DIFFUSION CONTROL OF SUCCESSIVE PRODUCT GENERATIONS WITH RECYCLING POTENTIAL

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

DIFFUSION CONTROL OF SUCCESSIVE PRODUCT GENERATIONS WITH RECYCLING POTENTIAL

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In this thesis, we study the sales planning problem of a producer who sells two successive generations of a durable good with recycling potential. Certain expensive materials can be recovered from consumer returns of the early-generation product and can be used in manufacturing of the new-generation product. Demands for the successive product generations arrive as a generalized Norton-Bass diffusion process and the recycling operations for the new-generation product are constrained by the early-generation product returns. In this setting, we investigate whether slowing down the new-generation product diffusion by partially satisfying its demand might be profitable for the producer who aims to maximize its total profit from the entire product line. Such manipulation of the diffusion process may improve the use of recycled content in production as well as the cross-generation repeat purchases over a sufficiently long selling horizon. The optimal sales plan involves partial demand fulfillment when the diffusion curves of the early- and new-generation products overlap substantially and the release of the new-generation product only moderately increases the customer base. However, partial demand fulfillment is less likely to be desirable if the product returns mostly arrive through trade-up programs rather than recycling programs such as free mail-back and physical drop-off options offered to consumers. Finally, partial demand fulfillment, if initiated too late, may escalate the overall consumption of virgin raw materials, making it environmentally undesirable.

Keywords: marketing-operations interface; multi-generation product diffusion; sales planning; closed-loop supply chains; recycling.

ÖZET

GERİ DÖNÜŞÜM POTANSİYELİ OLAN ARDIŞIK ÜRÜN NESİLLERİNİN YAYILIM KONTROLÜ

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Bu tezde, geri dönüşüm potansiyeli olan dayanıklı bir ürünün iki ardışık neslini satan bir üreticinin satış planlama problemi çalışılmaktadır. Bazı pahalı materyaller, müşterinin geri getirdiği eski-nesil üründen geri kazanılabilir ve yeni-nesil ürününün üretiminde kullanılabilir. Ardışık ürün nesillerinin talebi genelleştirilmiş Norton-Bass yayılım sürecine göre gerçekleşmektedir. Yeni-nesil ürünler için geri dönüşüm operasyonları, geri getirilen eski-nesil ürünlerin miktarıyla kısıtlanmaktadır. Bu modelde, yeni-nesil ürünün talebini *kısmen* karşılayarak yeni-nesil ürünün yayılım sürecini yavaşlatmanın, üreticinin tüm üretim hattından toplam kârını iyileştirip iyileştirmediği araştırılmıştır. Yayılım sürecine yapılan bu manipülasyon, yeterince uzun bir satış döneminde, üretimde geri dönüştürülmüş içerik kullanımının yanı sıra nesiller arası tekrarlayan satın alımını artırabilir. Eski-nesil ve yeni-nesil ürünlerin yayılım eğrileri büyük ölçüde örtüştüğünde ve yeni-nesil ürünün piyasaya sürümü müşteri tabanını kısıtlı miktarda arttırdığında en kârlı satış planında talep kısmen karşılanmaktadır. Ancak, eğer ürünler müşterilere sunulan postayla geri gönderme veya fiziksel teslim etme seçenekleri gibi geri dönüşüm programları yerine çoğunlukla takas programları aracılığıyla geri getiriliyorsa, kısmi talep karşılama daha az istenen bir durum olmaktadır. Son olarak, kısmi talep karşılama, eğer çok geç başlatılırsa, işlenmemiş hammaddenin toplam tüketiminin artmasına yol açarak çevresel açıdan zararlı olabilmektedir.

Anahtar sözcükler: pazarlama-faaliyetler arayüzü; çok nesilli ürün yayılımı; satış planlaması; kapalı devre tedarik zincirleri; geri dönüşüm.

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Chapter 1

Introduction

Many electronics producers now strive to build circular supply chains by increasing recycled content and renewable materials in their devices. In the smartphone industry, for example, Apple collects used iPhone devices through its trade-in and recycling program as well as its partner programs with Best Buy in the United States and KPN in the Netherlands. Apple's newly invented robots, Daisy and Dave, take apart the iPhone devices at the end of their life into distinct components and disassemble select components like the taptic engine and battery for recovery of materials such as rare earth elements, steel, tungsten, and cobalt. Several recovered materials are used to make brand-new iPhone devices (Apple [1], [2]). In the personal computer industry, Microsoft has committed to achieve 100% recyclable Surface devices and expand the consumer mail-back program for its own-brand products worldwide by 2030 (Microsoft [3]).

For many electronics products, as a result of short product life cycles due to fast evolution in technology, the end-of-life products are likely to be received from consumers of early-generation devices while the recycled materials are likely to be used in manufacturing of new-generation devices (Zhang and Zhang [4]). Therefore, the economic value of recovering the precious materials from the collected end-of-life items for use in manufacturing of new devices, if feasible, can be improved by taking a holistic view of the diffusion dynamics of successive generations of the device as well as the closed-loop dynamics of the supply chain. In this thesis, we investigate whether the producer who sells a durable good with recycling potential can increase its total profit from the sales of two successive product generations by endogenously shaping the diffusion process to integrate more recycled content into its devices in the long run.

Many papers in the marketing-operations interface have considered endogenous modeling of the diffusion process in forward supply chains (FSCs) in order to study the sales planning and/or pricing problem for a single product generation (Ho et al. [5], [6], Kumar and Swaminathan [7], Shen et al. [8], [9]) and the market entry timing and/or pricing problem for successive product generations (Wilson and Norton [10], Mahajan and Müller [11], Krankel et al. [12], Ke et al. [13], Guo and Chen [14], Jiang et al. [15]). Several other papers have incorporated endogenous modeling of the diffusion process into closed-loop supply chains (CLSCs) in order to study the sales planning and/or pricing problem for a single product generation (Debo et al. [16], Robotis et al. [17], Akan et al. [18], Nadar et al. [19]). All of the above papers have adopted, with slight modifications, the seminal Bass diffusion process [20] or its multi-generation extensions, in which the diffusion dynamics are governed by the word-of-mouth communication as well as the external sources of information such as mass advertising.

To our knowledge, however, endogenous modeling of the diffusion process has been overlooked for CLSCs of successive product generations. This thesis is the first attempt to fill this gap in the literature: in the presence of two successive generations of a device with recycling potential, we investigate whether delaying new-generation product diffusion by rejecting some demands and thus curbing the positive word-of-mouth feedback about its new device can be profitable for the producer. We consider a dynamic model in which demands for the successive generations of the device arrive as the generalized Norton-Bass diffusion process developed by Jiang and Jain [21]. The classical Norton-Bass model extends the Bass diffusion dynamics to multiple generations of the same product by capturing the substitution effect among these generations that may coexist in the market (Norton and Bass [22]). The generalized Norton-Bass model counts the number of buyers who substitute an early-generation product with a new-generation product by differentiating those who have bought the early-generation product (switching adopters) from those who have not (leapfrogging adopters). Including all these aspects of the product adoption process, our model posits that a customer whose demand is not satisfied communicates no feedback about the experience of the product, as widely recognized in the literature (see, for example, Ho et al. [5], [6], Kumar and Swaminathan [7], Shen et al. [8], [9]).

In our model, some of the consumers who buy the early-generation product return their end-of-life products to the producer in the future. Currently, many electronics producers accept the end-of-life returns from their consumers at no charge in retail stores (e.g., Hewlett-Packard [23], Apple [1], [2], Samsung [24]), while several states in the United States require the producers to collect the end-of-life devices by offering free mail-back programs to their consumers and/or supporting physical drop-off locations (OECD [25]). In this thesis, we consider two possible scenarios for the consumer returns: In the first scenario, the consumers are heterogeneous in their timing of end-of-life returns, which is independent of their timing of purchases of new-generation devices. This scenario is sensible for modeling the unpredictable product usage behavior of the consumers. This scenario may be realistic when most returns arise from the consumers who return their old devices through free mail-back programs and/or physical drop-off locations. In the second scenario, however, the consumers return their end-of-life devices only at the time of their purchases of new-generation devices, so that the return process is perfectly aligned with the diffusion process. This scenario may be realistic when most returns arise from the consumers who return their old devices to trade up to the next-generation product. Finally, in both scenarios, the producer can profitably recycle a certain amount of material from each collected early-generation device for use in manufacturing of new-generation devices.

We prove that delaying new-generation product diffusion via partial demand fulfillment raises the total number of switching adopters as well as the total number of cross-generation purchases over a sufficiently long selling horizon, potentially improving the total sales volume in the long run. Any deliberately delayed demand for the new-generation product is also more likely to be met by using the recycled content because more returns become available in later stages of the selling horizon. In both scenarios of the product returns, we establish the conditions that ensure the optimality of partial demand fulfillment: it is optimal to reject some demand from the customers who are only attracted by the new-generation product in some period with shortage of the recyclable material if

- the available amount of recyclable material is sufficiently large in later periods so that each unit of the new-generation product demand in these periods can benefit from the recycled content to the fullest extent possible and
- the revenue gain from improved use of recycled content and possible increase in cross-generation repeat purchases via delayed demand exceeds the revenue loss due to reduced new-generation product sales induced by partial fulfillment.

We demonstrate the optimality of the partial-fulfillment policy in fast-clockspeed industries when the diffusion curves of early- and new-generation products overlap substantially and the new-generation product moderately increases the market potential. However, if most returns arise from the switching adopters participating in the producer's trade-up program, rather than the producer's recycling program such as free mail-back and physical drop-off options, our results imply that the partial-fulfillment policy is less likely to be desirable. This is because the available amount of recyclable material obtained from such returns displays fluctuations over time similar to the diffusion curve of the new-generation product demand from the switching adopters, negating the need for a further alignment between the supply and demand for recycled content via partial fulfillment. Finally, we show that the partial-fulfillment policy, if initiated too late, may escalate the overall consumption of virgin raw materials in production; such a sales strategy should be approached with caution from an environmental perspective.

Chapter 2 of this thesis reviews the related literature for FSCs and CLSCs. Chapter 3 formulates our problem for the two scenarios of product returns. Chapter 4 presents our analytical results, their illustration via numerical experiments, and an extension of our analysis. Chapter 5 provides an exact solution algorithm for the optimal sales plan and additional numerical experiments conducted with this algorithm. Chapter 6 offers a summary and conclusion. Detailed versions and proofs of the analytical results are contained in Appendix A.

Chapter 2

Literature Review

Most of the existing multi-generation diffusion models build upon the seminal Bass diffusion process that partitions the market into two consumer segments: some consumers are influenced in their timing of initial product purchases by the word-of-mouth feedback spread by previous buyers (imitators) while the other consumers are not affected by previous buyers (innovators). See Bass ([20], [26]) for details. In their pioneering work on multi-generation diffusion, Norton and Bass [22] consider an extension of the Bass diffusion model in which each generation has its own market potential and market-penetration process whereas buyers of earlier generations can also buy newer generations. Jiang and Jain [21] extend the Norton-Bass model by explicitly identifying the consumers' purchasing behaviors: some of the potential adopters of the previous generation buy the new generation by skipping the previous generation (leapfrogging adopters) and some of the existing adopters of the previous generation also buy the new generation (switching adopters). The leapfrogging behavior cannibalizes the sales of earlier generations while the switching behavior raises the cross-generation repeat purchases. In this thesis, we adopt the generalized Norton-Bass model of Jiang and Jain [21] since it has a strong empirical support and offers the flexibility to conveniently capture the key consumer characteristics. This generalized Norton-Bass model is also mathematically consistent with the one proposed by Norton and Bass [22].

In the FSC literature, multi-generation extensions of the Bass diffusion model have been widely applied to study the market entry timing problem for successive generations: Wilson and Norton [10] show that introducing a new-generation product promptly or never introducing it -*the Now or Never policy*- is optimal when the objective is to maximize the total undiscounted profit from the entire product line and the new-generation product has a lower profit margin than the early-generation product. Mahajan and Müller [11] extend the work of Wilson and Norton [10] by relaxing the assumption of decreasing profit margins. They show that introducing a new-generation product promptly or delaying its launch time until the early-generation product reaches its maturity phase -*the Now or at Maturity policy*- is optimal in the discounted-profit case. Krankel et al. [12] consider a setting in which the early-generation product becomes completely obsolete once a new-generation product is introduced (a single product rollover strategy is adopted), the available product technology improves stochastically over time, and the firms incur a fixed cost upon introduction of a new-generation product. They prove that it is optimal to introduce the new-generation product when the technology level of the incumbent generation is below a threshold that varies depending on the cumulative sales volume and the available technology level. The optimal introduction times in their setting are later than in *the Now or at Maturity policy* of Mahajan and Müller [11]. Ke et al. [13] extend the work of Wilson and Norton [10] by incorporating the inventory holding costs. They show that *the Now or Never policy* is optimal under low inventory costs or frequent inventory replenishments while the sequential introduction is optimal under substantial inventory costs. Guo and Chen [14] study the market entry timing and pricing problem for successive generations by taking into account various types of behaviors of strategic consumers. They find that a higher performance improvement achieved with the new-generation product and a lower salvage value of the early-generation product tend to result in a higher optimal price and a later introduction time for the new-generation product as well as a larger price discount for the early-generation product. Jiang et al. [15] adopt the generalized Norton-Bass model of Jiang and Jain [21] and approach the problem for two types of products (purchase-to-own vs. subscribe-to-use) under two generation transitions strategies (phase-out vs. total). They prove the optimality of *the Now or*

Never policy for subscribe-to-use products regardless of the generation transition strategy.

Several other papers focus on the product rollover strategy and introduction frequency decisions for successive generations. Druehl et al. [27] extend the diffusion model proposed by Wilson and Norton [10] in order to investigate the factors that escalate the introduction frequency of successive generations. They show that a faster pace of product updates results from a faster diffusion rate and margin decline as well as higher contributions of newer generations to customer base. Liao and Seifert [28] also study the introduction frequency decisions by explicitly identifying the relation between the speed of industrial technology evolution and the pace of new-generation introductions. Koca et al. [29] concentrate on the product rollover strategy decisions of a firm whose product demands arrive as the Norton-Bass diffusion process. They highlight a variety of exogenous factors that affect the non-trivial decision of whether to choose a single or dual product rollover strategy.

We depart from all of the papers in the above two paragraphs by considering the new-product introduction time as an exogenous model input and studying the sales planning problem in a CLSC setting. Our approach is realistic for consumer electronics producers whose new-product launch decisions are driven primarily by strategic motives, rather than the operational considerations that we address in this study. Nevertheless, in this paper, we examine the impact of the new-product launch time on the optimal sales plans in a setting where the dual product rollover strategy is adopted.

