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Superposition of FLRW universes

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Abstract. We show that (1) the Einstein field equations with a perfect fluid source admit a nonlinear superposition of two distinct homogenous Friedman-Lemaitre-Robertson-Walker (FLRW) metrics as a solution, (2) the superposed solution is an inhomogeneous geometry in general, (3) it reduces to a homogeneous one in the two asymptotes which are the early and the late stages of the universe as described by two different FLRW metrics, (4) the solution possesses a scale factor inversion symmetry and (5) the solution implies two kinds of topology changes: one during the time evolution of the superposed universe and the other occurring in the asymptotic region of space.

Keywords: cosmic singularity, cosmological phase transitions, cosmology of theories beyond the SM, dark energy theory

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Contents

1 Introduction

An exact solution of the Einstein-perfect fluid system, representing an inhomogeneous cosmological model and admitting various interpretations, was recently rediscovered in [1] by the method of separation of variables. This solution is a subcase of the Kustaanheimo-Qvist class [2–9], and coincides with the other previously found solutions. See [2–9] and [10, 11] for the classification and interpretations of the previously found inhomogeneous cosmological solutions. One of the possible interpretations is that it is a generalization of the McVittie solution representing a black hole immersed in a FLRW universe [11, 13–18]. In this work we propose a different interpretation of our solution. The inhomogeneous cosmological model of [1] is a nonlinear superposition of generically two distinct FLRW universes with different spatial curvatures. Nonlinear superposition in general relativity has been studied in the context of Bäcklund transformations [19–22] for spacetimes that possess two Killing vector fields. Here we give a novel example of a nonlinear superposition in general relativity. The FLRW phases show up in the early and late stages of the universe. Since each FLRW metric has a different spatial curvature, then in the early and late eras of the universe the spatial curvatures are different in general and a topology change is possible. The superposed model has a scale factor inversion symmetry akin to the scale factory duality symmetry observed earlier in string cosmology [23–26].

The layout of the paper is as follows. We first recapitulate the field equations of the inhomogeneous model in section 2. In section 3, we introduce a solution to the field equations possessing the scale factor inversion symmetry. In section 4, we give two distinct interpretations of the obtained solution. We mainly discuss the second interpretation: superposition of two FLRW universes. In section 5, we address two examples and discuss the features of our solution in the context of Big Bang and cyclic cosmological models [27–34]. Section 6 is devoted to our conclusions.

2 Matter-coupled field equations

We consider the spherically symmetric metric in isotropic coordinates

$$ds^2 = -a^2(t,r)dt^2 + c^4(t,r)\left(dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2\right), \tag{2.1}$$

where $a(t,r)$ and $c(t,r)$ are generic differentiable functions of time t and radial coordinate r. In [1], it was proven that the Einstein-perfect fluid field equations for $\dot{c}(t,r) \neq 0$ reduce to

$$2rcc'' - 6rc'^2 - 2cc' - h(r) = 0, \tag{2.2}$$

$$a = 2q(t)\,\frac{\dot{c}}{c}, \tag{2.3}$$

where $h(r)$ is an arbitrary function of r and $q(t)$ is an arbitrary function of t; the fluid velocity is given as $u_\mu = a\delta^0_\mu$. Here a dot and a prime denote differentiations with respect to t and r, respectively. The energy density $\rho(t,r)$ and the pressure $p(t,r)$, respectively read as

$$8\pi\rho(t,r) = \frac{3}{q^2} - \frac{2}{rc^6}\left(6rc'^2 + 6cc' + h\right) - \Lambda, \tag{2.4}$$

$$8\pi p(t,r) = -\frac{3}{q^2} + \frac{1}{rq^3c^6\dot{c}}\left(2q^3\left(-rc'^2\dot{c} + rcc'\dot{c}' + cc'\dot{c} + c^2\dot{c}'\right) + rc^7\dot{q}\right) + \Lambda, \tag{2.5}$$

where Λ is the cosmological constant. In [1], we discussed that if the metric functions $c(t,r)$ and $a(t,r)$ vanish on some surfaces then either p or ρ diverges, and hence the Ricci scalar $R = 8\pi(\rho - 3p) + 4\Lambda$ diverges. These surfaces are defined as $\Sigma_1 = \{(t,r) \in U | c(t,r) = 0\}$ and $\Sigma_2 = \{(t,r) \in U | a(t,r) = 0\}$ where U is a part of spacetime and $0 < t < \infty$, $r \geq 0$. In particular, the surface Σ_2 represents namely the cosmological singularities [35] or sudden cosmological singularities [36–38].

