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Decision Support

# Competitive location and pricing on a line with metric transportation costs

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ABSTRACT

Consider a three-level non-capacitated location/pricing problem: a firm first decides which facilities to open, out of a finite set of candidate sites, and sets service prices with the aim of revenue maximization; then a second firm makes the same decisions after checking competing offers; finally, customers make individual decisions trying to minimize costs that include both purchase and transportation. A restricted two-level problem can be defined to model an optimal reaction of the second firm to known decision of the first.

For non-metric costs, the two-level problem corresponds to Envy-free Pricing or to a special Network Pricing problem, and is $\mathcal{APX}$-complete even if facilities can be opened at no fixed cost. Our focus is on the metric 1-dimensional case, a model where customers are distributed on a main communication road and transportation cost is proportional to distance. We describe polynomial-time algorithms that solve two- and three-level problems with opening costs and single $1^{st}$ level facility. Quite surprisingly, however, even the two-level problem with no opening costs becomes $\mathcal{NP}$-hard when two $1^{st}$ level facilities are considered.



## 1. Introduction

In this paper, we consider a multi-level location and pricing problem. More precisely, we model location, pricing and purchase policies of several decision-makers as a Stackelberg game (Stackelberg, 1952), formulate a consequent three-level optimization problem and describe methods for its solution under various assumptions. The model has applications in sectors where companies are considering candidate locations for opening a new service point as well as a pricing policy for capturing clients from an existing competitor.

In the proposed setting, we have a discrete market (finite set $C$ of $n$ customers) where a company (Firm A) wants to choose sites and prices for the service offered at its facilities so as to maximize revenue. When making its choice, Firm A has to consider that, at a second level, another company (Firm B) will offer the same service and, seeking to maximize its own utility, will react to Firm A by opening new facilities (possibly at fixed costs) and setting the relevant service prices. Both firms have finally to take into account that, at a third level, customers in $C$ will choose the facilities where to get serviced by minimizing the sum of service price and general access cost. In applications, access costs can either be metric, that is, proportional to a distance somehow defined, or general non-metric. A metric assumption is normally reasonable when the cost reflects transport, but also when it measures the distance, in terms of features, between the product/service offered by the firm and the one required by the customer.

When dealing with location and pricing, we assume negligible variability of prices, opening costs and demand in the planning horizon. If this is not the case, the strategic decision on location should be withdrawn from the tactical one on pricing. An appropriate setting, then, is when Firm A first places its facilities, then Firm B places its own ones, then prices are set by firms simultaneously: this leads to quite a different model that is not considered here. Also multi-echelon, multi-service, finite capacities can be envisaged as in classical location problems, but are not addressed here. Rather, as in many spatial competition models, we focus on the case in which both facilities and customers are *distributed along a line*. This assumption has two faces: first, it describes real situations where potential production or retail sites, as well as clients, are distributed with a linear topology along a main communication road; secondly, it shows how a simple assumption on the number of competing facilities can unexpectedly mark the boundary between “easy” and “hard” problems.

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Our contribution is twofold:

(i) On one hand, we focus on difficult two-level problems with Firm B reacting to prices of Firm A given in advance. We easily see (Section 3.3) that the non-metric problem is $\mathcal{APX}$-complete, although it can be approximated in polynomial time within a factor $O(\log(n))$; here we show (Theorem 13 of Section 6) that the metric problem remains $\mathcal{NP}$-hard even for markets scattered on a line, no opening costs for Firm B, and Firm A holding two facilities only.

(ii) On the other hand, we provide a polynomial-time algorithm for the three-level metric problem on the line, where Firm B may bear opening costs and makes its decision after A, which in turn holds one facility only (Section 5); simpler algorithms are devised for the two-level cases with or without opening costs. When Firm A holds more than one facility, a polynomial-time algorithm can easily be devised for the two-level metric problem with opening costs and a constant number of facilities scattered on the line (Corollary 3 of Section 4).

The paper is organized as follows. A review of literature is provided in Section 2. General assumptions are introduced and discussed in Section 3, along with the general statement of the two- and three-level problems. Special properties of the one-dimensional metric case, with a short algorithmic digression, are described in Section 4. The case in which Firm A holds a single facility is then tackled in Section 5, and illustrated with two examples. Section 6 treats the case in which Firm A has more than one facility. Conclusions are then drawn in Section 7.

## 2. Literature review

Location problems lie firmly in the OR domain, but completeness of presentation of the problem requires to consider also its interface with two fields: theoretical economics (in particular, spatial competition) and computer science (envy-free pricing).

In theoretical economics, models of spatial competition among firms have a long tradition that dates back to the seminal work of Hotelling (Hotelling, 1929). This is a fruitful setting for analyzing location and pricing policies in an oligopolistic market that is often distributed on a line, see e.g. (Fischer, 2002; Perez & Pelegrin, 2003; Pinto & Parreira, 2015; Teraoka, Osumi, & Hohjo, 2003) and the references therein. However, most of those models concentrate on conditions for a price equilibrium, whereas practical situations often call for a focus on optimal decisions, and on the complexity of their computation. In fact, the common assumption of complete information does not imply that information can easily be sorted out; in turn, the existence of an equilibrium does not mean that the relevant prices can easily be computed. More in general, computation deserves interest in view of the increasing role of information technology in commerce and logistics. As a matter of fact, getting good decisions from information entails costs (to collect and validate data, construct and compare models, compute results, etc.) to which firms are normally sensitive. And a practical question on computation arises even if one is not concentrated on equilibrium: for example when Firm A has already made a (possibly sub-optimal) decision, and one is interested in the optimal reaction of Firm B.

Computational complexity is actually the focus of a closely related problem: Envy-free Pricing, originally introduced in the area of theoretical computer science (Guruswami et al., 2005). Here, a retailer wants to sell a set of items, for which potential buyers have personal valuations or reserve prices.

An envy-free assignment of product items to buyers requires that the items obtained by every buyer be purchased at a price not larger than her/his valuation, and each buyer's welfare (difference between product value and price) be the largest possible. Under this condition, the problem of finding prices maximizing the seller's revenue is known to be $\mathcal{APX}$-hard even for unit-demand bidders, that is when each buyer wishes to buy at most one item.

As explained in Section 3.3, our two-level problem corresponds to the unit-demand envy-free pricing problem tackled in Guruswami et al. (2005). Fernandes, Ferreira, Franco, and Schouery (2016) investigate connections of this problem with Network Pricing (Heilporn, Labbé, Marcotte, & Savard, 2010b), compare mixed-integer linear programming formulations for the non-metric case, and provide an instance generator. Heilporn, Labbé, Marcotte, and Savard (2010a,b) introduced network pricing problems that, as we will see in Section 3.3, include our two-level problem as a special case. In Heilporn, Labbé, Marcotte, and Savard (2011), the same authors present a related problem that consists in selecting profit maximizing tolls on a clique subset of a multicommodity transportation network: a linear mixed-integer programming formulation is proposed, and then strengthened via effective valid inequalities, some of which define facets. All these models (and ours) are indebted to Labbé, Marcotte, and Savard (1988), and set a two-level Stackelberg game where the leader is a firm that tries to enter a market with given fares, and the clients are its followers.

The main difference between most network or envy-free pricing models and ours is that, in our case, access costs or reserve prices possess *metric properties*. These properties are clearly reasonable as long as costs derive from transportation. Note however that metrics may not only be introduced to relate logistics to physical distances: in modeling actors' behaviour, for instance, one may wish to measure the difference between desired and available product attributes. To quote an example, Crama et al. (Crama, Hansen, & Jaumard, 1995) address the complexity of the Product Positioning problem. It consists in choosing the attributes (not prices, though) of a new product so as to maximize its market share, i.e., to attract a maximum number of customers: a customer group switches to a new product if its attributes are closer, with respect to a metric, to the customers' ideal than those of the product currently chosen.

A recent contribution on envy-free pricing is however closely related to the present work and can be found in Chen, Ghosh, and Vassilvitskii (2011), where Metric Substitutability is considered. Both items (= facilities) and customers are located at every node of a symmetric graph: thus a customer in a node has to decide whether buying an item at the residence place, paying no substitution cost, or at another place, paying a cost for the substitution. Substitution cost is modeled as a distance between customer and item position, and therefore is metric; customer valuation instead depends on customer position only. Moving to our terms, customers' valuations can then be viewed as the price at which a competitor sells the item in the residence location. Note that this price does not depend on the competitor location. In our case instead, the competitors lie in some position, and the cost to purchase their service depends on that position. Due to this difference, our problem is $\mathcal{NP}$-hard even for a simple path topology and regardless of customers' valuations (even the same for all customers: in fact, minimizing price plus access cost, each customer maximizes its utility), while the problem in Chen et al. (2011) can be solved in polynomial time.

References Chen et al. (2011); Fernandes et al. (2016); Heilporn et al. (2010a, 2010b, 2011); Labbé et al. (1988), as well as Berglund and Kwon (2014); Karakitsiou and Migdalas (2017); Tóth and Kovács (2016) and many others dealing with pricing under equilibrium conditions, share a methodological vein typical of OR, namely one oriented towards problem resolution. Some OR papers focus on competitive facility location but base customers behaviour just on distance and not on price, see e.g. (Gentile, Pessoa, Poss, & Roboredo, 2018) and references therein. These and even simpler two-level optimization problems are all $\mathcal{NP}$-hard in general, but special assumptions or solution properties are often investigated, leading sometimes to encouraging computational results (for a general view on sequential competitive location the reader can refer to Eiselt (1997); Kress and Pesch (2012)).

Three-level problems are considerably harder in practice than two-level ones (Blair, 1992; Jeroslow, 1985); maybe this explains why relatively little work can be found in the literature, despite the indubitable interest. Several important studies, that we do not quote here, can be found in a bibliographical survey (Vicente & Calamai, 1994). Papers are mainly devoted to applications, ranging from supply-chain management (Xu, Meng, & Shen, 2013) to power network defence (Yao, Edmunds, Papageorgiou, & Alvarez, 2007) or energy generation and transmission (Jin & Ryan, 2014; Street, Moreira, & Arroyo, 2014). Among theoretical papers, we quote the pioneering single-follower linear problem in Bard (1984), and the multi-follower linear problem in Han, Lu, Hu, and Zhang (2015). In the latter, decision levels are arranged in a tree structure, with a single top-level leader and multiple middle- and bottom-level followers. The followers at the same level compete with one another in a non-hierarchical relation, thus their decisions do not tend to a Stackelberg equilibrium but to a Nash one (if any).

On one hand, our three-level case is simpler than (Han et al., 2015) not only because the middle-level contains a single firm, but also because — unlike (Han et al., 2015) — bottom-level followers do not compete with each other. On the other hand, our case is more complex as the top- and middle-level objectives are not linear but bilinear forms in the relevant decision variables. An implicit three-level location problem similar to ours is studied in Fischer (2002): the main difference lies in prices, which unlike our case are discriminatory and in one model are adjusted afterwards according to a Nash equilibrium; a second difference is in the result, that is focused on models and heuristics rather than on exact algorithms and complexity.

## 3. General assumptions and problem statement

In the following we let $F_A$, $F_B$, respectively, denote the potential sites where firms A and B already hold or can place a facility to sell a product or a service; we set $|F_k| = m_k, F = F_A \cup F_B, |F| = m$, and consider a discrete market distribution $C, |C| = n$, namely a finite numerable set of customers potentially interested in being serviced from the facilities opened in $F$. Firms and customers are assumed *rational*, in the sense that each actor seeks to maximize its own utility, and *non-cooperative*. Prior to problem statement (Section 3.3), in Sections 3.1 and 3.2 we specify the rules according to which firms' and customers' decisions are made.