Our work is also closely related to the CLSC literature (see Atasu et al. [30], Souza [31], and Govindan et al. [32] for comprehensive reviews). In this literature, several papers examine the sales planning and/or pricing problem for a single generation of a product and its end-of-use version that can be remanufactured. Debo et al. [16] study the pricing problem for new and remanufactured types of a product that are imperfect substitutes by allowing for cross-generation repeat purchases, variable market sojourn times, and supply constraints. They generalize the price-dependent Bass diffusion model in Bass [33] by formulating the coefficient of imitation as a function of the installed base of new products. They characterize the diffusion paths of new and remanufactured products, analyzing the impacts of the remanufacturability level, capacity structure, and reverse channel speed on profitability. Robotis et al. [17] consider a producer who leases new and remanufactured versions of a product that are perfect substitutes. The product demand arrives as a diffusion process that is controlled by the producer through the leasing price and duration. They investigate the effects of the remanufacturing savings and production capacity constraints on the optimal leasing price and duration. Akan et al. [18] consider a producer with ample production capacity who sells new and remanufactured versions of a product that are imperfect substitutes. The product demand arrives as a price-dependent diffusion process. They characterize the optimal pricing, production, and inventory policies of the producer, showing that partially satisfying demand for the remanufactured item is never optimal. Finally, Nadar et al. [19] employ the Bass diffusion process to study the sales planning problem for new and remanufactured versions of a product that are imperfect substitutes by allowing for partial backlogging and consumer heterogeneity in their timing of returns. They find that the optimal sales plan involves partial demand fulfillment when the product diffusion rate and the profit margin from remanufacturing are large and the remanufactured item is in limited demand. Unlike these papers, we study the sales planning problem in a CLSC of *successive* generations of a product in which *end-of-life* product returns have *recycling* potential.

The closest model to ours in the above literature is that of Nadar et al. [19], in which the diffusion dynamics are formulated for a single product generation in the absence of intergeneration product competition, and the end-of-use returns are remanufactured and remarketed over the single-generation selling horizon. In our model, however, the diffusion dynamics are jointly formulated for successive product generations, and the end-of-life returns from consumers of the early-generation product are recycled to be used in manufacturing of the new-generation product. While we provide additional support for the usefulness of the partial-fulfillment policy in CLSCs, the comprehensive account of successive product generations allows us to investigate the interplay between them.

Chapter 3

Problem Formulation

In a discrete-time framework, we study the sales planning problem of a producer that offers two successive generations of a durable good over a finite selling horizon of T periods: the early-generation product is available in the market in periods 1 through T, while the new-generation product is released in period τ and available in the market in periods τ through T. Demand evolves over time according to a slightly modified version of the generalized Norton-Bass model in Jiang and Jain [21]. A population of consumers of size m_1 is initially attracted by the early-generation product; these consumers gradually purchase the early-generation product in periods 1 through $\tau - 1$ and the early- or new-generation product in periods τ through T. Those adopters in periods 1 through $\tau - 1$ become the potential adopters of the new-generation product in periods τ through T (switching adopters). However, some of those adopters in periods τ through T choose to skip the early-generation product and buy the new-generation product (leapfrogging adopters), while the others still buy the early-generation product and become the potential adopters of the new-generation product in the future (switching adopters). Another population of consumers of size m_2 is only attracted by the new-generation product; these consumers gradually purchase the product in periods τ through T. Consumers in the population of size m_2 buy at most one unit of the new-generation product, while consumers in the population of size m_1 buy at most one unit of each of the two generations.

In the generalized Norton-Bass model, all demand for both product generations is immediately met in each period and the Bass diffusion dynamics hold separately for the two consumer populations. The Bass diffusion demand for the early-generation product in period $t \geq 1$, if the new-generation product were ignored, would take the following form:

$$\breve{d}_{1t}^b = \left(p_1 + \frac{q_1 \breve{D}_{1t}^b}{m_1} \right) \left(m_1 - \breve{D}_{1t}^b \right) \tag{3.1}$$

where p_1 and q_1 are the coefficients of innovation and imitation for the early-generation product, respectively, and $\breve{D}_{1t}^b$ is the total number of consumers from the population of size m_1 who have bought the product up to period t (i.e., $\breve{D}_{11}^b = 0$ and $\breve{D}_{1t}^b = \sum_{i=1}^{t-1} \breve{d}_{1i}^b$, $\forall t > 1$). See Bass [20], [26] for details. Likewise, the Bass diffusion demand for the new-generation product in period $t \geq \tau$, if the early-generation product were ignored, would take the following form:

$$\breve{d}_{2t}^b = \left(p_2 + \frac{q_2 \breve{D}_{2t}^b}{m_2} \right) \left(m_2 - \breve{D}_{2t}^b \right) \tag{3.2}$$

where p_2 and q_2 are the coefficients of innovation and imitation for the new-generation product, respectively, and $\breve{D}_{2t}^b$ is the total number of consumers from the population of size m_2 who have bought the product up to period t (i.e., $\breve{D}_{2t}^b = 0$, $\forall t \leq \tau$, and $\breve{D}_{2t}^b = \sum_{i=\tau}^{t-1} \breve{d}_{2i}^b$, $\forall t > \tau$). However, the generalized Norton-Bass model takes into account not only the Bass diffusion dynamics shown above but also the substitution effect among the two generations: the demand for the early-generation product in period $t \geq 1$ is formulated as

$$\breve{d}_{1t}^n = \breve{d}_{1t}^b - \frac{\breve{d}_{1t}^b \breve{D}_{2t}^b}{m_2} \tag{3.3}$$

and the demand for the new-generation product in period $t \geq \tau$ is formulated as

$$\breve{d}_{2t}^n = \breve{d}_{2t}^b + \frac{\breve{d}_{1t}^b \breve{D}_{2t}^b}{m_2} + \frac{\breve{D}_{1t}^b \breve{d}_{2t}^b}{m_2}. \tag{3.4}$$

In the above formulation, the terms $\frac{\breve{d}_{1t}^b \breve{D}_{2t}^b}{m_2}$ and $\frac{\breve{D}_{1t}^b \breve{d}_{2t}^b}{m_2}$ represent the numbers of leapfrogging and switching adopters from the population of size m_1 in period t, respectively. Notice that $\breve{d}_{1t}^n = \breve{d}_{1t}^b$ and $\breve{d}_{2t}^n = 0$ if $t < \tau$. The calculation of $\breve{d}_{1t}^b$ and $\breve{D}_{1t}^b$ implies that the word-of-mouth feedback can be spread in the population of size m_1 by adopters from this population, regardless of whether they buy the early- or new-generation product. See Jiang and Jain [21] for details.

In our sales planning problem, during each period $t \geq \tau$, the customers in the population of size m_1 arrive earlier than the customers in the population of size m_2 and the producer is able to reject any amount of demand from the customers in the population of size m_2. We define s^b_{2t} as the sales volume for the customers in the population of size m_2 in period $t \geq \tau$ and S^b_{2t} as the total sales volume for these customers up to period t (i.e., $S^b_{2t} = 0$, $\forall t \leq \tau$, and $S^b_{2t} = \sum_{i=\tau}^{t-1} s^b_{2i}$, $\forall t > \tau$). We slightly modify the generalized Norton-Bass model by incorporating the sales decisions into the Bass diffusion demand for the new-generation product and revising our notation for both product generations as follows:

$$d^b_{1t} = \left(p_1 + \frac{q_1 D^b_{1t}}{m_1}\right)\left(m_1 - D^b_{1t}\right) \tag{3.5}$$

and

$$d^b_{2t} = \left(p_2 + \frac{q_2 S^b_{2t}}{m_2}\right)\left(m_2 - D^b_{2t}\right) \tag{3.6}$$

where $D^b_{11} = 0$, $D^b_{1t} = \sum_{i=1}^{t-1} d^b_{1i}$, $\forall t > 1$, $D^b_{2t} = 0$, $\forall t \leq \tau$, and $D^b_{2t} = \sum_{i=\tau}^{t-1} d^b_{2i}$, $\forall t > \tau$. The demand for the early-generation product in period $t \geq 1$ is reformulated as

$$d^n_{1t} = d^b_{1t} - \frac{d^b_{1t} D^b_{2t}}{m_2} \tag{3.7}$$

and the demand for the new-generation product in period $t \geq \tau$ is reformulated as

$$d^n_{2t} = d^b_{2t} + \frac{d^b_{1t} D^b_{2t}}{m_2} + \frac{D^b_{1t} d^b_{2t}}{m_2}. \tag{3.8}$$

Our demand formulation in (3.6) is consistent with those studied in the sales planning literature; see, for instance, Ho et al. [5], [6], Kumar and Swaminathan [7], Shen et al. [8], [9], and Nadar et al. [19]. Notice that $d^n_{1t} = \breve{d}^n_{1t} = \breve{d}^b_{1t}$ and $d^n_{2t} = 0$ if $t < \tau$. If all demand from the customers in the population of size m_2 is met in each period, our diffusion model reduces to the generalized Norton-Bass model. See Figure 3.1 for an illustration of our diffusion model for two different sales plans.

We note that we could also allow the producer to reject demand from the customers in the population of size m_1. However, rejecting the early-generation product demand from these customers slows down the early-generation product diffusion as well as the product return process. Rejecting the new-generation product demand from these customers reduces the cross-generation repeat purchases if this demand arises from switching adopters, and slows down the early-generation product diffusion if this demand arises from leapfrogging adopters. Thus, intuitively, rejecting demand from these customers is not advisable.

We denote by s_{2t} the sales volume of the new-generation product in period $t \geq \tau$. Since all demand from the customers in the population of size m_1 is met in each period, we obtain $s_{2t} = s_{2t}^b + \frac{d_{1t}^b D_{2t}^b}{m_2} + \frac{D_{1t}^b d_{2t}^b}{m_2}$, $\forall t \geq \tau$. We assume that a fraction α of the unmet demand in period t is backlogged to be satisfied in period $t+1$ while the remaining fraction of the unmet demand is lost. In addition, the customers whose demands were rejected in earlier periods retain no memory of how long they have waited for the product adoption. These assumptions are standard in the sales planning literature; again, see Ho et al. [5], [6], Kumar and Swaminathan [7], Shen et al. [8], [9], and Nadar et al. [19]. We denote by b_t the accumulated number of backorders in period t from the customers in the population of size m_2. The sales volume of the new-generation product in period t is constrained to take values between the total number of leapfrogging and switching adopters in period t and the total demand observed for the new-generation product in period t:

$$\frac{d_{1t}^b D_{2t}^b}{m_2} + \frac{D_{1t}^b d_{2t}^b}{m_2} \leq s_{2t} \leq d_{2t}^n + b_t. \tag{3.9}$$

The above constraint implies that $0 \leq s_{2t}^b \leq d_{2t}^b + b_t$ in period t. Taking $b_\tau = 0$, we can calculate b_t, $\forall t > \tau$, with the following recursion:

$$b_{t+1} = \alpha(d_{2t}^n + b_t - s_{2t}). \tag{3.10}$$

In Chapters 3.1 and 3.2, we consider two possible scenarios for modeling the consumer behavior regarding the end-of-life product returns, formulating the producer's objective function in each scenario. In both scenarios, a certain type of

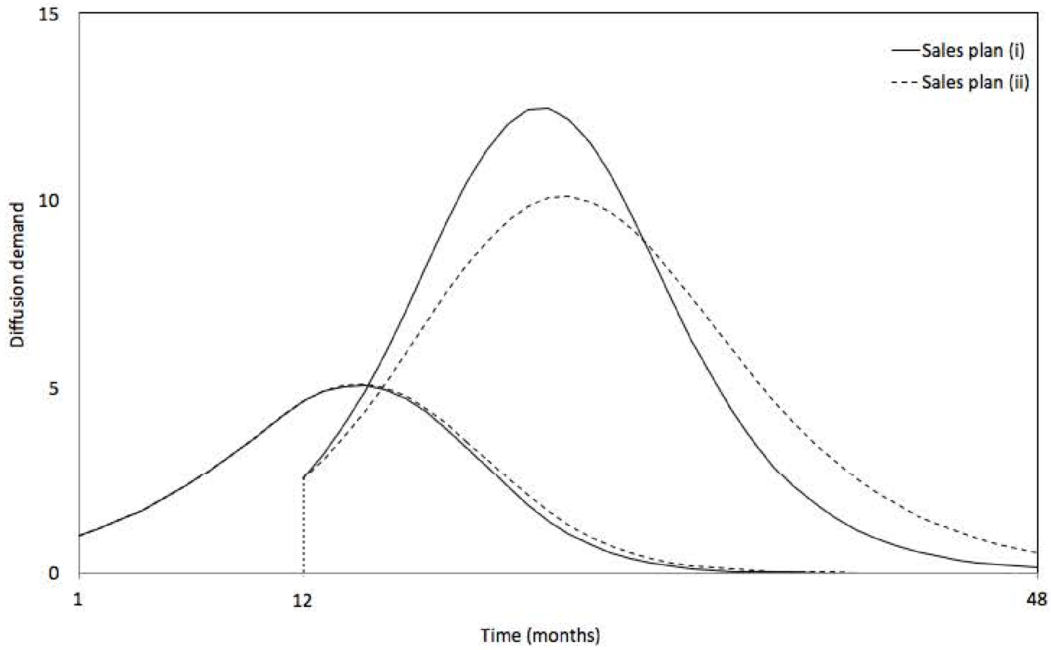

Figure 3.1: Multi-generation product diffusion when $\tau = 12$, $p_1 = 0.01$, $p_2 = 0.02$, $q_1 = q_2 = 0.20$, and $m_1 = m_2 = 100$ for two different sales plans: (i) all demand is met in each period and (ii) 75% of the diffusion demand from the customers unique to the new-generation product is met in each period but no backlogged demand is met at all. Sales plan (i) corresponds to the generalized Norton-Bass diffusion process.

material can be extracted from the end-of-life returns of the early-generation product and can be used in manufacturing of the new-generation product (e.g., cobalt for smartphones). We define θ_1 as the fixed amount of material of this type that can be recovered from one unit of the end-of-life product and θ_2 as the maximum possible amount of recycled material of this type that can be used to make one unit of the new-generation product. The total amount of material of this type required to make one unit of the new-generation product can exceed θ_2 because the use of the recycled content may be unacceptable in certain parts of the device that are in need of virgin raw materials. (Although we focus on a single material type in our analysis, our results in Chapter 4 continue to hold when multiple material types can be recycled and the ratio θ_1/θ_2 remains the same across these material types.) We assume that recycling is a profitable operation in both scenarios. This assumption is realistic for smartphones (Geyer and Blass [34], Atasu and Souza [35], Esenduran et al. [36]).

3.1 Product Returns via Recycling Programs

In this scenario, a fraction β_i of the early-generation products that have been sold in period t are returned by consumers to the producer at the end of their life and become available for material recovery and reuse in period $t+i$, $\forall i \geq 1$. We define $\beta \triangleq \sum_i \beta_i \leq 1$. The fraction $(1-\beta)$ of the products that have been sold in any specific period cannot be collected in any future period. This scenario may be realistic for producers who collect the bulk of the end-of-life returns through recycling programs such as free mail-back and physical drop-off options for consumers. (A similar scenario was also proposed by Nadar et al. [19] for end-of-use returns of a remanufacturable durable good.) We define ν_t as the total amount of material that can be recovered and used in period t. Taking $\nu_1 = 0$, we can calculate ν_t, $\forall t > 1$, with the following recursion:

$$\nu_{t+1} = \begin{cases} \theta_1 \sum_{i=1}^{t} \beta_i d^n_{1(t+1-i)} & \text{if } \nu_t < \theta_2 s_{2t}, \\ \nu_t - \theta_2 s_{2t} + \theta_1 \sum_{i=1}^{t} \beta_i d^n_{1(t+1-i)} & \text{if } \nu_t \geq \theta_2 s_{2t}. \end{cases} \tag{3.11}$$

We define c_1 as the unit manufacturing cost and r_1 as the unit selling price for the early-generation product. We also define c_2 as the unit manufacturing cost and r_2 as the unit selling price for the new-generation product. Lastly, we define p_r as the cost reduction achieved by integrating a unit amount of recycled content into new-generation product manufacturing. We assume $r_1 > c_1$ and $r_2 > c_2 > \theta_2 p_r$. Hence, the producer's problem of maximizing the total profit over the selling horizon of T periods can be formulated as

$$\max_{s_{2\tau},..,s_{2T}} \sum_{t=1}^{T} (r_1 - c_1) d^n_{1t} + \sum_{t=\tau}^{T} (r_2 - c_2) s_{2t} + p_r \sum_{t=\tau}^{T} \min\{\nu_t, \theta_2 s_{2t}\}$$

subject to (3.5)–(3.11). In the above formulation, the first summation represents the total profit from selling the early-generation product, the second summation represents the total profit from selling the new-generation product in the case of regular manufacturing, and the last summation represents the total cost savings in new-generation product manufacturing thanks to the use of recycled content.