3 A solution to the field equations possessing the scale factor inversion symmetry

In [1], it was shown that

$$c(t,r) = \frac{\sqrt{R(t)}}{\sqrt{c_0 + c_1 r^2}} + \frac{\gamma}{\sqrt{R(t)}}\frac{1}{\sqrt{c_2 + c_3 r^2}}, \tag{3.1}$$

is a solution to the ordinary nonlinear differential equation (2.2) with arbitrary constants c_0, c_1, c_2, c_3 and γ; and $R(t)$ is an at least twice differentiable function. For this solution, $h(r)$ is given by

$$h(r) = \frac{6\gamma(c_0\,c_3 - c_1\,c_2)^2 r^3}{(c_0 + c_1 r^2)^{\frac{5}{2}}(c_2 + c_3 r^2)^{\frac{5}{2}}}, \tag{3.2}$$

and furthermore choosing $q(t) = R(t)/\dot{R}(t)$, the lapse function reads as

$$a(t,r) = \frac{1 - \frac{\gamma}{R(t)}\sqrt{\frac{c_0 + c_1 r^2}{c_2 + c_3 r^2}}}{1 + \frac{\gamma}{R(t)}\sqrt{\frac{c_0 + c_1 r^2}{c_2 + c_3 r^2}}}. \tag{3.3}$$

Hence, the spacetime metric reads as

$$ds^2 = -\left(\frac{1 - \frac{\gamma}{R(t)}\sqrt{\frac{c_0+c_1r^2}{c_2+c_3\,r^2}}}{1 + \frac{\gamma}{R(t)}\sqrt{\frac{c_0+c_1r^2}{c_2+c_3r^2}}}\right)^2 dt^2$$
$$+ \left(\frac{\sqrt{R(t)}}{\sqrt{c_0+c_1r^2}} + \frac{\gamma}{\sqrt{R(t)}}\frac{1}{\sqrt{c_2+c_3r^2}}\right)^4 \left(dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2\right). \tag{3.4}$$

Substituting (3.1) in (2.4) and (2.5), one can find the asymptotic form when $R(t) \to 0$ and keeping the leading orders, $\rho(t,r)$ and $p(t,r)$ become homogeneous and reduce to

$$8\pi\rho(t) \to 3\frac{\dot{R}^2}{R^2} + \frac{12c_2c_3R^2}{\gamma^4} - \Lambda, \tag{3.5}$$
$$8\pi p(t) \to 2\frac{\ddot{R}}{R} - 5\frac{\dot{R}^2}{R^2} - \frac{4c_2c_3R^2}{\gamma^4} + \Lambda. \tag{3.6}$$

Similarly, $\rho(t,r)$ and $p(t,r)$ also become asymptotically homogeneous as $R(t) \to \infty$ and read as

$$8\pi\rho(t) \to 3\frac{\dot{R}^2}{R^2} + \frac{12c_0c_1}{R^2} - \Lambda, \tag{3.7}$$
$$8\pi p(t) \to -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{4c_0c_1}{R^2} + \Lambda. \tag{3.8}$$

Observe that the metric, i.e (3.1) and (3.3), is invariant under the scale factor inversion, as $R \to \gamma^2/R$, $c_0 \leftrightarrow c_2$ and $c_1 \leftrightarrow c_3$. This symmetry is akin to the scale factor duality symmetry in string cosmology [23–26]. This inversion symmetry can also be observed in the field equations: under this symmetry, the pair of equations (3.5) and (3.6) go to (3.7) and (3.8) and vice versa.

4 Interpretations of the solution

4.1 A generalized McVittie metric

Without losing any generality, one can choose the arbitrary constants c_0, c_1, c_2, c_3 and γ in such a way that the function $c(t,r)$ becomes

$$c(t,r) = \frac{\sqrt{R(t)}}{\sqrt{\mu + r^2}} + \frac{M}{2\sqrt{R(t)}}\frac{1}{\sqrt{1+kr^2}}, \tag{4.1}$$

with three constants M, k and μ representing the mass, spatial curvature of the background FLRW universe and the reduction parameter, respectively and $R(t)$ is the same function as in (3.1) up to a multiplicative constant. Letting $\mu = 0$, the solution reduces to the uncharged Vaidya-Shah and the McVittie solution [12–15], see also [10]. The McVittie solution can be interpreted as a black hole in a positively curved FLRW universe [16–18]. Next we shall give another interpretation of this metric.