### 3.1. Firms' decisions

A *two-level facility location problem* is a location problem where decisions are made according to an uncooperative Stackelberg game between Firm B and a set $C$ of customers. The game is *sequential*: the firm makes its decision first — i.e., decides which facility $k \in F_B \subset F$ to open and at which price $\pi_k$ to offer the service; then the customers make their own decision, according to local convenience. To exclude monopoly, facilities in $F_A = F - F_B$ are owned by a competitor, Firm A, that sells the same service at known prices. Prices can differ from place to place but are *non-discriminatory*, that is, the same price $\pi_k$ is asked from any of the clients served by facility $k$. Firm B's objective is to maximize its own utility — that is the total value of the service sold to customers — regardless of the decisions of followers, who in turn aim at optimizing their individual criteria.

Levels can be added to the model, increasing its complexity. So, in a *three-level facility location problem* Firm A and Firm B with candidate sites $F_A$, $F_B$ are in a leader-follower relation. Both firms aim at maximizing utility and operate with perfect knowledge of the other firm's objective, the customers' preferences, and the space of potential reactions. In particular, and unlike some traditional spatial competition models (P-P, Mayer, & J-F, 2008), each firm is assumed to know in advance whether a price $\pi_k$ is convenient or not for customer $i$. The firms' decision sequence is as follows:

1. First Firm A (the leader) decides which facilities to open in $F_A$, and at what price to offer their services.
2. Then Firm B (the middle follower) decides which facilities to open in $F_B$ and at what price to offer the service, playing a follower role vs. the top-level leader, and a leader role vs. the bottom-level followers, i.e., the customers.

Actors' *rationality* implies that no firm would accept a sub-optimal solution, in particular one with negative utility. In the models considered here, it can be shown in particular that no negative price can improve a firm's performance, so we do not lose generality by supposing

**Assumption 1** ($\pi_k \geq 0$ for all $k \in F$)**.** We already observed that firms behaviour is *non-cooperative*. We stress this issue by a specific assumption that uses the notion of vector dominance: a price vector $\pi$ dominates another $\pi'$ if $\pi_k \leq \pi'_k$ for all $k$ and $\pi_k < \pi'_k$ for at least one $k$. Then

**Assumption 2.** Firms are not allowed to artificially increase prices. More precisely, among all optimal solutions, each firm is obliged to choose a non-dominated one.

Assumption 2 is compliant with the existence of an authority that regulates competition and prevents monopolies, as well as with general markets freely accessible by new operators. Its role will be better clarified in Section 3.2.

After firms have made their decisions, customers make their own choices according to the assumptions described next.

### 3.2. Customers' decisions

User preference is quite a common way of disciplining bottom-level followers' behaviour, see (Hansen, Kochetov, & Mladenović, 2004): each customer holds a list that ranks the sites of $F$ from the most to the least preferred and, after the firms' decision, prioritizes the facility with the highest rank among those opened. If preference is defined after a measurable parameter $d^i_k$ that depends on customer $i$ and facility $k$ (e.g., the cost of reaching the facility) then we speak of *measurable preference*. In this case, the list of customer $i$ ranks the facilities from the smallest to the largest $d^i_k$ (with a specified tie-breaking rule in case of equally preferred facilities). A measurable preference is *metric* if $d^i_k$ fulfills the metric axioms of non-negativity, identity, symmetry, and sub-additivity. Otherwise it is *non-metric*. Notice that sub-additivity ($d^i_k \leq d^j_i + d^k_j$ for all $i$, $j$, $k \in C \cup F$) requires $d^i_k$ defined not only for $i \in C$ and $k \in F$, but for any $i$, $k \in C \cup F$.

The metric axioms are often fulfilled by such measures as the reaching cost. In this case, the space where firms and customers operate is either linear (often reduced to the closed interval [0,1], see (Hotelling, 1929; Pinto & Parreira, 2015; Teraoka et al., 2003)) or bi-dimensional (generally $\mathbb{R}^2$ (Fischer, 2002)). More in general, however, $d^i_k$ can be derived from any norm $\|.\|$ applied to a vector pair in $\mathbb{R}^p$. As suggested in Crama et al. (1995), vectors describe either user requirements $\mathbf{r}^i$ or service features $\mathbf{f}_k$ along $p$ different dimensions: in this way, for example, $d^i_k = \|\mathbf{r}^i - \mathbf{f}_k\|$ measures the distance between service offered at $k$ and user $i$ expectations.

In the problems dealt with here, no preferred order among facilities is a-priori given, but additional costs (for access, transportation, or adaptation to requirements etc.) are considered: as in Heilporn et al. (2010a,b); Hotelling (1929); Pinto and Parreira (2015), these costs are borne by customers, that is, we suppose that $d^i_k$ measures the cost customer $i$ must add to service price $\pi_k$ in order to reach the facility, or cover the gap between offer and expectation. Thus,

**Assumption 3.** Customer $i$ — that has complete information on both types of costs — accords its preference to the facility $k$ that minimizes, among those opened by the firms, the sum $\pi_k + d^i_k$.

Two more assumptions regulate tie-breaking on customers decisions:

**Assumption 4.** If facility $j \in F_A$ and facility $k \in F_B$ offer service to $i$ at the same total cost $\pi_j + d^i_j = \pi_k + d^i_k$, then customer $i$ prefers $k$.

In other words, the assumption reflects a pessimistic perspective of Firm A. This is not a limit of the two-level model: it can in fact be shown that the optimal utility of Firm B under Assumption 4 differs by an arbitrarily small $\epsilon > 0$ from that obtained with the assumption reversed in favour of Firm A.

It is convenient to explain the joint role of Assumptions 2 (Section 3.1) and 4 by an example: suppose that Firms A and B operate, respectively, facility 1 and 2 at reciprocal distance 1, and that a single customer is located on the segment joining the two facilities at distance (= cost) $d > \frac{1}{2}$ from facility 1. Then, in force of Assumption 4, for Firm B to capture the customer it is sufficient to set $\pi_2 + (1 - d) = \pi_1 + d$, that is, $\pi_2 = \pi_1 + 2d - 1$. In the worst case for Firm B we have $\pi_1 = 0$, and Firm B utility is $2d - 1 > 0$. But Firm A is indifferent to any price $\pi_1 \geq 0$, because by Assumption 4 it results in no utility in any case: so, should Assumption 2 not hold and co-operation be allowed, Firm B utility would be unbounded.

The case in which a client obtains the service from two facilities of the same firm at the same total cost is regulated by supposing that the client prefers the closest facility:

**Assumption 5.** If $d^i_j < d^i_k$ and both facilities $j$ and $k$ in $F_A$ (or both in $F_B$) offer service to $i$ at the same total cost $\pi_j + d^i_j = \pi_k + d^i_k$, then customer $i$ prefers $j$.

Note that this assumption is optimistic for both firms A and B because it means that, if total costs are equal, the customer chooses the facility with highest price. We remark that an opposite behaviour would not alter mathematical properties of the problem, since also in this case, under a pessimistic assumption, an $\varepsilon$-perturbation of a price vector, for an arbitrarily small $\varepsilon > 0$, would ensure an amount of revenue arbitrarily close to the optimistic setting.

### 3.3. Problem statement

Let $c_k$ be the cost of opening a facility at $k \in F_B$. From now on, when indexing a parameter, facilities will be denoted at subscripts and customers at superscripts.

Firm A (Firm B) decision variables are

- $x_k$: binary, set to 1 if a facility is opened in site $k \in F_A$ ($k \in F_B$), 0 otherwise;
- $\pi_k$: real, price applied to any customer served by facility $k \in F_A$ ($k \in F_B$).

Customers' decision variables are

- $y^i_k$: binary, set to 1 if customer $i \in C$ gets service from $k \in F$, 0 otherwise.

A two-level problem models the reaction of Firm B to the decision of Firm A:

**Problem 1.** Given prices $\pi_j$, $j \in F_A$, find values of $x_k$ and $\pi_k \in \mathbb{R}$, $k \in F_B$, that maximize Firm B total utility:

$$B(\mathbf{x}, \pi, \mathbf{y}) = \sum_{k \in F_B} x_k \sum_{i \in C} \pi_k y^i_k - \sum_{h \in F_B} c_k x_k$$

subject to

$$\pi_k \in \Pi_k,\ x_k \in \{0, 1\}\ k \in F_B \tag{1}$$

$$\mathbf{y}^i = \arg\min_{\mathbf{y}^i} \left\{ \sum_{h \in F} (\pi_h + d^i_h) y^i_h : \sum_{h \in F} y^i_h = 1, 0 \leq y^i_h \leq x_h, h \in F \right\}\ i \in C \tag{2}$$

$\Pi_k$ indicates a general set of positive real numbers corresponding to feasible prices. The *feasible region* of Firm B is $R_B = \Pi_1 \times \cdots \times \Pi_{m_B} \times \{0, 1\}^{m_B}$, and the *inducible region* of Problem 1 is $R_B \times Y$, where $Y \subseteq \{0, 1\}^{mn}$ is the so-called *rational reaction set* of the customers described by (2).

**Observation 1.** We can suppose $F_A \cap F_B = \emptyset$: by Assumption 4, if two facilities share a site and offer identical prices, then Firm A is ruled out (if the assumption is pessimistic for Firm B, any price $\pi_k > 0$ of Firm A is beaten by $\pi_k - \varepsilon$).

**Observation 2.** In the non-metric case we lose no generality by representing the top-level facilities as a single dummy facility 0 with price $\pi_0 = 0$. In fact, one can set $d^i_0 = \min_{k \in F_B} \{\pi_k + d^i_k\}$ for all $i \in C$. Adopting this simplification in the metric case will not preserve metric properties as sub-additivity is of course no longer guaranteed.

**Observation 3.** For $c_k = 0$, Problem 1 is a special case of the network pricing problem tackled in Heilporn et al. (2010b) and of several pricing problems in marketing (Heilporn et al., 2010a), and is equivalent to the $\mathcal{APX}$-complete Unit Demanded Envy-free Pricing problem (Guruswami et al., 2005) defined as follows: client $i$ has a reservation price $r^i_k$ for service in a facility $k$ that exhibit price $\pi_k$, and wishes to choose a $k$ that maximizes the difference $r^i_k - \pi_k$ (provided that it is $\geq 0$); on the other hand, the seller wishes to maximize the profit $\Sigma_{k \in F} n_k \pi_k$, where $n_k$ indicates the number of clients served by facility $k$. To transform this problem into ours, it suffices to choose $\hat{r}^i > \max_k \{r^i_k\}$ and set $d^i_k = \hat{r}^i - r^i_k$. Then, client $i$ objective is equivalent to $\min_{k \in F} \{\pi_k + \hat{r}^i - r^i_k\}$, that is $\min_{k \in F} \{\pi_k + d^i_k\}$.

The three-level problem, in general, regards Firm A as a leader that chooses sites and prices facilities while taking Firm B optimal reaction into account. However, we do not insist here on site selection because the two-level problem is already $\mathcal{APX}$-hard with a single competing facility in the non-metric case, and is $\mathcal{NP}$-hard with two competing facilities in the metric case on the line, as we will show in Section 6. Therefore, in the formulation of the three-level problem we assume $c_j = 0$ (and then $x_j = 1$) for all $j \in F_A$.