3.2 Product Returns via Switching Adopters

In this scenario, the end-of-life product returns in any specific period arise only from the switching adopters who bought the early-generation product in earlier periods and buy the new-generation product in this period. We define γ as the fraction of switching adopters who return their end-of-life products at the time of their purchases of the new-generation product. This scenario may be realistic for producers who collect the bulk of the end-of-life returns from the consumers who return their old devices to trade up to the new-generation product. Unlike the previous scenario, the consumers' timing of product returns is governed by the new-generation product diffusion process in the current scenario.

We assume that the recyclable content obtained from the switching adopters' returns in any period is sufficient, and can be immediately used, for fulfillment of these switching adopters' demand in the same period (i.e., $\gamma\theta_1 \geq \theta_2$) and that the remaining recyclable content in any period can be used for fulfillment of the new-generation product demand (other than the switching adopters' demand) in subsequent periods. Under these assumptions, we redefine ν_t as the total amount of material that can be recovered and used in fulfillment of the new-generation product demand, except the switching adopters' demand, in period t. Taking $\nu_\tau = 0$, we can calculate ν_t, $\forall t > \tau$, with the following recursion:

$$\nu_{t+1} = \begin{cases} (\gamma\theta_1 - \theta_2)\frac{D^b_{1t}d^b_{2t}}{m_2} & \text{if } \nu_t < \theta_2\left(s_{2t} - \frac{D^b_{1t}d^b_{2t}}{m_2}\right), \\ \nu_t - \theta_2\left(s_{2t} - \frac{D^b_{1t}d^b_{2t}}{m_2}\right) + (\gamma\theta_1 - \theta_2)\frac{D^b_{1t}d^b_{2t}}{m_2} & \text{if } \nu_t \geq \theta_2\left(s_{2t} - \frac{D^b_{1t}d^b_{2t}}{m_2}\right). \end{cases} \tag{3.12}$$

In this case, the producer's problem of maximizing the total profit over the selling horizon of T periods can be formulated as

$$\max_{s_{2\tau},..,s_{2T}} \quad \sum_{t=1}^{T}(r_1-c_1)d_{1t}^n+\sum_{t=\tau}^{T}(r_2-c_2)s_{2t}+\theta_2 p_r \sum_{t=\tau}^{T}\frac{D_{1t}^b d_{2t}^b}{m_2}$$
$$+p_r\sum_{t=\tau}^{T}\min\left\{\nu_t,\theta_2\left(s_{2t}-\frac{D_{1t}^b d_{2t}^b}{m_2}\right)\right\}$$

subject to (3.5)–(3.10) and (3.12).

In the above formulation, the third summation represents the total cost savings in manufacturing for the switching adopters' new-generation product demand, while the last summation represents the total cost savings in manufacturing for the remaining new-generation product demand, thanks to the use of recycled content. We note that if we were to assume $\gamma\theta_1 < \theta_2$, the last summation would equal zero and the third summation would have $\gamma\theta_1$ in place of θ_2. In this case, the total amount of recycled content used in manufacturing would linearly increase with the total new-generation product demand from the switching adopters. Partial demand fulfillment might be optimal in this case if it raises the total number of cross-generation repeat purchases.

Chapter 4

Analytical Results

In this chapter, we investigate whether slowing down the diffusion of the new-generation product by partially satisfying its demand might be profitable for the producer. For this purpose, we partition the feasible sales plans of the producer's optimization problem into two different classes:

i. All demand for the early- and new-generation products is met in each period. We call this sales plan the *immediate-fulfillment policy.* Such a policy involves a myopically optimal decision in each period and the resulting diffusion process is equivalent to the generalized Norton-Bass diffusion process. We use the breve ($\breve{}$) to denote the variables of the problem under this policy. Note that $\breve{s}^b_{2t} = \breve{d}^b_{2t}$, $\breve{S}^b_{2t} = \breve{D}^b_{2t}$, and $\breve{s}_{2t} = \breve{d}^n_{2t}$, $\forall t \geq \tau$.

ii. Some demand from the customers in the population of size m_2 (who are unique to the new-generation product) is rejected in some period while all the other demand is met in each period. We call this sales plan the *partial-fulfillment policy.* We use the hat ($\widehat{}$) to denote the variables of the problem under this policy.

Proposition 1. *For all* $t > \tau$, $\widehat{d}^n_{1t} \geq \breve{d}^n_{1t}$. *When* T *is sufficiently large,* $\sum_{t=\tau}^{T} \widehat{D}^b_{1t}\widehat{d}^b_{2t} > \sum_{t=\tau}^{T} \breve{D}^b_{1t}\breve{d}^b_{2t}$.

Proof. See Appendix A.

Proposition 1 highlights two major implications of our sales planning in the population of size m_2 for the demand structure in the population of size m_1: It states that the partial-fulfillment policy curbs the leapfrogging behaviour and leads to a larger diffusion demand for the early-generation product in each period after the new-generation product is introduced in the market. See Figure 3.1 for an example. It also implies that the partial-fulfillment policy induces a greater total number of switching adopters and a smaller total number of leapfrogging adopters, compared to the immediate-fulfillment policy, when T is sufficiently large so that all customers demand the new-generation product before the selling horizon ends. Thus, the partial-fulfillment policy has the advantage of increasing the cross-generation repeat purchases over a sufficiently long selling horizon. However, it still has the disadvantage of losing some of the unmet demand from the customers who are unique to the new-generation product. These results follow from the slowdown of the new-generation product diffusion induced by the partial-fulfillment policy.

Theorems 1 and 2 establish the conditions that ensure the optimality of the partial-fulfillment policy for the problems in Chapters 3.1 and 3.2, respectively. Theorem 3 extends Theorem 2 by inducing positive salvage revenue for the recyclable material. Despite the complex nonlinear nature of the sales planning problem, all these conditions are easy to check with the diffusion and closed-loop dynamics available under the immediate-fulfillment policy. We refer the reader to Appendix A for detailed versions of Theorems 1-3 that include lengthy mathematical expressions.

Theorem 1. *Suppose that the consumers' timing of end-of-life product returns is independent of their timing of new-generation product purchases (as in Chapter 3.1). Then, the partial-fulfillment policy is optimal if, under the immediate-fulfillment policy, there exists a period* $\kappa < T$ *such that* *(i) the total available amount of recyclable material exceeds the maximum amount of recycled material that can be used to fulfill all the new-generation product demand in each period $t > \kappa$ while the reverse is true in each period $t \leq \kappa$,*

(ii) the ratio of the amount of recyclable material that can be extracted from one unit of the early-generation product to the maximum amount of recycled material that can be used to make one unit of the new-generation product is above a certain threshold (detailed in Appendix A), and

(iii) rejecting a unit of the new-generation product demand in period κ induces a loss of diffusion demand in period $\kappa + 1$ that is below a certain threshold (detailed in Appendix A) and a backlogged demand for the new-generation product in period $\kappa+1$ that is above a certain threshold (detailed in Appendix A).

Suppose that the above conditions hold and T is sufficiently large. Then, the partial-fulfillment policy, if initiated after period κ, leads to no improvement in the total amount of recycled material used in manufacturing of the new-generation product.

Proof. See Appendix A.

For the problem in Chapter 3.1, Theorem 1 states that it is optimal to reject some demand from the customers unique to the new-generation product in some period with shortage of the recyclable material (i.e., in period κ) if the available amount of recyclable material is sufficiently large in later periods so that each unit of the new-generation product demand in these periods can benefit from the recycled content to the fullest extent allowable by the product design (conditions i and ii) and if the revenue gain from improved use of recycled content as well as possible increase in cross-generation repeat purchases via delayed demand is able to outweigh the revenue loss due to reduced new-generation product sales induced by partial fulfillment (condition iii). However, if the partial-fulfillment policy is optimally initiated in future periods with abundance of the recyclable material (i.e., in periods $t > \kappa$) over a sufficiently long selling horizon, Theorem 1 implies that the producer exploits the benefit of sales planning by means of increased cross-generation repeat purchases without relying on any improvement in the use of recycled content. Since the cross-generation repeat purchases are likely to escalate the consumption of virgin raw materials in manufacturing, such postponement of the partial-fulfillment policy is not advisable from an environmental perspective.

In the literature, for single-generation remanufacturable products, Nadar et al. [19] have found that the partial-fulfillment policy cannot be optimally initiated in future periods with abundant product returns: The partial-fulfillment policy in Nadar et al. [19] reduces the total sales volume. This policy can only be profitable if it improves the remanufacturing volume in the long run and the remanufacturing volume can only be improved if this policy is initiated in earlier periods. In our study, however, the existence of successive product generations provides an additional motivation for the partial-fulfillment policy: This policy may be optimally initiated even in future periods with abundant product returns because it has the potential to increase the cross-generation repeat purchases as well as the total sales volume.

We conduct numerical experiments to investigate the optimality of the partial-fulfillment policy as well as the environmentally critical time period κ for the optimal initiation of this policy, as characterized in Theorem 1, with respect to the key problem parameters. We construct a base scenario by choosing the parameter values that are calibrated for smartphones: $T = 48$ months, $\tau = 16$, $p = p_1 = p_2 = 0.05$, $q = q_1 = q_2 = 0.35$, $m_1 = m_2$, $\theta_1 = \theta_2$, $r_1 - c_1 = r_2 - c_2 = 5\theta_2 p_r$, $\alpha = 0.88$, $\beta = 0.12$, and $\beta_i = \beta \times \mathbb{P}\{i - 0.5 \leq X \leq i + 0.5\}$ where X has a Weibull distribution with scale parameter 25 and shape parameter 2 (implying a mean of 22.16), $\forall i \geq 1$. Our choice of constant values for p and q reflects the empirical evidence in Stremersch et al. [37] that indicates no significant change in these coefficients across generations for many consumer electronics products. Most of our parameter values are similar to those studied by Jiang et al. [15] and Nadar et al. [19] who consider consumer electronics products in their numerical experiments. We generate instances from the base scenario by varying the values of p and q, those of τ and m_1/m_2, those of τ and θ_1/θ_2, and those of τ and $\theta_2 p_r/(r_2 - c_2)$. (Our results are not affected by changes in m_1 and m_2 as long as m_1/m_2 remains the same, by changes in θ_1 and θ_2 as long as θ_1/θ_2 remains the same, and by changes in $\theta_2 p_r$ and $(r_2 - c_2)$ as long as $\theta_2 p_r/(r_2 - c_2)$ remains the same.) Figure 4.1 exhibits our results for a very large number of compiled instances; the partial-fulfillment policy is optimal in the vast majority of these instances.

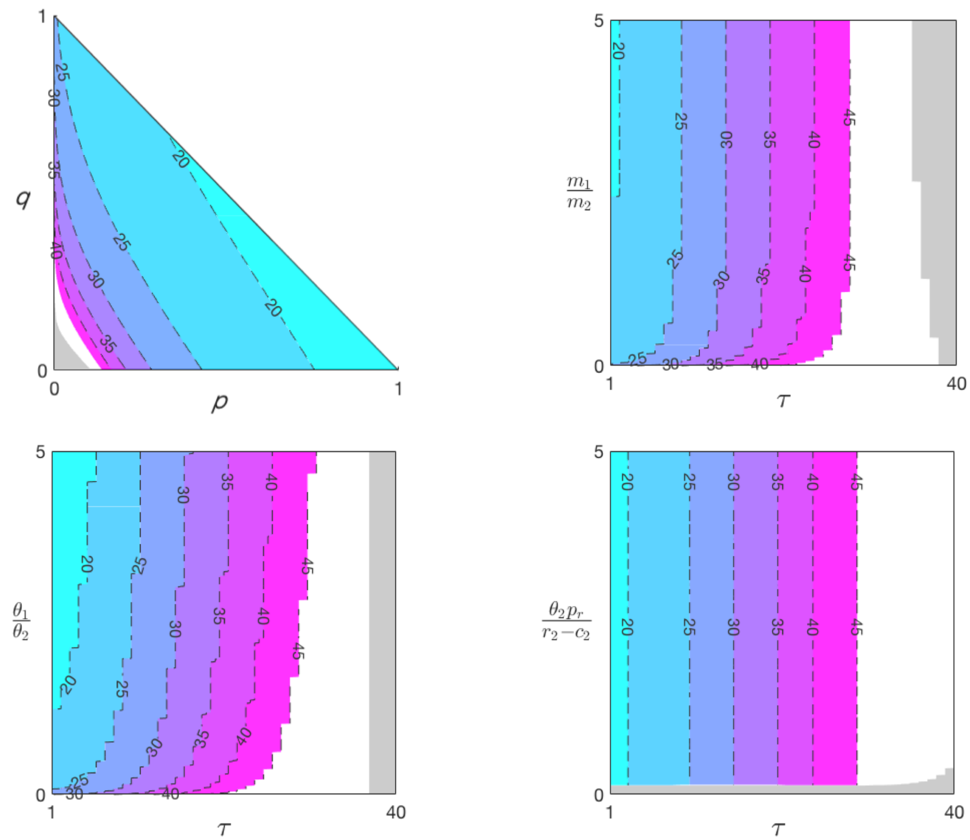

Figure 4.1: Contour plots of the environmentally critical time period κ in Theorem 1. The partial-fulfillment policy is optimal in colored regions. Condition (iii) of Theorem 1 is met in white regions while it is not met in gray regions. Note $p + q \leq 1$.

Figure 4.1 indicates that the partial-fulfillment policy is optimal, and the environmentally critical time period κ is sooner, when p and q are large: Increasing p and q not only speeds up the diffusion process for both generations but also provides larger return volumes during the selling horizon. As a result, when p

and q are large, the amount of recyclable material is likely to be sufficient for fulfillment of some delayed demand for the new-generation product, while the shifted diffusion demand can still arrive before the selling horizon ends thanks to the fast diffusion process. In addition, the cross-generation repeat purchases are likely to grow with the slowdown of the new-generation product diffusion. Figure 4.1 also indicates that the partial-fulfillment policy is optimal, and the critical time period κ is sooner, when m_1/m_2 and θ_1/θ_2 are large: When m_1/m_2 is small, the new-generation product has a much greater sales volume than the early-generation product so that the return volume of the early-generation product may not suffice to fulfill any delayed demand for the new-generation product. When θ_1/θ_2 is small, only a small amount of recyclable material can be extracted from one unit of the early-generation product while a large amount of recycled material can be used to make one unit of the new-generation product. Hence, the return volume may again not suffice to fulfill any delayed demand, reducing the need for partial demand fulfillment. Another important observation is that the partial-fulfillment policy is optimal when τ is small: When τ is large, the selling horizon falls short of complete market penetration for the new-generation product so that the slowdown of new-generation product diffusion is likely to induce a loss of diffusion demand at the end of the selling horizon. In addition, when τ is large, the new-generation product is in high demand toward the end of the selling horizon so that the recyclable material is likely to always be in shortage and thus the use of recycled content cannot be improved by partial fulfillment.

