International Journal of Mathematics
(2023) 2350111 (8 pages)

DOI: 10.1142/S0129167X23501112

Vector invariants of permutation groups in characteristic zero

Fabian Reimers
Zentrum Mathematik — M11, Technische Universität München
Boltzmannstrasse 3, 85748 Garching bei München, Germany
reimers@ma.tum.de

Müfit Sezer*
Department of Mathematics, Bilkent University
Cankaya, Ankara 06800, Turkey
sezer@fen.bilkent.edu.tr

Received 22 August 2023
Accepted 18 November 2023
Published 21 December 2023

Communicated by Chen-Bo Zhu

We consider a finite permutation group acting naturally on a vector space V over a field $\Bbbk$. A well-known theorem of Göbel asserts that the corresponding ring of invariants $\Bbbk[V]^G$ is generated by the invariants of degree at most $\binom{\dim V}{2}$. In this paper, we show that if the characteristic of $\Bbbk$ is zero, then the top degree of vector coinvariants $\Bbbk[V^m]_G$ is also bounded above by $\binom{\dim V}{2}$, which implies the degree bound $\binom{\dim V}{2}+1$ for the ring of vector invariants $\Bbbk[V^m]^G$. So, Göbel's bound almost holds for vector invariants in characteristic zero as well.

Keywords: Invariant theory; permutation groups; vector invariants.

Mathematics Subject Classification 2020: 13A50

1. Introduction

Let G be a finite group, $\Bbbk$ a field and V a finite-dimensional vector space over $\Bbbk$ on which G acts. The action of G on V induces an action on the symmetric algebra $\Bbbk[V]$ on V^* given by $gf(v) = f(g^{-1}v)$ for $g \in G$, $f \in \Bbbk[V]$ and $v \in V$. Let $\Bbbk[V]^G$ denote the ring of invariant polynomials in $\Bbbk[V]$. This is a finitely generated, graded subalgebra of $\Bbbk[V]$ and a central goal in invariant theory is to determine $\Bbbk[V]^G$ by computing the generators and relations. Let $\beta(G, V)$ denote the maximal degree of a polynomial in a minimal homogeneous generating set for $\Bbbk[V]^G$. It is well known by [4, 5, 9] that $\beta(G, V) \leq |G|$ if $|G| \in \Bbbk^*$. If the characteristic of $\Bbbk$ divides $|G|$,

*Corresponding author.

then the invariant ring is more complicated and there is no bound on $\beta(G,V)$ that applies to all V. It is, however, possible to bound $\beta(G,V)$ using both $|G|$ and dimension of V (see [11]). The Hilbert ideal

$$I(G,V) := \Bbbk[V]_+^G \, \Bbbk[V]$$

is the ideal in $\Bbbk[V]$ generated by all homogeneous invariants of positive degree. The algebra of coinvariants

$$\Bbbk[V]_G := \Bbbk[V]/I(G,V)$$

is the quotient ring by the Hilbert ideal. Both the Hilbert ideal and the algebra of coinvariants are subjects of interest as it is possible to extract information about the invariant ring from them. Since G is finite, $\Bbbk[V]_G$ is finite-dimensional as a vector space and the highest degree in which $\Bbbk[V]_G$ is nonzero is called the *top degree of coinvariants*. This degree plays an important role in computing the invariant ring and is closely related to $\beta(G,V)$ when $|G| \in \Bbbk^*$ (see [8]).

In this paper, we study the case where G is a permutation group acting naturally on V by permuting a fixed basis of V. By a well-known theorem of Göbel [6], $\beta(G,V) \leq \binom{n}{2}$, where n is the dimension of V with $n \geq 3$. This bound applies in all characteristics and it is known to be sharp as for the alternating group A_n we have $\beta(A_n,V) = \binom{n}{2}$. We consider the so-called *vector invariants* of V, which means we have m direct copies $V^m = V \oplus V \oplus \cdots \oplus V$ of V with the action of G extended diagonally. We show that, if $\Bbbk$ has the characteristic zero, then the top degree of the coinvariant ring $\Bbbk[V^m]_G$ is also bounded above by $\binom{n}{2}$. Our method relies on polarizing polynomials in the Hilbert ideal $I(G,V)$ and obtaining enough leading monomials in $I(G,V^m)$ to bound the top degree of $\Bbbk[V^m]_G$. This implies that

$$\beta(G,V^m) \leq \binom{n}{2} + 1.$$

If polarization of a generating set for $\Bbbk[V]^G$ gives a generating set for $\Bbbk[V^m]^G$, then a generating set for $I(G,V^m)$ can be obtained by polarizing any generating set for $I(G,V)$. But, in general, one should not expect to get a Gröbner basis of $I(G,V^m)$ from a Gröbner basis of $I(G,V)$ by polarization.