4.2 Superposition of two FLRW universes

One can arrange the constants c_0, c_1, c_2 and c_3 in (3.1) so that $c(t,r)$ takes the form

$$c(t,r) = \frac{\sqrt{R(t)}}{\sqrt{1+k_1 r^2}} + \frac{\gamma}{\sqrt{R(t)}} \frac{1}{\sqrt{1+k_2 r^2}}, \tag{4.2}$$

where k_1, k_2 and γ are arbitrary constants and $R(t)$ is the same function as in (3.1) up to a multiplicative constant.

To show that the two FLRW metrics are nonlinearly superposed, we need to use the FLRW metric in the isotropic coordinates which reads

$$ds^2 = -dt^2 + \frac{R^2(t)}{(1+kr^2)^2}\left(dr^2 + r^2\, d\Omega^2\right), \tag{4.3}$$

where $R(t)$ is the scale factor and k is $1/4$ of the curvature constant of 3-space (lets call it k_*); and after making the coordinate r unitless, k_* can take the values ± 1 and 0. To discuss that our new solution represents the superposition of two FLRW universes, we consider the following two different FLRW universes in the isotropic coordinates (t, r, θ, ϕ).

FLRW$_1$:

The metric for this case is given by

$$ds^2 = -dt^2 + c_1^4(t,r)\left(dr^2 + r^2\, d\Omega^2\right), \tag{4.4}$$

where

$$c_1(t,r) = \frac{\sqrt{R(t)}}{\sqrt{1+k_1 r^2}}. \tag{4.5}$$

This metric represents a FLRW universe with the scale factor $R(t)$ and the spatial curvature $k_* = 4k_1$. The homogeneous matter density and pressure profiles for this case read as

$$8\pi\rho_1(t) = 3\frac{\dot{R}^2}{R^2} + 3\frac{4k_1}{R^2} - \Lambda, \tag{4.6}$$

$$8\pi p_1(t) = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{4k_1}{R^2} + \Lambda. \tag{4.7}$$

FLRW$_2$:

The metric for this case is given by

$$ds^2 = -dt^2 + c_2^4(t,r)\left(dr^2 + r^2\, d\Omega^2\right), \tag{4.8}$$

where

$$c_2(t,r) = \frac{1}{\sqrt{R(t)}} \frac{1}{\sqrt{1+k_2 r^2}}. \tag{4.9}$$

Then it represents a FLRW metric with the scale factor $R^{-1}(t)$ and the spatial curvature $k_* = 4k_2$. Here, the matter density and pressure profiles are

$$8\pi\rho_2(t) = 3\frac{\dot{R}^2}{R^2} + 12k_2 R^2 - \Lambda, \tag{4.10}$$

$$8\pi p_2(t) = 2\frac{\ddot{R}}{R} - 5\frac{\dot{R}^2}{R^2} - 4k_2 R^2 + \Lambda. \tag{4.11}$$

Now, regarding the above two cases, we have the following theorem indicating the second novel interpretation of the general solution (3.1) which we mainly discuss below.

Theorem *The linear superposition of $c_1(t,r)$ and $c_2(t,r)$, i.e. $c(t,r) = c_1(t,r) + \gamma c_2(t,r)$, with $c_1(t,r)$ and $c_2(t,r)$ given in (4.5) and (4.9) solves the Einstein field equations (2.2), (2.3), (2.4) and (2.5). Here, γ is an arbitrary constant. Then, this represents a nonlinear superposition of two particular solutions (4.2) for which each part is generically a distinct* FLRW *universe with different spatial curvatures k_1 and k_2. If $k_1 = k_2 = k$, this general solution reduces to a single* FLRW *solution with the spatial curvature $k_* = 4k$.*

The model has a Big Bang singularity if $R \to 0$ as $t \to 0$ without further assumption on the energy density and pressure, this is the only constraint on the function R. Hence, each choice of R generates a different cosmological model. For instance, assuming the scale factor to be $R(t) = R_0 \left(e^{\lambda t} - 1\right)^n$ where R_0, λ and n are positive constants, then from (3.5) and (3.6), one finds $\rho(t) \to \infty$ and $p(t) \to -\infty$ as $t \to 0$, respectively. In the limit $t \to \infty$, $R(t) \to \infty$ and from (3.7) and (3.8), one finds $\rho(t) \to 3n^2\lambda^2 - \Lambda$ and $p(t) \to -3n^2\lambda^2 + \Lambda$, respectively, which corresponds to a (anti)de Sitter space. In this case, the universe starts from a Big Bang and evolves to a pure (anti)de Sitter phase at late times.