**Problem 2.** Find values for $y^i_j, \pi_j$, $j \in F_A$, $i \in C$, that maximize Firm A total utility

$$A(\pi, \mathbf{y}) = \sum_{j \in F_A} \sum_{i \in C} \pi_j y^i_j$$

subject to

$$\pi_j \in \Pi_j \quad j \in F_A \tag{3}$$

$$\pi_k = \arg\max_{\mathbf{x}, \pi, \mathbf{y}} \{B(\mathbf{x}, \pi, \mathbf{y}) : (1), (2)\}\ \ k \in F_B. \tag{4}$$

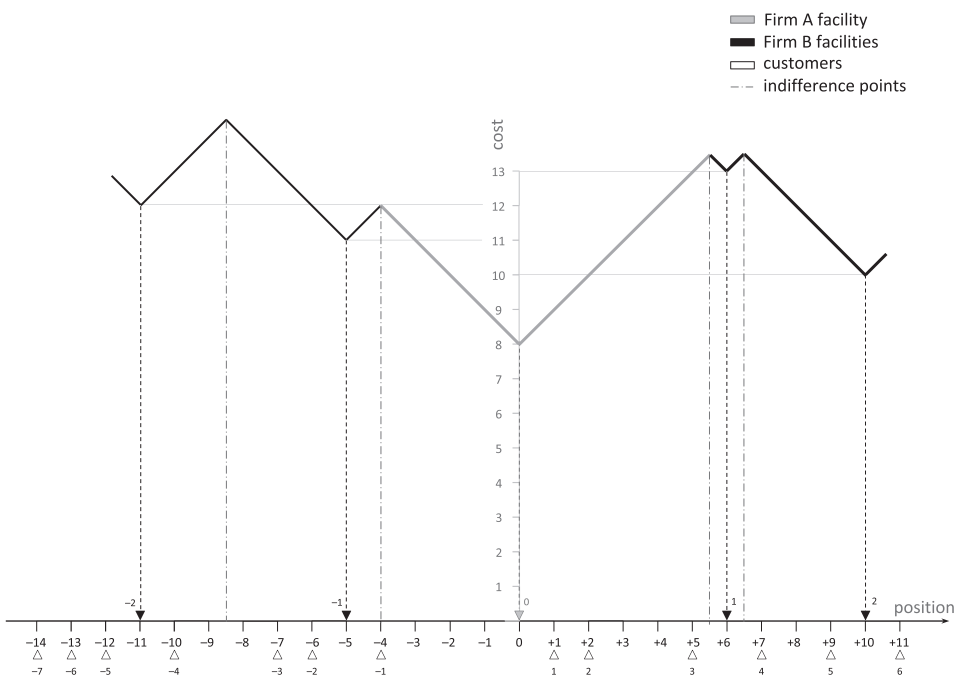


**Fig. 1.** Example of customer partitioning after a price assignment.

**Table 1**
Sample problem with $|C| = 13$ and $|F| = 5$.

| Facilities | | | Customers | |
|---|---|---|---|---|
| Index $k$ | Position $p_k$ | Price $\pi_k$ | Index $i$ | Position $p^i$ |
| −2 | −11 | 12 | −7 | −14 |
| −1 | −5 | 11 | −6 | −13 |
| 0 | 0 | 8 | −5 | −12 |
| 1 | 6 | 13 | −4 | −10 |
| 2 | 10 | 10 | −3 | −7 |
| | | | −2 | −6 |
| | | | −1 | −4 |
| | | | 1 | 1 |
| | | | 2 | 2 |
| | | | 3 | 5 |
| | | | 4 | 7 |
| | | | 5 | 9 |
| | | | 6 | 11 |

## 4. Metric properties on the line

From this section on we will focus on the case where facilities and customers are located in a *metric space* and $d_k^i$ is the *distance* between customer $i$ and facility $k$. The space is 1-*dimensional*, thus representable as a straight line on which a reference (origin) is defined: call $p_k$ the position (abscissa) of facility $k$, and $p^i$ that of customer $i$. In this case, for any $t = 1, 2, \ldots, \infty$, the $\mathcal{L}_t$-norm defines the distance between customer $i$ and facility $k$ as

$$d_k^i = \left|p_k - p^i\right|.$$

Consistently with notation, the distance between facilities (customers) $i$ and $j$ is denoted as $d_{ij}$ (as $d^{ij}$). Facility sites are distinct and ranked so that $p_j < p_k$, $p^j \le p^k$ for $j < k$. Recall (Observation 2) that metric properties are generally not retained when ascribing the facilities of Firm A to a single dummy facility 0.

To fix ideas, consider the situation depicted in Fig. 1 and summarized in Table 1. Triangles below the horizontal line represent customers, those over the line represent facilities: one, coloured red and indexed 0, belongs to firm A; the others, coloured blue, belong to firm B. Facilities in $F_B$ and customers are progressively numbered from left to right, with negative indexes to the left of facility 0 and positive to its right: in the example we e.g. see $p_{-1} = -5$, $p^3 = 5$, $d_{02} = 10$, $d_{-1}^2 = 6$, $d^{25} = 7$. The piecewise linear function gives the minimum cost borne by a customer in any point of the line, and its local minima correspond to facility prices.

Remember (Assumption 3) that client $i$ prefers a facility that minimizes $\pi_k + d_k^i$ in $F$. One can then use a plot like in Fig. 1 to assign customers to facilities for any given set of prices $\pi_k$, $k \in F$, and then observe

**Proposition 1.** *Let $k \in F$, and $i < j$ be two customers that use facility $k$. Then, any customer $h$ for which $i < h < j$ also uses facility $k$.*

Thus, for any price vector, each facility will serve exactly one integer interval $[i, j]$ of customers indexes on the line. In the example of Table 1 and Fig. 1, facility $-2$ serves interval $[-7, -4]$; facility $-1$, interval $[-3, -1]$; facility 0, interval [1,3]; facility 1 serves no customer and facility 2 the remaining ones. Note that customer $-1$ is assigned to facility $-1$ and not to 0 due to Assumption 4.

Moreover, one observes the following property:

**Proposition 2.** *Let $\pi$ be any price vector, $k \in F$ and $i$ be a customer served by $k$. Then a facility $l$ placed between $i$ and $k$ does not serve any customer.*

**Proof.** If a facility $l$ is placed between $i \in C$ and $k \in F$, then $d_k^i = d_{kl} + d_l^i$. Since $i$ is served by $k$, Assumption 3 requires in general $\pi_k + d_k^i \le \pi_l + d_l^i$; then $d_{kl} \le \pi_l - \pi_k$. For any client $j$, we know by sub-additivity that $d_k^j - d_l^j \le d_{kl}$. Combining the two inequalities we get:

$$d_k^j - d_l^j \le d_{kl} \le \pi_l - \pi_k \quad \Rightarrow \quad \pi_k + d_k^j \le \pi_l + d_l^j, \tag{5}$$

If $k \in F_A$, $l \in F_B$, then $l$ does not serve any customer. Otherwise, Assumptions 4,5 require $\pi_k + d_k^i < \pi_l + d_l^i$ and the second of (5) holds strict, leading to the same conclusion. □

From now on, for a given price vector $\pi$, we will indicate as $[\lambda_k, \rho_k]$ ($\lambda$ = left, $\rho$ = right) the range of customer indexes served by facility $k$, when such range is non-empty. The number of customers in the range $[\lambda_k, \rho_k]$ is given by $n_k = \rho_k - \lambda_k + 1$ for $\rho_k \lambda_k > 0$, and $n_k = \rho_k - \lambda_k$ otherwise.

A set $Y$ of pairs in $C \times F$ is an assignment of customers to facilities if $ik \in Y \Rightarrow ij \notin Y$ for any $j \neq k$; in other words, $Y$ is identified by the incidence vector $(\mathbf{y}^1, \ldots, \mathbf{y}^n)$ defining a feasible customer choice. Consider the following notion:

**Definition 1.** An assignment $Y$ is *stable* if there exists a price vector $\pi$ such that $\pi_k + d^i_k \le \pi_j + d^i_j$ for all $ik \in Y$ and $j \in F$.

Let

$$d = \max_{j,k \in F, i \in C} \{d^i_j - d^i_k\}.$$

Then a nice consequence of Definition 1 and Proposition 1 is

**Corollary 3.** *In the metric 1-dimensional case Problem 1 can be solved in* $O(n^{|F|-1}|F|^3 \log(|F|) \log(d|F|))$ *time, that is in polynomial time for a fixed number of facilities.*

**Proof.** Given a stable assignment $Y$, let $Y_k = \{ik \in Y : i \in C\}$ for any $k \in F$. An optimal price for this assignment is an optimal solution to the linear program

$$\begin{aligned} \max \quad & B_Y(\pi) = \sum_{k \in F_B} |Y_k| \pi_k \\ & \pi_k - \pi_j \le d^i_j - d^i_k \qquad \forall j \in F, ik \in Y \\ & \pi_k \ge 0 \qquad \forall k \in F_B. \end{aligned} \tag{6}$$

Note that if $Y$ is stable with $|Y_k| = 0$ for some $k$, then (6) has infinite optimal solutions, because $\pi_k$ can take infinite values without changing the utility value: we call *minimal* an optimal solution of (6) that has $\pi_k$ minimum for all such $k$, and point out that only minimal solutions are compliant with Assumption 2. Note also that allocations that are optimal for (6) are all equivalent from the viewpoint of the revenue guaranteed by the corresponding prices, see (Arbib, Karaşan, & Pınar, 2017). Problem 1 can then be solved by enumerating the assignments $Y$ of $C$ to $F$ and solving problem (6) for each of them. By Proposition 1, those assignments are as many as the partitions of the integer interval $[1, \ldots, n]$ into $|F|$ subintervals, i.e. $O(n^{|F|-1})$. Moreover, problem (6) is the dual of a min cost flow problem, and $d$ is the largest capacity in any of these problems. Thus (6) is solvable in $O(|F|^3 \log(|F|) \log(d|F|))$ by Tarjan's algorithm (Tarjan, 1997). □

## 5. One-dimensional metric problem, $|F_A| = 1$

We next describe an efficient algorithm for two- and three-level one-dimensional metric problems, limiting our attention to the case in which Firm A has *just one facility*, denoted by 0. Our notation is as in Fig. 1:

- Firm A facility is placed on the origin of the line, that is, $p_0 = 0$.
- Firm B can choose among $m_B$ facility sites (for simplicity of notation, we set $m_B = m$); those to the right (to the left) of 0 are numbered with positive (negative) integers, starting from 1 and increasing from left to right (from $-1$ and decreasing from right to left).
- Similarly, customers to the right (to the left) of 0 are numbered with positive (negative) integers, starting from 1 and increasing from left to right (from $-1$ and decreasing from right to left).

To solve the problem, we observe a property of *isolation at optimality* that derives from the following two results.