For more background on invariant theory, we refer the reader to [2] and as a reference for Gröbner bases and related concepts, we recommend [1].

2. Polarization and the Hilbert Ideal

An important ingredient in the proof of our main result is the computation of leading monomials of certain polarized polynomials. To that end, we show that in order to compute the leading monomial of a polarized polynomial it is sufficient to polarize the leading monomial of this polynomial. We identify $\Bbbk[V]$ with $\Bbbk[x_1,\ldots,x_n]$ and $\Bbbk[V^m]$ with $\Bbbk[x_i^{(j)} \mid i = 1,\ldots,n,\ j = 1,\ldots,m]$. We use lexicographic order on

$\Bbbk[V^m]$ with

$$\begin{aligned} & x_1^{(1)} > x_1^{(2)} > \cdots > x_1^{(m)} \\ & \quad > x_2^{(1)} > x_2^{(2)} > \cdots > x_2^{(m)} \\ & \qquad \ddots \\ & \qquad\quad > x_n^{(1)} > x_n^{(2)} > \cdots > x_n^{(m)}, \end{aligned} \tag{2.1}$$

and the order on $\Bbbk[V]$ is obtained by setting $m = 1$. For an ideal I, the lead term ideal of I is denoted by $\mathrm{L}(I)$ and the leading monomial of a polynomial f is denoted by $\mathrm{LM}(f)$. We introduce extra variables $t_1, \ldots, t_m$ and define the algebra homomorphism

$$\Phi : \Bbbk[V] \to \Bbbk[V^m][t_1, \ldots, t_m], \quad x_i \mapsto \sum_{j=1}^{m} x_i^{(j)} t_j.$$

For any $f \in \Bbbk[V]$, we write

$$\Phi(f) = \sum_{(k_1,\ldots,k_m)\in\mathbb{N}^m} f_{k_1,\ldots,k_m} t_1^{k_1} \cdot \ldots \cdot t_m^{k_m},$$

with the polynomials $f_{k_1,\ldots,k_m} \in \Bbbk[V^m]$. This process is known as polarization and for an m-tuple $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$, let $\mathrm{Pol}_{\underline{k}}(f)$ denote the coefficient $f_{k_1,\ldots,k_m}$. We set

$$\mathrm{Pol}(f) = \{\mathrm{Pol}_{\underline{k}}(f) \mid \underline{k} \in \mathbb{N}^m\} \setminus \{0\}.$$

The importance of polarization for invariant theory comes from the fact that $f \in \Bbbk[V]^G$ implies $\mathrm{Pol}(f) \subseteq \Bbbk[V^m]^G$. In addition, it was observed in [8] that for every polynomial $f \in I(G, V)$ we have $\mathrm{Pol}(f) \subseteq I(G, V^m)$.

Lemma 1. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *Let M, M' be monomials of degree d in $\Bbbk[V]$ with $M' < M$. Then for all $\underline{k} \in \mathbb{N}^m$ with $\sum_{j=1}^m k_j = d$, we have*

$$\mathrm{LM}(\mathrm{Pol}_{\underline{k}}(M')) < \mathrm{LM}(\mathrm{Pol}_{\underline{k}}(M)).$$

Proof. Fix $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ with $\sum_{j=1}^m k_j = d$ and let $M = x_1^{a_1} \cdots x_n^{a_n} \in \Bbbk[V]$. We have

$$\Phi(M) = (x_1^{(1)} t_1 + \cdots + x_1^{(m)} t_m)^{a_1} \cdots (x_n^{(1)} t_1 + \cdots + x_n^{(m)} t_m)^{a_n}. \tag{2.2}$$