We also note from Figure 4.1 that the partial-fulfillment policy is optimal in our instances when $\theta_2 p_r/(r_2 - c_2) \geq 0.12$ and $\tau \in \{1, 2, \ldots, 29\}$: If the use of recycled content in the production of new-generation devices can substantially reduce the consumption of an expensive virgin raw material (leading to a unit cost saving of at least 12% of the new-generation product margin in our instances), the partial-fulfillment policy helps amplify this potential benefit of recycling. We have also conducted additional experiments by varying the values of $(r_1 - c_1)/(r_2 - c_2)$ and τ in the base scenario. We have found that the partial-fulfillment policy is optimal when $(r_1 - c_1)/(r_2 - c_2) \in [0, 5]$ and $\tau \in \{1, 2, \ldots, 29\}$. Decreasing margins across generations (i.e., $(r_1 - c_1)/(r_2 - c_2) > 1$) may appear if the producer offers a lower selling price for the new-generation product due to competitive pressures arising from the other firms' market entry moves and/or if the new-generation product incurs a higher production cost due to major design changes. Increasing margins across generations (i.e., $(r_1 - c_1)/(r_2 - c_2) < 1$) may arise from lower production costs thanks to the experience curve effect and/or higher selling prices thanks to the strong brand loyalty built over time. See chapter 15 in Jain [38] and chapter 1 in Nahmias [39] for further detailed discussions. The partial-fulfillment policy appears to remain a viable sales strategy in both settings.

Theorem 2. *Suppose that the consumers' timing of end-of-life product returns coincides with their timing of new-generation product purchases (as in Chapter 3.2). Then, the partial-fulfillment policy is optimal if, under the immediate-fulfillment policy, there exists a period $\kappa < T$ such that*

(i) the total available amount of recyclable material exceeds the maximum amount of recycled material that can be used to fulfill all the new-generation product demand, except the switching adopters' demand, in each period $t > \kappa$ while the reverse is true in each period $t \leq \kappa$,

(ii) the amount of recyclable material that can be extracted from one unit of the early-generation product multiplied with the return rate of switching adopters is greater than the maximum amount of recycled material that can be used to make one unit of the new-generation, and

(iii) rejecting a unit of the new-generation product demand in period κ induces a loss of diffusion demand in period $\kappa + 1$ that is below a certain threshold (detailed in Appendix A) and a backlogged demand for the new-generation product in period $\kappa+1$ that is above a certain threshold (detailed in Appendix A).

Suppose that the above conditions hold and T is sufficiently large. Then, the partial-fulfillment policy, if initiated after period κ, reduces the total amount of recycled material used in manufacturing of the new-generation product for the customers who have not bought the early-generation product.

Proof. See Appendix A.

For the problem in Chapter 3.2, Theorem 2 states that the partial-fulfillment policy is optimal under conditions that have similar implications to those of the conditions in Theorem 1: conditions (i) and (ii) in Theorem 2 ensure that the available amount of recyclable material is sufficiently large in future periods so that each unit of the new-generation product demand in these periods can be met by using the recycled content to the fullest extent allowable by the product design, and condition (iii) in Theorem 2 is identical to condition (iii) in Theorem 1. Although the conditions in Theorems 1 and 2 have similar implications, conditions (i) and (ii) in Theorem 2 are less likely to hold than those in Theorem 1 because the product returns in Chapter 3.2 arrive only at times when the switching adopters buy the new-generation product and the vast amount of recyclable material obtained from these returns are used primarily for fulfillment of the switching adopters' demand. The partial-fulfillment policy, if initiated after period κ over a sufficiently long selling horizon, reduces the total amount of recycled material used to fulfill the new-generation product demand from the customers who have not bought the early-generation product, whereas it increases the total number of switching adopters as well as the total amount of recycled material used to fulfill the new-generation product demand from the switching adopters. Since the cross-generation repeat purchases are likely to escalate the consumption of virgin raw materials, such postponement of the partial-fulfillment policy is again environmentally inadvisable.

We extend our numerical experiments to Theorem 2 by taking $\theta_2 = (0.6)\gamma\theta_1$ (rather than $\theta_1 = \theta_2$) in the base scenario. Our modification of the base scenario allows us to observe the optimality of the partial-fulfillment policy and examine the comparative statics of the environmentally critical time period κ in our experiments. We generate instances from the base scenario by varying the values of p and q, those of τ and m_1/m_2, those of τ and $\gamma\theta_1/\theta_2$, and those of τ and $\theta_2 p_r/(r_2 - c_2)$. Figure 4.2 exhibits our results for a very large number of selected instances. Although condition (iii) is still met in the vast majority of these instances, conditions (i)-(iii) fail to simultaneously hold in a substantially greater number of instances, compared to conditions (i)-(iii) in Theorem 1.

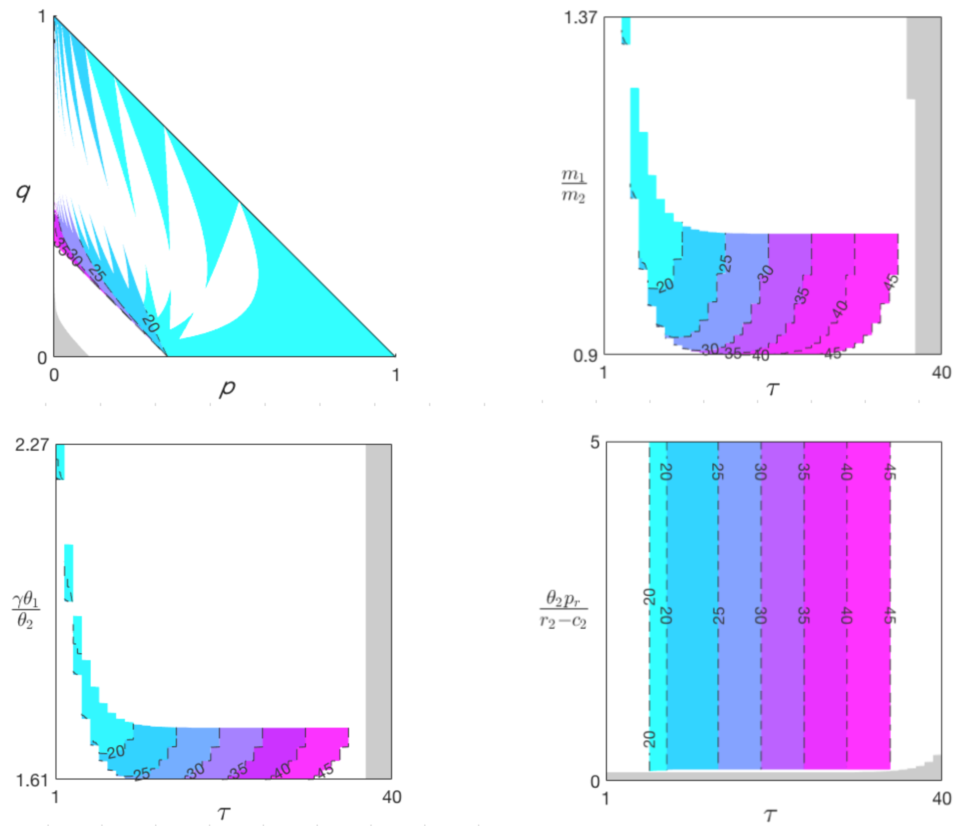

Figure 4.2: Contour plots of the environmentally critical time period κ in Theorem 2. The partial-fulfillment policy is optimal in colored regions. Condition (iii) of Theorem 2 is met in white regions while it is not met in gray regions. Note $p + q \leq 1$.

Contrary to our observations in Figure 4.1, Figure 4.2 indicates that conditions (i)-(iii) may fail to simultaneously hold even for larger p and q values. Our explanation for this counterintuitive result is as follows. When p and q are large, the customers in the population of size m_1 arrive earlier to buy the early-generation product, leading to a rapid growth of cross-generation repeat purchases. This induces the available amount of recyclable material to exceed the demand for this material in much earlier periods than in the problem defined in Chapter 3.1 where the early-generation product returns arrive after a positive market sojourn time. Although some delayed demand for the new-generation product in these early periods can be met by benefiting from the recycled content, rejecting a demand too early in the selling horizon has the potential to reduce the future diffusion demand, making the partial-fulfillment policy less at attractive. Hence, the timing of end-of-life returns plays a major role in the optimality of partial fulfillment. Likewise, Figure 4.2 indicates that conditions (i)-(iii) may fail to simultaneously hold when m_1/m_2 or $\gamma\theta_1/\theta_2$ is large: high values of m_1/m_2 and $\gamma\theta_1/\theta_2$ again induce the available amount of recyclable material to exceed its demand in very early periods, in which rejecting a demand may reduce the future diffusion demand. Finally, we note from Figure 4.2 that conditions (i)-(iii) may fail to simultaneously hold when τ is small: When the new-generation product is released too early, the number of leapfrogging adopters is higher and the number of switching adopters is lower. Hence, the recyclable material obtained from the switching adopters' product returns tends to always fall short in meeting the new-generation product demand from the customers in the population of size m_2 as well as the leapfrogging adopters, and the use of recycled content cannot be improved via delayed demand.

We now extend our model in Chapter 3.2 by including positive salvage revenue for the recyclable material. Specifically, we assume that the salvage revenue is linear in the available amount of recyclable material at the end of period T: the total salvage revenue is $p_s\left(\nu_T - \theta_2\left(s_{2T} - \frac{D^b_{1T}d^b_{2t}}{m_2}\right)\right)$ where p_s is the unit salvage value. We incorporate this revenue term into our calculation of the total profit:

$$\begin{aligned}\max_{s_{2\tau},..,s_{2T}} \quad & \sum_{t=1}^{T}(r_1-c_1)d^n_{1t} + \sum_{t=\tau}^{T}(r_2-c_2)s_{2t} + \theta_2 p_r \sum_{t=\tau}^{T}\frac{D^b_{1t}d^b_{2t}}{m_2} \\ & + p_r\sum_{t=\tau}^{T}\min\left\{\nu_t, \theta_2\left(s_{2t} - \frac{D^b_{1t}d^b_{2t}}{m_2}\right)\right\} + p_s\left[\nu_T - \theta_2\left(s_{2T} - \frac{D^b_{1T}d^b_{2t}}{m_2}\right)\right]\end{aligned}$$

subject to (3.5)–(3.10) and (3.12). We generalize Theorem 2 to this case as follows.

Theorem 3. *Suppose that the consumers' timing of end-of-life product returns coincides with their timing of new-generation product purchases and the recyclable materials extracted from the early-generation product returns are available for salvaging at the end of the selling horizon. Then, the partial-fulfillment policy is optimal if, under the immediate-fulfillment policy, there exists a period $\kappa < T$ such that* *(i) the total available amount of recyclable material exceeds the maximum amount of recycled material that can be used to fulfill all the new-generation product demand, except the switching adopters' demand, in each period $t > \kappa$ while the reverse is true in each period $t \leq \kappa$,*

(ii) the amount of recyclable material that can be extracted from one unit of the early-generation product multiplied with the return rate of switching adopters is greater than the maximum amount of recycled material that can be used to make one unit of the new-generation, and

(iii) rejecting a unit of the new-generation product demand in period κ induces a loss of diffusion demand in period $\kappa + 1$ that is below a certain threshold (detailed in Appendix A) and a backlogged demand for the new-generation product in period $\kappa+1$ that is above a certain threshold (detailed in Appendix A).

Proof. See Appendix A.

Although conditions (i) and (ii) of Theorem 3 are identical to conditions (i) and (ii) of Theorem 2, the threshold level for the backlogged demand in condition (iii) of Theorem 3 is higher than that in condition (iii) if Theorem 2 (see Appendix A for those threshold levels): the existence of positive salvage value dilutes the incentive for partial demand fulfillment. Our explanation for this result is that a smaller amount of the recyclable material over time under partial fulfillment, leading to a smaller amount of salvaged material at the end of the selling horizon and thus a lower salvage revenue in the presence of positive salvage value.

Chapter 5

Exact Solution Algorithm for the Optimal Sales Plan

We develop a dynamic programming (DP) algorithm that can be employed to exactly calculate the optimal sales plan as well as the optimal total profit in a special case of the problem discussed in Chapter 3.1 (Recall from Chapter 4 that the partial-fulfillment policy is more likely to be profitable for the problem discussed in Chapter 3.1.). In this special case, the number of newly available end-of-use item returns at the beginning of period t is determined by a fraction ζ of the total number of early-generation products sold that have not been returned to the producer prior to period t. (See Akan et al. [18], Nadar et al. [19] for a similar assumption.) The problem in Chapter 3.1 reduces to this case when $\beta_i = (1-\zeta)^{i-1}\zeta$, $\forall i \geq 1$. The state variables of the DP algorithm are as follows: We define U_t as the total number of buyers who continue to use their early-generation products at the beginning of period t. Taking $U_1 = 0$, we can calculate U_t, $\forall t \geq 2$, with the following recursion:

$$U_t = (1-\zeta)\left(U_{t-1} + d^n_{1(t-1)}\right). \tag{5.1}$$

We define V_t as the accumulated amount of material that can be recovered and reused in period t. Taking $V_1 = 0$, we can calculate V_t, $\forall t \geq 2$, with the following recursion:

$$V_t = \begin{cases} \theta_1 \zeta \left(U_{t-1} + d^n_{1(t-1)} \right) & \text{if } V_{t-1} < \theta_2 s_{2(t-1)}, \\ V_{t-1} - \theta_2 s_{2(t-1)} + \theta_1 \zeta \left(U_{t-1} + d^n_{1(t-1)} \right) & \text{if } V_{t-1} \geq \theta_2 s_{2(t-1)}. \end{cases} \tag{5.2}$$

We also require D^b_{2t} and S^b_{2t} in the state description of the DP algorithm for our calculation of the new-generation product diffusion demand in each period t. Taking $D^b_{2\tau} = 0$, we can calculate D^b_{2t}, $\forall t > \tau$, with the following recursion:

$$D^b_{2t} = D^b_{2(t-1)} + d^b_{2(t-1)}. \tag{5.3}$$

Likewise, taking $S^b_{2\tau} = 0$, we can calculate S^b_{2t}, $\forall t > \tau$, with the following recursion:

$$S^b_{2t} = S^b_{2(t-1)} + s^b_{2(t-1)}. \tag{5.4}$$

Lastly, we define B_t as the accumulated number of backorders in period t from the customers unique to the new-generation product. Taking $B_\tau = 0$, we can calculate B_t, $\forall t > \tau$, with the following recursion:

$$B_t = \alpha \left(B_{t-1} + d^n_{2(t-1)} - s_{2(t-1)} \right). \tag{5.5}$$

Notice that

$$\frac{d^b_{1t} D^b_{2t}}{m_2} + \frac{D^b_{1t} d^b_{2t}}{m_2} + s^b_{2t} = s_{2t} \leq d^n_{2t} + B_t, \ \forall t \geq 1. \tag{5.6}$$

We are now ready to formulate the DP recursion in this special case of the problem in Chapter 3.1:

$$\begin{aligned} f_t \left(U_t, V_t, D^b_{2t}, S^b_{2t}, B_t \right) = & \max_{s_{2t}} \{ (r_1 - c_1) d^n_{1t} + (r_2 - c_2) s_{2t} + p_r \min \{ V_t, \theta_2 s_{2t} \} \\ & + f_{t+1} \left(U_{t+1}, V_{t+1}, D^b_{2(t+1)}, S^b_{2(t+1)}, B_{t+1} \right) \}, \end{aligned}$$

$\forall t \in \{\tau, \tau{+}1, \ldots, T\}$, $f_t(U_t, V_t, 0, 0, 0) = (r_1 - c_1) d^n_{1t} + f_{t+1}(U_{t+1}, V_{t+1}, 0, 0, 0)$, $\forall t \in \{1, 2, \ldots, \tau{-}1\}$, and $f_{T+1}(\cdot) = 0$, subject to (3.5)–(3.8) and (5.1)–(5.6). Note that $f_1(0, 0, 0, 0, 0)$ is the optimal total profit over the T-period selling horizon. We also define $\breve{f}_t(\cdot)$ as the profit function in period t under the immediate-fulfillment policy. This function can be calculated via the above DP recursion by restricting the action s_{2t} to be equal to d^n_{2t} for all $t \geq \tau$.

