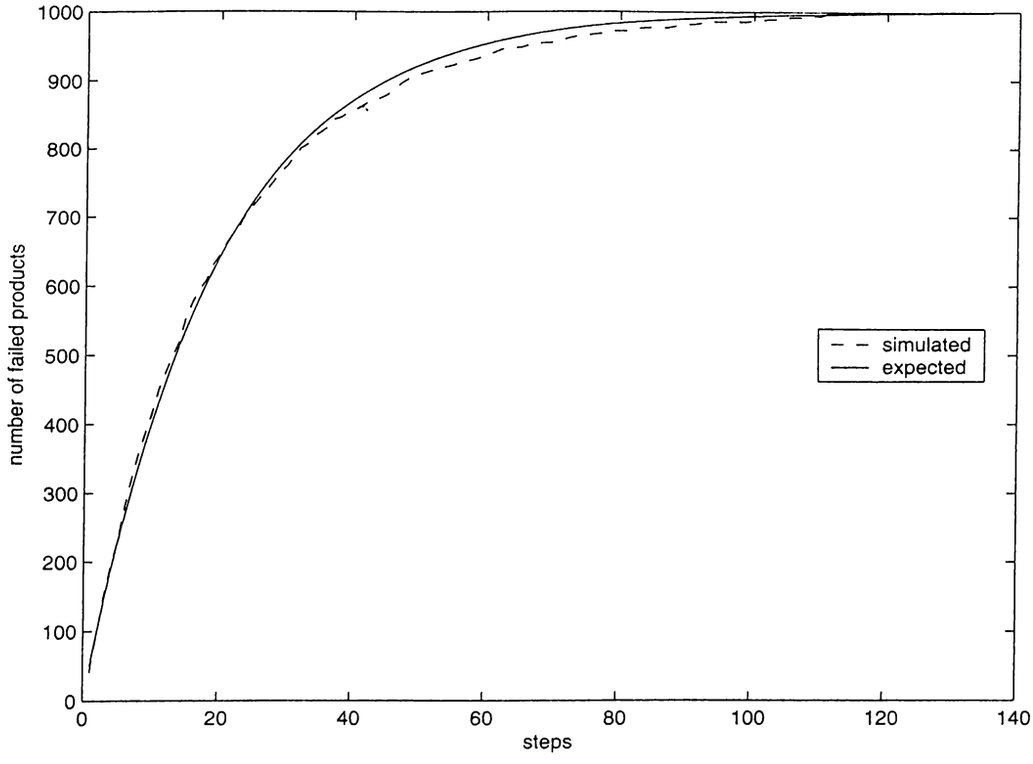


Figure 2.1: The graph of simulated and expected number of failed units for $N = 1000$ and $q = 0.05$

adding simulated binomial random variables, $Q_k$ at time $k$, $k = 1, 2, ...$, is almost fitting into the graph of expected number of failed items.

## 2.2 Total Control with Block Replacement

In this section, we study a control policy with modified block replacement. Again suppose that $N$ independent units with failure probability $q$ are installed. We call this control policy as "Total Control" because all installed units are controlled at time $jT$, $j = 1, 2, ...$ where $T$ is some positive integer. While making replacement we can make a mistake which means that we may change the failed item with good one only with probability $\alpha$. We assume that replacement, repair or inspection time is negligible. Replacement is allowed only at times $jT$, $j = 1, 2, ...$ for a fixed $T > 0$, that is, the failed units may be replaced only at discrete times $T, 2T, ....$

Consider the process at times $jT$, $j = 1, 2, ...$ and let $Q_{jT}$ be the number of failed items before control at time $jT$, $j = 1, 2, ...$ and

$$M_{jT} = E(Q_{jT}), \quad j = 1, 2, ...$$

and let $M_\infty$ be the expected number of failed items before control in the long-run, that is,

$$M_\infty = \lim_{j \to \infty} M_{jT}.$$

(We will prove that this limit exits.)

**Main Relations:**

We denote the number of failed units before control at time $jT$, $j = 1, 2, ...$ by $Q_{jT}$ and the number of failed items just after control at time $jT$, $j = 1, 2, ...$ by $Q^+_{jT}$. Then,

$$Q^+_{jT} = Q_{jT} - Bin(Q_{jT}, \alpha) \tag{2.2.1}$$

Let

$$E(Q^+_{jT}) = M^+_{jT}.$$

Then the expectation after control yields

$$M^+_{jT} = M_{jT} - M_{jT}\alpha.$$

Finally,

$$M^+_{jT} = (1 - \alpha)M_{jT}. \tag{2.2.2}$$

Rewriting Equation (2.1.1) by putting $Q_{jT}^{+}$ instead of $Q_k$, we have

$$Q_{(j+1)T} = Q_{jT}^{+} + Bin(N - Q_{jT}^{+}, 1 - p^T) \tag{2.2.3}$$

It follows that the recurrent relation for mean number of failed items is

$$M_{(j+1)T} = M_{jT}^{+} + (N - M_{jT}^{+})(1 - p^T). \tag{2.2.4}$$

Substitute $M_{jT}^{+}$ and rewrite $M_{(j+1)T}$,

$$\begin{aligned} M_{(j+1)T} &= (1-\alpha)M_{jT} + (N - (1-\alpha)M_{jT})(1 - p^T) \\ &= M_{jT}[1 - \alpha - (1-\alpha)(1-p^T)] + N(1-p^T). \end{aligned}$$

$M_{(j+1)T}$ depending on $M_{jT}$ is given by

$$M_{(j+1)T} = M_{jT}(1-\alpha)p^T + N(1-p^T). \tag{2.2.5}$$

In Equation (2.2.5), let

$$a = (1-\alpha)p^T, \quad 0 < a < 1$$

and

$$b = N(1-p^T).$$

Then

$$M_{(j+1)T} = aM_{jT} + b.$$

Here note that number of failed units at time 0 is greater than or equal to zero, $M_0 \geq 0$. We can get $M_{jT}$ depending on $M_0$, $a$ and $b$, that is,

$$\begin{aligned} M_T &= aM_0 + b. \\ M_{2T} &= a^2M_0 + ab + b. \\ M_{3T} &= a^3M_0 + a^2b + ab + b. \end{aligned}$$

$$\begin{aligned} M_{jT} &= a^j M_0 + a^{j-1} b + \ldots + ab + b \\ &= a^j M_0 + b \sum_{k=0}^{j-1} a^k \\ &= a^j M_0 + b \frac{(1-a^j)}{1-a}. \end{aligned}$$

In general at time $jT$, $j = 1, 2, \ldots$

$$M_{jT} = a^j \left( M_0 - \frac{1}{1-a} \right) + \frac{b}{1-a}$$

As the number of inspection times $j$ goes to $\infty$, since $0 < a < 1$, then we get

$$\lim_{j \to \infty} M_{jT} = \frac{b}{1-a} \tag{2.2.6}$$

Hence we can find average number of failed items before control in the long-run, $M_\infty$, by substituting a and b in Equation (2.2.6). Therefore,

$$M_\infty = \frac{N(1-p^T)}{1-(1-\alpha)p^T}. \tag{2.2.7}$$

The expected number failed units just after control in the long-run is the following:

$$\begin{aligned} M_\infty^+ &= M_\infty - \alpha M_\infty \\ &= (1-\alpha) M_\infty, \end{aligned}$$

substitute $M_\infty$ obtained in Equation (2.2.7) and get,

$$M_\infty^+ = \frac{N(1-p^T)(1-\alpha)}{1-(1-\alpha)p^T}. \tag{2.2.8}$$

We proved that limit, as the number of inspection times $j$ goes to infinity, exists. We can also find the limit directly from Equation (2.2.5). Thus,

$$\lim_{j \to \infty} M_{(j+1)T} = M_\infty$$

then,

$$M_\infty = (1-\alpha) M_\infty p^T + N(1-p^T),$$

that implies

$$M_\infty = \frac{N(1-p^T)}{1-(1-\alpha)p^T}. \tag{2.2.9}$$

In order to obtain the average long-run proportion of expected number of failures, we can use the following definitions and theorems:

**Definition 2.2** *If the random variable $\xi_n \xrightarrow{P} a_0$, then we can denote it as*

$$P \lim_{n\to\infty} \xi_n = a_0$$

*where $\xrightarrow{P}$ denotes converges in probability and $P\lim$ means a limit in probability.*

**Definition 2.3** *$R(N,T,p,\alpha)$ is the average long-run proportion of expected number of failed items in the system and it is represented as follows:*

$$R(N,T,p,\alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} EQ_k.$$

**Definition 2.4** *$Q(N,T,p,\alpha)$ is the long-run average proportion of failed items in the system and given by*

$$Q(N,T,p,\alpha) = P \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Q_k.$$

**Lemma 2.1** *The limit of the expected value of $Q_{lT+j}$, $j = 0,1,...,T-1$ and $l = 0,1,...$ exists and is given by*

$$\lim_{l\to\infty} EQ_{lT+j} = E_j^\infty$$

*where*

$$E_j^\infty = M_\infty^+ + (N - M_\infty)(1-p^j).$$

**Proof:**

$EQ_{lT+j}$ can be expressed by using Equation (2.1.3) as

$$EQ_{lT+j} = M_{lT}^+ + (N - M_{lT}^+)(1-p^j)$$

where $M_{lT}^{+}$ is the expected number of failed items just after control at time $lT$, $l = 0, 1, \ldots$ Therefore, as $l \to \infty$, we have

$$\lim_{l\to\infty} EQ_{lT+j} = M_{\infty}^{+} + (N - M_{\infty}^{+})(1 - p^{j}). \qquad \square$$

**Lemma 2.2** *If $a_k \to a_\infty$ as $k \to \infty$ then,*

$$\frac{1}{n}\sum_{k=1}^{n} a_k \to a_\infty \quad as \quad n \to \infty.$$

**Proof:**

$$\begin{aligned} |\frac{1}{n}\sum_{k=1}^{n} a_k - a_\infty| \le \frac{1}{n}\sum_{k=1}^{n} |a_k - a_\infty| &= \frac{1}{n}\sum_{k=1}^{L} |a_k - a_\infty| + \frac{1}{n}\sum_{k=L+1}^{n} |a_k - a_\infty| \\ &< \epsilon\frac{n}{n} = \epsilon. \end{aligned}$$

As $a_k$ goes to $a_\infty$, then for any $\epsilon > 0$, there exists $L$ such that $k > L$, $|a_k - a_\infty| < \epsilon$, then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \to a_\infty. \qquad \square$$

**Theorem 2.1** *Let $EQ_k$ be the expected number of failed units at time $k$, $k = 1, 2, \ldots$. Then, we have*

$$R(N, T, p, \alpha) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} EQ_k = \frac{1}{T}\sum_{j=0}^{T-1} E_j^{\infty}$$

*where $E_j^{\infty}$ is obtained by stochastic relation (2.2.5) in the long-run, that is,*

$$E_j^{\infty} = [M_{\infty}^{+} + (N - M_{\infty}^{+})(1 - p^{j})].$$

**Proof:**

If $n = Tm$, $\frac{1}{n}\sum_{k=1}^{n} EQ_k$ can be written in the following form:

$$\frac{1}{T}\left(\frac{1}{m}\sum_{l=1}^{m} EQ_{lT}^{+} + \frac{1}{m}\sum_{l=1}^{m} EQ_{lT+1} + \ldots + \frac{1}{m}\sum_{l=1}^{m} EQ_{lT+T-1}\right)$$

Consider $EQ_{lT+j}$ for $0 < j < T$, and refer to Lemma 2.1. As $l \to \infty$, we get

$$EQ_{lT+j} \to E_j^{\infty}.$$

Using Lemma 2.2 we get for any $j$ as $m \to \infty$,

$$\frac{1}{m}\sum_{l=1}^{m} EQ_{lT+j} \to E_j^{\infty}.$$

where

$$E_j^{\infty} = [M_{\infty}^{+} + (N - M_{\infty}^{+})(1 - p^j)].$$

$M_{\infty}^{+}$ is the average long-run number of failed items just after control and $(N - M_{\infty}^{+})(1 - p^j)$ is average number of items failed during time $(lT, lT + j)$ in the long-run. Consider the case $n = Tm + i$ where $0 < i < T$. $\frac{1}{n}\sum_{k=1}^{n} EQ_k$ can be written in the following form:

$$\frac{1}{Tm+i}\left[\left(\sum_{l=1}^{m} EQ_{lT}^{+} + \sum_{l=1}^{m} EQ_{lT+1} + \ldots + \sum_{l=1}^{m} EQ_{lT+T-1}\right) + \sum_{j=Tm}^{Tm+i} EQ_j\right]$$

Rewrite the equation then we get

$$\left(\frac{Tm}{Tm+i}\right)\frac{1}{T}\left(\frac{1}{m}\sum_{l=1}^{m} EQ_{lT}^{+} + \frac{1}{m}\sum_{l=1}^{m} EQ_{lT+1} + \ldots + \frac{1}{m}\sum_{l=1}^{m} EQ_{lT+T-1}\right)$$
$$+\frac{1}{Tm+i}\sum_{j=Tm}^{Tm+i} EQ_j$$

Note that

$$\frac{1}{Tm+i}\sum_{j=Tm}^{Tm+i} EQ_j \leq \frac{Ni}{Tm+i} \tag{2.2.10}$$

As $m$ goes to infinity, the term (2.2.10) goes to zero and

$$\frac{Tm}{Tm+i} \to 1.$$

Therefore, we get the same situation when the case $n = Tm$. □

**Theorem 2.2** *In the long-run, average proportion of number of failed units $Q(N, T, p, \alpha)$ exists and it is given by*

$$Q(N, T, p, \alpha) = \frac{1}{T}\sum_{k=0}^{T-1} E_j^{\infty}$$

**Proof:**

Let us represent the value $1/n\sum_{k=1}^{n} Q_k$ in the following way: Consider at fixed $j = 0, 1, ..., T-1$ a sequence $Q_{lT+j}$, $l = 0, 1, ...$ and the case when $n = Tm$, then

$$\frac{1}{n}\sum_{k=1}^{n} Q_k = \frac{1}{T}\left(\frac{1}{m}\sum_{l=1}^{m} Q_{lT} + \frac{1}{m}\sum_{l=1}^{m} Q_{lT+1} + ... + \frac{1}{m}\sum_{l=1}^{m} Q_{lT+T-1}\right)$$