Three consequences of the above theorem are as follows: (i) the exact solution unifies two different homogeneous FLRW solutions in a single superposed solution which generally is inhomogeneous, (ii) regarding (3.5)–(3.8), and (4.6), (4.7), (4.10) and (4.11), the universe is approximately FLRW$_2$ as $R(t) \to 0$ and approximately FLRW$_1$ as $R(t) \to \infty$, see the item (iv) in the next section for an expanding universe for more detail. Then the solution represents a phase transition from FLRW$_2$ to FLRW$_1$ with possibly a topology change: If we denote the state of the universe with the FLRW parameters as (R, k) where R is scale factor and k is the normalized curvature of 3-space. According to our model, during the time evolution, the universe undergoes a change of state from $(1/R, k_2)$ to a state (R, k_1). In other words, the universe starts with a state $(1/R, k_2)$ and ends with a different state (R, k_1). We call this change of state as a "phase transition", and (iii) there exists a scale factor inversion symmetry in the metric as $R \to \gamma^2/R$ and $k_1 \leftrightarrow k_2$. Under this symmetry, FLRW$_2 \leftrightarrow$ FLRW$_1$.

Let us expound on the difference of the early and late eras (as $R \to 0$ and $R \to \infty$) of the universe as different FLRW universes. We can normalize only one of the parameters k_1 or k_2 by scaling the coordinate r. By such a scaling k_1, k_2 and r become unitless. If the beginning of the universe is FLRW$_2$ with a normalized k_2 and ends as FLRW$_1$ with spatial curvature k_1, we have two possibilities, either sign $(k_1) =$ sign (k_2) (two universes having the same topology) and sign $(k_1) \neq$ sign (k_2) indicating a change of topology, see the item (iv) in the next section for an expanding universe for more detail. In these asymptotic stages, the universe is approximately homogeneous.

In the following section, we elaborate on the properties of the solution for $k_1 \neq k_2$ with two specific cosmological scenarios.

5 Two specific cosmological scenarios

5.1 Expanding universe scenario

For an ever expanding universe with $R(t) \to 0$ as $t \to 0$ and $R(t) \to \infty$ as $t \to \infty$, or a universe with a minimum size $R_{\rm min}$ ($R_{\rm min} \ll 1$) for $t \to 0$ and a maximum size $R_{\rm max}$ ($R_{\rm max} \gg 1$) for $t \to \infty$, we note the following points.

(i) In the early era ($t \to 0$), $c_2(t,r)$ dominates in $c(t,r)$. Then, the universe is effectively FLRW$_2$ and hence becomes homogeneous according to the matter density and pressure