We conduct numerical experiments by coding the discrete-state and discrete-action version of our DP algorithm in the Java programming language on a system with 1.8 GHz CPU and 8 GB of RAM. We consider 108 instances in which $T = 16$ quarters, $m_1 = 160$, $m_2 = 40$, $\theta_2 = 1$, $r_1 - c_1 = r_2 - c_2 = 5$, $\alpha = 0.88$, $\zeta = 0.02$, $p = p_1 = p_2 \in \{0.03, 0.05, 0.07\}$, $q = q_1 = q_2 \in \{0.30, 0.40, 0.50\}$, $\tau \in \{4, 5, 6\}$, $\theta_1 \in \{1, 1.5\}$, and $p_r \in \{1, 2\}$. In all scenarios, $T = 16$ is sufficiently large so that we can observe the maturity stages of both generations: When $p \in \{0.03, 0.05, 0.07\}$ and $q \in \{0.30, 0.40, 0.50\}$, more than 80% of the total diffusion demand arrives before the selling horizon ends, under the immediate-fulfillment policy. Although $f_1(0, 0, 0, 0, 0)$ and $\breve{f}_1(0, 0, 0, 0, 0)$ are the total profits over the entire selling horizon, we compare $f_\tau(U_\tau, V_\tau, 0, 0, 0)$ and $\breve{f}_\tau(U_\tau, V_\tau, 0, 0, 0)$ in our experiments as we initiate our sales planning in period τ. Table 5.1 illustrates the profit improvement achieved with the partial-fulfillment policy in each of these instances (i.e., $100 \times [f_\tau(U_\tau, V_\tau, 0, 0, 0) - \breve{f}_\tau(U_\tau, V_\tau, 0, 0, 0)] / \breve{f}_\tau(U_\tau, V_\tau, 0, 0, 0)$). We observe that the partial-fulfillment policy can improve the total profit under the immediate-fulfillment policy by up to 5.6% on our test bed.

A closer examination of the numerical results has revealed that the partial-fulfillment policy improves the total profit under the immediate-fulfillment policy in the vast majority of our instances primarily due to increased cross-generation repeat purchases. We observe that the partial-fulfillment policy becomes less beneficial as τ increases: When τ is smaller, the total number of leapfrogging adopters is larger, and thus there is a greater opportunity to significantly stimulate the switching behavior via slowdown of the new-generation product diffusion. We also observe that the greatest profit improvements (larger than %5) occur when p and q are higher: Revenue loss due to rejecting a demand is small under larger p and q values since almost all of the diffusion demand can still arrive, despite the slowdown of the product diffusion, before the selling horizon ends. Lastly, we note that the partial-fulfillment policy tends to become more beneficial when p_r is increased: The partial-fulfillment policy helps improve the use of recycled content in manufacturing.

Table 5.1: Profit improvements via the partial-fulfillment policy (in percentages).

p	q	τ	$\theta_1 = 1$ & $p_r = 1$	$\theta_1 = 1$ & $p_r = 2$	$\theta_1 = 1.5$ & $p_r = 1$	$\theta_1 = 1.5$ & $p_r = 2$
0.03	0.30	4	0.39	0.39	0.39	0.38
		5	0.81	0.80	0.79	0.80
		6	0.44	0.43	0.44	0.42
	0.40	4	2.43	2.39	*	*
		5	0.35	0.35	0.56	0.75
		6	0.37	0.37	0.37	0.36
	0.50	4	2.56	2.79	*	*
		5	1.15	1.26	1.00	0.97
		6	0.00	0.00	0.00	0.00
0.05	0.30	4	2.24	2.20	2.22	2.16
		5	1.54	1.52	1.53	1.49
		6	1.23	1.21	1.38	1.50
	0.40	4	4.78	4.85	5.17	5.62
		5	2.26	2.29	2.30	2.38
		6	1.23	1.28	1.13	1.10
	0.50	4	5.02	5.50	*	*
		5	3.53	3.82	3.43	3.61
		6	1.18	1.23	1.09	1.06
0.07	0.30	4	1.11	1.09	1.10	1.07
		5	0.77	0.76	0.76	0.74
		6	0.98	1.12	0.80	0.77
	0.40	4	3.98	4.21	4.47	5.16
		5	2.57	2.82	2.54	2.75
		6	1.66	1.70	1.56	1.51
	0.50	4	2.63	3.08	2.89	3.58
		5	1.86	2.19	2.06	2.58
		6	1.03	1.24	1.10	1.36
Average			1.78	1.88	1.62	1.75

* The DP recursion could not be solved to optimality since the memory limit was reached.

Chapter 6

Conclusion

We have considered a durable-good producer who aims to jointly optimize its sales for two successive generations of a device with recycling potential. We have derived sufficient conditions under which it is optimal to partially satisfy the new-generation product demand by modeling the demand for early- and new-generation products as a generalized Norton-Bass diffusion process. If the product returns arrive mostly through recycling programs such as free mail-back and physical drop-off options, the partial-fulfillment policy is optimal when the product diffusion rate is high, the diffusion curves of successive product generations overlap substantially, and the new-generation product makes a limited contribution to the market potential. We have also conducted numerical experiments in this case to examine the exact value of the partial-fulfillment policy. We have found that the partial-fulfillment policy can significantly improve the total profit under the immediate-fulfillment policy (by up to 5.6%. However, if the product returns arise mostly from the switching adopters' participation in trade-up programs, the partial-fulfillment policy may bring no benefit even when the product diffusion rate is high, the diffusion curves have a large overlap, or the new-generation product only slightly increases the market potential.

Taken together, these results suggest that durable-good producers in fast-clockspeed industries may benefit from the partial-fulfillment policy if they focus on the recycling programs that allow consumers to conveniently return their old items at any time, while the benefit of the partial-fulfillment policy diminishes as the trade-up programs contribute more to the supply of the product returns. Nevertheless, the trade-up programs are often more appealing to consumers who bring their end-of-use devices for credits toward their next purchase, compared to those who bring their end-of-life devices for no such credit, motivating many producers to view their trade-up programs as a supply source for remanufacturing rather than recycling. The partial-fulfillment policy is still potentially profitable for such producers, provided that they also collect the end-of-life items through other channels. In a different setting, Nadar et al. [19] have sounded a note of caution with regard to the partial-fulfillment policy: the sales divisions are often motivated to sell as many items as possible over the entire selling horizon, posing a serious barrier to successful implementation of the partial-fulfillment policy in practice. Since the partial-fulfillment policy in our setting has the potential to boost not only the use of recycled content in manufacturing but also the total sales volume, such a barrier may be overcome more easily in our problem.

This thesis is not without limitations like other research. Future extensions of this study may study the sales planning problem for successive generations of a durable good when both remanufacturing and recycling are two possible disposition decisions for used items. Nadar et al. [19] study the sales planning problem for a remanufacturable durable good over a single-generation selling horizon, demonstrating that the partial-fulfillment policy can be desirable in fast-clockspeed industries. Combining their findings with ours, we intuitively expect partially satisfying demand for the *early*-generation product to be less desirable in the multi-generation case than in the single-generation case of Nadar et al. [19] if the used early-generation products become available for both remanufacturing and recycling: the recycling option here favors a faster diffusion process for the early-generation product that enables more returns to arrive earlier throughout the selling horizon. However, we expect delaying demand for the *new*-generation product to be more valuable in the multi-generation case than in the single-generation case of Nadar et al. [19] if the used early-generation products become available for recycling and the used new-generation products become available for remanufacturing: the recycling option here favors a slower diffusion process for the new-generation product that enables more returns to be used in new-generation product manufacturing.

Another direction for future work is to extend our analysis to a setting where a single product rollover strategy is adopted. Such an extension may provide further insights into the sales planning problem for CLSCs by comparing two different rollover strategies (single vs. dual). Future research may allow for more than two successive product generations; the generalized Norton-Bass diffusion model can still be employed in this research direction. Future work may also incorporate acquisition costs for collected end-of-life items, variable used-item conditions, and procurement decisions into the sales planning problem. Lastly, future work may include pricing as an additional lever to control the diffusion process for either generation and/or may consider competition against other producers' products.

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Appendix A

Detailed Versions and Proofs of the Analytical Results

We provide first the proof of Proposition 1 and then the detailed versions of Theorems 1, 2, and 3 (Theorems A.1, A.2, and A.3) along with their proofs. For the generalized Norton-Bass model, the term $A \triangleq q_2 - \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$ represents the loss of diffusion demand from the customers unique to the new-generation product in period $\kappa + 1$ when a unit demand from these customers is rejected in period κ. This term appears in several places in Theorems A.1, A.2, and A.3.

Proof of Proposition 1. Let t_p denote the earliest time period in which some demand from the customers unique to new-generation product is rejected. We know from Lemma 2 of Nadar et al. (2021) [19] that $\widehat{D}^b_{2t} = \breve{D}^b_{2t}$ if $t \leq t_p + 1$ and $\widehat{D}^b_{2t} < \breve{D}^b_{2t}$ otherwise. Since $\widehat{d}^b_{1t} = \breve{d}^b_{1t}$, we obtain $\widehat{d}^n_{1t} = \widehat{d}^b_{1t}\left(1 - \frac{\widehat{D}^b_{2t}}{m_2}\right) \geq \breve{d}^b_{1t}\left(1 - \frac{\breve{D}^b_{2t}}{m_2}\right) = \breve{d}^n_{1t}, \forall t$. Also, note that $\widehat{D}^n_{2(T+1)} \triangleq \widehat{D}^b_{2(T+1)} + \sum_{t=\tau}^{T} \frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2} + \sum_{t=\tau}^{T} \frac{\widehat{d}^b_{1t}\widehat{D}^b_{2t}}{m_2} \cong \breve{D}^n_{2(T+1)} \triangleq \breve{D}^b_{2(T+1)} + \sum_{t=\tau}^{T} \frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2} + \sum_{t=\tau}^{T} \frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2} \cong m_1 + m_2$ and $\widehat{D}^b_{2(T+1)} \cong \breve{D}^b_{2(T+1)} \cong m_2$ when T is sufficiently large. Since $\sum_{t=\tau}^{T} \breve{d}^b_{1t}\breve{D}^b_{2t} > \sum_{t=\tau}^{T} \widehat{d}^b_{1t}\widehat{D}^b_{2t}$, we obtain $\sum_{t=\tau}^{T} \widehat{D}^b_{1t}\widehat{d}^b_{2t} > \sum_{t=\tau}^{T} \breve{D}^b_{1t}\breve{d}^b_{2t}$ when T is sufficiently large. $\square$

Theorem A.1. *Suppose that the consumers' timing of end-of-life product returns is independent of their timing of new-generation product purchases (as in Chapter 3.1).*

(a) Suppose that $r_1 - c_1 \geq r_2 - c_2$*. For optimality of partial demand fulfillment in some period over the* T*-period selling horizon, it is sufficient that* $\exists \kappa \in \{\tau, .., T-2\}$ *s.t.* $\breve{\nu}_t < \theta_2 \breve{d}^n_{2t}$ *for* $\tau \leq t \leq \kappa$*,* $\breve{\nu}_t > \theta_2 \breve{d}^n_{2t}$ *for* $t > \kappa$*,* $\frac{\theta_1}{\theta_2} \geq \frac{\breve{s}_{2t}}{\sum_{i=1}^{t-1} \beta_i \breve{d}^n_{1(t-i)}}$ *for* $t \geq \kappa + 3$*,* $\alpha > \frac{(1+A)(1-2B)}{1-B}$*, and*

$$\alpha \;>\; 1 + A + \frac{\frac{A\breve{D}^b_{1(\kappa+1)}}{m_2} - \frac{(r_1-c_1)-(r_2-c_2)}{\theta_2 p_r + r_2 - c_2}\left(\frac{A\breve{d}^b_{1(\kappa+2)}}{m_2}\right)}{1 + AC - ABC}$$
$$- \frac{\frac{\theta_2 p_r}{\theta_2 p_r + r_2 - c_2}\left(1 - \frac{A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right) + ABC\,(1+A)}{1 + AC - ABC},$$

where $A = q_2 - \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$*,* $B = p_2 + \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$*, and* $C = 1 + \frac{\breve{D}^b_{1(\kappa+2)}}{m_2}$*.*

(b) Suppose that $r_1 - c_1 \leq r_2 - c_2$*. For optimality of partial demand fulfillment in some period over the* T*-period selling horizon, it is sufficient that* $\exists \kappa \in \{\tau, .., T-2\}$ *s.t.* $\breve{\nu}_t < \theta_2 \breve{d}^n_{2t}$ *for* $\tau \leq t \leq \kappa$*,* $\breve{\nu}_t > \theta_2 \breve{d}^n_{2t}$ *for* $t > \kappa$*,* $\frac{\theta_1}{\theta_2} \geq \frac{\breve{s}_{2t}}{\sum_{i=1}^{t-1} \beta_i \breve{d}^n_{1(t-i)}}$ *for* $t \geq \kappa + 3$*,* $\alpha > \frac{(1+A)(1-2B)}{1-B}$*, and*

$$\alpha \;>\; 1 + A + \frac{\frac{A\breve{D}^b_{1(\kappa+1)}}{m_2} + \frac{\theta_2 p_r + (r_2-c_2)-(r_1-c_1)}{\theta_2 p_r + r_2 - c_2}\left(\frac{A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right)}{1 + AC - ABC}$$
$$- \frac{\frac{\theta_2 p_r}{\theta_2 p_r + r_2 - c_2} + ABC\,(1+A)}{1 + AC - ABC},$$

where $A = q_2 - \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$*,* $B = p_2 + \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$*, and* $C = 1 + \frac{\breve{D}^b_{1(\kappa+2)}}{m_2}$*.*