**Lemma 4.** *(Isolation) Let $-1$ (let 1) denote the first facility opened by Firm B on the left (on the right) of facility 0, and let $\pi \in \mathbb{R}^m$ be a vector of prices for $F_B$. If $\pi$ is optimal, then any customer $i$ with $p^i \ge p_1$ or $p^i \le p_{-1}$ will be served by a facility owned by Firm B.*

**Proof.** Suppose by contradiction that $\pi$ is optimal and that a customer $i$ located at $p^i \ge p_1$ is served by the competing facility 0 (Fig. 2). But (Assumptions 3 and 4) $i$ served by 0 means $\pi_1 + d^i_1 > \pi_0 + d^i_0$. Since $p^i \ge p_1$, the distance between $i$ and 0 is $d^i_0 = p_1 + d^i_1$, hence the cost inequality is rewritten

$$\pi_1 > \pi_0 + p_1. \tag{7}$$

Inequality (7) implies that, according to the optimal price vector $\pi$, facility 1 serves no clients. But reducing $\pi_1$ to $\bar{\pi}_1 = \pi_0 + p_1$, customer $i$ will instead use facility 1, so increasing Firm B utility by at least $p_1$: this contradicts the optimality of $\pi$. Symmetrically, the reasoning is repeated for customers $i$ for which $p^i \le p_{-1}$. □

While Lemma 4 establishes the destiny of customers out of $(p_{-1}, p_1)$ under an optimal pricing, a question arises on what happens to those in $(p_{-1}, p_1)$. In fact, we can prove that $\pi_{-1}$ and $\pi_1$ are barrier prices for all the other facilities in $F_B$:

**Lemma 5.** *(Barrier) Let $\pi \in \mathbb{R}^m$ be a price vector for $F_B$. If $\pi$ is optimal, then no facility $k > 1$ ($k < -1$) will serve clients to the left (to the right) of 0.*

**Proof.** Suppose by contradiction that facility $k > 1$ serves a customer $j < 0$ (Fig. 3, left). Since $k$ is to the right of 1 and the metric is on a line, $d^j_k = d_{1k} + d^j_1$, and since $k$ serves $j$ we also have $\pi_1 > \pi_k + d_{1k} \ge \pi_k + d^i_k - d^i_1$ (sub-additivity) for any customer $i$, meaning that facility 1 serves no customers. But then Firm B can serve $j$ (and all other customers to the left of 1 possibly served by $k$) at a higher price by setting $\bar{\pi}_1 = \pi_k + d_{1k}$, not affecting the service costs of all other customers. This contradicts the optimality of $\pi$ (Fig. 3, right). The case $k < -1$ is symmetric. □

For ease of presentation, we first illustrate (Section 5.1) an algorithm to solve the simpler case of Problem 1 with no opening costs, and then (Section 5.2) extend it to the case $c_k \ge 0$. Next we focus on Problem 2, by first considering zero opening costs (Section 5.3) and then (Section 5.4) the more general case with non-negative opening costs.

### 5.1. *Problem 1 with $|F_A| = 1$ and $c_k = 0$*

Here we consider the case in which the cost $c_k$ of opening facility $k$ is 0 for any $k \in F_B$. In the following, we let $r_k$ (let $l_k$) denote the index of the first customer to the right (to the left) of $k \in F_B$: for example, in Fig. 1 we have $l_{-2} = -5, l_{-1} = -2, r_1 = 4$. As announced, we first describe how to solve Problem 1, and then extend the solution to Problem 2: thus we initially suppose $\pi_0$ given. We also begin with two ground cases, where facility 0 is respectively placed to the *left* or to the *right* of all the sites in $F_B$.

Let us first look at the *left ground case*, and consider the leftmost facility in $F_B$, that is 1. If facility 1 sets $\pi_1 = \pi_0 + p_1$, then Firm B will offer service to every customer to its right at the same cost as A: so the entire range $[r_1, n]$ can be served by Firm B at price $\pi_1$. However, by reducing $\pi_1$, one can enable Firm B to serve customers to the left of facility 1, so possibly increasing its revenue. Not only this decision involves the customers to the left of 1, but also all the prices in $F_B$, and a compromise is to be sought. For instance, with facility 2 coming into play, Firm B can increase the gain by letting 2 cover the interval $[r_2, n]$ at a price higher than that fixed by facility 1: the corresponding gain is $(\pi_2 - \pi_1)(n + 1 - r_2)$.

In general, it is convenient to express the profits of Firm B in terms of price differences between consecutive facilities. To this aim, define

$$\Delta\pi_k = \begin{cases} \pi_1 & \text{if } k = 1 \\ \pi_k - \pi_{k-1} & \text{if } k > 1. \end{cases}$$

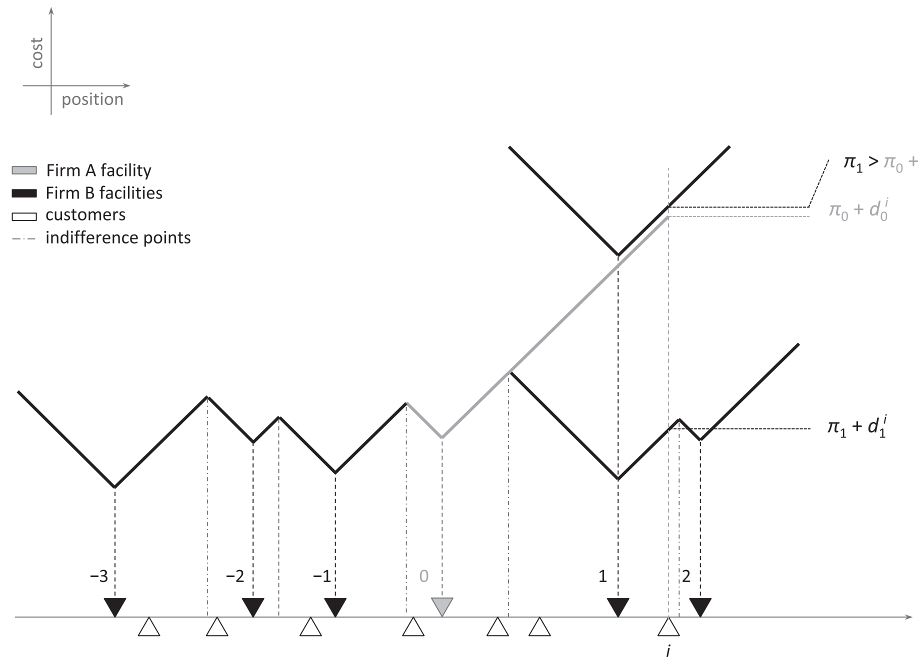


**Fig. 2.** Proof of Lemma 4.

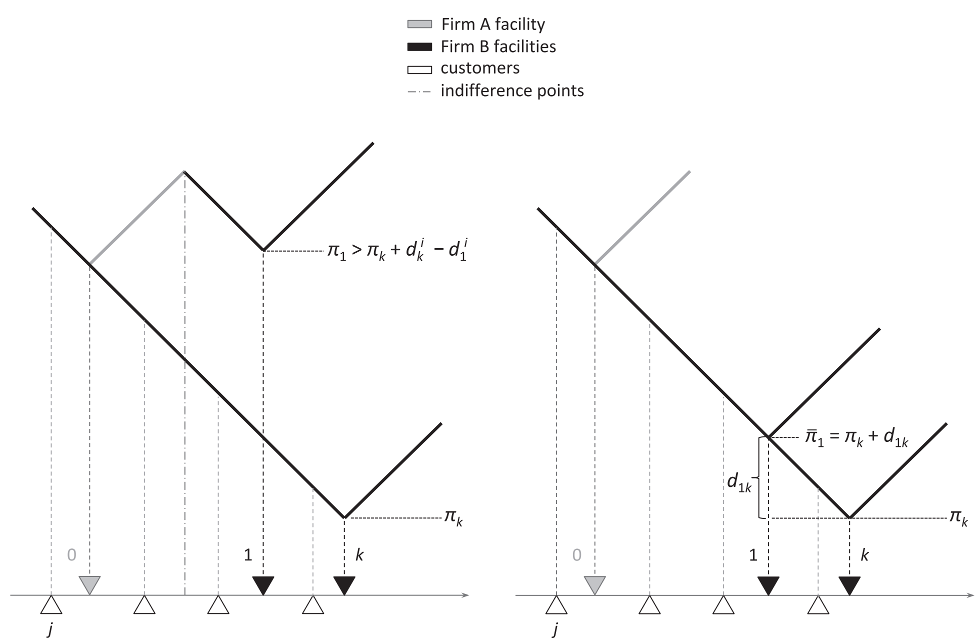


**Fig. 3.** Proof of Lemma 5.

By the Isolation Lemma 4, the number of customers that use facility $k$ is:

$$n_k = \lambda_{k+1} - \lambda_k,$$

where we recall that $\lambda_k$ indicates the index of the leftmost customer served by $k$ (we write for uniformity $\lambda_{m+1} := n+1$ and, if facility $k$ serves no customer, $\lambda_{k+1} = \lambda_k$). With this notation, we can rewrite the objective function:

$$\sum_{k\in F_B} n_k\pi_k = \sum_{k\in F_B} (n+1-\lambda_k)\Delta\pi_k = (n+1)\pi_m - \sum_k \lambda_k\Delta\pi_k. \tag{8}$$

The generic term of the right-hand summation (8) only depends on $\Delta\pi_k$ and $\lambda_k$. One can prove the following:

**Theorem 6.** *If $\pi$ is a vector of optimal prices then $r_{k-1} \le \lambda_k \le r_k$, for all $k = 1, \ldots, m$.*

**Proof.** Suppose by contradiction $j = \lambda_k < r_{k-1}$. Hence, being $r_{k-1}$ the first customer on the right of $k-1$, $j$ lies on the left of $k-1$. And since the metric is on a line, $d_k^j = d_{k-1}^j + p_k - p_{k-1}$. If $k$ serves $j$, Assumption 5 implies the strict inequality $\pi_k + d_k^j < \pi_{k-1} + d_{k-1}^j$, that is

$$\pi_{k-1} - \pi_k > p_k - p_{k-1} = d_{k-1,k}. \tag{9}$$

But for any $i \in C$ we have

$$\pi_{k-1} + d_{k-1}^i > \pi_k + d_{k-1,k} + d_{k-1}^i \ge \pi_k + d_k^i,$$

where the first inequality derives from (9) and the second from sub-additivity. Hence $k-1$ serves no customers. But then one can increase Firm B utility setting $\pi_{k-1} = \pi_k + p_k - p_{k-1}$, serving $j$ at $k-1$ at a price higher than that of $k$, and so contradict the optimality of $\pi$.

Now let us suppose $j = \lambda_k > r_k$ and distinguish two cases.

If $k$ does not serve any customer, $\lambda_k = \lambda_{k+1}$. Since by Lemma 4 all the customers to the right of facility 1 are served by facilities owned by Firm B, $r_k$ is served by a facility $l$ to the left of $k$ with price $\pi_l < \pi_k + p_k - p_l$. But, setting $\pi_k = \pi_l + p_k - p_l$, facility $k$ serves $r_k$ at a higher price, so increasing the firm utility and contradicting the optimality of $\pi$.