It can be easily see that this sequence $Q_{lT+j}$, $l = 0, 1, ...$ forms a Markov process which is embedded for $Q_k$. Lets find a limit of the value $1/n\sum_{k=1}^{n} Q_k$. Consider first the sequence $Q_{lT}$, $l = 0, 1, ...$ This sequence forms an irreducible Markov chain with finite state space $\{0, 1, ..., N\}$ and corresponding transition probabilities $p_{ij}$, $i, j = 0, 1, ..., N$. Denote for any $i = 0, 1, ..., N$

$$\xi_i = i + Bin(N - i, 1 - p^T) \qquad i = 0, 1, ...N.$$

Then, transition probabilities are obtained by

$$p_{ij} = P(\xi_i = j) > 0 \qquad i, j = 0, 1, ...N.$$

Let $\pi(i)$, $i = 0, 1, ..., N$ denote the stationary distribution of the Markov chain $Q_{lT}$. According to weak law of large numbers for Markov chain, as $m \to \infty$, we get

$$\frac{1}{m}\sum_{l=1}^{m} Q_{lT} \xrightarrow{P} E_\pi Q^+.$$

where

$$E_\pi Q^+ = \sum_{i=0}^{N} i\pi(i).$$

Now let us show that

$$E_\pi Q^+ = M_\infty^+.$$

Remember that

$$M_\infty^+ = \lim_{k\to\infty} EQ_{kT}^+$$

Let $M_{kT}^+$ be the expected number of failed items just after control at time $kT$, $k = 1, 2, ....$ Hence,

$$M_{kT}^+ = EQ_{kT}^+ = \sum_{i=1}^{n} iP_{kT}(i),$$

where $P_{kT}(i) = P(Q_{kT} = i)$. $Q_{kT}$ is ergodic Markov chain then as $k \to \infty$,

$$P_{kT}(i) \to \pi(i) \qquad i = 0, 1, ..., N.$$

Thus, we have

$$M_\infty^+ = \lim_{k\to\infty} EQ_{kT}^+ = \sum_{i=1}^{n} i\pi(i) = E_\pi Q^+$$

In the same way, we can prove that each $j = 1, 2, ..., T-1$, $Q_{lT+j}$ forms an ergodic Markov chain and

$$\frac{1}{m}\sum_{l=1}^{m} Q_{lT+j} \xrightarrow{P} E_j^\infty$$

Finally we get,

$$Q(N, T, p, \alpha) = \frac{1}{T}\sum_{k=0}^{T-1} E_j^\infty$$

Similarly as in Theorem 2.1 the case when $n = Tm + i$, $0 < i < T$ is the same as the case $n = Tm$. □

**Corollary 2.1** *The average long-run proportion of failed items in the system is equal to the average long-run proportion of expected number of failed items in the system. That is,*

$$Q(N, T, p, \alpha) = R(N, T, p, \alpha)$$

**Proof:**

It is obviously follows from Theorem 2.1 and Theorem 2.2. □

**Corollary 2.2** *The average long-run proportion of average number of failed items for total control in the system is the following:*

$$R(N, T, p, \alpha) = N\left[1 - \frac{1}{T}\left(\frac{\alpha}{1-(1-\alpha)p^T}\right)\left(\frac{1-p^T}{1-p}\right)\right]. \qquad (2.2.11)$$

**Proof:**

Proof of this theorem based on Theorem 2.1.

$$R(N, T, p, \alpha) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} EQ_{kT} \to \frac{1}{T}\sum_{k=0}^{T-1}[M_\infty^+ + (N - M_\infty^+)(1 - p^k)].$$

Equivalently,

$$\begin{aligned}
R(N,T,p,\alpha) &= M_\infty^+ + \frac{1}{T}\sum_{k=0}^{T-1}(N - M_\infty^+)(1-p^k) \\
&= M_\infty^+ + \frac{1}{T}\sum_{k=0}^{T-1} N - \frac{1}{T}\sum_{k=0}^{T-1} M_\infty^+ - \frac{1}{T}\sum_{k=0}^{T-1}(N - M_\infty^+)p^k \\
&= N - \frac{1}{T}(N - M_\infty^+)\sum_{k=0}^{T-1} p^k
\end{aligned}$$

substitute $M_\infty^+$ obtained at Equation (2.2.8), then

$$\begin{aligned}
R(N,T,p,\alpha) &= N - \frac{1}{T}\left(N - \frac{N(1-p^T)(1-\alpha)}{1-(1-\alpha)p^T}\right)\left(\frac{1-p^T}{1-p}\right) \\
&= N\left[1 - \frac{1}{T}\left(\frac{1-p^T+\alpha p^T-1+\alpha+p^T-\alpha p^T}{1-(1-\alpha)p^T}\right)\left(\frac{1-p^T}{1-p}\right)\right]
\end{aligned}$$

Finally we have,

$$R(N,T,p,\alpha) = N\left[1 - \frac{1}{T}\left(\frac{\alpha}{1-(1-\alpha)p^T}\right)\left(\frac{1-p^T}{1-p}\right)\right] \qquad \square$$

## 2.3 Partial (Group) Control with Block Replacement

In practical models it is difficult to inspect all units at the control times $jT$, $j = 1, 2, ...$ when the number of installed units are large at the beginning. Control of all operating units requires a large amount of spare units, money and workers. In order to avoid high cost, in this section, we will introduce a modified block replacement which depends on control of a partial choice of whole units in the system.

Assume that there are $N$ units in the system. Each item may fail with probability $q$. At each control time $jT$, $j = 1, 2, ...$ we take a sample of size $n$ where $n < N$ units are selected randomly from the all units in the system. Assume that the probability of changing the failed unit with a good one in the sample is $\beta$. Let $Q_{jT}$ be the total number of failed items at time $jT$, $j = 1, 2, ...$ and $Q^+_{jT}$ be the total number of failed products just after control. The sample contains $HG(N, n, Q_{jT})$ faulty units at the inspection time, where $HG(N, n, Q_{jT})$ is a double stochastic hypergeometric random variable with parameters $N, n$ and $Q_{jT}$, that is,

$$P\{HG(N, n, Q_{jT}) = k\} = \frac{\binom{Q_{jT}}{k}\binom{N - Q_{jT}}{n - k}}{\binom{N}{n}}$$

where $k = 0, 1, 2, ..., n$ and $E(K) = n(M_{jT}/N)$
(Inspection technique is based on the model of Anisimov and Sereda [1].)

Suppose that $M_{jT} = E(Q_{jT})$ and $M^+_{jT} = E(Q^+_{jT})$. The following stochastic relation contains again a double stochastic binomial random variable as explained in Definition 1.1. Then, we get

$$Q^+_{jT} = Q_{jT} - Bin(HG(N, n, Q_{jT}), \beta) \tag{2.3.1}$$

If we take the expectation of both sides in Equation (2.3.1), then we have

$$M_{jT}^{+} = M_{jT} - M_{jT}\frac{n}{N}\beta = \left(1 - \frac{n}{N}\beta\right) M_{jT}. \tag{2.3.2}$$

If we consider the number of failed items at time $(j+1)T$ then,

$$Q_{(j+1)T} = Q_{jT}^{+} + Bin(N - Q_{jT}^{+}, 1 - p^{T}) \tag{2.3.3}$$

The stochastic relation between the average number of failed items before control at time $(j+1)T$ and and the average number of failed units just after control at time , $jT$ is the following:

$$\begin{aligned} M_{(j+1)T} &= M_{jT}^{+} + (N - M_{jT}^{+})(1 - p^{T}), \\ &= M_{jT}\left(1 - \frac{n}{N}\beta\right) p^{T} + N(1 - p^{T}). \end{aligned} \tag{2.3.4}$$

**Corollary 2.3** *Consider the partial (group) control with block replacement. The long-run average number of failed items before control in the system is given by*

$$\lim_{j\to\infty} M_{jT} = M_{\infty},$$

*where*

$$M_{\infty} = \frac{N(1 - p^{T})}{1 - (1 - \beta\frac{n}{N})p^{T}}.$$

*The expected number of failed units just after control in the long-run is the following:*

$$M_{\infty}^{+} = \left(1 - \beta\frac{n}{N}\right) M_{\infty}.$$

*substitute $M_{\infty}$ then,*

$$M_{\infty}^{+} = \frac{N(1 - p^{T})(1 - \beta\frac{n}{N})}{1 - (1 - \beta\frac{n}{N})p^{T}}.$$

*The average long-run proportion of number of failed products is the following:*

$$R(N,T,p,n,\beta) = N\left[1 - \frac{1}{T}\left(\frac{\beta\frac{n}{N}}{1-(1-\beta\frac{n}{N})p^T}\right)\left(\frac{1-p^T}{1-p}\right)\right]. \qquad (2.3.5)$$

**Proof:**

Equation (2.3.4) is basic stochastic relation for partial control to obtain the long-run values given the theorem. When we compare the Equation (2.3.4) with Equation (2.2.5) which is basic for total control, $\alpha$ is replaced by $\beta n/N$. So, in Equations (2.2.2) and (2.2.7), $\alpha$ is replaced by the probability $\beta(n/N)$ and get analogous results for this model. □

**Example:**

If all failed units in the sample are changed with good ones, that is, $\beta = 1$, we have

$$Q_{jT}^{+} = Q_{jT} - HG(N,n,Q_{jT})$$

taking expectation of both sides yields,

$$M_{jT}^{+} = \left(1 - \frac{n}{N}\right) M_{jT}.$$

The stochastic relation between the time $jT$ and $(j+1)T$ is

$$Q_{(j+1)T} = Q_{jT}^{+} + Bin(N - Q_{jT}^{+}, 1 - p^T)$$

and

$$M_{(j+1)T} = M_{jT}\left(1 - \frac{n}{N}\right)p^T + N(1-p^T).$$

## 2.4 Cyclic Control with Block Replacement

In the previous model, the sample chosen for the control was selected randomly. In this model, we assume that $N$ installed units are divided into fixed $r$ groups (parties) each of which has $n$ independent items (that is, $N = nr$). Figure 2.2 shows the partition of $N$ items. Each item in the system may fail with

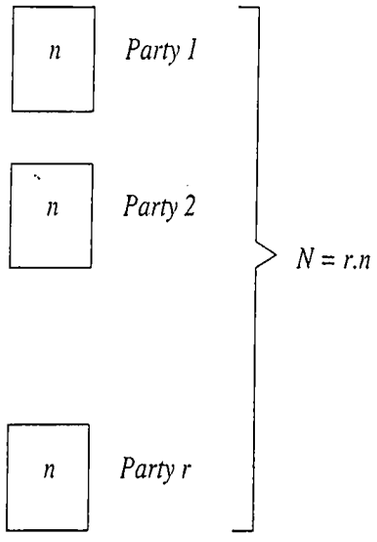


Figure 2.2: The partition of $N$ units.

probability $q$ in each step. Let $jT$, $j = 1, 2, ...$ be the control times. Party 1 is controlled at time $T$, party 2 is controlled at time $2T$, party 3 is controlled at time $3T$, this continues until inspection of party $r$ at time $rT$. The first "cycle of control" is completed when party 1 is controlled again at time $(r+1)T$ and is equal to $rT$. Then, the second control cycle begins. Note that all units are inspected in party $k$, $k = 1, 2, ..., r$. Figure 2.3 summarizes the control times of parties. We change the failed unit with good one with probability $\alpha$ only

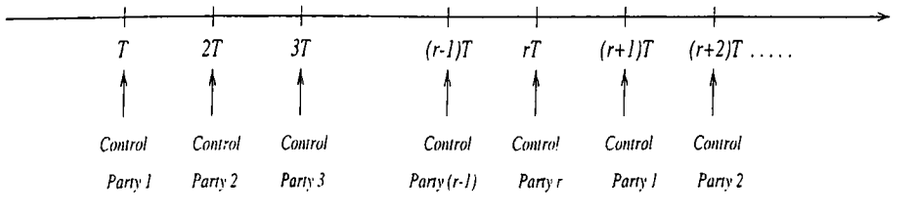


Figure 2.3: Control times of parties, $jT$, $j = 1, 2, ....$

at times $jT$, $j = 1, 2, ...$ for a fixed $T > 0$. In this model we assume that replacement, repair or inspection time is negligible.
Before starting calculations, let us introduce the notation used. For $j = 1, 2, ...$, $k = 1, 2, ..., r$ and fixed $T > 0$:
*Notation*

| | |
|---|---|
| $M_0^{(k)}$ | expected number of failed units for party $k$ at time $(k-1)T$. |
| $q$ | probability of failure for each item on each step. |
| $r$ | number of parties in the system. |
| $n$ | total number of items in party $k$. |
| $\alpha$ | probability of changing the failed unit with a a good one at time control $jT$. |

Under assumption that all items are in the good condition at time 0, expected number of failed units for party $k$ at time $(k-1)T$. We get the following relations:

$$\begin{aligned} M_0^{(1)} &= 0 \\ M_0^{(2)} &= n(1-p^T) \\ M_0^{(3)} &= n(1-p^{2T}) \\ &\vdots \\ M_0^{(r)} &= n(1-p^{(r-1)T}) \end{aligned}$$

If we know the average number of failed units for party 1 at any time $jT$, $j = 1, 2, ...$ we can easily calculate that for other parties by changing $M_0^{(k)}$, $k = 2, 3, 4, ...$ with $M_0^{(1)}$. Note that $kT$, $(r+k)T$, $(2r+k)T$, ..., $(lr+k)T$ are the control times of party $k$, $k = 1, 2, ..., r$.