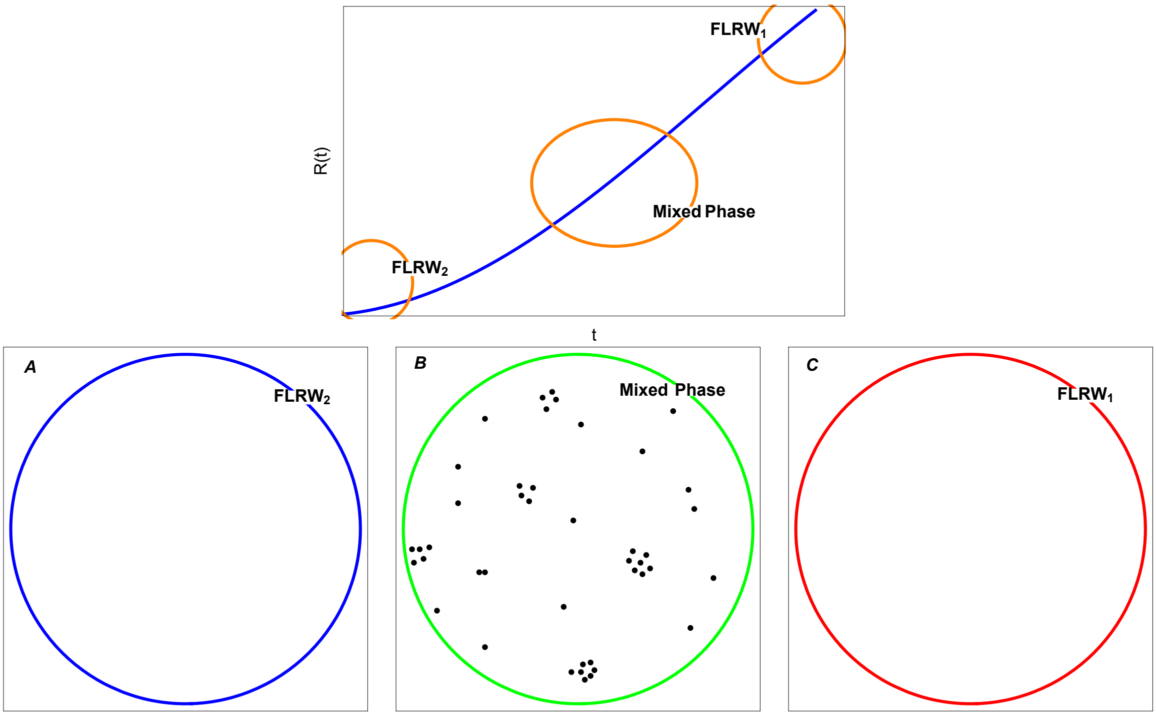

Figure 1. The upper plot represents two different initial and final FLRW universes. The plots A and C represent two different initial and final homogeneous universes while the plot B denotes growing inhomogeneities in the intermediate mixed phase.

given by (4.10) and (4.11), respectively. See the upper plot in figure 1 for $k_1 \geq 0$ and $k_2 > 0$.

(ii) In the intermediate era ($0 < t < \infty$), both $c_1(t,r)$ and $c_2(t,r)$ are effective and hence we have a mixture of FLRW$_1$ and FLRW$_2$ universes which indeed is an inhomogeneous universe according to the matter density and pressure given by (2.4) and (2.5), respectively (inhomogeneity can also be seen from any of the non-vanishing curvature invariants such as the Ricci scalar which has position dependence). In this case, cosmological inhomogeneities emerge and contribute to the formation of structures in the universe. See figure 1 for $k_1 \geq 0$ and $k_2 > 0$ where the plots A and C represent initial and final homogeneous FLRW universe while the plot B indicates a mixed inhomogeneous universe. The dots in the plot B depict symbolically the inhomogeneities in the mixed phase.

(iii) In the late times ($t \to \infty$), $c_1(t,r)$ dominates in $c(t,r)$. Therefore, the universe tends effectively to FLRW$_1$ and hence becomes homogeneous according to (4.6) and (4.7). See the upper plot in figure 1 for $k_1 \geq 0$ and $k_2 > 0$.

(iv) An important consequence of the phase transition between FLRW$_2$ and FLRW$_1$ is the topology change in the universe. Since FLRW$_2$ dominating at early times and FLRW$_1$ dominating at late times have different spatial curvatures in general, the universe may undergo a topology change from k_2 to k_1 during its evolution. For this topological transition, one realizes the following two points.

1. If $k_1 \geq 0$ and $k_2 > 0$ (or $k_1 > 0$ and $k_2 \geq 0$), hence $r \in [0, \infty)$. This case represents the topology change from a closed universe to another flat or closed universe (or vice versa). To observe this topology change, considering $k_1 \geq 0$ and $k_2 > 0$ in (4.2), we have

$$c(t,r) = \frac{\sqrt{R(t)}}{\sqrt{1+k_1 r^2}} + \frac{\gamma}{\sqrt{R(t)}} \frac{1}{\sqrt{1+k_2 r^2}}. \tag{5.1}$$

The dominant term in $c(t,r)$ at the asymptotic limit $t \to 0$ and thus $R(t) \to 0$ is the second term with the topology $k_2 > 0$. As time progresses toward the late times, i.e as $t \to \infty$ and thus $R(t) \to \infty$, the first term in (5.1) with the topology $k_1 \geq 0$ dominates and consequently a change of topology occurs.