If instead $k$ serves at least one customer, it surely serves $j = \lambda_k$, which by definition is the leftmost customer served by $k$. Because $\lambda_k > r_k$, $r_k$ is not served by $k$ but by $k-1$. Therefore

$$\pi_{k-1} + d_{k-1}^{r_k} < \pi_k + d_k^{r_k}.$$

Summing $d^{r_k\lambda_k}$ to both members we then obtain

$$\pi_{k-1} + d_{k-1}^{\lambda_k} < \pi_k + d_k^{\lambda_k},$$

that is, $k-1$ serves $\lambda_k$ too: a contradiction. In conclusion, neither $\lambda_k < r_{k-1}$, nor $\lambda_k > r_k$, therefore $r_{k-1} \le \lambda_k \le r_k$. □

Theorem 6 gives a lower and an upper bound to the value that $\lambda_k$ can take in an optimal solution. Now, for the largest possible value of $\Delta\pi_k$, two cases can occur depending on $\lambda_k$:

- If $\lambda_k = r_k$, price $\pi_k$ must verify the restriction:

  $$\pi_k + p^{\lambda_k} - p_k \le \pi_{k-1} + p^{\lambda_k} - p_{k-1}.$$

  Thus:

  $$\Delta\pi_k \le p_k - p_{k-1}. \tag{10}$$

- If $r_{k-1} \le \lambda_k < r_k$, the restriction on $\pi_k$ to guarantee that $k$ serves customer $\lambda_k$ is:

  $$\pi_k + p_k - p^{\lambda_k} \le \pi_{k-1} + p^{\lambda_k} - p_{k-1}.$$

  Thus:

  $$\Delta\pi_k \le 2p^{\lambda_k} - p_k - p_{k-1}. \tag{11}$$

From Theorem 6 we also know that, in an optimal solution, $r_{k-1} \le \lambda_k \le r_k$ (consider that, if facility $k$ serves no customer, $\lambda_{k+1} = \lambda_k$). We then conclude:

**Proposition 7.** *The optimal values of $\Delta\pi_k$ do not bind one another, and can independently be determined from the positions of facilities and customers.*

To find an optimum price vector $\pi^*$ it is then sufficient to calculate, independently for each $k \in F_B$, the index $\lambda_k^*$ that maximizes the $k$th term of the sum (8). In formulæ, according to (10), (11):

$$\lambda_k^* = \arg\max_t \left\{ \begin{array}{ll} (n+1-t)(2p^t - p_k - p_{k-1}) : & t \in \{r_{k-1}, r_{k-1}+1, \ldots, l_k\} \\ (n+1-t)(p_k - p_{k-1}) : & t = r_k \end{array} \right\}, \tag{12}$$

$$\Delta\pi_k^* = \begin{cases} 2p^{\lambda_k^*} - p_k - p_{k-1} & \text{if } r_{k-1} \le \lambda_k^* < r_k \\ p_k - p_{k-1} & \text{if } \lambda_k^* = r_k < r_k. \end{cases} \tag{13}$$

Optimal prices are then immediately derived as

$$\pi_k^* = \pi_{k-1}^* + \Delta\pi_k^*, \tag{14}$$

with initial value $\pi_0^* = \pi_0$, and the whole computation requires $O(m+n)$ steps. Prices are symmetrically computed in the *right ground case* with $l_k$ in place of $r_k$ and $\rho_k$ in place of $\lambda_k$.

In the following, we say that facility $h \in F$ *rules out* facility $k \in F$ if a client in any point of the line is more conveniently served by $h$ than by $k$ (recalling Assumptions 4, in a metric case this means $\pi_h \le \pi_k - d_{hk}$ for $h \in F_B$, $k \in F_A$).

In case $F_B$ contains both facilities to the left and to the right of 0, optimal prices can be separately computed for the left and right ground cases only if $\pi_0 = 0$. To tackle $\pi_0 > 0$, we resort to the Barrier Lemma 5, according to which the customers lying in $[p_{-1}, p_1]$ (e.g., customers $-1, 1, 2, 3$ in Fig. 1) are contended by facilities $-1, 0$ and $1$ only. In an optimal solution, one out of three cases occurs: in one, facility 1 (facility $-1$) does not capture clients to the left (to the right) of the competitor; in the remaining two, the competitor is ruled out either by facility 1 or by $-1$. Note that when facility 1 (facility $-1$) rules out the competitor in this way, in an optimal solution we have $\pi_k = \pi_0 - d_{0k} \ge 0$. Should this price be negative, in fact, facility $k$ would get a negative contribution from all the customers it serves, and moreover the contribution of all the customers served by the other facilities of $F_B$ would be reduced by $|\pi_1|$: thus it is preferable for facility $k$ to set $\pi_1 = 0$. An optimum assignment to $\{-1, 0, 1\}$ of the customers lying in $[p_{-1}, p_1]$ is then found in $O(m+n)$ time by separately considering these three cases, and then choosing the best solution:

1. Neither facility 1 nor $-1$ rule out the competitor: in this case, check all $\lambda_1 \in [1, r_1]$ and $\rho_{-1} \in [l_{-1}, -1]$. This requires $O(n)$ time.
2. The competitor is ruled out by facility $-1$, which happens by setting $\pi_{-1} = \pi_0 - |p_{-1}|$. Then choose $\lambda_1$ to maximize $\pi_1(n^r - \lambda_1 + 1) + \pi_{-1}(\lambda_1 - 1)$, where $n^r$ denotes the number of customers to the right of facility 0 and

   $$\pi_1 = \begin{cases} p_1 & \text{if } \lambda_1 = r_1 \\ 2p^{\lambda_1} - p_1 & \text{otherwise.} \end{cases}$$

   Also this step requires $O(n)$ time.
3. The competitor is ruled out by facility 1: this situation is symmetric to the previous.

Then, combining the formulae (12)–(14) for the left ground case with the equivalent for the right ground case, we can again find $\pi^*$ in further $O(m)$ steps.

### 5.2. Problem 1 with $|F_A| = 1$ and $c_k \ge 0$

The left (as well as the right) ground case of Problem 1 with non-zero fixed costs can be solved by dynamic programming or, equivalently, as a longest $(s, t)$-path on a complete directed acyclic graph. In fact, the $k$th term of objective function (8) depends only on: the position of facility $k$, that of the rightmost facility $h$ open on the left of $k$, and the positions of the customers that lie between $r_h$ and $r_k$ (analogous of Theorem 6). Since each term can be maximized independently, one can compute the maximum utility $w_{jk}$ generated by opening $k$ when $j$ is the rightmost facility opened by B to the left of $k$. Call $w_{sk}$ the utility generated by $k$ when no facility of B is opened to the left of $k$; then both $w_{jk}$ and $w_{sk}$ can be computed in $O(n)$ steps through:

$$w_{jk} = \begin{cases} \max_{i=1,\ldots,r_k}\{(\pi_0 + f(i))(n+1-i)\} - c_k & \text{if } j = s \\ \max_{i=r_j,\ldots,r_k}\{(f(i) - p_j)(n+1-i)\} - c_k & \text{if } j \in F_B \\ 0 & \text{if } k = t, \end{cases} \tag{15}$$

where

$$f(i) = \begin{cases} 2p^i - p_k & \text{if } i < r_k \\ p_k & \text{if } i = r_k. \end{cases} \tag{16}$$

Now define a graph $G = (F_B \cup \{s, t\}, E)$, where $jk \in E$ if and only if $p_j < p_k$, or $j = s$, or $k = t$. Arc $jk$ is weighted by $w_{jk}$ and, by the weights definition, the weight of an $(s, t)$-path $P \subseteq E$ of $G$ equals the optimal revenue obtained when firm B opens the facilities represented by the nodes touched by $P$.

With an approach similar to Section 5.1, the two ground cases can be composed in order to find an optimal solution for the general case in which $F_B$ has elements on both the left and the right of

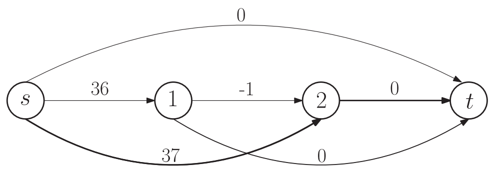


**Fig. 4.** Digraph used to solve Problem 1 in the numerical example.

**Table 2**
Computation of weights for arcs ($s$, 1), ($s$, 2) and (1,2).

| Index $i$ | Expression: first of (15) | Numerical value |
|---|---|---|
| Computation for ($s$, 1) | | |
| 1 | $(\pi_0 + 2p^1 - p_1)(n + 1 - 1)$ | $(10 + 2 - 6)6 = 36$ |
| 2 | $(\pi_0 + 2p^2 - p_1)(n + 1 - 2)$ | $(10 + 4 - 6)5 = 40$ |
| **3** | $(\pi_0 + 2p^3 - p_1)(n + 1 - 3)$ | $(10 + 10 - 6)4 = \mathbf{56}$ |
| 4 | $(\pi_0 + p_1)(n + 1 - 4)$ | $(10 + 6)3 = 48$ |
| Result: $w_{s,1} = 56 - 20 = \mathbf{36}$ | | |
| Computation for ($s$, 2) | | |
| 1 | $(\pi_0 + 2p^1 - p_2)(n - 1 + 1)$ | $(10 + 2 - 10)6 = 12$ |
| 2 | $(\pi_0 + 2p^2 - p_2)(n - 2 + 1)$ | $(10 + 4 - 10)5 = 20$ |
| 3 | $(\pi_0 + 2p^3 - p_2)(n - 3 + 1)$ | $(10 + 10 - 10)4 = 40$ |
| **4** | $(\pi_0 + 2p^4 - p_2)(n - 4 + 1)$ | $(10 + 14 - 10)3 = \mathbf{42}$ |
| 5 | $(\pi_0 + 2p^5 - p_2)(n - 5 + 1)$ | $(10 + 18 - 10)2 = 36$ |
| 6 | $(\pi_0 + p_2)(n - 6 + 1)$ | $(10 + 10)1 = 20$ |
| Result: $w_{s,2} = 42 - 5 = \mathbf{37}$ | | |
| Computation for (1,2) | | |
| 4 | $(2p^4 - p_2 - p_1)(n^r - 4 + 1)$ | $(14 - 6 - 10)3 = -6$ |
| **5** | $(2p^5 - p_2 - p_1)(n^r - 5 + 1)$ | $(18 - 6 - 10)2 = \mathbf{4}$ |
| 6 | $(p_2 + p_1)(n^r - 6 + 1)$ | $(10 - 6)1 = 4$ |
| Result: $w_{1,2} = 4 - 5 = \mathbf{-1}$ | | |

0. More precisely, one can guess all the $O(m^2)$ facility pairs opened next to Firm A facility in an optimal solution, compute for each pair the optimal prices, and then decide which other facilities to open as described above. Now $G$ has $O(m^2)$ arcs and — as previously seen — $n$ basic steps are required to compute arc weights; moreover, finding the longest ($s$, $t$)-path requires a number of steps linear in the number of arcs, that is $O(m^2)$: hence the ground case is solved in $O(m^2 n)$ steps. To solve the general problem, the ground cases must be solved for each facility pair, so an optimal solution is found in $O(m^4 n)$ elementary steps.

*Simple numerical example*. We use data from Table 1 and Fig. 1. When Firm B bears opening costs, the solution of Problem 1 requires to construct a weighted digraph $G$ as explained in Section 5.2. Graph $G$ has a node per Firm B facility, plus a source node $s$ and a sink node $t$, see Fig. 4. Here we suppose $\pi_0 = 10$, and that opening facilities 1, 2 costs $c_1 = 20$, $c_2 = 5$, respectively.

Using formulae (15), (16) we first obtain the graph weights $w_{s,t} = w_{1,t} = w_{2,t} = 0$. According to (15), the weight $w_{s,k}$ of arc ($s$, $k$) is computed by subtracting $c_k$ from

$$\max_{i=1,\dots,r_k} \{(\pi_0 + f(i))(n + 1 - i)\},$$

where $f(i) = 2p^i - p_1$ for $i \le 3$ and $f(i) = p_1$ for $i = 4$. Table 2 shows the explicit computation of the terms within brackets for the three arcs ($s$, 1), ($s$, 2), (1, 2) (maxima in boldface). With these weights, the longest ($s$, $t$)-path of $G$ is {($s$, 2), (2, $t$)}. This means that in an optimal solution Firm B will only open facility 2, getting a revenue of 37.

### 5.3. Problem 2 with $|F_A| = 1$ and $c_k = 0$

For any given price $\pi_0$ set by Firm A, the results of Section 5.1 enable us to find an optimal price vector $\pi^*$ for Firm B.

With zero opening costs, facility 1 (facility −1) serves the first customer on its right (left), if any (Isolation Lemma).