**Corollary 2.4** *Consider the cyclic control with block replacement. The long-run average number of failed units before control is the following,*

$$M_\infty = \frac{N(1-p^{rT})}{1-(1-\alpha)p^{rT}}$$

*and the average number of failed items just after control,*

$$M_\infty^+ = (1-\alpha)M_\infty.$$

*The average long-run proportion of expected number of failed units is given by*

$$rR(N, rT, p, n, \alpha) = N - \frac{1}{T}\left(\frac{\alpha n}{1 - p^{rT}(1-\alpha)}\right)\left(\frac{1 - p^{rT}}{1 - p}\right) \tag{2.4.1}$$

**Proof:**

In cyclic control we divide the items into $r$ different parties each of which has $n$ items. Cyclic control can be thought as $r$ different total control models having $n$ units with operation time interval $rT$. So, if we put $n$ and $rT$ instead of $N$ and $T$, respectively, in the formulae of total control, we obtain the formulae for only one party of cyclic control. That is,

$$M_\infty = \frac{n(1-\alpha)(1-p^{rT})}{1-(1-\alpha)p^{rT}}$$

and

$$R(N, rT, p, n, \alpha) = n\left[1 - \frac{1}{rT}\left(\frac{\alpha}{1 - p^{rT}(1-\alpha)}\right)\left(\frac{1 - p^{rT}}{1 - p}\right)\right]$$

Since there are $r$ parties in the system, we multiply $M_\infty$ and $R(N, rT, p, n, \alpha)$ by $r$ to obtain final the formulae for the system. □

## 2.5 Comparison of Partial (Group) Control and Cyclic Control

In previous sections some discrete time control models are introduced. As we stated before, in both partial (group) control and cyclic control we control $n$ units of the system at prescribed control times $jT$, $j = 1, 2, \ldots$. A question may arise:

> *Which model is better if we consider the average number of failed units?*

In order to answer this question average long run proportions of expected number failed items can be used. Recall Equations (2.3.5) and (2.4.1) under the assumption that all parameters $N, T, p$ and $n$ are the same for both models and probability of changing failed unit with good one $\beta$ and $\alpha$ are equal to each other for partial choice control and cyclic control, respectively:

Partial (Group) Control

$$R(N,T,p,n,\alpha) = N\left[1 - \frac{1}{T}\left(\frac{\alpha\frac{n}{N}}{1-(1-\alpha\frac{n}{N})p^T}\right)\left(\frac{1-p^T}{1-p}\right)\right]$$

Cyclic control

$$\begin{aligned} rR(N,rT,p,n,\alpha) &= N - \frac{1}{T}\left(\frac{\alpha n}{1-p^{rT}(1-\alpha)}\right)\left(\frac{1-p^{rT}}{1-p}\right) \\ &= N\left[1 - \frac{1}{T}\left(\frac{\alpha\frac{n}{N}}{1-p^{rT}(1-\alpha)}\right)\left(\frac{1-p^{rT}}{1-p}\right)\right] \end{aligned}$$

Now we claim that the average number of failed units for "Cyclic Control" is less than the average number of failed units for "Partial Control" at any time.

If we examine Equations (2.3.5) and (2.4.1), it is enough to check the following inequality,

$$\frac{1}{T}\left(\frac{\alpha n}{1-p^{rT}(1-\alpha)}\right)\left(\frac{1-p^{rT}}{1-p}\right) \geq \frac{1}{T}\left(\frac{\alpha n}{1-(1-\alpha\frac{n}{N})p^{T}}\right)\left(\frac{1-p^{T}}{1-p}\right)$$

As defined before $N$ is equal to $rn$ so we may put $1/r$ instead of $n/N$. After cancellations, previous inequality yields,

$$\frac{1-p^{rT}}{1-p^{rT}(1-\alpha)} \geq \frac{1-p^{T}}{1-(1-\alpha/r)p^{T}}$$

equivalently,

$$(1-p^{rT})[1-(1-\alpha/r)p^{T}] \geq (1-p^{T})[1-p^{rT}(1-\alpha)]$$

after making some cancellations,

$$\frac{p^{T}}{r} - \frac{p^{(r+1)T}}{r} \geq p^{rT} - p^{(r+1)T}$$

Let put terms on the right to the left and denote this function $f(T)$

$$f(T) = \frac{p^{T}}{r} - \frac{p^{(r+1)T}}{r} - p^{rT} + p^{(r+1)T}.$$

Thus, if our claim is true, $f(T) \geq 0$ should be satisfied.

**Theorem 2.3** *If $\alpha \in [0,1]$ and for any $p \in [0,1]$, fixed $T > 0$, $n < N$ where $N = 1, 2, \ldots$ and $r = N/n$, then,*

$$rR(N, rT, p, n) \leq R(N, T, p, n) \tag{2.5.1}$$

*that means the average long-run proportion of failed items for cyclic control is less than the average long-run proportion of failed items for partial (group) control.*

**Proof:**

Consider the function $f(T)$

$$f(T) = \frac{p^{T}}{r} - \frac{p^{(r+1)T}}{r} - p^{rT} + p^{(r+1)T}.$$

To show $f(T) > 0$ we may take derivative with respect to $T$ and check whether it is greater than 0, that is, $f'(T) > 0$. However, it is not visible to show in

this way. So, let

$$x = p^T$$

where $0 \leq p \leq 1$ and $T > 0$, then $0 \leq x \leq 1$. Function depending on $x$ will be the following:

$$h(x) = \frac{x}{r} - x^r + \left(1 - \frac{1}{r}\right) x^{r+1}$$

it yields,

$$h(x) = \frac{x}{r}[1 - rx^{r-1} + x^r(r-1)]$$

where $x/r$ is always positive since $0 \leq x \leq 1$ and $r = 2, 3, 4, \ldots$. Now let $h(x) = (x/r)g(x)$

$$g(x) = 1 - rx^{r-1} + x^r(r-1)$$

Note that on boundary points of $x$, $g(0) = 1$ and $g(1) = 0$. Also, $g(x)$ is continuous on $[0, 1]$. Thus,

$$\begin{aligned} g'(x) &= r(r-1)x^{r-1} - r(r-1)x^{r-2} \\ &= r(r-1)x^{r-2}(x-1) \end{aligned}$$

Since $r(r-1)x^{r-2} > 0$ and $(x-1) < 0$, this means that $g'(x)$ is less than zero. We can conclude that $g(x)$ is strictly decreasing function on $[0, 1]$. Therefore,

$$x \in [0, 1] \stackrel{g(x)}{\rightarrow} [0, 1]$$

Since $0 \leq g(x) \leq 1$ for every $x \in [0, 1]$, in conclusion, we can say that $h(x) > 0$, $x \in (0, 1)$. Therefore our claim that the average number of failed items in "Cyclic Control" is less than or equal to the average number of failed items in "Partial Control" is true. Note that if $0 < p < 1$, we have strict inequality in (2.5.1). □

# Chapter 3

# CONTINUOUS TIME MODELS

In this chapter some modified replacement policies will be studied with multi-component units in continuous time. We will consider their basic stochastic characteristics, such as the number of failed items at any replacement time $jT, j = 1, 2, ...$, where $T$ is fixed, its mean value and long-run proportion. In the first section, we will construct a model without replacement in order to determine the number of failed units at any time $t, t > 0$. In the second section, total control with periodic replacement will be studied. In this model, all failed units are replaced with good ones at time $jT, j = 1, 2, ....$ In sequel sections, partial choice control and cyclic control will be discussed for continuous time.

## 3.1 A Continuous Time Model without Replacement

In this section, we construct a model which constitutes the core of our replacement models in the succeeding sections. $N$ identical units are installed at the beginning. Each unit in the system is assumed to fail with rate $\lambda$.

Let $Q(t)$ denote the number of failed items at time $t$. Suppose that $Q(0) = Q_0$ and let

$$m(t) = E[Q(t)]$$

we will determine $m(t)$ by deriving and then solving a differential equation.

We start by deriving an equation for $m(t+h)$ by conditioning on $Q(t)$. This yields

$$m(t+h) = E[Q(t+h)] = E[E[Q(t+h)|Q(t)]]$$

Now, given the size of the system at time $t$ then, ignoring events whose probability $o(h)$, the system at time $t+h$ will increase in size 1 if a failure occurs in $(t, t+h)$ or remain the same if no failures occurs. That is, given $Q(t)$

$$Q(t+h) = \begin{cases} Q(t) & \text{with probability } 1 - \lambda(N - Q(t))h + o(h) \\ Q(t)+1 & \text{with probability } \lambda(N - Q(t))h + o(h) \end{cases}$$

Therefore,

$$E[Q(t+h)|Q(t)] = Q(t) + \lambda(N - Q(t))h + o(h)$$

Taking expectations yields

$$m(t+h) = m(t) + \lambda(N - m(t))h + o(h)$$

or equivalently

$$\frac{m(t+h) - m(t)}{h} = \lambda(N - m(t)) + \frac{o(h)}{h}$$

Taking the limit as $h \to 0$ yields the differential equation

$$m'(t) = \lambda(N - m(t)) \tag{3.1.1}$$

then

$$\frac{\lambda m'(t)}{\lambda[N - m(t)]} = \lambda.$$

Integration yields

$$-ln[\lambda(N - m(t))] = \lambda t + c$$

Therefore,

$$\lambda(N - m(t)) = Ke^{-\lambda t}$$
$$m(t) = N - \frac{Ke^{-\lambda t}}{\lambda}$$

To determine the value of the constant $K$, we use the fact that $m(0) = m_0$ and evaluate the preceding at $t = 0$.

This gives

$$\lambda(N - m_0) = K$$

Substituting this back in the preceding equation for $m(t)$ yields the following

$$m(t) = N - (N - m_0)e^{-\lambda t}$$

If the number of failed products at the beginning is zero, that is, $Q(0) = 0$, then its expectation will be $m(0) = 0$. Therefore, we get

$$m(t) = N(1 - e^{-\lambda t}), \quad t \geq 0 \tag{3.1.2}$$

When we compare Equation (3.1.2) with Equation (2.1.3), for discrete time $p^k$ is changed with $e^{-\lambda t}$ for continuous case. In conclusion, the number of failed units at time $t, t \geq 0$ has binomial distribution with parameters $N$ and $1 - e^{-\lambda t}$ for $t \geq 0$.

## 3.2 Total Control with Block Replacement

In this section we consider multi-component total control with periodic replacement for continuous time. At time zero $N$ units are in the good condition. After a fixed time period $T$, the replacement of some units are made with probability $\alpha$. That is, while making replacement we make mistake with probability $1-\alpha$. Total control is different from the block replacement in such a way that replacements are allowed only at times $jT, j = 1, 2, ...$ where $T$ is a positive real value. Figure 3.1 summarizes total control. Marks between control times $jT$ and $(j+1)T$ represent the failures.

Unlike block replacement, total control cannot be represented as a renewal

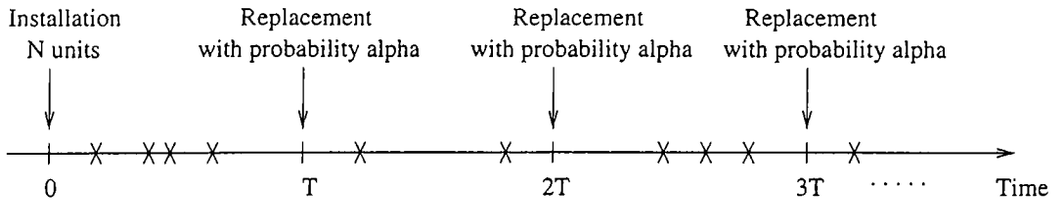


Figure 3.1: Summary of Total Control.

process because of making mistake in replacements at control times. We assume that replacement, repair or inspection are made immediately.

Before starting to calculate some stochastic characteristics of this model, we will define the notation. Note that the following notation will be used for all other continuous time models in this chapter.

*Notation:*

For $j = 1, 2, ...,$

| | |
|---|---|
| $N$ | total number of items in the system. |
| $\alpha$ | probability of changing the failed unit with good one. |
| $\lambda$ | intensity of failure of each item. |
| $Q(jT)$ | number of failed items in the system at time $jT$ before control. |
| $M(jT)$ | expected number of failed units in the system at time $jT$ before control. |
| $Q^+(jT)$ | number of failed units in the system just after control at time $jT$. |

$M^+(jT)$ expected number of failed units in the system just after control at time $jT$.

**Corollary 3.1** *Consider the total control policy in continuous time. The recurrent relation for the number of failed items is given by*

$$Q[(j+1)T] = Q^+(jT) + Bin(N - Q^+(jT), 1 - e^{-\lambda T}) \tag{3.2.1}$$

*where $Bin(N - Q^+(jT), 1 - e^{-\lambda T})$ denotes the number of failed products during operation time $(jT, (j+1)T)$, $j = 1, 2, \ldots$. And the recurrent relation for average number of failed items by using expected value of Equation (3.2.1) is the following,*

$$M[(j+1)T] = (1-\alpha)e^{-\lambda T}M(jT) + N(1 - e^{-\lambda T}) \tag{3.2.2}$$

**Proof:**
Since there is a one to one correspondence between discrete time and continuous time, we simply put $e^{-\lambda T}$ instead of $p^T$ in Equations (2.2.3) and (2.2.5). □

**Corollary 3.2** *The long-run average number of failed items before control for total control is the following:*

$$M_\infty = \frac{N(1 - e^{-\lambda T})}{1 - (1-\alpha)e^{-\lambda T}} \tag{3.2.3}$$

*and the average number of faulty units in the system just after control is given by*

$$M_\infty^+ = (1-\alpha)M_\infty.$$

**Proof:**
We know that the limits exist when the number of inspection times goes to infinity. In Equation (3.2.2) $j$ goes to infinity, we have

$$M_\infty = (1-\alpha)e^{-\lambda T}M_\infty + N(1 - e^{-\lambda T}).$$

And then we get the final average long-run equation. Since we change the failed item with good one with probability $\alpha$, we have $(1-\alpha)M_\infty$ failed items just after control. □

**Theorem 3.1** *The average long-run proportion of the average number of failed items is given by*

$$\frac{1}{T}\int_0^T \left[M_\infty^+ + (N - M_\infty^+)(1 - e^{-\lambda t})\right] dt$$

*as* $t \to \infty$.