2. If $k_1 \geq 0$ and $k_2 < 0$ or ($k_1 < 0$ and $k_2 \geq 0$). This case represents a topology change from a spatially open to a flat or closed universe (or vice versa). To observe the topology change in this case, we have

$$c(t,r) = \frac{\sqrt{R(t)}}{\sqrt{1+k_1 r^2}} + \frac{\gamma}{\sqrt{R(t)}} \frac{1}{\sqrt{1-|k_2| r^2}}. \tag{5.2}$$

Here, there is a restriction on the coordinate patch as $r \in [0, \frac{1}{\sqrt{|k_2|}})$. If $k_1 = 0$, in the asymptotic limit $t \to 0$ and hence $R(t) \to 0$, regardless of whatever r is, the second term in (5.2) dominates. Then, at the early time asymptotic state, the topology is $k_2 < 0$. At the late times, there are two possibilities as (i) in the region $0 \leq r \ll \frac{1}{\sqrt{|k_2|}}$, as $t \to \infty$ and $R(t) \to \infty$, the first term in (2) dominates and a topology change occurs in time. In this region, the universe starts with the open topology $k_2 < 0$ and evolves toward the state with a flat topology $k_1 = 0$, and (ii) in the asymptotic region, i.e. $r \to \frac{1}{\sqrt{|k_2|}}$, the second term survives in the limit $t \to \infty$ and $R(t) \to \infty$. Thus, both the terms in (5.2) can be effective that preserves the inhomogeneity at the asymptotic region. For this case, the topological structure of the asymptotic region can be more complicated than the previous cases and it may be different than both of the k_1 and k_2. It is interesting that here the topologies for the internal and asymptotic regions of spacetime can be different which implies another kind of topology change. A similar topology change occur also for $k_1 < 0$ and $k_2 \geq 0$. In particular, the case $k_1 < 0$ and $k_2 > 0$ represents the topology change of the spacetime akin to the "bag of gold" geometry of Wheeler [39].

A particular example: initially inflating and finally accelerating expanding universe scenario. Considering the scale factor $R(t) = R_0 \left(e^{\sqrt{\frac{\Lambda}{3}} t} - 1 \right)$, one observes that in the early universe as $R(t) \to 0$, FLRW$_2$ (4.8) dominates in the superposition. Then, the evolution of the universe is governed by (4.10) and (4.11). According to (4.10) and (4.11), we have $\rho_2(t) \to \infty$ and $p_2(t) \to -\infty$ as $t \to 0$, respectively. The first represents the initial singularity in the matter sector while the latter represents a self driven inflation in the early times. It the late times, as $t \to \infty$, $R(t) \to R_0 e^{\sqrt{\frac{\Lambda}{3}} t}$, FLRW$_1$ (4.8) is dominant in the superposition and the universe evolves according to (4.6) and (4.7) to an accelerating