Suppose to know (as by an oracle) that $\lambda$ is the leftmost customer captured by facility 1 in an optimal solution of Problem 2. Conditioned to this fact, the maximum utility $B_1^\lambda(\pi_0)$ locally obtained by facility 1 is then

$$B_1^\lambda(\pi_0) = \begin{cases} (n^r - \lambda + 1)(\pi_0 + p_1) & \text{if} \quad \lambda = r_1 \\ (n^r - \lambda + 1)(\pi_0 + 2p^\lambda - p_1) & \text{if} \quad 1 \le \lambda < r_1 \\ (n^r + |\lambda|)(\pi_0 - p_1) & \text{if} \quad \lambda < 0, \end{cases} \tag{17}$$

where $n^r$ denotes the number of customers to the right of facility 0. Similarly, we obtain the utility $B^\rho_{-1}(\pi_0)$ locally obtained by facility −1 conditioned to capture $\rho$ as the rightmost customer:

$$B^\rho_{-1}(\pi_0) = \begin{cases} (n^l - |\rho| + 1)(\pi_0 + |p_{-1}|) & \text{if} \quad \rho = l_{-1} \\ (n^l - |\rho| + 1)(\pi_0 + 2|p^\rho| - |p_{-1}|) & \text{if} \quad l_{-1} < \rho \le -1 \\ (n^l + \rho)(\pi_0 - |p_{-1}|) & \text{if} \quad \rho > 0, \end{cases} \tag{18}$$

(here $n^l$ denotes the number of customers to the left of facility 0). Problem 2 can be solved by enumerating the $O(n^2)$ pairs $\lambda$, $\rho$ of customers closest to 0 and possibly served by 1 and −1, respectively. For each pair, one can easily find the value of $\pi_0$ that maximizes the utility of Firm A with $B_1^i, B_{-1}^i$ conditioned as explained above, and then select the best:

$$\max_{\pi_0} (\lambda + |\rho| - 2)\pi_0 \tag{19}$$
$$\begin{aligned} B_1^i(\pi_0) &\le B_1^\lambda(\pi_0) & \lambda \ne i, 1 \le i \le r_1 \\ B_{-1}^i(\pi_0) &\le B^\rho_{-1}(\pi_0) & \rho \ne i, l_{-1} \le i \le -1 \\ B_{-1}^i(\pi_0) + B_1^{i+1}(\pi_0) &\le B_1^\lambda(\pi_0) + B^\rho_{-1}(\pi_0) & 1 \le i < r_1 \text{ or } l_{-1} \le i \le -2 \\ \pi_0 &\ge 0. \end{aligned}$$

The meaning of (19) is that Firm A seeks a price $\pi_0 \ge 0$ that forces Firm B to an optimum where $\lambda$ (resp., $\rho$) is the leftmost (resp., the rightmost) customer served by facility 1 (resp., −1). The first two constraints impose that any other choice $i$ for the leftmost or rightmost customer would not result in a revenue improvement for the relevant facility. The remaining constraints impose that Firm B does not rule Firm A out of market by capturing customer $i$ with facility −1 and customer $i + 1$ with facility 1. Note that, according to the Barrier Lemma 5, the prices $\pi_k^*$ adopted for $|k| \ne 1$ by Firm B in order to maximize its utility depend on $\pi_1^*, \pi_{-1}^*$ only through the system of differences (14), and therefore do not influence the optimal price for Firm A.

In case problem (19) is infeasible for all $\lambda$, $\rho$ pairs, two cases occur for any $\pi_0$: either Firm A is ruled out because $\pi_0$ violates a constraint in the third set, or $\pi_0 < 0$. In both cases, the maximum revenue of Firm A is evidently 0. Note that, in case $\pi_0$ violates constraints in the first or second set, one can take the most violated of those constraints and exchange $\rho$ or $\lambda$ with $i$, so getting a feasible problem.

**Proposition 8.** *For $|F_A| = 1$, $c_k = 0$ and any values of $p^i$, $p_k$, Problem 2 is bounded from above.*

**Proof.** In case Firm A serves no customers, as observed its utility is 0. We have to prove the claim for any $\lambda > 1$ or $\rho < -1$ for which problem (19) has a feasible solution. Assume $\lambda > 1$. In this case, $\pi_0$ must verify $B_1^1(\pi_0) \le B_1^\lambda(\pi_0)$. Then

$$\begin{cases} n^r(\pi_0 + 2p^1 - p_1) \le (n^r - \lambda + 1)(\pi_0 + 2p^\lambda - p_1) & \text{if} \quad \lambda < r_1 \\ n^r(\pi_0 + 2p^1 - p_1) \le (n^r - \lambda + 1)(\pi_0 + p_1) & \text{if} \quad \lambda = r_1. \end{cases}$$

For $\lambda < r_1$, the constraint is fulfilled if and only if

$$\pi_0 \le 2n^r \frac{(p^\lambda - p^1)}{(\lambda - 1)} + p_1 - 2p^\lambda$$

**Table 3**
Utility of facilities $1, -1$ as a function of $\pi_0$.

| Computation for facility 1 | | |
|---|---|---|
| Index $\lambda$ | Expression: (17) | Numerical value |
| $-1$ | $B_1^{-1}(\pi_0) = (n^r + \lvert\lambda\rvert)(\pi_0 - d_0 1)$ | $B_1^{-1}(\pi_0) = 7(\pi_0 - 6)$ |
| 1 | $B_1^1(\pi_0) = (n^r - \lambda + 1)(\pi_0 + 2p^\lambda - p_1)$ | $B_1^1(\pi_0) = 6(\pi_0 - 4)$ |
| 2 | $B_1^2(\pi_0) = (n^r - \lambda + 1)(\pi_0 + 2p^\lambda - p_1)$ | $B_1^2(\pi_0) = 5(\pi_0 - 2)$ |
| 3 | $B_1^3(\pi_0) = (n^r - \lambda + 1)(\pi_0 + 2p^\lambda - p_1)$ | $B_1^3(\pi_0) = 4(\pi_0 + 4)$ |
| 4 | $B_1^4(\pi_0) = (n^r - \lambda + 1)(\pi_0 + p_1)$ | $B_1^4(\pi_0) = 3(\pi_0 + 6)$ |
| Computation for facility $-1$ | | |
| Index $\rho$ | Expression: (18) | Numerical value |
| $-2$ | $B_{-1}^{-2}(\pi_0) = (n^l - \lvert\rho\rvert + 1)(\pi_0 + \lvert p_{-1}\rvert)$ | $B_{-1}^{-2}(\pi_0) = 7(\pi_0 + 3)$ |
| $-1$ | $B_{-1}^{-1}(\pi_0) = (n^l - \lvert\rho\rvert + 1)(\pi_0 + 2\lvert p^\rho\rvert - \lvert p_{-1}\rvert)$ | $B_{-1}^{-1}(\pi_0) = 6(\pi_0 + 5)$ |
| 1 | $B_{-1}^{1}(\pi_0) = (n^l + \rho)(\pi_0 - \lvert p_{-1}\rvert)$ | $B_{-1}^{1}(\pi_0) = 8(\pi_0 - 5)$ |
| 2 | $B_{-1}^{2}(\pi_0) = (n^l + \rho)(\pi_0 - \lvert p_{-1}\rvert)$ | $B_{-1}^{2}(\pi_0) = 9(\pi_0 - 5)$ |
| 3 | $B_{-1}^{3}(\pi_0) = (n^l + \rho)(\pi_0 - \lvert p_{-1}\rvert)$ | $B_{-1}^{3}(\pi_0) = 10(\pi_0 - 5)$ |

while for $\lambda = r_1$ if and only if

$$\pi_0 \le 2n^r \frac{(p_1 - p^1)}{(\lambda - 1)} - p_1.$$

In both cases $\pi_0$ is bounded and utility of Firm A cannot be increased without bound.

The case $\rho < -1$ is dealt with by the same argument, considering constraint $B_{-1}^{-1}(\pi_0) \le B_{-1}^{\rho}(\pi_0)$. $\square$

*Simple numerical example*. The problem is solved by first rewriting formulae (17) and (18) and then solving (19) for all possible $\rho$, $\lambda$ pairs (see Section 5.3). In our case the $\rho$, $\lambda$ pairs are $\{(-1,1), (-1,2), (-1,3), (-1,4), (-2,1), (-2,2), (-2,3), (-2,4)\}$. The utility of facilities 1 and $-1$, computed in this way as a function of $\pi_0$, is indicated in Table 3. Then, for each possible $(\rho, \lambda)$, we compute the maximum price that makes $\rho$ and $\lambda$ optimal for Firm B (in boldface), and the corresponding revenue for Firm A.

- $(-1,1) \to$ no client for Firm A. Revenue of Firm A = 0.
- $(-1,2) \to$ 1 client for Firm A

$$\begin{cases} 7\pi_0 + 21 & \ge & 6\pi_0 + 30 & \Rightarrow & \pi_0 & \ge & 9 \\ 5\pi_0 - 10 & \ge & 6\pi_0 - 24 & \Rightarrow & \pi_0 & \le & 14 \\ 5\pi_0 - 10 & \ge & 4\pi_0 + 16 & \Rightarrow & \pi_0 & \ge & 26 \end{cases}$$

Infeasible.

- $(-1,3) \to$ 2 clients for Firm A

$$\begin{cases} 7\pi_0 + 21 & \ge & 6\pi_0 + 30 & \Rightarrow & \pi_0 & \ge & 9 \\ 4\pi_0 + 16 & \ge & 6\pi_0 - 24 & \Rightarrow & \pi_0 & \le & \mathbf{20} \\ 4\pi_0 + 16 & \ge & 5\pi_0 - 10 & \Rightarrow & \pi_0 & \le & 26 \\ 4\pi_0 + 16 & \ge & 3\pi_0 + 18 & \Rightarrow & \pi_0 & \ge & 2 \\ 11\pi_0 + 37 & \ge & 13\pi_0 - 12 & \Rightarrow & \pi_0 & \le & 24,5 \\ 11\pi_0 + 37 & \ge & 13\pi_0 - 50 & \Rightarrow & \pi_0 & \le & 43,5 \\ 11\pi_0 + 37 & \ge & 13\pi_0 - 29 & \Rightarrow & \pi_0 & \le & 33 \\ 11\pi_0 + 37 & \ge & 13\pi_0 - 32 & \Rightarrow & \pi_0 & \le & 24,5 \end{cases}$$

$\pi_0 = 20$. Revenue of Firm A $= 20 \cdot 2 = 40$.

- $(-1,4)$

$$\begin{cases} 7\pi_0 + 21 & \ge & 6\pi_0 + 30 & \Rightarrow & \pi_0 & \ge & 9 \\ 3\pi_0 + 18 & \ge & 6\pi_0 - 24 & \Rightarrow & \pi_0 & \le & 14 \\ 3\pi_0 + 18 & \ge & 5\pi_0 - 10 & \Rightarrow & \pi_0 & \le & 14 \\ 3\pi_0 + 18 & \ge & 4\pi_0 + 16 & \Rightarrow & \pi_0 & \le & 2 \\ 12\pi_0 + 39 & \ge & 13\pi_0 - 12 & \Rightarrow & \pi_0 & \le & 51 \\ 12\pi_0 + 39 & \ge & 13\pi_0 - 50 & \Rightarrow & \pi_0 & \le & 89 \\ 12\pi_0 + 39 & \ge & 13\pi_0 - 29 & \Rightarrow & \pi_0 & \le & 68 \\ 12\pi_0 + 39 & \ge & 13\pi_0 - 32 & \Rightarrow & \pi_0 & \le & 71 \end{cases}$$

Infeasible.

- $(-2,1) \to$ 1 client for Firm A

$$\begin{cases} 6\pi_0 + 30 & \ge & 7\pi_0 + 21 & \Rightarrow & \pi_0 & \le & 9 \\ 6\pi_0 - 24 & \ge & 5\pi_0 + 10 & \Rightarrow & \pi_0 & \ge & 14 \end{cases}$$

Infeasible.