**Proof:**
The average proportion of number of failed items at time $t$ is the following:

$$\frac{1}{t}\int_0^t E[Q(t)]dt$$

and when $t = nT$ we can write it in the following form:

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{T}\int_{kT}^{(k+1)T}\left\{E[Q(kT)^+] + (N - E[Q(kT)^+])(1 - e^{-\lambda u})\right\} du.$$

As $n$ goes to $\infty$, then $t$ goes to $\infty$. Now we can use Lemma 2.2 and take into account that as $k \to \infty$,

$$E[Q(kT)^+] \to M_\infty^+.$$

Then as $t \to \infty$, we have

$$\frac{1}{t}\int_0^t E[Q(t)]dt \to \frac{1}{T}\int_0^T \left\{M_\infty^+ + (N - M_\infty^+)(1 - e^{-\lambda u})\right\} du.$$

The case when $t = nT + i$, $i < T$ is similar to the situation in Theorem 2.1. Therefore, it is the same with the case $t = nT$ in the long-run. □

**Corollary 3.3** *The average log-run proportion of failed units in the system for total control is obtained as*

$$R(N, T, \lambda, \alpha) = N\left[1 - \frac{1}{T}\left(\frac{\alpha}{1 - (1-\alpha)e^{-\lambda T}}\right)\frac{1}{\lambda}(1 - e^{-\lambda T})\right]. \qquad (3.2.4)$$

**Proof:**

By Theorem 3.1 the average long-run proportion of the average number of failed items can be calculated using average number of faulty units just after control in the long-run.

$$\begin{aligned} R(N,T,\lambda,\alpha) &= \frac{1}{T}\int_0^T \left[M_\infty^+ + (N - M_\infty^+)(1 - e^{-\lambda t})\right] dt \\ &= M_\infty^+ + \frac{1}{T}(N - M_\infty^+)\left[t + \frac{e^{-\lambda t}}{\lambda}\right]_{t=0}^{t=T} \end{aligned}$$

After taking integration,

$$\begin{aligned} R(N,T,\lambda,\alpha) &= M_\infty^+ + \frac{1}{T}(N - M_\infty^+)\left(T + \frac{e^{-\lambda T}}{\lambda} - \frac{1}{\lambda}\right) \\ &= M_\infty^+ + N - M_\infty^+ + (N - M_\infty^+)\frac{1}{\lambda T}(e^{-\lambda T} - 1) \end{aligned}$$

Substitute $M_\infty^+$ in the long-run equation. Then we get,

$$R(N,T,\lambda,\alpha) = N - \left[N - \frac{N(1 - e^{-\lambda T})(1 - \alpha)}{1 - (1 - \alpha)e^{-\lambda T}}\right]\frac{1}{\lambda T}(1 - e^{-\lambda T}).$$

Rewrite the equation,

$$\begin{aligned} R(N,T,\lambda,\alpha) &= N - N\left[\frac{1 - (1 - \alpha)e^{-\lambda T} - (1 - e^{-\lambda T})(1 - \alpha)}{1 - (1 - \alpha)e^{-\lambda T}}\right]\frac{1}{\lambda T}(1 - e^{-\lambda T}) \\ &= N\left[1 - \left(\frac{\alpha}{1 - (1 - \alpha)e^{-\lambda T}}\right)\frac{1}{\lambda T}(1 - e^{-\lambda T})\right]. \end{aligned}$$

Then we get the final formula for the long-run proportion. □

**Remark:**

Note that the average long-run proportion of expected number of failed items for discrete time and that for continuous time are different, [See Equations (2.2.11) and (3.2.4)].

## 3.3 Partial (Group) Control with Block Replacement

In this section a multi-component modified block replacement policy is introduced for continuous time. Assume that there are $N$ components in the system and each unit may fail with intensity $\lambda$. At time $jT$ where $j = 1, 2, ...$ we take a random sample of size $n$, $n < N$ and then replace faulty units with probability $\beta$. Figure 3.2 shows the summary of partial (group) control. Remark that this model also cannot be represented as a renewal process.

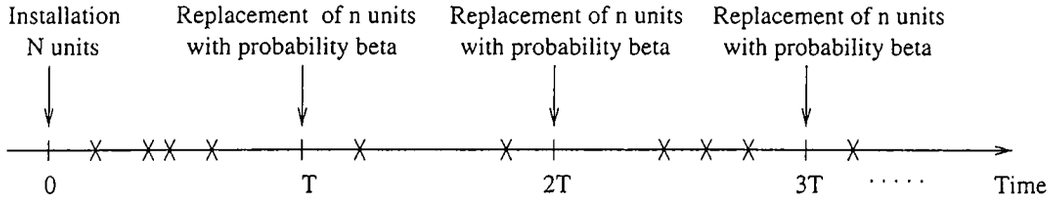


Figure 3.2: Summary of Partial (Group) Control.

Additional notation which we will use:

$n$ size of random sample taken from $N$ units.

$\beta$ probability of changing failed item with good one.

**Corollary 3.4** *Suppose that $Q(jT)$ is the number of failed units at time $jT$, $j = 1, 2, ...$ with expectation $M(jT)$ for partial control. The number of failed units just after control is given by*

$$Q^+(jT) = Q(jT) - Bin\{HG[N, n, Q(jT)], \beta\}$$

*after taking expectations of both sides,*

$$\begin{aligned} M^+(jT) &= M(jT) - \frac{n}{N}\beta M(jT) \\ &= \left(1 - \frac{n}{N}\beta\right) M(jT) \end{aligned}$$

*The recurrent relation is given by*

$$Q[(j+1)T] = Q^+(jT) + Bin(N - Q^+(jT), 1 - e^{-\lambda T}). \qquad (3.3.1)$$

*After taking expectation of both parts,*

$$\begin{aligned} M[(j+1)T] &= M^+(jT) + (N - M^+(jT))(1 - e^{-\lambda T}) \\ &= M(jT)\left(1 - \frac{n}{N}\beta\right)e^{-\lambda T} + N(1 - e^{-\lambda T}) \end{aligned} \tag{3.3.2}$$

**Proof:**

Since there is a one to one correspondence between discrete time and continuous time, we simply put $e^{-\lambda T}$ instead of $p^T$ in Equations (2.3.1), (2.3.2), (2.3.3) and (2.3.4).

**Corollary 3.5** *Consider the partial control in continuous time. The average number of failed items before control in the long-run is obtained as*

$$M_\infty = \frac{N(1 - e^{-\lambda T})}{1 - \left(1 - \beta\frac{n}{N}\right)e^{-\lambda T}} \tag{3.3.3}$$

*The average long-run number of faulty units just after control is given by*

$$M_\infty^+ = (1 - \beta n/N)M_\infty.$$

*The average long-run proportion of failed items in the system is given by*

$$R(N, T, \lambda, n, \beta) = N\left[1 - \frac{1}{T}\left(\frac{\beta\frac{n}{N}}{1 - (1 - \beta\frac{n}{N})e^{-\lambda T}}\right)\frac{1}{\lambda}(1 - e^{-\lambda T})\right]. \tag{3.3.4}$$

**Proof:**

When we compare the basic formulae (3.3.1) and (3.3.2) for partial (group) control with that for total control (3.2.1) and (3.2.2) for continuous time, $\alpha$ in total control is replaced by $\beta n/N$ in partial control for expectations. Therefore, if we want to find the average long-run values for partial control, it is enough to put the coefficient $\beta n/N$ instead of $\alpha$. □

**Remark:**

Note that the average long-run proportion of expected number of failed items for discrete time and that for continuous time are different, [See Equations (2.3.5) and (3.3.4)].

## 3.4 Cyclic Control with Block Replacement

In this section, cyclic control with modified block replacement for continuous time is studied. Total number of units, $N$, in the system is divided into $r$ independent groups each of that has $n$ items. Replacements are allowed only at time $jT$ where $j = 1, 2, ....$ We assume that faulty units are changed immediately, that is, no time is spent for replacement. The first control of group 1 is at time $T$. Failed units in group 2 are replaced at time $2T$. As for time $3T$, control of group 3 is made. Each failed unit is replaced with probability $\alpha$. In general we can say that at time $jT$, $j = 1, 2, ...$ where $j \equiv k$ (modulus $r$), $k = 0, 1, 2, ..., r-1$,

$$\text{Control of Party} \rightarrow \begin{cases} k & \text{if } k \neq 0, \\ r & \text{if } k = 0. \end{cases}$$

Note that this model also cannot be represented as a renewal process, because failed units in party $k$, $k = 1, 2, ..., r$ are changed with probability $\alpha$.

**Corollary 3.6** *Consider the cyclic control in continuous time. The long-run average number of failed units before control is the following,*

$$M_\infty = \frac{N(1 - e^{-\lambda r T})}{1 - (1-\alpha)e^{-\lambda r T}}$$

*and the average number of failed items just after control,*

$$M_\infty^+ = (1-\alpha)M_\infty.$$

*The average long-run proportion of expected number of failed units is given by*

$$rR(N, rT, \lambda, n, \alpha) = N\left[1 - \frac{1}{T}\left(\frac{1 - e^{-r\lambda T}}{\lambda}\right)\left(\frac{\alpha/r}{1 - e^{-r\lambda T}(1-\alpha)}\right)\right]. \quad (3.4.1)$$

**Proof:**

As we stated before in discrete time, cyclic control model can be expressed as $r$ different total control model. If we put in formulae of total control $n$ and $rT$ instead of $N$ and $T$, respectively, we get the formulae of cyclic control for

a party. Since there are $r$ parties in the system, we multiply the formulae obtained for only one party by $r$ to get the long-run average values in the system. □

**Remark:**

Note that the average long-run proportion of expected number of failed items for discrete time and that for continuous time are different, [See Equations (2.4.1) and (3.4.1)].

## 3.5 Comparison of Partial (Group) Control and Cyclic Control

In the the last section of previous chapter we have proved that cyclic control is better than partial (group) control if we consider the number of failed units in the system for the discrete time. In this chapter the following question may arise:

> *Is cyclic control better than partial (group) control for continuous time if we consider the number of failed units?*

We assume that probability of changing failed unit with good one is $\alpha$ for both models for comparison. Therefore, the average long-run proportion of faulty units in the system for partial (group) control is given by

$$R(N,T,\lambda,n,\alpha) = N\left[1 - \frac{1}{T}\left(\frac{\alpha/r}{1-(1-\alpha/r)e^{-\lambda T}}\right)\frac{1}{\lambda}(1-e^{-\lambda T})\right]$$

and for cyclic control,

$$rR(N,rT,\lambda,n,\alpha) = N\left[1 - \frac{1}{T}\left(\frac{1-e^{-r\lambda T}}{\lambda}\right)\left(\frac{\alpha/r}{1-e^{-r\lambda T}(1-\alpha)}\right)\right]$$

If the cyclic control is better than partial control, it is enough to check the following:

$$\frac{1}{T}\left(\frac{1-e^{-r\lambda T}}{\lambda}\right)\left(\frac{\alpha/r}{1-e^{-r\lambda T}(1-\alpha)}\right) > \frac{1}{T}\left(\frac{\alpha/r}{1-(1-\alpha/r)e^{-\lambda T}}\right)\frac{1}{\lambda}(1-e^{-\lambda T})$$

Rewrite the statement after cancellations,

$$\frac{e^{-\lambda T}}{r} - e^{-r\lambda T} + \left(1-\frac{1}{r}\right)e^{-(r+1)\lambda T} > 0$$

Let $x$ be $e^{-\lambda T}$. Since $\lambda > 0$ and $T > 0$, $0 < e^{-\lambda T} < 1$. Let $y(x)$ be a function of $x$ and it should be greater than 0 in order to show our claim being true.

$$y(x) = \frac{x}{r} - x^r + \left(1-\frac{1}{r}\right)x^{r+1}, \qquad 0 < x < 1.$$

Note that $y(x)$ is exactly the same with function $h(x)$ obtained for discrete time. We have already proved in Theorem 2.5 that this function is greater than 0 analytically. So, cyclic control is better than partial (group) control if we consider the number of failed units in the system.

## 3.6 Correspondence Between Discrete and Continuous Time

Consider time $t_k = kh$, $k = 1, 2, \ldots$. $q$ is the probability of failure of a unit in discrete time. Suppose

$$q = q_h = \lambda h + o(h) \tag{3.6.1}$$

and

$$T = T_h = [T/h].$$

It can be easily seen that from our formulae that as $h \to 0$, a model in discrete time is equivalent to a model in continuous time. Let us show what it means. Consider the average long-run proportion of failed items given in (2.2.8) in discrete time. That is,

$$M_\infty = \frac{N(1-p^T)}{1-(1-\alpha)p^T}.$$

Now, if we take $q$ as in Equation (3.6.1), then we have

$$p = p_h = 1 - \lambda h + o(h).$$

Let $T = T_h = [T/h]$. Then, we get

$$M_\infty = \frac{N(1-(1-\lambda h + o(h))^{[T/h]})}{1-(1-\alpha)(1-\lambda h + o(h))^{[T/h]}}.$$

If we multiply $[T/h]$ by $\lambda$ and divide by $\lambda$, then, as $h \to 0$,

$$\begin{aligned} \lim_{h\to 0} M_\infty &= \frac{N(1-(1-\lambda h + o(h))^{[T\lambda/h\lambda]})}{1-(1-\alpha)(1-\lambda h + o(h)^{[T\lambda/h\lambda]})} \\ &= \frac{N(1-e^{-\lambda T})}{1-(1-\alpha)e^{-\lambda T}}. \end{aligned} \tag{3.6.2}$$

Remark that Equation (3.6.2) is the long-run average number of failed items in continuous time. In the same way we can see that when $h \to 0$ there is a one to one correspondence between discrete time and continuous time in all other formulae such as long-run average number failed units just after control, the long-run proportion of failed items in the system etc.