Let $h = \mathrm{LM}(\mathrm{Pol}_{\underline{k}}(M))$ and write $h = \prod_{i=1}^n \prod_{j=1}^m (x_i^{(j)})^{b_{i,j}}$. Note that h contains, counted with multiplicities, exactly k_1 variables from the set $\{x_1^{(1)}, \ldots, x_n^{(1)}\}$. Since $x_1^{(1)}$ is the highest ranked variable among them, we have $b_{1,1} = \min\{k_1, a_1\}$. More generally, h contains, counted with multiplicities, exactly k_j variables from the set $\{x_1^{(j)}, \ldots, x_n^{(j)}\}$. Moreover, out of a_i factors $(x_i^{(1)} t_1 + \cdots + x_i^{(m)} t_m)^{a_i}$, $b_{i,l}$ of them contribute $x_i^{(l)}$ to h for $1 \leq l < j$. Since $x_i^{(j)}$ is the highest ranked monomial in $\{x_i^{(j)}, \ldots, x_n^{(j)}\}$, we get a recursive relation

$$b_{i,j} = \min\left\{k_j - \sum_{l=1}^{i-1} b_{l,j},\ a_i - \sum_{l=1}^{j-1} b_{i,l}\right\}. \tag{2.3}$$

Equation (2.2) also leads to

$$\Phi(M) = \left(\sum_{\substack{l_1,\ldots,l_m \in \mathbb{N} \\ l_1+\cdots+l_m=a_1}} \frac{a_1!}{l_1!\cdots l_m!}(x_1^{(1)}t_1)^{l_1} \cdot \ldots \cdot (x_1^{(m)}t_m)^{l_m}\right)$$
$$\cdots \left(\sum_{\substack{l_1,\ldots,l_m \in \mathbb{N} \\ l_1+\cdots+l_m=a_n}} \frac{a_n!}{l_1!\cdots l_m!}(x_n^{(1)}t_1)^{l_1} \cdot \ldots \cdot (x_n^{(m)}t_m)^{l_m}\right)$$

by the multinomial theorem. Note that the coefficient of h in $\mathrm{Pol}_{\underline{k}}(M)$ is

$$\prod_{i=1}^{n} \frac{a_i!}{b_{i,1}!\cdots b_{i,m}!}, \tag{2.4}$$

which is nonzero because $\mathrm{char}(\mathbb{k}) = 0$.

We may take $M' = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \cdot x_k^{a'_k} \cdots x_n^{a'_n}$ with $a_{k'} < a_k$. Set $h' = \mathrm{LM}(\mathrm{Pol}_{\underline{k}}(M'))$ and write $h' = \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i^{(j)})^{b'_{i,j}}$. As in the case for $b_{i,j}$, the exponents $b'_{i,j}$ depend only on k_l for $1 \leq l \leq j$ and a_l for $1 \leq l \leq i$. Since the multiplicities of the variables $x_1, \ldots, x_{k-1}$ in M and M' are the same, we get that $b_{i,j} = b'_{i,j}$ for $i < k$. On the other hand, since $\sum_{l=1}^{m} b_{k,l} = a_k > a_{k'} = \sum_{l=1}^{m} b'_{k,l}$, the equality $b_{k,l} = b'_{k,l}$ fails for some $1 \leq l \leq m$. Let j denote the smallest index such that $b_{k,j} \neq b'_{k,j}$. Then we have

$$\begin{aligned} b'_{k,j} &= \min\left\{k_j - \sum_{l=1}^{k-1} b'_{l,j},\ a'_k - \sum_{l=1}^{j-1} b'_{k,l}\right\} \\ &= \min\left\{k_j - \sum_{l=1}^{k-1} b_{l,j},\ a'_k - \sum_{l=1}^{j-1} b_{k,l}\right\} \\ &\leq \min\left\{k_j - \sum_{l=1}^{k-1} b_{l,j},\ a_k - \sum_{l=1}^{j-1} b_{k,l}\right\} \\ &= b_{k,j}. \end{aligned}$$

It follows that $b'_{k,j} < b_{k,j}$. So we have that $h' < h$. □

Remark 2. The coefficient of the highest ranked monomial h which is given by Eq. (2.4) is nonzero over all fields $\mathbb{k}$ with $a_i! \in \mathbb{k}^*$ for all i. So the assertion of

Lemma 1 is true for all pairs of monomials M', M with $M = x_1^{a_1} \cdots x_n^{a_n}$ satisfying $a_i < \mathrm{char}(\Bbbk)$ for all i.

Remark 3. Consider the lexicographic monomial order with a slightly different ordering of the variables

$$x_1^{(1)} > x_2^{(1)} > \cdots > x_n^{(1)} > x_1^{(2)} > x_2^{(2)} > \cdots > x_n^{(2)} > \cdots > \cdots > x_n^{(m)}.$$

For the fixed $1 \le i \le n$ and $1 \le j \le m$, the sets of variables in $\{x_1^{(j)}, \ldots, x_n^{(j)}\}$ and in $\{x_i^{(1)}, \ldots, x_i^{(m)}\}$ that are smaller than $x_i^{(j)}$ remain unchanged. So the recursive description of the leading monomial $\mathrm{LM}(\mathrm{Pol}_{\underline{k}}(M))$ in Eq. (2.3) and consequently the assertion of Lemma 1 carry over to this ordering as well.