(c) Suppose that the conditions in part (a) or (b) hold and T *is sufficiently large. Then, the partial-fulfillment policy, if initiated after period* κ*, leads to no improvement in the total amount of recycled material used in manufacturing of the new-generation product.*

Proof of Theorem A.1. We will prove that $\exists \epsilon > 0$ s.t. rejecting a demand of size ϵ from the customers unique to the new-generation product in period $\kappa \geq \tau$

while meeting all the remaining demand for both generations in period κ and all demand for both generations in each period $t \neq \kappa$ (sales plan i) is more profitable than meeting all demand for both generations in each period (sales plan ii). We use the hat ($\widehat{\ }$) and the breve ($\breve{\ }$) to denote the variables of sales plans (i) and (ii), respectively. Let $\widehat{D}^n_{1t} = \sum_{i=1}^{t-1} \widehat{d}^n_{1i}$ for $t > 1$ and $\widehat{D}^n_{11} = 0$, $\breve{D}^n_{1t} = \sum_{i=1}^{t-1} \breve{d}^n_{1i}$ for $t > 1$ and $\breve{D}^n_{11} = 0$, $\widehat{D}^n_{2t} = \sum_{i=\tau}^{t-1} \widehat{d}^n_{2i}$ for $t > \tau$ and $\widehat{D}^n_{2\tau} = 0$, and $\breve{D}^n_{2t} = \sum_{i=\tau}^{t-1} \breve{d}^n_{2i}$ for $t > \tau$ and $\breve{D}^n_{2\tau} = 0$. It can be shown that $\exists t_r \geq \kappa + 1$ s.t. $\widehat{D}^n_{1(t+1)} + \widehat{D}^n_{2(t+1)} \leq \breve{D}^n_{1(t+1)} + \breve{D}^n_{2(t+1)}$, $\forall t \leq t_r$, and $\widehat{D}^n_{1(t+1)} + \widehat{D}^n_{2(t+1)} \geq \breve{D}^n_{1(t+1)} + \breve{D}^n_{2(t+1)}$, $\forall t \geq t_r + 1$. We will provide a proof of this statement later. We make the following observations:

(1) Periods $t < \kappa$: $\widehat{d}^n_{1t} = \breve{d}^n_{1t}$ and $\widehat{s}_{2t} = \widehat{d}^n_{2t} = \breve{d}^n_{2t} = \breve{s}_{2t}$.

(2) Period κ: $\widehat{d}^n_{1\kappa} = \breve{d}^n_{1\kappa}$ and $\widehat{s}_{2\kappa} = \widehat{d}^n_{2\kappa} - \epsilon = \breve{d}^n_{2\kappa} - \epsilon = \breve{s}_{2\kappa} - \epsilon$. Since $\breve{\nu}_\kappa < \theta_2 \breve{d}^n_{2\kappa}$, $\exists \epsilon > 0$ s.t. a demand of size ϵ from the customers unique to the new-generation product cannot be met in period κ by using the recycled content.

(3) Period $\kappa + 1$: Note that $\widehat{d}^n_{1(\kappa+1)} = \widehat{d}^b_{1(\kappa+1)} \left(1 - \frac{\widehat{D}^b_{2(\kappa+1)}}{m_2}\right) = \widehat{d}^b_{1(\kappa+1)} \left(1 - \sum_{i=\tau}^{\kappa} \frac{\widehat{d}^b_{2i}}{m_2}\right) = \breve{d}^b_{1(\kappa+1)} \left(1 - \sum_{i=\tau}^{\kappa} \frac{\breve{d}^b_{2i}}{m_2}\right) = \breve{d}^n_{1(\kappa+1)}$.
Also, note that

$$\begin{aligned}
\widehat{d}^n_{2(\kappa+1)} &= \widehat{d}^b_{2(\kappa+1)} + \frac{\widehat{D}^b_{1(\kappa+1)} \widehat{d}^b_{2(\kappa+1)}}{m_2} + \frac{\widehat{d}^b_{1(\kappa+1)} \widehat{D}^b_{2(\kappa+1)}}{m_2} \\
&= \widehat{d}^b_{2(\kappa+1)} + \frac{\breve{D}^b_{1(\kappa+1)} \widehat{d}^b_{2(\kappa+1)}}{m_2} + \frac{\breve{d}^b_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} \\
&= \widehat{d}^b_{2(\kappa+1)} \left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right) + \frac{\breve{d}^b_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} \\
&= \left(p_2 + \frac{q_2 \widehat{S}^b_{2(\kappa+1)}}{m_2}\right) \left(m_2 - \widehat{D}^b_{2(\kappa+1)}\right) \left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^b_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} \\
&= \left(p_2 + \frac{q_2 \left(\breve{D}^b_{2(\kappa+1)} - \epsilon\right)}{m_2}\right) \left(m_2 - \breve{D}^b_{2(\kappa+1)}\right) \left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^b_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2}
\end{aligned}$$

$$
\begin{aligned}
&= \left(p_2 + \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}\right)\left(m_2 - \breve{D}^b_{2(\kappa+1)}\right)\left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^b_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^b_{2(\kappa+1)}\right)\left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right) \\
&= \breve{d}^n_{2(\kappa+1)} - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^b_{2(\kappa+1)}\right)\left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right) \\
&= \breve{d}^n_{2(\kappa+1)} - \epsilon\left(q_2 - \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}\right)\left(1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}\right).
\end{aligned}
$$

Letting $E = 1 + \frac{\breve{D}^b_{1(\kappa+1)}}{m_2}$, we obtain $\widehat{s}_{2(\kappa+1)} = \widehat{d}^n_{2(\kappa+1)} + \alpha\epsilon = \breve{d}^n_{2(\kappa+1)} - AE\epsilon + \alpha\epsilon = \breve{s}_{2(\kappa+1)} - AE\epsilon + \alpha\epsilon$. Also, $\widehat{S}^b_{2(\kappa+2)} = \sum_{i=\tau}^{\kappa+1} \widehat{d}^b_{2i} - \epsilon + \alpha\epsilon = \sum_{i=\tau}^{\kappa+1} \breve{d}^b_{2i} - \epsilon - A\epsilon + \alpha\epsilon = \breve{D}^b_{2(\kappa+1)} + \breve{d}^b_{2(\kappa+1)} - \epsilon - A\epsilon + \alpha\epsilon$ and $\widehat{D}^b_{2(\kappa+1)} + \widehat{d}^b_{2(\kappa+1)} + A\epsilon = \breve{D}^b_{2(\kappa+1)} + \breve{d}^b_{2(\kappa+1)}$. Lastly, since $\breve{\nu}_{\kappa+1} = \theta_1 \sum_{i=1}^{\kappa} \beta_i \breve{d}^n_{1(\kappa+1-i)} > \theta_2 \breve{d}^n_{2(\kappa+1)}$, $\exists \epsilon > 0$ s.t. $\widehat{\nu}_{\kappa+1} = \theta_1 \sum_{i=1}^{\kappa} \beta_i \widehat{d}^n_{1(\kappa+1-i)} = \theta_1 \sum_{i=1}^{\kappa} \beta_i \breve{d}^n_{1(\kappa+1-i)} > \theta_2 \breve{d}^n_{2(\kappa+1)} - \theta_2 AE\epsilon + \theta_2\alpha\epsilon = \theta_2 \widehat{d}^n_{2(\kappa+1)} + \theta_2\alpha\epsilon$.

(4) Period $\kappa + 2$: Note that $\widehat{d}^b_{1t} = \breve{d}^b_{1t}$, $\forall t$. Thus:

$$
\begin{aligned}
\widehat{d}^n_{1(\kappa+2)} &= \widehat{d}^b_{1(\kappa+2)}\left(1 - \frac{\widehat{D}^b_{2(\kappa+2)}}{m_2}\right) \\
&= \widehat{d}^b_{1(\kappa+2)}\left(1 - \frac{\sum_{i=\tau}^{\kappa} \widehat{d}^b_{2i}}{m_2} - \frac{\widehat{d}^b_{2(\kappa+1)}}{m_2}\right) \\
&= \breve{d}^b_{1(\kappa+2)}\left(1 - \frac{\sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} - \left(p_2 + \frac{q_2 \widehat{S}^b_{2(\kappa+1)}}{m_2}\right)\left(\frac{m_2 - \widehat{D}^b_{2(\kappa+1)}}{m_2}\right)\right) \\
&= \breve{d}^b_{1(\kappa+2)}\left(1 - \frac{\sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} - \left(p_2 + \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2} - \frac{q_2\epsilon}{m_2}\right)\left(\frac{m_2 - \breve{D}^b_{2(\kappa+1)}}{m_2}\right)\right) \\
&= \breve{d}^b_{1(\kappa+2)}\left(1 - \frac{\sum_{i=\tau}^{\kappa} \breve{d}^b_{2i}}{m_2} - \left(p_2 + \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}\right)\left(\frac{m_2 - \breve{D}^b_{2(\kappa+1)}}{m_2}\right)\right) \\
&\quad + \frac{A\epsilon \breve{d}^b_{1(\kappa+2)}}{m_2} \\
&= \breve{d}^n_{1(\kappa+2)} + \frac{A\epsilon \breve{d}^b_{1(\kappa+2)}}{m_2}.
\end{aligned}
$$

Also, note that

$$\begin{aligned}
\widehat{s}_{2(\kappa+2)} = \widehat{d}^{n}_{2(\kappa+2)} &= \widehat{d}^{b}_{2(\kappa+2)} + \frac{\widehat{D}^{b}_{1(\kappa+2)}\widehat{d}^{b}_{2(\kappa+2)}}{m_2} + \frac{\widehat{d}^{b}_{1(\kappa+2)}\widehat{D}^{b}_{2(\kappa+2)}}{m_2} \\
&= \widehat{d}^{b}_{2(\kappa+2)}\left(1 + \frac{\widehat{D}^{b}_{1(\kappa+2)}}{m_2}\right) + \frac{\widehat{d}^{b}_{1(\kappa+2)}\widehat{D}^{b}_{2(\kappa+2)}}{m_2} \\
&= \widehat{d}^{b}_{2(\kappa+2)}\left(1 + \frac{\breve{D}^{b}_{1(\kappa+2)}}{m_2}\right) + \frac{\breve{d}^{b}_{1(\kappa+2)}\widehat{D}^{b}_{2(\kappa+2)}}{m_2} \\
&= \widehat{d}^{b}_{2(\kappa+2)}\left(1 + \frac{\breve{D}^{b}_{1(\kappa+2)}}{m_2}\right) + \frac{\breve{d}^{b}_{1(\kappa+2)}\breve{D}^{b}_{2(\kappa+2)}}{m_2} - \frac{A\epsilon\breve{d}^{b}_{1(\kappa+2)}}{m_2}
\end{aligned}$$

where

$$\begin{aligned}
\widehat{d}^{b}_{2(\kappa+2)} &= \left(p_2 + \frac{q_2\widehat{S}^{b}_{2(\kappa+2)}}{m_2}\right)\left(m_2 - \widehat{D}^{b}_{2(\kappa+1)} - \widehat{d}^{b}_{2(\kappa+1)}\right) \\
&= \left(p_2 + \frac{q_2\left(\breve{D}^{b}_{2(\kappa+1)} + \breve{d}^{b}_{2(\kappa+1)}\right)}{m_2}\right)\left(m_2 - \breve{D}^{b}_{2(\kappa+1)} - \breve{d}^{b}_{2(\kappa+1)}\right) \\
&\quad + \left(\frac{q_2\epsilon\left(\alpha - 1 - A\right)}{m_2}\right)\left(m_2 - \breve{D}^{b}_{2(\kappa+1)} - \breve{d}^{b}_{2(\kappa+1)}\right) \\
&\quad + \left(p_2 + \frac{q_2\left(\breve{D}^{b}_{2(\kappa+1)} + \breve{d}^{b}_{2(\kappa+1)}\right)}{m_2} + \frac{q_2\epsilon\left(\alpha - 1 - A\right)}{m_2}\right)A\epsilon.
\end{aligned}$$

We will show that $\exists\epsilon > 0$ s.t. $\widehat{d}^{b}_{2(\kappa+2)} \geq \breve{d}^{b}_{2(\kappa+2)}$. Let $\overline{a} = p_2 + \frac{q_2\left(\breve{D}^{b}_{2(\kappa+1)}+\breve{d}^{b}_{2(\kappa+1)}\right)}{m_2}$ and $\underline{b} = m_2 - \breve{D}^{b}_{2(\kappa+1)} - \breve{d}^{b}_{2(\kappa+1)}$. Note that $\breve{d}^{b}_{2(\kappa+2)} = \overline{a}\underline{b}$. Thus:

$$\widehat{d}^{b}_{2(\kappa+2)} = \breve{d}^{b}_{2(\kappa+2)} + \overline{a}A\epsilon + \frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2} + \frac{q_2A\epsilon^2(\alpha-1-A)}{m_2}.$$

Note that $\overline{a} < 1$, $\alpha - 1 - A < 0$, and $\breve{d}^{b}_{2(\kappa+2)} + A\epsilon \geq \widehat{d}^{b}_{2(\kappa+2)}$. We need to show that $\exists\epsilon > 0$ s.t.

$$\overline{a}A\epsilon + \frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2} + \frac{q_2A\epsilon^2(\alpha-1-A)}{m_2} > 0.$$

To this end, it suffices to show that $\overline{a} + \frac{\underline{b}(\alpha-1-A)}{m_2-\breve{D}^{b}_{2(\kappa+1)}} > 0$, or equivalently,

$$\left(p_2 + \frac{q_2\breve{D}^{b}_{2(\kappa+1)}}{m_2}\right)(1+A) + \left(1 - p_2 - \frac{q_2\breve{D}^{b}_{2(\kappa+1)}}{m_2}\right)(\alpha-1-A) > 0.$$

The above inequality holds as we assume $\alpha(1-B) > (1+A)(1-2B)$. Thus $\widehat{d}^b_{2(\kappa+2)} \geq \breve{d}^b_{2(\kappa+2)}$ and $\widehat{s}_{2(\kappa+2)} = \widehat{d}^n_{2(\kappa+2)} \geq \breve{d}^n_{2(\kappa+2)} - \frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2} = \breve{s}_{2(\kappa+2)} - \frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}$. Moreover, note that

$$\begin{aligned}
\widehat{d}^n_{2(\kappa+2)} &= \breve{d}^n_{2(\kappa+2)} + \left(\overline{a}A\epsilon + \frac{bq_2\epsilon(\alpha-1-A)}{m_2}\right)\left(1+\frac{\breve{D}^b_{1(\kappa+2)}}{m_2}\right) \\
&\quad + \left(\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2}\right)\left(1+\frac{\breve{D}^b_{1(\kappa+2)}}{m_2}\right) - \frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}.
\end{aligned}$$

Since $\alpha-1-A<0$, we obtain $\widehat{d}^n_{2(\kappa+2)} \leq \breve{d}^n_{2(\kappa+2)} + \overline{a}A\epsilon\left(1+\frac{\breve{D}^b_{1(\kappa+2)}}{m_2}\right)$. Since $\breve{\nu}_{\kappa+2} > \theta_2\breve{d}^n_{2(\kappa+2)}$, $\exists\epsilon>0$ s.t.