## 3.7 Cyclic Partial Control with Block Replacement

In this section, we consider a cyclic choice control policy with block replacement. Assume that there are $N$ items in the system. All items are divided into $r$ parties each of which has $n$ units, (that is, $N = nr$.) At inspection times $jT, j = 1, 2, ...$, we take a sample of size $m < n$ from a party according to inspection times of parties in cyclic control policy and controlled with probability $\beta$. That is, a sample is taken from party 1 and inspected at time $T$, another one is taken from party 2 and inspected at time $2T$,...

**Corollary 3.7** *Consider the cyclic partial control with block replacement. The long-run average number of failed units just before control is the following,*

$$M_\infty = \frac{N(1 - e^{-\lambda r T})}{1 - (1 - \beta m/n)e^{-\lambda r T}}$$

*and the average number of failed items just after control is expressed as*

$$M_\infty^+ = (1 - \beta m/n)M_\infty.$$

*The average long-run proportion of expected number of failed units is given by*

$$rR(N, rT, \beta, \lambda, m, n) = N\left[1 - \frac{1}{rT}\left(\frac{\beta m/n}{1 - (1 - \beta m/n)e^{-\lambda r T}}\right)\frac{1}{\lambda}(1 - e^{-\lambda r T})\right].$$

**Proof:**
Since cyclic partial control is a combination of cyclic control and partial control, it can be reduced to partial control of $r$ groups. Then, if we want to obtain the formulae of cyclic partial control, it is enough to put in the formulae of total control $\beta m/n$ and $rT$ instead of $\alpha$ and $T$, respectively. □

**Theorem 3.2** *If $\beta \in [0, 1]$ and for any $\lambda > 0$, fixed $T > 0$, $m < n < N$ where $N = 1, 2, ...$ and $r = N/n$. Then, we have*

$$rR(N, rT, \beta, \lambda, m, n) < R(N, T, \lambda, m, \beta)$$

*means that the average long-run proportion of failed items for cyclic partial control is less than the average long-run proportion of failed items for partial (group) control.*

**Proof:**

Consider the equations of long-run proportion for cyclic choice control and partial control. If the cyclic choice control is better than partial control when we consider the average number of failed items, it is enough to check the following:

$$\frac{1}{rT}\left(\frac{\beta m/n}{1-(1-\beta m/n)e^{-\lambda rT}}\right)\frac{1}{\lambda}(1-e^{-\lambda rT}) > \frac{1}{T}\left(\frac{\beta m/N}{1-(1-\beta m/N)e^{-\lambda T}}\right)\frac{1}{\lambda}(1-e^{-\lambda T}).$$

After making cancellations, we have

$$\frac{1-e^{-\lambda rT}}{1-(1-\beta m/n)e^{-\lambda T}} > \frac{1-e^{-\lambda T}}{1-(1-\beta m/N)e^{-\lambda T}},$$

equivalently,

$$(1-e^{-\lambda rT})[1-e^{-\lambda T}+(\beta m e^{-\lambda T})/N] > (1-e^{-\lambda T}[1-e^{-\lambda rT}+(\beta m e^{-\lambda rT})/n].$$

After making cancellations, we get

$$\frac{e^{-\lambda T}}{N}-\frac{e^{-\lambda(r+1)T}}{N} > \frac{e^{-\lambda rT}}{n}-\frac{e^{-\lambda(r+1)T}}{n}.$$

Let

$$x = e^{-\lambda T}, \qquad x \in [0,1].$$

Then, we have

$$\frac{1}{N}(x-x^{r+1})-\frac{1}{n}(x^r-x^{r+1}) > 0,$$

and $f(x)$ denotes the function of $x$, then

$$f(x) = \frac{x}{N}(1-x^r-rx^{r-1}+rx^r) > 0.$$

Let $g(x)$ denote a function which is a part of the $f(x)$. And it is given by

$$g(x) = 1-x^r-rx^{r-1}+rx^r.$$

Note that

$$g(0) = 1 \qquad \text{and} \qquad g(1) = 0.$$

If we take the derivative of $g(x)$ with respect to $x$, we get

$$\begin{aligned} g'(x) &= -rx^{r-1} - r(r-1)x^{r-2} + r^2x^{r-1} \\ &= rx^{r-2}(x-1)(r-1) \end{aligned}$$

Since $r(r-1)x^{r-2}$ is always greater than zero for $r > 1$ and $(x-1) < 0$, then $g'(x)$ is always less than zero. Therefore, $g(x)$ is strictly monotonically decreasing function on $(0,1)$. Finally we can say that since $0 \leq g(x) \leq 1$ for every $x \in [0,1]$, $f(x) > 0$ for every $x \in (0,1)$. Then, cyclic choice control is better than partial control when we consider the average number of failed items. $\square$

# Chapter 4

# COST CONSIDERATIONS

In this chapter, we consider the cost functionals and optimal replacement interval $T^*$. In usual block replacement policies, units are replaced at times $jT$, $j = 1, 2, ...$ and also at failure times. Thus, an almost new item may be replaced by a good one. However, in our models replacement is allowed only at prescribed times. If a unit is failed, it remains inactive until planned control time $jT$, $j = 1, 2, ....$ Now, we will analyze cost functionals for total control, partial control and cyclic control in continuous time.

## 4.1 Formulation and Analysis

In all block replacement models to be considered, a fixed cost $C_0$ is incurred for the system inspection. A cost $C_1$ is suffered for each failed item which is replaced. And a cost $C_2$ is incurred for each failed item staying inactive until the next planned replacement time per unit of time. We denote the expected cost during $[0, t]$ as $B(t)$. The expected long-run cost per unit time for an infinite time span may be expressed as

$$C(T) = \lim_{t \to \infty} \frac{B(t)}{t}$$

(See Barlow and Proschan [6] p.85)

**Theorem 4.1** *The long-run expected cost $C(T)$ per unit of time following a modified block replacement policy during time interval $T$ over an infinite time span is given by*

$$C(T) = \frac{1}{T}\left[C_0 + C_1 E_\infty + C_2 \int_0^T m(t)dt\right]$$

*where $E_\infty = \alpha M_\infty$ is the average number of failed items which is replaced in the long-run and $m(t)$ may be expressed as*

$$m(t) = M_\infty^+ + (N - M_\infty^+)(1 - e^{-\lambda t})$$

*where $M_\infty^+$ is the long-run average number of failed items just after control.*

**Proof:**
Consider the case $t = nT$. The expected cost per unit of time for a time interval $[0, t]$ is the following:

$$\begin{aligned}\frac{B(t)}{t} &= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{T}\{C_0 + C_1\alpha M[(k+1)T] \\ &+ C_2\int_0^t [M(kT)^+ + (N - M(kT)^+)(1 - e^{-\lambda t})]dt\}\end{aligned}$$

as $n \to \infty$, then $t \to \infty$. Now we can use Lemma 2.2. Note that

$$\lim_{k\to\infty} M[(k+1)T] = M_\infty$$

and

$$\lim_{k\to\infty} M(kT)^+ = M_\infty^+.$$

Finally, we have

$$\lim_{t\to\infty}\frac{B(t)}{t} = \frac{1}{T}\left[C_0 + C_1\alpha M_\infty + C_2\int_0^T [M_\infty^+ + (N - M_\infty^+)(1 - e^{-\lambda t}]dt\right]$$

where $E_\infty = \alpha M_\infty$. Consider the case $T = nT + i$, $0 < i < T$. It is the same with the case $t = nT$ as $n$ goes to infinity. (Similar to proof of Theorem 2.1)
□

**Proposition 4.1** *The long-run expected cost per unit of time for total control is given by*

$$C(T) = C_2N + \frac{1}{T}\left[C_0 + \frac{\alpha N(1-e^{-\lambda T})}{1-(1-\alpha)e^{-\lambda T}}\left(C_1 - \frac{C_2}{\lambda}\right)\right]. \tag{4.1.1}$$

**Proof:**

For total control

$$E_\infty = \alpha M_\infty$$

and note that

$$\begin{aligned}\frac{C_2}{T}\int_0^T m(t)dt &= C_2R(N,T,\alpha,\lambda)\\ &= C_2N\left[1 - \frac{1}{T\lambda}\left(\frac{\alpha}{1-(1-\alpha)e^{-\lambda T}}\right)(1-e^{-\lambda T})\right].\end{aligned}$$

$R(N,T,\alpha,\lambda)$ can be written in the following form:

$$R(N,T,\alpha,\lambda) = N - \frac{\alpha}{\lambda T}M_\infty.$$

Then, $C(T)$ is equal to the following:

$$\begin{aligned}C(T) &= \frac{C_0}{T} + \frac{C_1\alpha M_\infty}{T} + C_2N - \frac{C_2\alpha M_\infty}{\lambda T}\\ &= C_2N + \frac{1}{T}\left[C_0 + \frac{M_\infty\alpha}{T}\left(C_1 - \frac{C_2}{\lambda}\right)\right].\end{aligned}$$

If we put $M_\infty$ obtained in Equation (3.2.3) again and get the final long-run cost function. □

**Proposition 4.2** *The long-run expected cost per unit of time for partial control is given by*

$$C(T) = C_2N + \frac{1}{T}\left[C_0 + \frac{\frac{\beta n}{N}N(1-e^{-\lambda T})}{1-(1-\frac{\beta n}{N})e^{-\lambda T}}\left(C_1 - \frac{C_2}{\lambda}\right)\right].$$

**Proof:**

Similar to proof of Proposition 4.1, if we put $\frac{\beta n}{N}$ instead of $\alpha$ and we use $M_\infty$ obtained in Equation (3.3.3) we get the long-run cost for partial control.

□

**Proposition 4.3** *The long-run expected cost per unit of time for cyclic control is given by*

$$C(T) = C_2 N + \frac{1}{T}\left[C_0 + \frac{\alpha n(1-e^{-\lambda rT})}{1-(1-\alpha)e^{-\lambda rT}}\left(C_1 - \frac{C_2}{\lambda}\right)\right].$$

**Proof:**

In cyclic control, each party in the system is inspected after $rT$ units of time. Thus, the system may be thought as $r$ different total control model with $n$ items and operational interval $rT$. Then, the long-run average number of failed items for a party is the following,

$$M_\infty = \frac{n(1-e^{-\lambda rT})}{1-(1-\alpha)e^{-\lambda rT}}.$$

The long-run expected cost for a party is given by

$$c(T) = C_2 n + \frac{1}{rT}\left[C_0 + \frac{n\alpha(1-e^{-\lambda rT})}{1-(1-\alpha)e^{-\lambda rT}}\left(C_1 - \frac{C_2}{\lambda}\right)\right] \tag{4.1.2}$$

Since there are $r$ different parties in the system, we multiply Equation (4.1.2) by $r$ to find the long-run expected cost for the system. Hence, we get the final long-run expected cost. □

**Theorem 4.2** *The long-run expected cost per unit of time for cyclic control is less than that for partial control when $C_1 - (C_2/\lambda) < 0$. Otherwise, the long-run expected cost per unit of time for partial control is less than that for cyclic control.*

**Proof:**

Let us check that $C(T)_d = C(T)_{partial} - C(T)_{cyclic} > 0$ when $C_1 - (C_2/\lambda) < 0$.

$$\begin{aligned} C_d &= \frac{\alpha n(1-e^{-\lambda T})}{T[1-(1-\alpha/r)e^{-\lambda T}]}\left(C_1 - \frac{C_2}{\lambda}\right) \\ &- \frac{\alpha n(1-e^{-\lambda rT})}{T[1-(1-\alpha)e^{-\lambda rT}]}\left(C_1 - \frac{C_2}{\lambda}\right) \\ &= \left(C_1 - \frac{C_2}{\lambda}\right)\frac{\alpha n}{T}\left[\frac{(1-e^{-\lambda T})}{1-(1-\alpha/r)e^{-\lambda T}} - \frac{(1-e^{-\lambda rT})}{1-(1-\alpha)e^{-\lambda rT}}\right] \end{aligned}$$

Since $C_1 - (C_2/\lambda) < 0$ and $(\alpha n)/T > 0$, the remaining part should be less than zero. Hence,

$$\frac{(1-e^{-\lambda T})}{1-(1-\alpha/r)e^{-\lambda T}} < \frac{(1-e^{-\lambda r T})}{1-(1-\alpha)e^{-\lambda r T}}$$

Let

$$x = e^{-\lambda T}, \qquad x \in (0,1),$$

then,

$$\frac{1-x}{1-(1-\alpha/r)x} < \frac{1-x^r}{1-(1-\alpha)x^r},$$

$$1-(1-\alpha)x^r - x + (1-\alpha)x^{r+1} < 1 - x^r - (1-\alpha/r)x + (1-\alpha/r)x^{r+1}$$