Proposition 4. *Let $f \in I(G, V)$ be a homogeneous polynomial in the Hilbert ideal. Assume that* $\mathrm{char}(\Bbbk) = 0$ *or* $\mathrm{char}(\Bbbk) > \deg(f)$. *Then we have*

$$\mathrm{LM}(\mathrm{Pol}_{\underline{k}}(\mathrm{LM}(f))) \in \mathrm{L}(I(G, V^m))$$

for all $\underline{k} \in \mathbb{N}^m$.

Proof. With $M := \mathrm{LM}(f)$, we can write f as

$$f = c \cdot M + \sum_{M' < M} c_{M'} \cdot M,$$

where the sum runs over all monomials in $\Bbbk[V]$ that are smaller than M. Since $\mathrm{Pol}_{\underline{k}}$ is linear, we get

$$\mathrm{Pol}_{\underline{k}}(f) = c \cdot \mathrm{Pol}_{\underline{k}}(M) + \sum_{M' < M} c_{M'} \cdot \mathrm{Pol}_{\underline{k}}(M').$$

By Lemma 1 and Remark 2, we get

$$\mathrm{LM}(\mathrm{Pol}_{\underline{k}}(\mathrm{LM}(f))) = \mathrm{LM}(\mathrm{Pol}_{\underline{k}}(f)).$$

Since $\mathrm{Pol}(f) \subseteq I(G, V^m)$ for all $f \in I(G, V)$ by [8, Lemma 12], the result follows. $\square$

We now prove our main result.

Theorem 5. *Let G be a permutation group acting naturally on $V = \Bbbk^n$. Assume that* $\mathrm{char}(\Bbbk) = 0$ *or* $\mathrm{char}(\Bbbk) > n$. *Then we have*

$$\beta(G, V^m) \le \binom{n}{2} + 1.$$

Proof. A Gröbner basis for $I(S_n, V)$ has been computed in [12]. From this source, we get that

$$\mathrm{L}(I(S_n, V)) = (x_1, x_2^2, \ldots, x_n^n).$$

Since $I(S_n, V) \subseteq I(G, V)$, we get $(x_1, x_2^2, \ldots, x_n^n) \subseteq \mathrm{L}(I(G, V))$. For $1 \le i \le n$ and the vector $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ with $k_1 + k_2 + \cdots + k_m = i$, we consider

$\mathrm{Pol}_{\underline{k}}(x_i^i)$. Since $x_i^i \in \mathrm{L}(I(G,V))$, Proposition 4 yields $\mathrm{LM}(\mathrm{Pol}_{\underline{k}}(x_i^i)) \in \mathrm{L}(I(G,V^m))$. To identify $\mathrm{LM}(\mathrm{Pol}_{\underline{k}}(x_i^i))$, we write $\Phi(x_i^i) = (x_i^{(1)}t_1 + \cdots + x_i^{(m)}t_m)^i$. Then we get

$$\mathrm{Pol}_{\underline{k}}(x_i^i) = \frac{i!}{k_1! \cdots k_m!} \prod_{j=1}^{m} (x_i^{(j)})^{k_j}.$$

Note that since $i \leq n$, the coefficient is nonzero. It follows that $\mathrm{L}(I(G,V^m))$ contains the set of monomials

$$\left\{ \prod_{j=1}^{m} (x_i^{(j)})^{k_j} \,\middle|\, 1 \leq i \leq n,\ \sum_{j=1}^{m} k_j = i \right\}.$$