$$\begin{aligned}
\widehat{\nu}_{\kappa+2} &= \widehat{\nu}_{\kappa+1} - \theta_2\widehat{d}^n_{2(\kappa+1)} - \theta_2\alpha\epsilon + \theta_1\sum_{i=1}^{\kappa+1}\beta_i\breve{d}^n_{1(\kappa+2-i)} \\
&= \breve{\nu}_{\kappa+1} - \theta_2\breve{d}^n_{2(\kappa+1)} + \theta_2AE\epsilon - \theta_2\alpha\epsilon + \theta_1\sum_{i=1}^{\kappa+1}\beta_i\breve{d}^n_{1(\kappa+2-i)} \\
&= \breve{\nu}_{\kappa+2} - \theta_2\alpha\epsilon + \theta_2AE\epsilon \\
&> \theta_2\breve{d}^n_{2(\kappa+2)} + \theta_2\overline{a}A\epsilon\left(1+\frac{\breve{D}^b_{1(\kappa+2)}}{m_2}\right) \\
&= \theta_2\left(\breve{d}^n_{2(\kappa+2)} + \overline{a}A\epsilon\left(1+\frac{\breve{D}^b_{1(\kappa+2)}}{m_2}\right)\right) \geq \theta_2\widehat{d}^n_{2(\kappa+2)}.
\end{aligned}$$

(5) Periods $t>\kappa+2$: We know from Proposition 1 that $\widehat{d}^n_{1t} \geq \breve{d}^n_{1t}$, $\forall t$. We now consider the new-generation product sales:

$$\begin{aligned}
\widehat{s}_{2(\kappa+3)} = \widehat{d}^n_{2(\kappa+3)} &= \widehat{d}^b_{2(\kappa+3)} + \frac{\widehat{D}^b_{1(\kappa+3)}\widehat{d}^b_{2(\kappa+3)}}{m_2} + \frac{\widehat{d}^b_{1(\kappa+3)}\widehat{D}^b_{2(\kappa+3)}}{m_2} \\
&= \widehat{d}^b_{2(\kappa+3)}\left(1+\frac{\widehat{D}^b_{1(\kappa+3)}}{m_2}\right) + \frac{\widehat{d}^b_{1(\kappa+3)}\widehat{D}^b_{2(\kappa+3)}}{m_2} \\
&= \widehat{d}^b_{2(\kappa+3)}\left(1+\frac{\breve{D}^b_{1(\kappa+3)}}{m_2}\right) + \frac{\breve{d}^b_{1(\kappa+3)}\widehat{D}^b_{2(\kappa+3)}}{m_2} \\
&= \widehat{d}^b_{2(\kappa+3)}\left(1+\frac{\breve{D}^b_{1(\kappa+3)}}{m_2}\right) + \frac{\breve{d}^b_{1(\kappa+3)}\breve{D}^b_{2(\kappa+3)}}{m_2} \\
&\quad - \left(\widehat{d}^n_{1(\kappa+3)} - \breve{d}^n_{1(\kappa+3)}\right)
\end{aligned}$$

where

$$\widehat{d}^b_{2(\kappa+3)} = \left(p_2 + \frac{q_2\widehat{S}^b_{2(\kappa+2)}}{m_2} + \frac{q_2\widehat{d}^b_{2(\kappa+2)}}{m_2}\right)\left(m_2 - \widehat{D}^b_{2(\kappa+2)} - \widehat{d}^b_{2(\kappa+2)}\right).$$

We will show that $\widehat{d}^b_{2(\kappa+3)} \geq \breve{d}^b_{2(\kappa+3)}$. Recall that $\widehat{D}^b_{1(\kappa+3)} = \breve{D}^b_{1(\kappa+3)}$, $\overline{a} = p_2 + \frac{q_2\breve{D}^b_{2(\kappa+2)}}{m_2}$, $\underline{b} = m_2 - \breve{D}^b_{2(\kappa+2)}$, and $\breve{d}^b_{2(\kappa+2)} = \overline{a}\underline{b}$. Let $\underline{a} = p_2 + \frac{q_2\widehat{S}^b_{2(\kappa+2)}}{m_2}$ and $\overline{b} = m_2 - \widehat{D}^b_{2(\kappa+2)}$. Thus $\widehat{d}^b_{2(\kappa+2)} = \underline{a}\overline{b}$. Since $\overline{a} \geq \underline{a}$, $\overline{b} \geq \underline{b}$, and $\widehat{d}^b_{2(\kappa+2)} \geq \breve{d}^b_{2(\kappa+2)}$:

$$\begin{aligned}
\widehat{d}^b_{2(\kappa+3)} = \left(\underline{a} + \frac{q_2\underline{a}\overline{b}}{m_2}\right)\left(\overline{b} - \underline{a}\overline{b}\right) &= \underline{a}\overline{b}\left(1 + \frac{q_2\overline{b}}{m_2}\right)(1 - \underline{a}) \\
&\geq \overline{a}\underline{b}\left(1 + \frac{q_2\underline{b}}{m_2}\right)(1 - \overline{a}) \\
&= \left(\overline{a} + \frac{q_2\overline{a}\underline{b}}{m_2}\right)\left(\underline{b} - \overline{a}\underline{b}\right) = \breve{d}^b_{2(\kappa+3)}.
\end{aligned}$$

Thus, $\widehat{d}^b_{2(\kappa+3)} \geq \breve{d}^b_{2(\kappa+3)}$ and $\widehat{s}_{2(\kappa+3)} = \widehat{d}^n_{2(\kappa+3)} \geq \breve{d}^n_{2(\kappa+3)} - \left(\widehat{d}^n_{1(\kappa+3)} - \breve{d}^n_{1(\kappa+3)}\right) = \breve{s}_{2(\kappa+3)} - \left(\widehat{d}^n_{1(\kappa+3)} - \breve{d}^n_{1(\kappa+3)}\right)$. Proceeding similarly, it can be shown that $\widehat{d}^b_{2t} \geq \breve{d}^b_{2t}$ and $\widehat{s}_{2t} = \widehat{d}^n_{2t} \geq \breve{d}^n_{2t} - \left(\widehat{d}^n_{1t} - \breve{d}^n_{1t}\right) = \breve{s}_{2t} - \left(\widehat{d}^n_{1t} - \breve{d}^n_{1t}\right)$, $\forall t \geq \kappa+3$. Also, recall that $\breve{D}^b_{2(\kappa+2)} - \widehat{D}^b_{2(\kappa+2)} = A\epsilon$ and $\widehat{D}^b_{2t} < \breve{D}^b_{2t}$, $\forall t \geq \kappa+3$. Since $\widehat{d}^b_{2t} \geq \breve{d}^b_{2t}$, $\forall t \geq \kappa+2$, we obtain $\breve{D}^b_{2t} - \widehat{D}^b_{2t} \leq A\epsilon$ and $\widehat{d}^n_{1t} - \breve{d}^n_{1t} = \frac{\breve{d}^b_{1t}\left(\breve{D}^b_{2t} - \widehat{D}^b_{2t}\right)}{m_2} \leq \frac{A\epsilon\breve{d}^b_{1t}}{m_2}$, $\forall t \geq \kappa+3$. We now consider two possible scenarios depending on the value of t_r. First, suppose that $\kappa+3 \leq t \leq t_r$. Since $\breve{D}^n_{1(\kappa+4)} + \breve{D}^n_{2(\kappa+4)} \geq \widehat{D}^n_{1(\kappa+4)} + \widehat{D}^n_{2(\kappa+4)}$, $\widehat{D}^n_{1(\kappa+2)} = \breve{D}^n_{1(\kappa+2)}$, and $\widehat{D}^n_{2(\kappa+1)} = \breve{D}^n_{2(\kappa+1)}$:

$$\begin{aligned}
\breve{D}^n_{1(\kappa+4)} + \breve{D}^n_{2(\kappa+4)} &= \sum_{i=1}^{\kappa+1}\breve{d}^n_{1i} + \breve{d}^n_{1(\kappa+2)} + \breve{d}^n_{1(\kappa+3)} + \sum_{i=1}^{\kappa}\breve{d}^n_{2i} + \breve{d}^n_{2(\kappa+1)} \\
&\quad + \breve{d}^n_{2(\kappa+2)} + \breve{d}^n_{2(\kappa+3)} \\
&= \breve{D}^n_{1(\kappa+2)} + \breve{d}^n_{1(\kappa+2)} + \breve{d}^n_{1(\kappa+3)} + \breve{D}^n_{2(\kappa+1)} + \widehat{d}^n_{2(\kappa+1)} + AE\epsilon \\
&\quad + \breve{d}^n_{2(\kappa+2)} + \breve{d}^n_{2(\kappa+3)} \\
&\geq \widehat{D}^n_{1(\kappa+2)} + \widehat{d}^n_{1(\kappa+2)} + \widehat{d}^n_{1(\kappa+3)} + \widehat{D}^n_{2(\kappa+1)} + \widehat{d}^n_{2(\kappa+1)} \\
&\quad + \widehat{d}^n_{2(\kappa+2)} + \widehat{d}^n_{2(\kappa+3)} \\
&= \sum_{i=1}^{\kappa+1}\widehat{d}^n_{1i} + \widehat{d}^n_{1(\kappa+2)} + \widehat{d}^n_{1(\kappa+3)} + \sum_{i=1}^{\kappa+1}\widehat{d}^n_{2i} + \widehat{d}^n_{2(\kappa+2)} + \widehat{d}^n_{2(\kappa+3)} \\
&= \widehat{D}^n_{1(\kappa+4)} + \widehat{D}^n_{2(\kappa+4)}.
\end{aligned}$$

Hence, $\breve{d}^n_{2(\kappa+3)} + \breve{d}^n_{2(\kappa+2)} + AE\epsilon - \left(\widehat{d}^n_{1(\kappa+2)} - \breve{d}^n_{1(\kappa+2)}\right) - \left(\widehat{d}^n_{1(\kappa+3)} - \breve{d}^n_{1(\kappa+3)}\right) \geq \widehat{d}^n_{2(\kappa+2)} + \widehat{d}^n_{2(\kappa+3)}$, implying that $\breve{d}^n_{2(\kappa+3)} + \breve{d}^n_{2(\kappa+2)} + AE\epsilon \geq \widehat{d}^n_{2(\kappa+2)} + \widehat{d}^n_{2(\kappa+3)}$. Proceeding similarly, it can be shown that $\breve{d}^n_{2t} + \cdots + \breve{d}^n_{2(\kappa+2)} + AE\epsilon \geq \widehat{d}^n_{2t} + \cdots + \widehat{d}^n_{2(\kappa+2)}$, $\forall t \in \{\kappa+3, \kappa+4, \ldots, t_r\}$. Since $\widehat{d}^n_{1(\kappa+2)} \geq \breve{d}^n_{1(\kappa+2)}$ and $\breve{\nu}_{\kappa+3} > \theta_2 \breve{d}^n_{2(\kappa+3)}$, $\exists \epsilon > 0$ s.t.

$$
\begin{aligned}
\widehat{\nu}_{\kappa+3} &= \widehat{\nu}_{\kappa+2} - \theta_2 \widehat{d}^n_{2(\kappa+2)} + \theta_1 \beta_1 \widehat{d}^n_{1(\kappa+2)} + \theta_1 \sum_{i=2}^{\kappa+2} \beta_i \breve{d}^n_{1(\kappa+3-i)} \\
&= \breve{\nu}_{\kappa+2} - \theta_2 \alpha \epsilon + \theta_2 AE\epsilon - \theta_2 \widehat{d}^n_{2(\kappa+2)} + \theta_1 \beta_1 \widehat{d}^n_{1(\kappa+2)} + \theta_1 \sum_{i=2}^{\kappa+2} \beta_i \breve{d}^n_{1(\kappa+3-i)} \\
&= \breve{\nu}_{\kappa+3} + \theta_2 \breve{d}^n_{2(\kappa+2)} + \theta_1 \beta_1 \widehat{d}^n_{1(\kappa+2)} - \theta_1 \beta_1 \breve{d}^n_{1(\kappa+2)} - \theta_2 \widehat{d}^n_{2(\kappa+2)} - \theta_2 \alpha \epsilon + \theta_2 AE\epsilon \\
&= \breve{\nu}_{\kappa+3} - \theta_2 \alpha \epsilon + \theta_2 AE\epsilon + \theta_2 \left(\breve{d}^n_{2(\kappa+2)} - \widehat{d}^n_{2(\kappa+2)}\right) + \theta_1 \beta_1 \left(\widehat{d}^n_{1(\kappa+2)} - \breve{d}^n_{1(\kappa+2)}\right) \\
&\geq \breve{\nu}_{\kappa+3} - \theta_2 \alpha \epsilon + \theta_2 AE\epsilon + \theta_2 \left(\breve{d}^n_{2(\kappa+2)} - \widehat{d}^n_{2(\kappa+2)}\right) \\
&> \theta_2 \breve{d}^n_{2(\kappa+3)} + \theta_2 AE\epsilon + \theta_2 \left(\breve{d}^n_{2(\kappa+2)} - \widehat{d}^n_{2(\kappa+2)}\right) \\
&= \theta_2 \left(\breve{d}^n_{2(\kappa+3)} + \breve{d}^n_{2(\kappa+2)} - \widehat{d}^n_{2(\kappa+2)} + AE\epsilon\right) \\
&\geq \theta_2 \widehat{d}^n_{2(\kappa+3)}.
\end{aligned}
$$

Proceeding similarly, it can be shown that $\widehat{\nu}_t > \theta_2 \widehat{d}^n_{2t}$, $\forall t \in \{\kappa+3, \kappa+4, \ldots, t_r\}$. Now, suppose that $t > t_r$. In this case, $\widehat{D}^n_{1(t+1)} + \widehat{D}^n_{2(t+1)} \geq \breve{D}^n_{1(t+1)} + \breve{D}^n_{2(t+1)}$.

We next prove that $\exists t_r \geq \kappa+1$ s.t. $\widehat{D}^n_{1(t+1)} + \widehat{D}^n_{2(t+1)} \leq \breve{D}^n_{1(t+1)} + \breve{D}^n_{2(t+1)}$, $\forall t \leq t_r$, and $\widehat{D}^n_{1(t+1)} + \widehat{D}^n_{2(t+1)} \geq \breve{D}^n_{1(t+1)} + \breve{D}^n_{2(t+1)}$, $\forall t \geq t_r + 1$. Note that $\widehat{D}^n_{1t} + \widehat{D}^n_{2t} = \breve{D}^n_{1t} + \breve{D}^n_{2t}$, $\forall t \leq \kappa + 1$. Also, note that $\widehat{d}^n_{1(\kappa+1)} = \breve{d}^n_{1(\kappa+1)}$, $\widehat{d}^n_{2(\kappa+1)} < \breve{d}^n_{2(\kappa+1)}$, and $\widehat{D}^n_{1(\kappa+2)} + \widehat{D}^n_{2(\kappa+2)} < \breve{D}^n_{1(\kappa+2)} + \breve{D}^n_{2(\kappa+2)}$. Since $\widehat{d}^n_{1t} + \widehat{d}^n_{2t} = \breve{d}^n_{1t} + (\widehat{d}^n_{1t} - \breve{d}^n_{1t}) + \widehat{d}^n_{2t} \geq \breve{d}^n_{1t} + (\widehat{d}^n_{1t} - \breve{d}^n_{1t}) + \breve{d}^n_{2t} - (\widehat{d}^n_{1t} - \breve{d}^n_{1t}) = \breve{d}^n_{1t} + \breve{d}^n_{2t}$ and $\widehat{D}^n_{1(t+1)} > \breve{D}^n_{1(t+1)}$, $\forall t \geq \kappa + 2$, and since $\widehat{D}^n_{2(t+1)} \cong \breve{D}^n_{2(t+1)} \cong m_1 + m_2$ when t is sufficiently large, $\exists t_r \geq \kappa + 1$ s.t. $\widehat{D}^n_{1(t+1)} + \widehat{D}^n_{2(t+1)} \geq \breve{D}^n_{1(t+1)} + \breve{D}^n_{2(t+1)}$, $\forall t \geq t_r + 1$.