After making necessary cancellations, we have

$$\alpha x^r - \alpha x^{r+1} < \frac{\alpha x}{r} - \frac{\alpha x^{r+1}}{r}$$

if we take all terms to the left, then

$$\alpha x\left(x^{r-1} - x^r - \frac{1}{r} + \frac{x^r}{r}\right) < 0.$$

Let $f(x)$ be a function of $x$ given by

$$f(x) = x^{r-1} - \frac{1}{r} - x^r\left(1 - \frac{1}{r}\right).$$

On boundary points of $x$,

$$f(0) = -1/r \quad \text{and} \quad f(1) = 0.$$

If we take the derivative with respect to $x$ then we get,

$$\begin{aligned} f'(x) &= (r-1)x^{r-2} - (r-1)x^{r-1} \\ &= (r-1)x^{r-2}(1-x) \end{aligned}$$

Since $f'(x) > 0$ for $x \in (0,1)$, $f(x)$ is strictly monotonically increasing. Thus, the value of $f(x)$ is less zero. Note that $f(x) = 0$ only if $x = 1$, but for any $T > 0$ and $\lambda > 0$, $e^{-\lambda T} < 1$. That means $C(T)_{partial} > C(T)_{cyclic}$ for every $T > 0$, $\lambda > 0$ when $C_1 - (C_2/\lambda) < 0$. □

**Theorem 4.3** *The long-run expected cost per unit of time for total control is less than that for partial control when $C_1 - (C_2/\lambda) < 0$. Otherwise, the long-run expected cost per unit of time for partial control is less than that for total control.*

**Proof:**
Let us check that $C_d(T) = C(T)_{total} - C(T)_{partial} > 0$ when $C_1 - (C_2/\lambda) > 0$ and for every $\alpha \in [0,1]$ and our case $\alpha = \beta$.

$$\begin{aligned} C_d(T) &= \frac{\alpha N(1-e^{-\lambda T})}{T[1-(1-\alpha)e^{-\lambda T}]}\left(C_1 - \frac{C_2}{\lambda}\right) - \frac{\frac{\alpha n}{N}N(1-e^{-\lambda T})}{T\left[1-\left(1-\frac{\alpha n}{N}\right)e^{-\lambda T}\right]}\left(C_1 - \frac{C_2}{\lambda}\right) \\ &= \left(C_1 - \frac{C_2}{\lambda}\right)\frac{\alpha N(1-e^{-\lambda T})}{T}\left[\frac{1}{1-(1-\alpha)e^{-\lambda T}} - \frac{n/N}{1-[1-(\alpha n)/N]e^{-\lambda T}}\right] \end{aligned}$$

Since $C_1 - (C_2/\lambda) > 0$ and $\alpha N(1-e^{-\lambda T}) > 0$ for every $\lambda > 0$ and $T > 0$, the remaining part should be greater than zero. Thus,

$$\frac{1}{1-(1-\alpha)e^{-\lambda T}} > \frac{n/N}{1-[1-(\alpha n)/N]e^{-\lambda T}}$$

After making necessary cancellations, we have

$$1 - e^{-\lambda T} > \frac{n}{N}(1-e^{-\lambda T}),$$

then

$$(1-e^{-\lambda T})(1-n/N) > 0$$

Since, $1 - e^{-\lambda T} > 0$ and $1 - n/N > 0$ for every $\lambda > 0$, $T > 0$ and $n < N$. Then $C(T)_{total} > C(T)_{partial}$ when $C_1 - (C_2/\lambda) > 0$. □

**Theorem 4.4** *The long-run expected cost per unit of time for total control is less than that for cyclic control when $C_1 - (C_2/\lambda) < 0$. Otherwise, the long-run expected cost per unit of time for cyclic control is less than that for total control.*

**Proof:**
Let us check that $C_d(T) = C(T)_{cyclic} - C(T)_{total} > 0$ when $C_1 - (C_2/\lambda) < 0$.

$$C_d(T) = [C_1 - (C_2/\lambda)]\alpha n\left[\frac{1-e^{-\lambda r T}}{1-(1-\alpha)e^{-\lambda r T}} - \frac{r(1-e^{-\lambda T})}{1-(1-\alpha)e^{-\lambda T}}\right].$$

Since $C_1 - (C_2/\lambda) < 0$ and $\alpha n > 0$ for every $\lambda > 0$ and $T > 0$, the remaining part should be less than zero. That is,

$$\frac{1 - e^{-\lambda r T}}{1 - (1-\alpha)e^{-\lambda r T}} < \frac{r(1 - e^{-\lambda T})}{1 - (1-\alpha)e^{-\lambda T}}$$

Denote

$$x = e^{-\lambda T}, \qquad 0 < x < 1.$$

Then,

$$\frac{1 - x^r}{1 - (1-\alpha)x^r} < \frac{r(1-x)}{1 - (1-\alpha)x} \tag{4.1.3}$$

If we take the inverses of the both sides, then we get

$$\frac{1 - (1-\alpha)x^r}{1 - x^r} > \frac{1 - (1-\alpha)x}{r(1-x)}$$

Note that

$$1 - (1-\alpha)x^r > 1 - (1-\alpha)x$$

and

$$1 - x^r < r(1-x)$$

or equivalently,

$$1 + x + x^2 + ... + x^{r-1} < r, \quad 0 < x < 1, \quad r > 1.$$

Therefore, inequality (4.1.3) holds for every $0 < x < 1$ and $r > 1$. In conclusion, when $C_1 - (C_2/\lambda) < 0$, $C(T)_{cyclic} > C(T)_{total}$. Otherwise, $C(T)_{cyclic} < C(T)_{total}$ holds. □

**A Numerical Example:**

Suppose that $N = 50$, $n = 10$ and $C_0 = 1$. In Table 4.1, long-run expected costs for total control, partial control and cyclic control for different values of parameters are presented where

$$K = C_1 - \frac{C_2}{\lambda},$$

*TC:* Long-run expected cost for total control,

*PC:* Long-run expected cost for partial control,

*CC:* Long-run expected cost for cyclic control.

**Theorem 4.5** *If $C_0 < N\alpha(C_2/\lambda - C_1)$ where $\lambda > 0$, $N = 1, 2, \ldots$ and $C_0, C_1, C_2 > 0$, then the point of minimum $T^*$ of the long-run cost function $C(T)$ for total control exits and is unique. If this condition is not satisfied, then there is no sense to provide a control. (Note that necessary and sufficient condition for partial control and cyclic control is $C_0 < n\alpha(C_2/\lambda - C_1)$.)*

**Proof:**

Consider the total control policy. In order to find the optimal $T^*$ which minimizes the cost, we take derivative of long-run cost function $C(T)$ with respect to $T$. Then we have,

$$\frac{dC(T)}{dT} = -\frac{C_0}{T^2} + \alpha\left(C_1 - \frac{C_2}{\lambda}\right)\frac{m'_\infty T - m_\infty}{T^2} = 0 \tag{4.1.4}$$

where $m'_\infty$ is the derivative of $m_\infty$ respect to $T$. Hence,

$$m'_\infty = \frac{N\lambda e^{-\lambda T}[1-(1-\alpha)e^{-\lambda T}] - N\lambda(1-\alpha)e^{-\lambda T}(1-e^{-\lambda T})}{[1-(1-\alpha)e^{-\lambda T}]^2}.$$

From Equation (4.1.4) and we get

$$\frac{C_0\lambda}{\alpha(C_1\lambda - C_2)} = m'_\infty T - m_\infty. \tag{4.1.5}$$

Put $m'_\infty$ and $m_\infty$ back into the Equation (4.1.5), then,

$$\begin{aligned}\frac{C_0\lambda}{\alpha(C_1\lambda - C_2)} &= \frac{NT\lambda e^{-\lambda T}[1-(1-\alpha)e^{-\lambda T}] - NT\lambda(1-\alpha)e^{-\lambda T}(1-e^{-\lambda T})}{[1-(1-\alpha)e^{-\lambda T}]^2} \\ &- \frac{N(1-e^{-\lambda T})}{1-(1-\alpha)e^{-\lambda T}}.\end{aligned} \tag{4.1.6}$$

Let

$$C = \frac{C_0\lambda}{\alpha(C_1\lambda - C_2)}.$$

After making necessary cancellations, we have

$$\begin{aligned}-C &= N - e^{-\lambda T}[2C(1-\alpha) + NT\lambda - NT\lambda(1-\alpha) + N + N(1-\alpha) \\ &- e^{-2\lambda T}[-C(1-\alpha)^2 - N(1-\alpha)].\end{aligned}$$

We divide both sides by $N$ and let

$$\widehat{C} = \frac{-C}{N} > 0.$$

Then,

$$\begin{aligned} \widehat{C} &= 1 - e^{-\lambda T}[-2\widehat{C}(1-\alpha) + 2 - \alpha + \lambda\alpha T] \\ &- e^{-2\lambda T}(1-\alpha)[\widehat{C}(1-\alpha) - 1]. \end{aligned} \tag{4.1.7}$$

Let

$$a_1 = 2 - \alpha - 2\widehat{C}(1-\alpha) > 0,$$

$$a_2 = \lambda\alpha > 0,$$

and

$$b = 1 - \widehat{C}(1-\alpha) > 0.$$

If we denote by $f(T)$ the function on the right hand side of the Equation (4.1.7), we get

$$f(T) = 1 - e^{-\lambda T}(a_1 + a_2 T) + e^{-2\lambda T}(1-\alpha)b.$$

Note that

$$\begin{aligned} f(0) &= 1 - e^{-\lambda T}(a_1 + a_2 T) + e^{-2\lambda T}(1-\alpha)b \\ &= \widehat{C}(1-\alpha^2) < \widehat{C} \end{aligned}$$

and

$$lim_{T\to\infty} f(T) = 1.$$

If we take the derivative of $f(T)$ with respect to $T$, we have

$$f'(T) = e^{-\lambda T}(\lambda a_1 - a_2 + a_2 T) - e^{-2\lambda T}(1-\alpha)2\lambda b$$

or equivalently,

$$f'(T) = e^{-\lambda T}\varphi(T),$$

where

$$\varphi(T) = \lambda a_1 - a_2 + a_2 T - e^{-\lambda T}(1-\alpha)2\lambda b.$$

Note that $e^{-\lambda T} > 0$ for every $\lambda > 0$ and $T > 0$. Then,

$$\varphi(0) = \lambda a_1 - a_2 - (1-\alpha)2\lambda b,$$

after substituting $a_1, a_2$ and $b$, we get

$$\varphi(0) = -2\alpha\widehat{C}\lambda(1-\alpha) < 0.$$

Now, if we take the derivative of $\varphi(T)$ with respect to $T$, we have

$$\varphi'(T) = a_2 + \lambda e^{-\lambda T}(1-\alpha)2\lambda b > 0.$$

Therefore, $\varphi(T)$ is monotonically increasing function $T$. That means that $f(T)$ is increasing. Finally if $0 < \widehat{C} < 1$, root of $f(T) = \widehat{C}$ is unique. As $C(0) = +\infty$ and $T^*$ is a unique point of solution of $C'(T) = 0$. Then, $T^*$ is the point of minimum. Therefore, optimal $T^*$ exists for total control. Since the parameters in the condition that $C_0 < n(C_2/\lambda - C_1)$ where $\lambda > 0$, $n < N$ and $C_0, C_1, C_2 > 0$, are common for all other control policies, the unique optimal solution $T^*$ exists for partial control and cyclic control also exists.
Suppose now that

$$\widehat{C} > 1.$$

Denote

$$\widehat{C} = \frac{C_0}{\alpha N D}$$

where

$$D = C_2/\lambda - C_1 \quad \text{and} \quad x = e^{-\lambda T}.$$

Then the second term in $C(T)$ of total control without $1/T$ can be written as

$$F(x) = \frac{1}{\alpha N D}\left[\frac{\widehat{C} - \widehat{C}(1-\alpha)x - 1 + x}{1-(1-\alpha)x}\right].$$

Denote the numerator of $F(x)$ as

$$\varphi(x) = \widehat{C} - \widehat{C}(1-\alpha)x - 1 + x$$

which is a linear function of $x$, $0 < x < 1$. Remark that

$$\varphi(0) = \widehat{C} - 1 \geq 0,$$

and

$$\varphi(1) = \alpha\hat{C} > 0.$$

Thus, $\varphi(x) \geq 0$ for every $0 < x < 1$ that means $F(x) > 0$, $0 < x < 1$. In conclusion, for any $0 < T < \infty$,

$$\frac{1}{T}\left[C_0 + \frac{\alpha N(1-e^{-\lambda T})}{1-(1-\alpha)e^{-\lambda T}}\left(C_1 - \frac{C_2}{\lambda}\right)\right] > 0.$$

But this means that $C(T)$ can reach its minimal value only when $T = +\infty$, that is, there is no optimal $T^*$. □

**Example:**

Consider the case $\alpha = 1$. Then,

$$\begin{aligned}\frac{C_0\lambda}{(C_1\lambda - C_2)} &= NT\lambda e^{-\lambda T} - N(1-e^{-\lambda T}) \\ &= NT\lambda e^{-\lambda T} - N + Ne^{-\lambda T}\end{aligned}$$

and finally we have

$$\frac{C_0\lambda}{N(C_2 - C_1\lambda)} = 1 - e^{-\lambda T}(\lambda T + 1). \tag{4.1.8}$$

Note that Equation (4.1.8) is obtained by setting $\alpha$ equal to 1 in Equation (4.1.6).

Let us study the existence conditions of the root of Equation (4.1.8). (See Anisimov and Gürler [2] p. 15-16) Now, assume that

$$0 < \frac{C_0\lambda}{N(C_2 - C_1\lambda)} < 1. \tag{4.1.9}$$

(If this condition is not satisfied, there is no optimal point.)