Therefore, the top degree of the coinvariants $\Bbbk[V^m]_G$ is bounded above by $\binom{n}{2}$. It follows that every monomial of degree $\binom{n}{2}+1$ in $\Bbbk[V^m]$ is in $I(G,V^m)$. So we get that $I(G,V^m)$ is generated by the polynomials of degree at most $\binom{n}{2}+1$. Now we apply a standard argument to get a bound for $\beta(G,V^m)$ as follows. Let $f_1,\dots,f_s$ be homogeneous generators for $I(G,V^m)$ of degree at most $\binom{n}{2}+1$. We may assume these generators lie in $\Bbbk[V^m]^G$. Let $f \in \Bbbk[V^m]^G$ be homogeneous of degree $> \binom{n}{2}+1$. Write $f = \sum_{i=1}^{s} q_i f_i$ with $q_i \in \Bbbk[V^m]_+$. Let $\mathrm{Tr} : \Bbbk[V^m] \to \Bbbk[V^m]^G$ denote the transfer map defined by $\mathrm{Tr}(h) = \sum_{\sigma\in G} \sigma(h)$ for $h \in \Bbbk[V^m]$. Then we have $\mathrm{Tr}(f) = |G|f = \sum_{i=1}^{s} \mathrm{Tr}(q_i) f_i$. Therefore, f is in the algebra generated by invariants of strictly smaller degree. So we get $\beta(G,V^m) \leq \binom{n}{2}+1$ as desired. $\square$

Remark 6. Theorem 5 is no longer true when the characteristic of $\Bbbk$ divides $|G|$. In fact, by establishing a lower bound on the maximal degree of a polynomial in a minimal generating set, it was proven in [10] that $\lim_{m\to\infty} \beta(G,V^m) = \infty$, where G is a nontrivial group acting faithfully on V. This lower bound on $\beta(G,V^m)$ was later sharpened for the full symmetric group [3] and general permutation groups [7].

A natural question is when the polarization of a generating set for $I(G,V)$ gives a generating set for $I(G,V^m)$. We note that this will be always the case if polarization of a generating set for $\Bbbk[V]^G$ gives a generating set for $\Bbbk[V^m]^G$.

Proposition 7. *Assume that $\Bbbk[V^m]^G$ is generated by polarizations of a homogeneous generating set for $\Bbbk[V]^G$. Then $I(G,V^m)$ is generated by polarizations of any homogeneous generating set for $I(G,V)$.*

Proof. Assume that $\Bbbk[V]^G = \Bbbk[g_1,\dots,g_d]$ and

$$\Bbbk[V^m]^G = \Bbbk[\mathrm{Pol}_{\underline{k}}(g_1),\dots,\mathrm{Pol}_{\underline{k}}(g_d) \mid \underline{k} \in \mathbb{N}^m].$$

Let $f_1,\dots,f_r$ be a homogeneous generating set for $I(G,V)$. Then there exists $q_{j,i} \in \Bbbk[V]$ such that $g_i = \sum_j q_{j,i} f_j$ for $1 \leq i \leq d$ and $1 \leq j \leq r$. Since Φ is an algebra homomorphism, we have

$$\operatorname{Pol}_{\underline{k}}(g_i) = \sum_{\underline{k_1}+\underline{k_2}=\underline{k},\, j} \operatorname{Pol}_{\underline{k_1}}(q_{j,i}) \operatorname{Pol}_{\underline{k_2}}(f_j)$$

for $1 \leq i \leq d$. Since polarizations of g_i for $1 \leq i \leq d$ generate $\Bbbk[V^m]^G$, it then also follows that polarizations of f_j for $1 \leq j \leq r$ generate $I(G, V^m)$, as desired. $\square$

The hypothesis of Proposition 7 is satisfied for the natural action of the symmetric group in characteristic zero (see [13]). But it does not remain true for all permutation groups as we show in the following example that polarization of a generating set for the Hilbert ideal of the natural action of the alternating group does not give a generating set for the Hilbert ideal of two copies of the action. The symmetric group case also has a flaw: for example, a polarization of a Gröbner basis for $I(S_3, V)$ does not give a Gröbner basis for $I(S_3, V^2)$.

Example 8. We take $V = \Bbbk^3$ and $m = 2$, where $\Bbbk$ is a field of characteristic zero. Then we have $\Bbbk[V] = \Bbbk[x_1, x_2, x_3]$ and for simplicity, we identify $\Bbbk[V^2]$ with $\Bbbk[x_1, x_2, x_3, y_1, y_2, y_3]$. For one copy, Gröbner bases for the Hilbert ideals of the symmetric group S_3 and the alternating group A_3 are given by $I(S_3, V) = (h_1, h_2, h_3)$ and $I(A_3, V) = (h_1, h_2, h_3, f)$, respectively, where

$$\begin{aligned} h_1 &= x_1 + x_2 + x_3, \\ h_2 &= x_2^2 + x_2x_3 + x_3^2, \\ h_3 &= x_3^3, \\ f &= x_1^2x_2 + x_1x_3^2 + x_2^2x_3 \end{aligned}$$