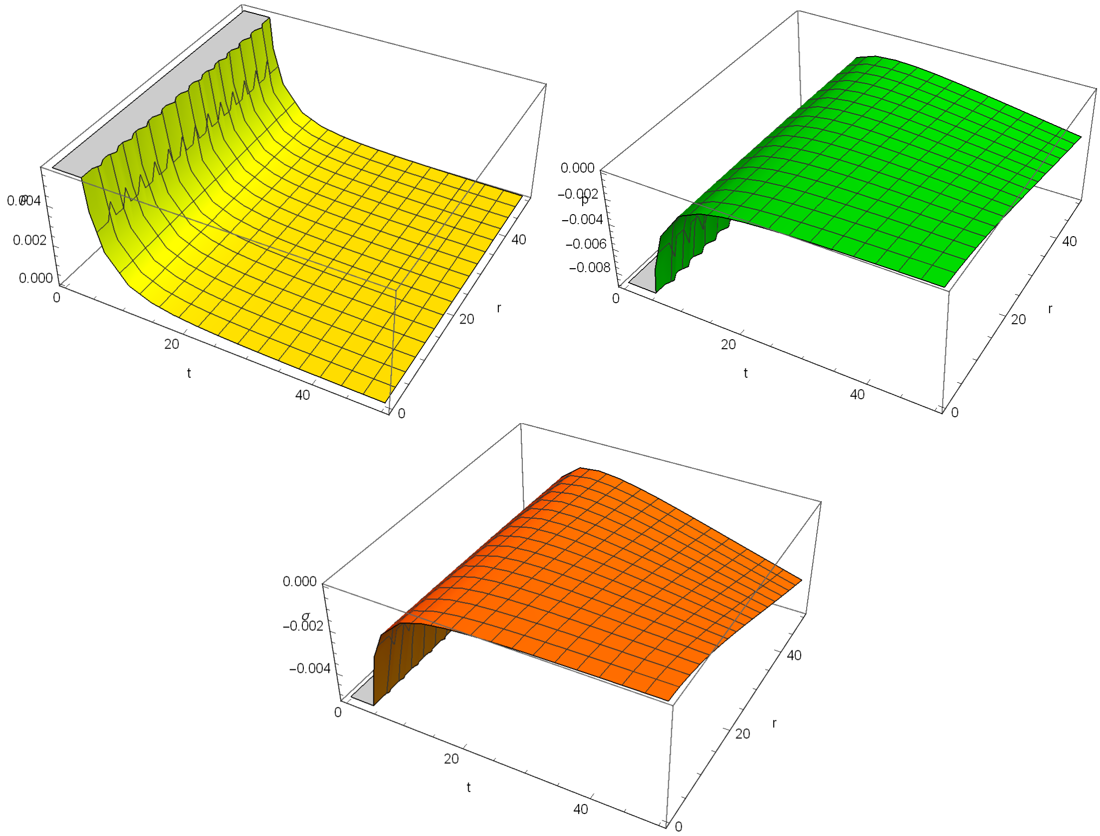

Figure 2. The evolution of the density $\rho(t,r)$, pressure $p(t,r)$ and $\sigma(t,r) = \rho + p$, respectively, for the parameter values $k_1 = 0$, $k_2 = 1$, $\gamma = 1$, $R_0 = 1$, $\sqrt{\Lambda/3} = 0.0001$, and $R(t) = R_0\left(e^{\sqrt{\frac{\Lambda}{3}}t} - 1\right)$.

expanding a pure de Sitter phase, i.e. $\rho_1(t) \to 0$ and $p_1(t) \to 0$ as $t \to \infty$. Then, one observes an interesting property in this exact solution to the Einstein field equations. Indeed, this solution provides a scenario including the early time inflation, inhomogeneous structure formation in the intermediate era, and the late time accelerating expansion in a unified model possessing a possible topology change. In the intermediate phase where both FLRW_1 and FLRW_2 contribute effectively to the evolution of the universe, the inhomogeneities emerge. This possibility cannot be achieved in the usual standard FLRW cosmology with only time dependent field equations. One may argue about the physical nature of the solution by addressing the matter density, pressure and energy conditions. In figure 2, we have plotted the density $\rho(t,r)$ and pressure $p(t,r)$ in (2.4) and (2.5), respectively, as well as $\sigma(t,r) = \rho + p$ for some typical values of the parameters. It is seen that for a superposed universe undergoing a topology change from a closed to flat topology with the scale factor $R(t) = R_0\left(e^{\sqrt{\frac{\Lambda}{3}}t} - 1\right)$, there are regions where the density remains positive and pressure and σ are negative. The negativeness of σ represents the violation of weak energy condition that is consistent with the expanding nature of the cosmos in the context of this solution.

5.2 Cyclic universe scenarios

There are various cyclic universe scenarios, as exemplified in [27–34]. The old cyclic scenario is based on the possibility that the scale factor $R(t)$ of the universe oscillates at regularly

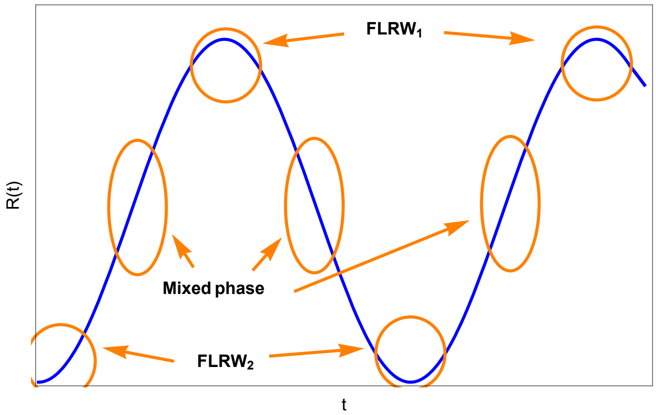

Figure 3. The evolution of a cyclic universe undergoing a topology change during the oscillation between subsequent global maximum and minimum values of its scale factor $R(t)$.

spaced intervals of time between maximum and minimum values [27, 28]. Considering such a scale factor, the superposed universe may undergo a topology change during each of its expansion and contraction phases between those minimum and maximum values of the scale factor $R(t)$, see figure 3. As the scale factor reaches its minimum and maximum values, the universe becomes approximately homogeneous FLRW_2 and FLRW_1, respectively, and it is in the mixed phase between these extremum points where inhomogeneities emerge.