- $(-2,2) \to$ 2 clients for Firm A

$$\begin{cases} 6\pi_0 + 30 & \ge & 7\pi_0 + 21 & \Rightarrow & \pi_0 & \le & 9 \\ 5\pi_0 + 10 & \ge & 6\pi_0 + 24 & \Rightarrow & \pi_0 & \le & 14 \\ 5\pi_0 + 10 & \ge & 4\pi_0 + 16 & \Rightarrow & \pi_0 & \ge & 26 \end{cases}$$

Infeasible.

- $(-2,3) \to$ 3 clients for Firm A

$$\begin{cases} 6\pi_0 + 30 & \ge & 7\pi_0 + 21 & \Rightarrow & \pi_0 & \le & \mathbf{9} \\ 4\pi_0 + 16 & \ge & 6\pi_0 - 24 & \Rightarrow & \pi_0 & \le & 20 \\ 4\pi_0 + 16 & \ge & 5\pi_0 - 10 & \Rightarrow & \pi_0 & \le & 26 \\ 4\pi_0 + 16 & \ge & 3\pi_0 + 18 & \Rightarrow & \pi_0 & \ge & 2 \\ 10\pi_0 + 46 & \ge & 13\pi_0 - 12 & \Rightarrow & \pi_0 & \le & 19,\overline{3} \\ 10\pi_0 + 46 & \ge & 13\pi_0 - 50 & \Rightarrow & \pi_0 & \le & 32 \\ 10\pi_0 + 46 & \ge & 13\pi_0 - 29 & \Rightarrow & \pi_0 & \le & 25 \\ 10\pi_0 + 46 & \ge & 13\pi_0 - 32 & \Rightarrow & \pi_0 & \le & 27 \end{cases}$$

$\pi_0 = 9$. Revenue of Firm A $= 9 \cdot 3 = 27$.

- $(-2,4) \to$ 4 clients for Firm A

$$\begin{cases} 6\pi_0 + 30 & \ge & 7\pi_0 + 21 & \Rightarrow & \pi_0 & \le & 9 \\ 3\pi_0 + 18 & \ge & 6\pi_0 - 24 & \Rightarrow & \pi_0 & \le & 14 \\ 3\pi_0 + 18 & \ge & 5\pi_0 - 10 & \Rightarrow & \pi_0 & \le & 14 \\ 3\pi_0 + 18 & \ge & 4\pi_0 + 16 & \Rightarrow & \pi_0 & \le & \mathbf{2} \\ 12\pi_0 + 48 & \ge & 13\pi_0 - 12 & \Rightarrow & \pi_0 & \le & 60 \\ 12\pi_0 + 48 & \ge & 13\pi_0 - 50 & \Rightarrow & \pi_0 & \le & 98 \\ 12\pi_0 + 48 & \ge & 13\pi_0 - 29 & \Rightarrow & \pi_0 & \le & 77 \\ 12\pi_0 + 48 & \ge & 13\pi_0 - 32 & \Rightarrow & \pi_0 & \le & 80 \end{cases}$$

$\pi_0 = 2$. Revenue of Firm A $= 2 \cdot 4 = 8$.

The price that maximizes Firm A utility is $\pi_0 = 20$; Firm B will then react according to the model developed.

### 5.4. Problem 2 with $|F_A| = 1$ and $c_k \ge 0$

Let us finally focus on Problem 2 when Firm B bears opening costs. The problem calls for finding an optimal price $\pi_0$ for Firm A at facility 0, taking all possible reactions of Firm B into account.

In Section 5.2 we saw that the revenue $B(.)$ of Firm B can be expressed as the length of a longest $(s, t)$-path in a directed acyclic graph $G$, the nodes of which (apart from the origin $s$ and the destination $t$) denote the facilities in $F_B$. On the other hand, in Section 5.1 we showed that Firm A affects (via $\pi_0$) the revenue of Firm B only by affecting the prices settled at the two facilities, $+1$ and $-1$, that are adjacent to facility 0.

The basic difference in this case is that, because of opening costs, we do not know in advance which facilities will be open in the optimal reaction of firm $B$. Notice however that $\pi_0$ only affects the weights of the edges outgoing from $s$, see formula (15). Now, since $G$ is acyclic, for any $u \neq s$ the length $w(u)$ of the longest $(u, t)$-path of $G$ does not depend on $\pi_0$. As a consequence, it is possible to compute the length of the longest path as a function of $\pi_0$, and consequently the optimal revenue achievable by B, as the minimum of polynomially many linear functions of $\pi_0$. For any facility pair, then, we can as well express by linear constraints on $\pi_0$ the conditions under which the pair is adjacent to 0 and opened by B in its optimal reaction.

Formally speaking, let $\kappa$ be the first facility open to the right of 0, and $\lambda$ be the leftmost customer served by $\kappa$. For any given $\kappa$, $\lambda$, we can express the dependence on $\pi_0$ of the optimal revenue of Firm B that derives from the facilities to the right of Firm A as follows:

$$B_\kappa^\lambda(\pi_0)=\begin{cases}(n^r-\lambda+1)(\pi_0+p_\kappa)+w(\kappa) & \text{if} \quad \lambda=r_\kappa\\ (n^r-\lambda+1)(\pi_0+2p^\lambda-p_\ell)+w(\kappa) & \text{if} \quad 1\le\lambda<r_1\\ (n^r+|\lambda|)(\pi_0-p_\kappa)+w(\kappa) & \text{if} \quad \lambda<0.\end{cases}$$

Similarly, let $\ell$ be the first facility open to the left of 0, and $\rho$ be the rightmost customer served by $\ell$. For any given $\ell$, $\rho$, the analogous of the previous expression is

$$B_\ell^\rho(\pi_0)=\begin{cases}(n^l-|\rho|+1)(\pi_0+|p_\ell|)+w(\ell) & \text{if} \quad \rho=l_\ell\\ (n^l-|\rho|+1)(\pi_0+2|p^\rho|-|p_\ell|)+w(\ell) & \text{if} \quad l_\ell<\rho\le -1\\ (n^l+\rho)(\pi_0-|p_\ell|)+w(\ell) & \text{if} \quad \rho>0.\end{cases}$$

As in Section 5.3, problem (19), we can compute the optimal revenue of Firm A by finding, for all possible $\kappa$, $\ell$, $\lambda$, $\rho$, the maximum $\pi_0$ (if one exists) that induces the corresponding Firm B solution:

$$\begin{aligned} &\max_{\pi_0}\,(\lambda+|\rho|-2)\pi_0 && (20)\\ &B_h^i(\pi_0)\le B_\kappa^\lambda(\pi_0) && \lambda\ne i,\ \kappa\ne h,\ 1\le i\le r_\kappa,\ h>0\\ &B_h^i(\pi_0)\le B_\ell^\rho(\pi_0) && \rho\ne i,\ \ell\ne h,\ l_\ell\le i\le -1,\ h<0\\ B_j^i(\pi_0)+B_h^{i+1}(\pi_0)&\le B_\kappa^\lambda(\pi_0)+B_\ell^\rho(\pi_0) && 1\le i<r_1,\ j<0,\ h>0\\ B_j^{i-1}(\pi_0)+B_h^i(\pi_0)&\le B_\kappa^\lambda(\pi_0)+B_\ell^\rho(\pi_0) && l_{-1}<i,\le 1\ j<0,\ h>0,\end{aligned}$$

where the former two constraint classes impose the guess on $r_k$ and $l_k$ (i.e., the first customer to the right and to the left of $k\in F_B$); the latter, that $\lambda$ and $\rho$ are the first two facilities opened by B to left and to the right of 0. The algorithm makes a guess for all open facility pairs adjacent to 0, and all possible number of customers served by 0 on its left and on its right, for a total amount of $O(m^2n^2)$ guesses. Problem (20) has just one variable and $m^2n$ linear constraints, so for fixed $\kappa$, $\ell$, $\lambda$, $\rho$, an optimal price is found in $O(m^2n)$ steps. Then the overall complexity of the method is $O(m^4n^3)$.

## 6. One-dimensional metric problem, $|F_A|>1$

Although apparently much simpler than Problem 1 introduced in its general form in Section 3.3, the one-dimensional case is still hard to solve. In this section we illustrate a reduction from 0-1 KNAPSACK. Let

$$\begin{aligned} \max \quad & \sum_{i=1}^n v_i z_i && (21)\\ & \sum_{i=1}^n w_i z_i \le w\\ & z_i\in\{0,1\}\ i=1,\dots,n,\end{aligned}$$

with $v_i, w_i\in\mathbb{R}_+$, be an instance $\mathcal{K}$ of 0-1 KNAPSACK. With no loss of generality, assume $\sum_{i=1}^n v_i=1$, and let $W=1+2\sum_{k=1}^n(n-k+1)w_k$. Associate with $\mathcal{K}$ the following metric 1-dimensional instance $\mathcal{P}$ of Problem 1:

- Firm A holds two facilities $0, n+2$ placed at the extremes of an interval $I=[p_0,p_{n+2}]$ of the real axis, with

  $$p_0=0 \qquad p_{n+2}=2(W+w),$$

  and offer the service at prices $\pi_0=\pi_{n+2}=0$
- Firm B holds $n+1$ facilities. The first $n$ are placed in $I$ at positions

  $$p_k=p_{k-1}+v_k+2(n-k+1)w_k$$

  for $k=1,\dots,n$: in particular, $p_n=W$. Facility $n+1$ is placed at position $p_{n+1}=W+2w$.
- There are $n+1$ customers. The first $n$ are placed at positions $p^i=p_i-w_i$. Customer $n+1$ is placed at $p^{n+1}=p_{n+1}$.

Fig. 5 shows the construction. Let $\pi^*\in\mathbb{R}^{n+1}$ be an optimal price vector. The following properties hold:

**Lemma 9.** *Under $\pi^*$, facility $n+1$ serves customer $n+1$ and no other customers.*

**Proof.** First suppose by contradiction that facility $n+1$ serves a customer $i\le n$. By construction, $p^i<p_i\le p_n$, and by Proposition 1 facility $n$ serves no customers. This means that $\pi_n^*>\pi_{n+1}^*+p_{n+1}-p_n$. Since $n$ is closer than $n+1$ to the customers with index smaller than $n$, by setting $\pi_n=\pi_{n+1}^*+p_{n+1}-p_n$, facility $n$ could serve all clients $j$ such that $i\le j\le n$ without changing the service costs of any customer, and thus increasing the objective function by $(p_{n+1}-p_n)(n-i+1)$: this contradicts the assumption of $\pi^*$ optimal. Therefore, $n+1$ does not serve any customer $i\le n$.

Let us now suppose by contradiction that $n+1$ serves no customer at all. Then customer $n+1$ could be served by either ($i$) facility $n+2$, or ($ii$) a facility $i\le n$. In case ($i$) it is sufficient (Assumption 4) to set $\pi_{n+1}=p_{n+2}-p_{n+1}=W$ to make facility $n+1$ appealing for $n+1$, so increasing Firm B revenue without decreasing the service costs of any other client (and consequently the relevant prices). Similarly, in case ($ii$) facility $n+1$ becomes appealing for customer $n+1$ as soon $\pi_{n+1}=p_{n+1}-p_i$; by doing so, the firm's revenue is increased by the same amount as in case ($i$). Thus, the optimality of $\pi^*$ is contradicted in either cases. □

**Lemma 10.** *Suppose that, under $\pi^*$, facility $k$ serves at least one customer and let $\lambda_k$ denote the leftmost customer served by $k$. Then $\lambda_k=k$ or $\lambda_k=k+1$.*

**Proof.** Let us first prove that $\lambda_k\ge k$. Indirectly, suppose that $\lambda_k=i<k$. We know by construction that $p^i<p_{k-1}<p_k$. Then by Proposition 2 facility $k-1$ does not serve any client. But then, setting $\pi_{k-1}=\pi_k+p_k-p_{k-1}$, facility $k-1$ could serve client $i$ without decreasing the serving cost of any other client, and increasing the objective function by at least $p_k-p_{k-1}$. This contradicts the optimality of $\pi^*$.