(see [12]). We show that $T := x_1y_2 + x_2y_3 + x_3y_1$ in $I(A_3, V^2)$ does not lie in the ideal generated by polarizations of the generators of $I(A_3, V)$. Since polarizations preserve degrees, it is enough to show that T does not lie in the ideal $J := (\operatorname{Pol}_{\underline{k}}(h_i) \,|\, \underline{k} \in \mathbb{N}^2, i = 1, 2)$ generated by polarizations of h_1 and h_2. A routine computation gives that $J = (x_1+x_2+x_3, y_1+y_2+y_3, x_2^2+x_2x_3+x_3^2, 2x_2y_2+x_2y_3+y_2x_3+2x_3y_3, y_2^2+y_2y_3+y_3^2)$. Note that the s-polynomials of any pair of polynomials with leading monomials which are not relatively prime in the above generating set are of degree three or more. It follows that J has a Gröbner basis consisting of the generators of J listed above together with polynomials of degree at least three. But T has degree two and the normal form T' of T with respect to this Gröbner basis is

$$\begin{aligned} T' &= T - y_2(x_1 + x_2 + x_3) - x_3(y_1 + y_2 + y_3) + \frac{1}{2}(2x_2y_2 + x_2y_3 + y_2x_3 + 2x_3y_3) \\ &= \frac{3}{2}(x_2y_3 - y_2x_3), \end{aligned}$$

which does not reduce any further, so we get that $T \notin J$.

Now, we consider $I(S_3, V^2)$ which is generated by the polarizations of h_1, h_2, h_3 by Proposition 7. A quick computation yields that the set of leading monomials of these polarizations is $M := \{x_1, y_1, x_2^2, x_2y_2, y_2^2, x_3^3, x_3^2y_3, x_3y_3^2, y_3^3\}$. We also have

$$(4y_2 + 2y_3)\operatorname{Pol}_{(2,0)}(h_2) + (-x_3 - 2x_2)\operatorname{Pol}_{(1,1)}(h_2) = -3x_2x_3y_3 + 3y_2x_3^2.$$

So $x_2x_3y_3 \in \mathrm{L}(I(S_3, V^2))$ but $x_2x_3y_3$ is not divisible by any monomial in M.

ORCID

Fabian Reimers https://orcid.org/0000-0003-4552-8186

Müfit Sezer https://orcid.org/0000-0002-6704-5399

References

[1] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics, Vol. 3 (American Mathematical Society, Providence, 1994).

[2] H. Derksen and G. Kemper, *Computational Invariant Theory*, 2nd enlarged edn., Encyclopaedia of Mathematical Sciences: "Invariant Theory and Algebraic Transformation Groups", Vol. 130, No. VIII (Springer, Heidelberg, 2015) [With two appendices by Vladimir L. Popov, and an addendum by Norbert A'Campo and Popov].

[3] P. Fleischmann, A new degree bound for vector invariants of symmetric groups, *Trans. Amer. Math. Soc.* **350**(4) (1998) 1703–1712.

[4] P. Fleischmann, The Noether bound in invariant theory of finite groups, *Adv. Math.* **156**(1) (2000) 23–32.

[5] J. Fogarty, On Noether's bound for polynomial invariants of a finite group, *Electron. Res. Announc. Amer. Math. Soc.* **7** (2001) 5–7.

[6] M. Göbel, Computing bases for rings of permutation-invariant polynomials, *J. Symb. Comput.* **19**(4) (1995) 285–291.

[7] G. Kemper, Lower degree bounds for modular invariants and a question of I. Hughes, *Transform. Groups* **3**(2) (1998) 135–144.

[8] M. Kohls and M. Sezer, On the top degree of coinvariants, *Int. Math. Res. Not.* **2014**(22) (2014) 6079–6093.

[9] E. Noether, Der endlichkeitssatz der invarianten endlicher gruppen, *Math. Ann.* **77**(1) (1915) 89–92.

[10] D. R. Richman, Invariants of finite groups over fields of characteristic p, *Adv. Math.* **124**(1) (1996) 25–48.

[11] P. Symonds, On the Castelnuovo–Mumford regularity of rings of polynomial invariants, *Ann. Math.* **174**(1) (2011) 499–517.

[12] T. Wada and H. Ohsugi, Gröbner bases of Hilbert ideals of alternating groups, *J. Symb. Comput.* **41**(8) (2006) 905–908.

[13] H. Weyl, *The Classical Groups*: *Their Invariants and Representations* (Princeton University Press, Princeton, 1939).