Another possible type of cyclic universe scenario was proposed recently by Ijjas and Steinhardt [29]. In this model, instead of the scale factor $R(t)$, the Hubble parameter $H(t) = \dot{R}(t)/R(t)$ oscillates periodically during the evolution of the universe. In the context of this model, the scale factor $R(t)$ increases substantially during each cosmological era and then undergoes an ultra-slow contraction phase at the end of each cycle. Then, the next cycle of the universe begins with a non-singular bounce. Figure 4 represents a typical plot of this type of cyclic universe scenario. In the first cycle, from t_0 to t_2, the universe expands from FLRW_2 at t_0 with the spatial curvature k_2 to FLRW_1 at the local maximum of $R(t)$ at t_1 with the spatial curvature of k_1. Then, it contracts between t_1 to t_2 and recovers its previous curvature k_2 of the FLRW_2 state provided that $R_2 \ll R_1$. In both the expansion and contraction phases, the universe becomes inhomogeneous in between the local maximum and minimum values of the scale factor. One also notes that by flattening the contraction phases, i.e. in the second cycle where $R_3 \sim R_4$, the universe keeps its topology in the contraction phase as in the past local maximum point but still evolves inhomogeneously toward the non-singular bouncing point.

As a final remark, let us note that although in Penrose's conformal cyclic cosmological picture [30–34] there is no contraction phase, since in each aeon the universe is ever expanding $(R(t_f) \gg R(t_i)$ where i and f denote the initial and final states of an arbitrary aeon) the universe undergoes a change of spatial curvature from k_2 to k_1 during the evolution of each aeon. This requires a sudden change of spatial curvature and possibly topology change at the transition point from the past aeon to the present aeon from k_1 to k_2. This sudden transition point corresponds to the Big Bang of the present aeon. However, since the metric of the past aeon at its null infinity and the metric of the present aeon at its Big Bang surface are conformally related, occurrence of such a sudden topology change is forbidden in the conformal cyclic cosmology picture.

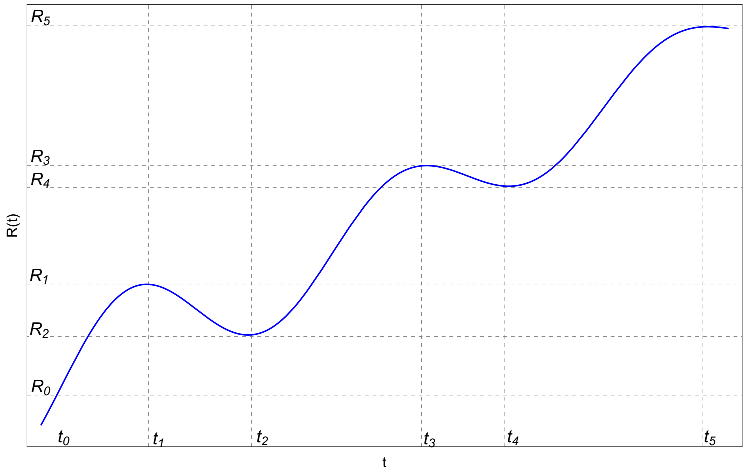

Figure 4. The evolution of a cyclic universe undergoing a topology change in transition through the local maximum and minimum values of a typical scale factor $R(t) = e^{\lambda t} J_0^2(t) + 4$ where J_0 is the zeroth order Bessel function and we have set the dimensionless parameter $\lambda = 0.009$.

6 Conclusions

A summary of what we propose in this work is as follows: (1) we give a new example of nonlinear superposition in general relativity. This is a superposition of two different homogenous FLRW universes yielding an inhomogeneous cosmological model. (2) The metric is invariant under the scale factor inversion. (3) If the scale factor R is zero in the beginning of the universe and goes to infinity as $t \to \infty$ then the universe starts approximately as a FLRW universe and ends as a different FLRW universe. (4) During such a phase transition the spatial curvature of 3-space changes in both magnitude and sign. If the sign changes then the topology of the 3-space also changes but if the sign remains intact, then the spatial curvature of the 3-space either increases or decreases.

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