Let us prove now that $\lambda_k\le k+1$. Suppose by contradiction that $\lambda_k=i>k+1$. Then facility $k$ does not serve client $k+1$, so let $k+1$ be served by facility $l$. If $l$ is left of facility $k$, then by Proposition 2, $k$ does not serve any client (in particular, it does not serve $i$). If instead $l\ge k+1$ is right of facility $k$, then again by Proposition 2 $p_l>p^i$ (if not, because facility $k$ serves $i$, $l$ would serve no client). Thus

$$d_k^i=|p_k-p^i|>|p_k-p^{k+1}|=d_k^{k+1},$$

$$d_l^{k+1}=|p_l-p^{k+1}|>|p_l-p^i|=d_l^i.$$

Since $l$ serves client $k+1$, prices must verify

$$\pi_l-\pi_k\le d_k^{k+1}-d_l^{k+1}<d_k^i-d_l^i.$$

This implies that client $i$ is not served by $k$. So in both cases the hypothesis $\lambda_k=i$ is contradicted. □

To exemplify Lemma 10, see Fig. 5 and observe that $\lambda_1=1, \lambda_2=2, \lambda_4=4$.

Summarizing, in the assignment induced by an optimal $\pi$ facility $n+1$ serves only $n+1$ (Lemma 9). Moreover the optimal price fixed by facility $n+1$ does not decrease the serving cost of any client. Therefore, the serving costs of the first $n$ clients are influenced by the price fixed by the first $n$ facilities only: then we can compute the revenues from the first $n$ facilities by expression (8), as we do for a single competing facility (Section 5.1).

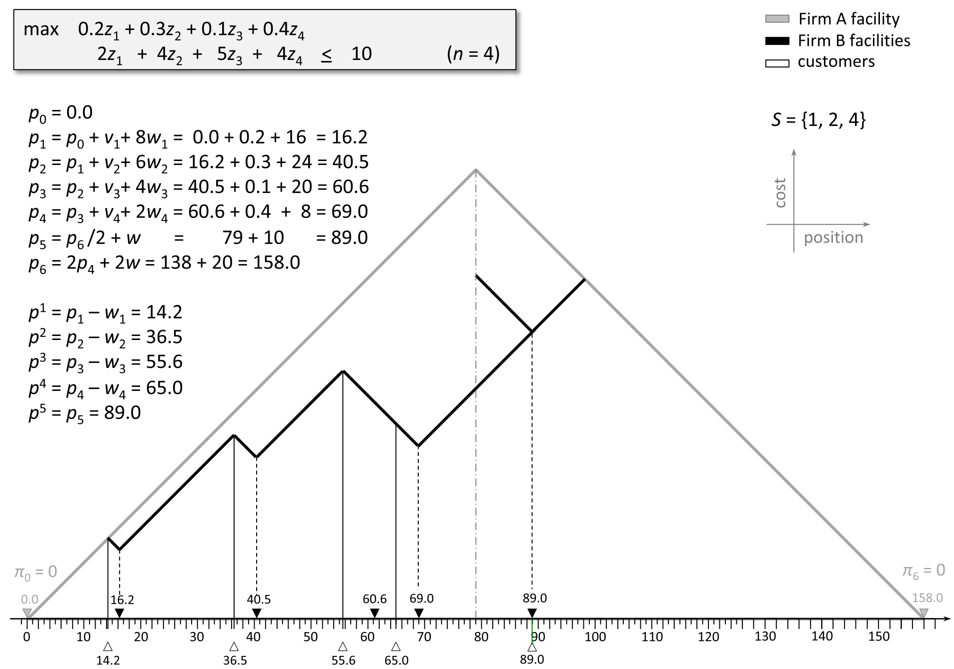


**Fig. 5.** Instance of Problem 2 associated with an instance of 0-1 KNAPSACK.

The next lemma characterizes the feasible solutions of knapsack problem $\mathcal{K}$.

**Lemma 11.** *Let $Y$ be an assignment of $C$ to $F$, and $\pi^*$ the associated minimal optimum price. Then the set*

$$S = \{k : \lambda_k = k\}$$

*identifies a feasible solution of problem $\mathcal{K}$ if and only if customer $n+1$ is assigned (to facility $n+1$) at price $\pi^*_{n+1} = W$.*

**Proof.** Since $d^{n+1}_{n+1} = |p^{n+1} - p_{n+1}| = 0$ and $\pi^*$ is minimal, $\pi^*_{n+1}$ must be equal to the minimum between the serving costs of facilities $n$ and $n+2$:

$$\pi^*_{n+1} = \min\{\pi^*_n + d^{n+1}_n, \pi_{n+2} + d^{n+1}_{n+2}\} = \min\{\pi^*_n + 2w, W\}. \tag{22}$$

Let us now consider the prices fixed by the other facilities in an optimal solution. Note that Firm A facility $n+2$ is not competitive for the customers lying to the left of the middle point $W+w$. So for $k \le n$ the optimal prices $\pi^*_k$ are determined as without facility $n+2$, and we can express them using (8), (13). Lemma 10 then allows us to say

$$\pi^*_k = \begin{cases} \pi^*_{k-1} - p_{k-1} + p_k - 2w_k & \text{if } \lambda_k = k \\ \pi^*_{k-1} - p_{k-1} + p_k & \text{otherwise.} \end{cases} \tag{23}$$

Unfolding (23) we obtain

$$\pi^*_n = \sum_{k=1}^{n} [p_k - p_{k-1} - 2w_k z_k] = W - 2\sum_{k=1}^{n} w_k z_k, \tag{24}$$

where $z_k$ is equal to 1 if $\lambda_k = k$ and 0 otherwise. In other words, $\mathbf{z}$ is the incidence vector of $S$. On the other hand, by (22) we know that $\pi^*_{n+1} = W$ if and only if $\pi^*_n + 2w \ge W$, that after (24) yields $\sum_{k=1}^n w_k z_k \le w$, that is $\mathbf{z}$ feasible for $\mathcal{K}$. □

By (24), the optimal price (22) becomes

$$\pi^*_{n+1} = \min\left\{W + 2\left(w - \sum_{k=1}^{n} w_k z_k\right), W\right\}. \tag{25}$$

**Lemma 12.** *Let $\pi^*$ be an optimal vector of prices, then $\pi^*_{n+1} = W$.*

**Proof.** Using (23), let us evaluate the difference in terms of revenue of the $k$-th term of (8) in the cases $\lambda_k = k$ and $\lambda_k \ne k$.

$$(n-k+1)(p_k - p_{k-1} - 2w_k) - (n-k)(p_k - p_{k-1})$$
$$= p_k - p_{k-1} - 2w_k(n-k+1) = v_k.$$

Using (8), we can write the revenue from the facilities $\ne n+1$ as

$$\sum_{k=1}^{n} (n+1-\lambda_k)\Delta\pi^*_k = \sum_{k=1}^{n} [(n-k)(p_k - p_{k-1}) + v_k z_k] \tag{26}$$

where $z_k$ is equal to 1 if $\lambda_k = k$ and 0 otherwise. Let us now include the contribution of facility $n+1$, and suppose by contradiction $\pi^*_{n+1} < W$. From Lemma 11 this happens only if $\sum_{k=1}^n w_k z_k > w$. Combining (25) and (26) we can compute the total revenue:

$$\begin{aligned} B(\pi^*) &= \sum_{k=1}^{n}(n-k)(p_k - p_{k-1}) + \sum_{k=1}^{n} v_k z_k + W + 2\left(w - \sum_{k=1}^{n} w_k z_k\right) \\ &\le \sum_{k=1}^{n}(p_k - p_{k-1})(n-k) + \sum_{k=1}^{n} v_k z_k + W - 2 \\ &\le \sum_{k=1}^{n}(p_k - p_{k-1})(n-k) + W - 1. \end{aligned} \tag{27}$$

The first passage derives from being $w, w_k, z_k \in \mathbb{N}$, and therefore $\sum_{k=1}^n w_k z_k \ge w+1$. The second, because by assumption $\sum_{k=1}^n v_k = 1$.

Let us now consider an optimal price vector $\bar{\pi}$ subject to $\lambda_k \ne k$ for all $k$. The relevant revenue can be computed by (26) setting $z_k = 0$ for all $k$, and adding $W$ to the result (in fact, $z_k = 0$ is feasible for $\mathcal{K}$, and we can therefore apply Lemma 11 to compute $\bar{\pi}_{n+1}$):

$$B(\bar{\pi}) = \sum_{k=1}^{n}(n-k)(p_k - p_{k-1}) + W. \tag{28}$$

Comparing (27) and (28) we then see the optimality of $\pi^*$ contradicted. □

**Theorem 13.** *Problem 1 is $\mathcal{NP}$-hard even for the case of one-dimensional metric, $c_k = c_{ik} = 0$ and $|F_B| = 2$.*

**Proof.** Summarizing the results stated so far:

- Lemma 12 ensures that, in an optimal price vector, $\pi_{n+1} = W$.
- From Lemma 11, we know that this happens only if $\mathbf{z}$ is feasible for $\mathcal{K}$
- Consequently, the objective function takes the form:

$$\sum_{k=1}^{n} (p_k - p_{k+1})(n-k) + \sum_{k=1}^{n} v_k z_k + W.$$

- Since $p_k$, $n$, $k$ and $W$ are constant, to find an optimal $\pi$ is equivalent to find a feasible solution $\mathbf{z}$ of $\mathcal{K}$ that maximizes $\sum_{k=1}^{n} v_k z_k$, that is, an optimum of $\mathcal{K}$. □

## 7. Conclusions

In this paper, we undertook an investigation of a Stackelberg pricing/location model strictly related to network and envy-free pricing problems. Two firms A and B compete along a line on determining prices to attract customers, who bear additional transportation costs and make rational access decisions after observing market prices. The leader firm, A, decides prices in its facilities first. Then the follower firm, B, decides both the sites where to open a facility and the relevant service prices, to which customers finally react by choosing a facility to receive service from.

We considered the particular case of metric transportation costs on a line, and concentrated on the computational complexity and numerical solvability of the optimization problems faced by the agents rather than on conditions for the existence of equilibrium. We established that maximizing the utility of B is $\mathcal{NP}$-hard even with two levels, no opening costs and just two opponent facilities held by A and offering service at known fixed costs. For a single opponent facility, the problem is instead solvable in polynomial time with three levels and fixed opening costs.

An investigation of the two-dimensional case will be the subject of a subsequent study. In this case, the indifference point — i.e., the point where, for a given price vector, a customer bears the same cost for service from two different facilities — evolves into a line or a more complex set: for example, under $\mathcal{L}_2$-norm, a hyperbola whose axis passes by the facility positions. In this case the function $c(\alpha, \beta) = \min_{k \in F}\{\pi_k + \sqrt{(\alpha_k - \alpha)^2 + (\beta_k - \beta)^2}\}$ expressing the cost borne by a customer in position $(\alpha, \beta) \in \mathbb{R}^2$ is obtained by composition of right circular conic surfaces pointed in $(\alpha_k, \beta_k, \pi_k) \in \mathbb{R}^3$. Even the analysis for a single facility in $F_A$ appears then quite complicated. For instance, an interesting question would be whether the Barrier Lemma can or not be extended in $\mathbb{R}^2$. Possible research might however consider dynamic programming to solve special cases.

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