Denote

$$\phi(T) = 1 - e^{-\lambda T}(\lambda T + 1).$$

After taking derivative with respect to $T$, we get

$$\phi'(T) = \lambda^2 T e^{-\lambda T}$$

Since $\lambda > 0$ and $T > 0$, $\phi'(T)$ is positive. Thus, $\phi(T)$ is strictly monotonically increasing from 0 to 1. If the condition (4.1.9) satisfied then there exists a

root $T^*$ of (4.1.8) and it is unique. Also, it is easy to see that the sign of the derivative of $C(T)$ changes from negative to positive which implies that $T^*$ is unique point of minimum of $C(T)$. Assume that $C_0 = 1$, $C_1 = 0$, $C_2 = 0.0125$, $N = 100$ and $\lambda = 1$. Analytical solution of Equation (4.1.8) gives the optimal point $T^* \cong 3$. Also in Figure 4.1, the graph of $C(T)$ is given and point of minimum occurs when $T \cong 3$.

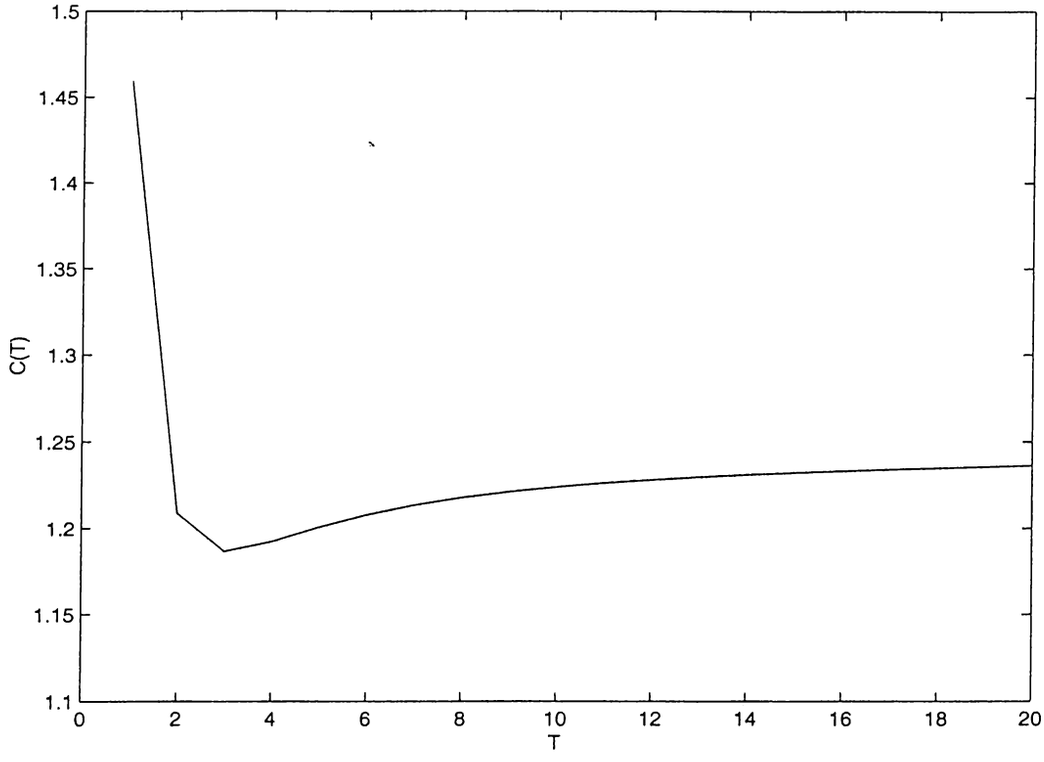


Figure 4.1: The graph of $C(T)$ when $C_0 = 1$, $C_1 = 0$, $C_2 = 0.0125$, $N = 100$ and $\lambda = 1$.

**Results of Simulation:**

Since $\alpha = 1$, there is no failed unit just after control at time $jT$, $j = 1, 2, \ldots$. That is, beginning of each operation time there are $N = 100$ good units. Assume that initial time is zero. Because each item may fail with rate $\lambda$, the first item will fail after $\tau_0$ units of time where $\tau_0$ has exponential distribution with parameter $\lambda_0 = 100\lambda$. The second item will fail after $\tau_1$ units of time after the time of failure of the first item where $\tau_1$ has exponential distribution with parameter $\lambda_1 = 99\lambda$. In the same way, the $(k+1)th$ item will fail after $\tau_k$

units of time after the time of failure of the $k$th item where $\tau_k$ has exponential distribution with parameter $\lambda_k = (100-k)\lambda$. Therefore, if there are $\nu(T) = k$ failed items during time interval $(0,T)$ , then we can get the proportion of expected number of failed items per unit of time in $(0,T)$ by the following:

$$E_{[0,T]} = \frac{1}{T}\left[\sum_{i=0}^{k-1} i\tau_i + k\left(T - \sum_{i=0}^{k-1} i\tau_i\right)\right]$$

Then the expected cost which is suffered for the $k$ failed items staying inactive until the next replacement time can be obtained by

$$C(i) = \frac{C_2}{T}\left[\sum_{i=0}^{k-1} i\tau_i + k\left(T - \sum_{i=0}^{k-1} i\tau_i\right)\right]$$

Finally, when $C_1 = 0$ we can generate $C(T)$ in the following form:

$$C(T) = \frac{1}{T}\left\{C_0 + C_2\left[\sum_{i=0}^{\nu(T)-1} i\tau_i + \nu(T)\left(T - \sum_{i=0}^{\nu(T)-1} i\tau_i\right)\right]\right\} \qquad (4.1.10)$$

where $\nu(T)$ is the number of failed items in $(0,T)$. Then, algorithm for the simulation of $C(T)$ is the following:

**Algorithm:**

*Step* 1: Set $T = T_0$, $t_0 = 0$, $k = 0$.
*Step* 2: Generate $U_k \sim UNIFORM(0,1)$.
*Step* 3: $\lambda_k = (100-k)\lambda$.
*Step* 4: $\tau_k = -(1/\lambda_k)lnU_k$.
*Step* 5: $t_{k+1} = t_k + \tau_{k+1}$.
*Step* 6: If $t_{k+1} < T$ then, $k = k+1$ and go to *Step* 2.
*Step* 7: Calculate $C(T) = \frac{1}{T}\left\{C_0 + C_2\left[\sum_{i=0}^{k-1} i\tau_i + k\left(T - \sum_{i=0}^{k-1} i\tau_i\right)\right]\right\}$

Now, optimal interval $T^*$ can be estimated by simulation. For different values of $T = 0.1, 0.2, ..., 4$, we simulate $C(T)$ as given in Equation (4.1.10). Then, we run the program written in Matlab 100 times and take the average of all

runs for each result of optimal $T^*$. Table 4.2 shows some results. Note that all results are around 2.9. To get more reliable results we repeated this procedure 60 times. If we take the average of runs, we get $T^* = 2.8663$ which is close to analytical solution.

Let us consider now a realistic system when we have to pay a cost for each inspected item in the system at time $kT$, $k = 1, 2, \ldots$. That means if we take a sample of size $n$ at time $kT$, $k = 1, 2, \ldots$, we pay $C_0 n$ as an inspection cost. Now consider the comparison between cost functions of partial and total control. From Theorem 4.3 it is easy to obtain that in case if we pay a fixed cost $C_0$ incurred for the system inspection, then the long-run expected cost per unit of time for partial control is less than that for total control when $C_1 - (C_2/\lambda) > 0$. That means it is enough to consider the case when $C_1 - (C_2/\lambda) < 0$.

**Theorem 4.6** *If $C_0$ is the inspection cost for each item in the system at time $kT$, $k = 1, 2, \ldots$ Denote*

$$D = -K = (C_2/\lambda) - C_1, \quad \textit{where} \quad K < 0$$

*and*

$$\gamma = n/N$$

*and put*

$$g = (1 - \gamma)\frac{C_0}{\alpha D} > 0.$$

- *If $\frac{C_0}{\lambda D} \geq 1$, partial control is better than total control.*
- *If $\frac{C_0}{\lambda D} < 1$, then there exists some $T_*$, $0 < T_* < \infty$ such that when $T > T_*$, total control is better than partial control and when $T < T_*$, partial control is better than total control.*

**Proof:**

The the long-run expected cost per unit of time for total control is given by

$$C_T(T) = C_2 N + \frac{1}{T}\left[NC_0 + \frac{\alpha N(1 - e^{-\lambda T})}{1 - (1 - \alpha)e^{-\lambda T}}K\right].$$

And that for partial control is given by

$$C_T(T) = C_2N + \frac{1}{T}\left[nC_0 + \frac{\alpha n(1-e^{-\lambda T})}{1-(1-\alpha\gamma)e^{-\lambda T}}K\right].$$

Let us investigate the case when $C_d(T) = C_T(T) - C_P(T) > 0$. Then,

$$\begin{aligned} C_d(T) &= (N-n)C_0 - \frac{\alpha N(1-e^{-\lambda T})D}{1-(1-\alpha)e^{-\lambda T}} + \frac{\alpha\gamma N(1-e^{-\lambda T})D}{1-(1-\alpha\gamma)e^{-\lambda T}} \\ &= \alpha ND\left[(1-\gamma)\frac{C_0}{\alpha D} - \frac{1-e^{-\lambda T}}{1-(1-\alpha)e^{-\lambda T}} + \frac{\gamma N(1-e^{-\lambda T})}{1-(1-\alpha\gamma)e^{-\lambda T}}\right] \end{aligned}$$

Denote

$$g = (1-\gamma)\frac{C_0}{\alpha D} > 0,$$
$$x = e^{-\lambda T}, \qquad x \in (0,1).$$

and

$$f(x) = g - \frac{1-x}{1-(1-\alpha)x} + \frac{\gamma(1-x)}{1-(1-\alpha\gamma)x}.$$

Note that

$$f(0) = g - 1 + \gamma,$$
$$f(1) = g.$$

If we rewrite $f(x)$, we have

$$\begin{aligned} f(x) &= \frac{g[1-(1-\alpha\gamma)x-(1-\alpha)x(1-\alpha\gamma)(1-\alpha)x^2)}{[1-(1-\alpha)x][1-(1-\alpha\gamma)x]} \\ &+ \frac{\gamma(1-x)[1-(1-\alpha)x]-(1-x)[1-(1-\alpha\gamma)x]}{[1-(1-\alpha)x][1-(1-\alpha\gamma)x]} \end{aligned}$$

Let $\varphi(x)$ be the numerator of $f(x)$.

$$\begin{aligned} \varphi(x) &= g[1-(1-\alpha\gamma)x-(1-\alpha)x+(1-\alpha\gamma)(1-\alpha)x^2] \\ &- 1+(1-\alpha\gamma)x-(1-\alpha\gamma)x^2+\gamma(1-x)-(1-\alpha)\gamma x(1-x) \end{aligned}$$

Remark that

$$\varphi(0) = g - 1 + \gamma$$

where $\varphi(0)$ can be less than 0 or greater than 0 and,

$$\varphi(1) = \alpha^2\gamma g > 0.$$

In order to determine the behaviour of the function $f(x)$, we take the derivative of $\varphi(x)$ with respect to $x$.

$$\varphi'(x) = (-2 + \alpha\gamma + \alpha)g + 2g(1 - \alpha\gamma)(1 - \alpha)x + 2(1 - x)(1 - \gamma)$$

Then,

$$\varphi'(0) = (-2 + \alpha\gamma + \alpha)g + 2(1 - \gamma)$$

which can be less than 0 or greater than 0.

$$\varphi'(1) = g\alpha(2\alpha\gamma - 1 - \gamma).$$

Since

$$\begin{aligned} 2\alpha\gamma - 1 - \gamma &= \alpha\gamma + \alpha\gamma - 1 - \gamma \\ &= \gamma(\alpha - 1) + (\alpha\gamma - 1) < 0, \end{aligned}$$

$\varphi'(1)$ is always less than 0.

Note that the sign of $f(x)$ is defined by sign of $\varphi(x)$. Table 4.3 shows the sign of $f(x)$. We can summarize them as follows:

1. If $\varphi'(0) < 0$ and $\varphi(0) > 0$,

   $$g > \frac{2(1-\gamma)}{2 - \alpha(1+\gamma)}$$

   and

   $$g > 1 - \gamma$$

   then $\varphi(x) > 0 \qquad 0 < x \leq 1$.

2. If $\varphi'(0) > 0$ and $\varphi(0) \geq 0$,

   $$g < \frac{2(1-\gamma)}{2 - \alpha(1+\gamma)}$$

   and

   $$g > 1 - \gamma$$

   then $\varphi(x) > 0$.

3. If $\varphi'(0) > 0$ and $\varphi(0) < 0$,

$$g < \frac{2(1-\gamma)}{2-\alpha(1+\gamma)}$$

and

$$g < 1-\gamma$$

then since $\varphi(x)$ is continuous and a quadratic function, there exists $x_*$: $\varphi(x_*) = 0$ and $\varphi(x) < 0$ when $0 \leq x < x_*$, $\varphi(x) > 0$ when $x_* \leq x < 1$. Thus, $\varphi(x) < 0$ when $x < x_*$ (that is, $T > T_*$), $\varphi(x) > 0$ and when $x > x_*$ (that is, $T > T_*$).

Note that

$$\frac{2(1-\gamma)}{2-\alpha(1+\gamma)} > 1-\gamma$$

is always true. Then, if we combine the results 1 and 2, we get $g > 1-\gamma$. The result 3 implies that $g < 1-\gamma$. If $g = 1-\gamma$ then $f(0) = 0$ and $f(x) > 0$.

**Results:**

- If $\frac{C_0}{\lambda D} \geq 1$, $f(x) > 0$, $0 < x < 1$. Partial control is better than total control for any $T$.
- If $\frac{C_0}{\lambda D} < 1$, $f(x) < 0$ when $x < x_*$, total control is better than partial control when $T > T_*$. And $f(x) > 0$ when $x > x_*$ that means partial control is better than total control when $T < T_*$, where

$$T_* = -\frac{1}{\lambda} ln x_*. \qquad \square$$

| $\lambda$ | $\alpha$ | $T$ | $C_1$ | $C_2$ | K | TC | PC | CC |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.6 | 2 | 3 | 2 | 1 | 114.2123 | 103.4470 | 103.4999 |
| 1 | 0.8 | 2 | 3 | 2 | 1 | 118.2744 | 104.4023 | 104.4999 |
| 1 | 0.7 | 3 | 4 | 1 | 3 | 84.0951 | 57.2824 | 57.3333 |
| 1 | 0.6 | 2 | 2 | 3 | -1 | 136.7877 | 147.5553 | 147.5001 |
| 1 | 0.8 | 2 | 2 | 3 | -1 | 132.7256 | 146.5977 | 146.5001 |
| 1 | 0.7 | 3 | 1 | 4 | -3 | 165.716 | 193.3843 | 193.3333 |
| 1 | 0.9 | 3 | 5 | 3 | 2 | 178.9824 | 156.2773 | 156.3333 |
| 2 | 0.8 | 3 | 4 | 1 | 3.5 | 96.9074 | 56.6630 | 59.6667 |
| 2 | 0.8 | 3 | 1 | 4 | -3.5 | 187.0265 | 197.6677 | 197.6667 |
| 3 | 0.8 | 3 | 2 | 3 | 1 | 163.6654 | 152.9999 | 153.0000 |
| 3 | 0.8 | 3 | 3 | 2 | 2.33 | 131.4414 | 106.5554 | 106.5556 |

Table 4.1: Long-run Expected Costs of Total Control, Partial Control and Cyclic Control for Different Values of Parameters.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2.801 | 2.933 | 2.851 | 2.836 | 2.893 | 2.851 | 2.839 | 2.831 | 2.875 | 2.846 |
| 2.916 | 2.887 | 2.883 | 2.852 | 2.811 | 2.905 | 2.862 | 2.817 | 2.963 | 2.864 |
| 2.907 | 2.841 | 2.840 | 2.900 | 2.831 | 2.818 | 2.886 | 2.773 | 2.810 | 2.799 |
| 2.971 | 2.901 | 2.847 | 2.869 | 2.910 | 2.763 | 2.831 | 2.862 | 2.865 | 2.863 |
| 3.020 | 2.827 | 2.824 | 2.914 | 2.854 | 2.924 | 2.887 | 2.800 | 2.814 | 2.995 |
| 2.889 | 2.935 | 2.875 | 2.988 | 2.850 | 2.754 | 2.786 | 2.866 | 2.887 | 2.868 |

Table 4.2: Simulated $T^*$ values when $\alpha = 1$, $C_0 = 1$, $C_1 = 0$, $C_2 = 0.0124867$, $N = 100$ and $\lambda = 1$.

| | $\varphi'(0) < 0$ | $\varphi'(0) > 0$ |
|---|---|---|
| $f(0) > 0$ | $f(x) > 0$ | $f(x) > 0$ |
| $f(0) < 0$ | not defined. | $f(x) < 0$ when $x < x^*$<br>$f(x) > 0$ when $x > x^*$ |

Table 4.3: Signs of $f(x)$. (If $f(x) > 0$, partial control is better than total control.)

**Comparisons with Some Models in the Literature:**

There are many studies about block replacement policy and its modifications in the literature. Similarities and differences between the models in this study and the models in the literature are the following:

- The studies in the literature usually are devoted to the analysis of single-unit systems (See examples in Section 1.1). We consider more than one unit (multi-unit) in our models.
- Since all units in the system are replaced by new ones at prescribed times $jT$, $j = 1, 2, ...$ in the literature, the known models are based on the renewal theory. In our models in the case of partial control or replacement with some probability, the points of control are not renewal points in our models. That is why the method of renewal theory is not applicable and we use asymptotic results for Markov process with discrete interference of chance.
- In our models, if an item is failed in an operating time, the system can continue to operate without replacement or repair unless all items in the system are failed. In contrast, if an item is failed during operation, it is usually replaced by a new one or perfectly or minimally repaired immediately in the literature. So the system can not continue to operate without renewal of the system.
- We found some analogies with the models in the following papers:
  *Comparison of the model of Sheu [28]:*
  - The model in [28] has the single-unit system whereas our models has the multi-unit system.
  - Age of the unit determines the repair procedure in [28]. If the system fails at age $y$ in $[(k-1)T, (k-1)T + T_0)$ it is either replaced by a new one with probability $p(y)$ or minimally repaired with probability $q(y)$, and if $[(k-1)T + T_0, kT)$ it is either minimally repaired with probability $p(y)$ or remains inactive until the next

planned replacement with probability $q(y)$. In our models, we allow the failed items to stay inactive until the next replacement time $jT$, $j = 1, 2, ...$ without taking into account of age of the unit.

- If $p(y) = 1$ and $T_0 = T$ in [28], we get in particular total control policy when $\alpha = 1$ for a single-unit system.

*Comparison with the model of Assaf and Shanthikumar [3]:*

- In both models the system is multi-unit and the lifetimes are exponential.
- A repaired unit is considered as good as new in [3]. In our models a failed unit is considered as good as new after replacement.
- In the model [3], it is assumed that the number of failed units in the system is unknown unless an inspection is carried out. And at the time of the inspection the current state of the system (the number of failed units) is known. In our models, the number of failed items in the system is unknown in an operation time and also at times of replacement $jT$, $j = 1, 2, ...$
- In [3], upon an inspection, a decision must be made on whether to repair the failed machines or not based on the number of failed units in the system (a threshold value). In contrast there is no threshold value for the number of failed units in our models because we don't have full information about the state of the system. And also each failed unit is replaced by new one with probability $\alpha$ at times $jT$, $j = 1, 2, ...$
- In both models all considered costs are the same except for an overhead cost is performed in [3]. That means that the mutual part (intersection) of the results of paper [3] and our study is only the model of total control in the case $\alpha = 1$. And if we compare the Equation (4.1.11) without an overhead cost ($C_0$) in the paper [3] (see formula (24) in [3]) with the formula (4.1.1) for total control, in the case $\alpha = 1$, $\tau = T$, $C_0^I = C_0$ and $C_0 = 0$, we have the same expression.

$$W(r) = \frac{C_0^I + C_0(1 - e^{-N\lambda\tau}) + N(C_1 - (C_2/\lambda))(1 - e^{-\lambda\tau})}{\tau} + NC_2 \tag{4.1.11}$$

* If $C_1 - (C_2/\lambda) > 0$, partial control is better than total control under the same cost considerations (proved in Theorem 4.3).
* If we pay a cost for each inspected item, when $C_1 - (C_2/\lambda) < 0$ and under some cost relations (see in Theorem 4.6), the cost of partial control and cyclic control is less than the cost of total control.

That means that partial and cyclic control models are important for study and application since they decrease the cost of maintenance and control of the system.

In conclusion, our study is oriented in the direction when we don't have total information about the current state of the system (the number of failed units). And we want to stress that models with the partial and cyclic control to the best of our knowledge were not considered in the literature in general are more efficient.

# Chapter 5

# CONCLUSION

- In this thesis, we proposed modified multicomponent block replacement policies that are oriented in the direction when we don't have total information about the current state of the system (number of failed items). Three replacement models which can not be represented in terms of renewal processes were presented: Total control, partial (group) control and cyclic control.

- Main relations for any time $t$, $t > 0$ and the expected number of failed items just after control and just before control were obtained in the long-run for discrete and continuous time.

- In both discrete and continuous time, comparison of partial (group) control and cyclic control was given. We concluded that cyclic control is better than partial (group) control when we consider the number of failed items in the system.

- Cost functionals for these models were obtained. The costs of partial control, total control and cyclic control are compared. We concluded that

  - The long-run expected cost per unit of time for cyclic control is less than that for partial control when $C_1 - (C_2/\lambda) < 0$. Otherwise, the

expected cost per unit of time for partial control is less than that for cyclic control.

- The long-run expected cost per unit of time for total control is less than that for partial control when $C_1 - (C_2/\lambda) < 0$. Otherwise, the expected cost per unit of time for partial control is less than that for total control.
- The long-run expected cost per unit of time for total control is less than that for cyclic control when $C_1 - (C_2/\lambda) < 0$. Otherwise, the long-run expected cost per unit of time for cyclic control is less than that for total control.

Comparison is also illustrated by a numerical example for different values parameters.

- Necessary and sufficient conditions of the existence of optimal interval $T^*$ which minimizes the long-run expected cost functions for all types of control policies are obtained. A numerical example with a simulation study for some values of parameters is included.
- If $C_0$ is the inspection cost for each item in the system at time $kT$, $k = 1, 2, ...$
  - If $C_0 \geq C_2 - C_1\lambda$, partial control is better than total control.
  - If $C_0 < C_2 - C_1\lambda$, then there exists some $T_*$, $0 < T_* < \infty$ such that when $T > T_*$, total control is better than partial control and when $T < T_*$, partial control is better than total control.
- Comparison with existing models in the literature shows that some similar results for multi-unit systems are known only for total control case. Under some assumptions about the relations between parameters partial and cyclic control models are better (in the sense of cost functions) than total control models that means that our approach can give better results than the known multi-unit models in the literature.

# Bibliography


[1] V. V. Anisimov, V. I. Sereda, Sampling Inspection in Semi-Markov Systems, Cybernetics, **25**, 400-408 (1989).

[2] V. V. Anisimov, Ü. Gürler, Asymptotic Analysis of a Maintenance Policy for a Multicomponent System, (Unpublished paper).

[3] D. Assaf, J. G. Shanthikumar, Optimal Group Maintenance Policies with Continuous and Periodic Inspections, Management Science, **33**, 1440-1452 (1987).

[4] T. Aven, R. Dekker, A Useful Framework for Optimal Replacement Models, **58**, 61-67 (1997).

[5] R. E. Barlow, L. C. Hunter, Optimum Preventive Maintenance Policies, Operations Research, **8**, 90-100 (1960).

[6] R. E. Barlow, F. Proschan, Mathematical Theory of Reliability, SIAM edition of the work first published by John Wiley and Sons Inc., New York 1965 (1996).

[7] F. Beichelt, A Generalized Block-Replacement Policy, IEEE Trans. on Rel., **30**, 171-172 (1981).

[8] M. Berg, B. Epstein, Comparison of Age, Block and Failure Replacement Policies, IEEE Trans. on Rel., **27**, 25-29 (1979).

[9] E. Berk, K. Moinzadeh, Analysis of Maintenance Policies for $M$ Machines with Deteriorating Performance, IIE Transactions, **5**, 433-444 (2000).

[10] H. W. Block, N. A. Langberg, T. H. Savits, Repair Replacement Policies, J. Appl. Prob., **30**, 194-206 (1993).

[11] M. Brown, F. Proschan, Imperfect Maintenance, J. Appl. Prob., **20**, 851-859 (1983).

[12] D. I. Cho, M. Parlar, A Survey of Maintenance Models for Multi-unit Systems, European Journal of Operational Research, **51**, 1-23 (1991).

[13] H. L. Demirel, Optimal Replacement Policies with Minimal Repair and Random Cost, Bilkent University, M. S. Thesis (1993).

[14] R. A. Fontenot, F. Proschan, Some Imperfect Maintenance Models, Stochastic Failure Models, Optimal Maintenance Policies, Life Testing, and Structure, **10**, 83-101, Academic Press Inc. (1983).

[15] A. Haurie, P. L'ecuyer, A Stochastic Control Approach to Group Preventive Replacement in a Multicomponent System, IEEE Trans. Auto. Control, **27**, 387-393 (1982).

[16] N. Kaio, S. Osaki, Review of Discrete and Continuous Distributions in Replacement Models, Int. J. Systems Sci., **19**, 171-177 (1988).

[17] E. P. C. Kao, Optimal Replacement Rules When Changes of State are Semi-Markovian, Operations Research, **21**, 1231-1249 (1973).

[18] N. A. Langberg, Comparisons of Replacement Policies, J. Appl. Prob., **25**, 780-788 (1988).

[19] T. Nakagawa, Imperfect Preventive-Maintenance, IEEE Trans. on Rel., **28** 402 (1979).

[20] T. Nakagawa, Optimum Policies when Preventive Maintenance is Imperfect, IEEE Trans. on Rel., **28**, 331-332 (1979).

[21] T. Nakagawa, A Summary of Imperfect Maintenance Policies with Minimal Repair, R. A. I. R. O. Operations Research, **14**, 249-255 (1980).

[22] T. Nakagawa, Modified, Discrete Replacement Models, IEEE Trans. on Reliability, **36**, 243-245 (1987).

[23] S. Özekici, Optimal Replacement of Multicomponent Reliability Systems, Operations Research, **36**, 542-552 (1988).

[24] P. Ritchken, P. G. Wilson, $(m, T)$ Group Maintenance Policies, Management Science, **36**, 632-639 (1990).

[25] S. M. Ross, Introduction to Probability Methods, Academic Press, Inc., San Diego, California (1993).

[26] M. Sheked, H. Zhu, Some Results on Block Replacement Policies and Renewal Theory, J. Appl. Prob., **29**, 932-946 (1992).

[27] S. Sheu, Optimal Block Replacement Policies with Multiple choice at Failure, J. Appl. Prob., **29**, 129-141 (1992).

[28] S. Sheu, A Modified Block Replacement Policy with Two Variables and General Random Minimal Repair Cost, J. Appl. Prob., **33**, 557-572 (1996).

[29] S. Sheu, W. S. Griffith, Multivariate Imperfect Repair, J. Appl. Prob., **29**, 947-956 (1992).

[30] T. Tango, A Modified Block Replacement Policy Using Less Reliable Items, IEEE Trans. on Rel., **28**, 400-401 (1979).

[31] F. Van Der Duyn Schouten, S. G. Vanneste,Two Simple Control Policies for Multicomponent Maintenance System, Operations Research, **41**, 194-201 (1993).