In order to prove part (a) of Theorem A.1, suppose that $r_1 - c_1 \geq r_2 - c_2$. We will show that sales plan (i) is more profitable than sales plan (ii). As we assume $\frac{\theta_1}{\theta_2} \geq \frac{\breve{s}_{2t}}{\sum_{i=1}^{t-1} \beta_i \breve{d}^n_{1(t-i)}}$, $\forall t \geq \kappa + 3$, we obtain $\widehat{\nu}_t \geq \theta_1 \sum_{i=1}^{t-1} \beta_i \widehat{d}^n_{1(t-i)} \geq$

$\theta_1 \sum_{i=1}^{t-1} \beta_i \breve{d}^n_{1(t-i)} \geq \theta_2 \breve{s}_{2t}$, $\forall t \geq \kappa+3$. Combination of this result and our observations in $(1)-(5)$ gives:

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\}\\
&=(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\sum_{t=\tau}^{t_r}\min\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\}+p_r\sum_{t=t_r+1}^{T}\min\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\}\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{t_r}\widehat{s}_{2t}+\sum_{t=t_r+1}^{T}\min\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\}\right]\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{t_r}\widehat{s}_{2t}+\sum_{\substack{t:\widehat{\nu}_t<\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\widehat{\nu}_t+\theta_2\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\widehat{s}_{2t}\right]\\
&\geq(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{t_r}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\quad+\theta_2p_r\left[\sum_{\substack{t:\widehat{\nu}_t<\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\breve{s}_{2t}+\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+p_r\left[\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\theta_2\sum_{t=\kappa+3}^{t_r}(\widehat{d}^n_{1t}-\breve{d}^n_{1t})-\theta_2\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\left(\widehat{d}^n_{1t}-\breve{d}^n_{1t}\right)\right]\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}-\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\right]\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)+\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]\\
&-p_r\left[\theta_2\sum_{t=\kappa+3}^{t_r}(\widehat{d}^n_{1t}-\breve{d}^n_{1t})+\theta_2\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\left(\widehat{d}^n_{1t}-\breve{d}^n_{1t}\right)\right]\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]\\
&-p_r\left[\theta_2\sum_{t=\kappa+3}^{t_r}(\widehat{d}^n_{1t}-\breve{d}^n_{1t})+\theta_2\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\left(\widehat{d}^n_{1t}-\breve{d}^n_{1t}\right)\right]\\
&\geq(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right.\\
&-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq (r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\breve{d}^n_{1(\kappa+2)}+\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&\quad-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}-(r_2-c_2)\left[\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)\right]\\
&\quad+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\theta_2\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}^b_{1t}}{m_2}\right]\\
&\quad+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&= (r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\breve{d}^n_{1(\kappa+2)}+\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&\quad-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}-(r_2-c_2)\left[\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)\right]\\
&\quad+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\frac{\theta_2A\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right]\\
&\quad+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&= (r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\breve{s}_{2t}\right\}\\
&\quad+[(r_1-c_1)-(r_2-c_2)]\left(\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right)\\
&\quad+(r_2-c_2)\left(\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)-\epsilon-AE\epsilon+\alpha\epsilon\right)\\
&\quad+\theta_2p_r\left(\alpha\epsilon-AE\epsilon\right)+\theta_2p_r\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&\quad-\frac{\theta_2p_rA\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}.
\end{aligned}
$$

In order to show that

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\right\}\\
&\geq(r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\breve{s}_{2t}\right\},
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&\frac{(r_1-c_1)\,A\breve{d}^b_{1(\kappa+2)}}{m_2}+(r_2-c_2)\left(\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A}{m_2}+\frac{bq_2(\alpha-1-A)}{m_2^2}\right)\right)\\
&+(r_2-c_2)\left((\alpha-1-AE)-\frac{A\breve{d}^b_{1(\kappa+2)}}{m_2}\right)\\
&+\theta_2p_r\left((\alpha-AE)-\frac{A\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right)\\
&+\theta_2p_r\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A}{m_2}+\frac{bq_2(\alpha-1-A)}{m_2^2}\right)>0.
\end{aligned}
$$

The above inequality holds as we assume

$$
\begin{aligned}
\alpha\quad>\quad&1+A+\frac{\frac{A\breve{D}^b_{1(\kappa+1)}}{m_2}-\frac{(r_1-c_1)-(r_2-c_2)}{\theta_2p_r+r_2-c_2}\left(\frac{A\breve{d}^b_{1(\kappa+2)}}{m_2}\right)}{1+AC-ABC}\\
&-\frac{\frac{\theta_2p_r}{\theta_2p_r+r_2-c_2}\left(1-\frac{A\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right)+ABC\,(1+A)}{1+AC-ABC}.
\end{aligned}
$$

In order to prove part (b) of Theorem A.1, suppose that $r_1-c_1\leq r_2-c_2$. We will show that sales plan (i) is more profitable than sales plan (ii). With similar arguments to those used in the proof of part (a):

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\right\}\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{t_r}\widehat{s}_{2t}+\sum_{\substack{t:\widehat{\nu}_t<\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\widehat{\nu}_t+\theta_2\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\T\geq t\geq t_r+1}}\widehat{s}_{2t}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq (r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\left(\breve{d}^n_{1t}+(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}\right]\\
&\quad+\theta_2 p_r\left[\sum_{t=\kappa+3}^{t_r}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)+\sum_{\substack{t:\widehat{\nu}_t<\theta_2\widehat{s}_{2t}\\ T\geq t\geq t_r+1}}\breve{s}_{2t}+\sum_{\substack{t:\widehat{\nu}_t\geq\theta_2\widehat{s}_{2t}\\ T\geq t\geq t_r+1}}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\geq (r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\left(\breve{d}^n_{1t}+(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
&\geq (r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\breve{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^n_{1t}\right]+[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}^b_{1t}}{m_2}\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]\\
&\quad+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\theta_2\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}^b_{1t}}{m_2}\right]\\
&\quad+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&= (r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\breve{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^n_{1t}\right]\\
&\quad+[(r_1-c_1)-(r_2-c_2)]\left(\frac{A\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right)\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]\\
&\quad+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\frac{\theta_2A\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&=(r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\{\breve{\nu}_t,\theta_2\breve{s}_{2t}\}\\
&+(r_2-c_2)\left(\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right)\\
&+\theta_2p_r\left(\alpha\epsilon-AE\epsilon\right)+\theta_2p_r\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&+\frac{\left((r_1-c_1)-(r_2-c_2)-\theta_2p_r\right)A\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}+(r_2-c_2)\left(-\epsilon-AE\epsilon+\alpha\epsilon\right).
\end{aligned}
$$

In order to show that

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\{\widehat{\nu}_t,\theta_2\widehat{s}_{2t}\}\\
&\geq(r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\{\breve{\nu}_t,\theta_2\breve{s}_{2t}\},
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&\frac{\left((r_1-c_1)-(r_2-c_2)\right)A\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}-\frac{\theta_2p_rA\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\\
&+(r_2-c_2)\left(\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A}{m_2}+\frac{bq_2(\alpha-1-A)}{m_2^2}\right)+(\alpha-1-AE)\right)\\
&+\theta_2p_r\left(\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\bar{a}A}{m_2}+\frac{bq_2(\alpha-1-A)}{m_2^2}\right)+(\alpha-AE)\right)>0.
\end{aligned}
$$

The above inequality holds as we assume

$$
\begin{aligned}
\alpha\;>\;&1+A+\frac{\frac{A\breve{D}^b_{1(\kappa+1)}}{m_2}+\frac{\theta_2p_r+(r_2-c_2)-(r_1-c_1)}{\theta_2p_r+r_2-c_2}\left(\frac{A\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right)}{1+AC-ABC}\\
&-\frac{\frac{\theta_2p_r}{\theta_2p_r+r_2-c_2}+ABC\left(1+A\right)}{1+AC-ABC}.
\end{aligned}
$$

(c) Suppose that the conditions in part (a) or (b) hold and T is sufficiently large. Thus the partial-fulfillment policy is optimal. Also, suppose that the partial-fulfillment policy is initiated in period $\kappa'>\kappa$ at optimality. We use the tilde $\tilde{}$ to denote the variables of this sales plan. Note that $\widetilde{D}^n_{2(T+1)}\cong\breve{D}^n_{2(T+1)}\cong m_1+m_2$, and $\widetilde{D}^n_{2t}=\breve{D}^n_{2t}$, $\forall t\leq\kappa'+1$. Hence $\sum_{t=\kappa'+1}^{T}\widetilde{d}^n_{2t}=\widetilde{D}^n_{2(T+1)}-\widetilde{D}^n_{2(\kappa'+1)}\cong\breve{D}^n_{2(T+1)}-\breve{D}^n_{2(\kappa'+1)}=\sum_{t=\kappa'+1}^{T}\breve{d}^n_{2t}$. The total amount of recycled material used in manufacturing equals $\sum_{t=1}^{\kappa}\breve{\nu}_t+\sum_{t=\kappa+1}^{T}\theta_2\breve{d}^n_{2t}$ under the immediate-fulfillment policy, while it cannot exceed $\sum_{t=1}^{\kappa}\widetilde{\nu}_t+\sum_{t=\kappa+1}^{T}\theta_2\widetilde{d}_{2t}^n=\sum_{t=1}^{\kappa}\breve{\nu}_t+\sum_{t=\kappa+1}^{\kappa'}\theta_2\breve{d}_{2t}^n+\sum_{t=\kappa'+1}^{T}\theta_2\widetilde{d}_{2t}^n$ under the partial-fulfillment policy. Since $\sum_{t=\kappa'+1}^{T}\theta_2\widetilde{d}_{2t}^n=\sum_{t=\kappa'+1}^{T}\theta_2\breve{d}_{2t}^n$, the total amount of recycled material used in manufacturing cannot be improved with the partial-fulfillment policy. $\square$

Theorem A.2. *Suppose that the consumers' timing of end-of-life product returns coincides with their timing of new-generation product purchases (as in Chapter 3.2).*

(a) Suppose that $r_1-c_1\geq r_2-c_2$ *and* $\gamma\theta_1\geq\theta_2$*. For optimality of partial demand fulfillment in some period over the* T*-period selling horizon, it is sufficient that* $\exists\kappa\in\{\tau,..,T-2\}$ *s.t.* $\breve{\nu}_t<\theta_2\left(\breve{d}_{2t}^b+\frac{\breve{d}_{1t}^b\breve{D}_{2t}^b}{m_2}\right)$ *for* $\tau\leq t\leq\kappa$*,* $\breve{\nu}_t>\theta_2\left(\breve{d}_{2t}^b+\frac{\breve{d}_{1t}^b\breve{D}_{2t}^b}{m_2}\right)$ *for* $t>\kappa$*,* $\alpha>\frac{(1+A)(1-2B)}{1-B}$*, and*

$$\alpha \;>\; 1+A+\frac{\frac{A\breve{D}_{1(\kappa+1)}^b}{m_2}-\frac{(r_1-c_1)-(r_2-c_2)}{\theta_2p_r+r_2-c_2}\left(\frac{A\breve{d}_{1(\kappa+2)}^b}{m_2}\right)}{1+AC-ABC}$$
$$-\frac{\frac{\theta_2p_r}{\theta_2p_r+r_2-c_2}\left(1-\frac{A\left(\breve{D}_{1(T+1)}^b+\breve{D}_{1(\kappa+2)}^b\right)}{m_2}\right)-ABC\,(1+A)}{1+AC-ABC},$$

where $A=q_2-\frac{q_2\breve{D}_{2(\kappa+1)}^b}{m_2}$*,* $B=p_2+\frac{q_2\breve{D}_{2(\kappa+1)}^b}{m_2}$*, and* $C=1+\frac{\breve{D}_{1(\kappa+2)}^b}{m_2}$*.*

(b) Suppose that $r_1-c_1\leq r_2-c_2$ *and* $\gamma\theta_1\geq\theta_2$*. For optimality of partial demand fulfillment in some period over the* T*-period selling horizon, it is sufficient that* $\exists\kappa\in\{\tau,..,T-2\}$ *s.t.* $\breve{\nu}_t<\theta_2\left(\breve{d}_{2t}^b+\frac{\breve{d}_{1t}^b\breve{D}_{2t}^b}{m_2}\right)$ *for* $\tau\leq t\leq\kappa$*,* $\breve{\nu}_t>\theta_2\left(\breve{d}_{2t}^b+\frac{\breve{d}_{1t}^b\breve{D}_{2t}^b}{m_2}\right)$ *for* $t>\kappa$*,* $\alpha>\frac{(1+A)(1-2B)}{1-B}$*, and*

$$\alpha \;>\; 1+A+\frac{\frac{A\breve{D}_{1(\kappa+1)}^b}{m_2}+\frac{\theta_2p_r+(r_2-c_2)}{\theta_2p_r+r_2-c_2}\left(\frac{A\left(\breve{D}_{1(T+1)}^b-\breve{D}_{1(\kappa+2)}^b\right)}{m_2}\right)}{1+AC-ABC}$$
$$-\frac{\frac{(r_1-c_1)}{\theta_2p_r+r_2-c_2}\left(\frac{A\left(\breve{D}_{1(T+1)}^b-\breve{D}_{1(\kappa+2)}^b\right)}{m_2}\right)+\frac{\theta_2p_r}{\theta_2p_r+r_2-c_2}+ABC\,(1+A)}{1+AC-ABC},$$

where $A=q_2-\frac{q_2\breve{D}_{2(\kappa+1)}^b}{m_2}$*,* $B=p_2+\frac{q_2\breve{D}_{2(\kappa+1)}^b}{m_2}$*, and* $C=1+\frac{\breve{D}_{1(\kappa+2)}^b}{m_2}$*.*

(c) Suppose that the conditions in part (a) or (b) hold and T *is sufficiently large. Then, the partial-fulfillment policy, if initiated after period* κ*, reduces the total*

amount of recycled material used in manufacturing of the new-generation product for the customers who have not purchased the early-generation product.

Proof of Theorem A.2. We consider sales plans (i) and (ii) in the proof of Theorem A.1 We again show that sales plan (i) is more profitable than sales plan (ii). We make the following observations:

(1) Periods $t < \kappa$: Recall from the proof of Theorem A.1 that $\widehat{d}^n_{1t} = \breve{d}^n_{1t}$ and $\widehat{s}_{2t} = \breve{s}_{2t}$.

(2) Period κ: Recall from the proof of Theorem A.1 that $\widehat{d}^n_{1\kappa} = \breve{d}^n_{1\kappa}$ and $\widehat{s}_{2\kappa} = \breve{s}_{2\kappa} - \epsilon$. Since $\breve{\nu}_\kappa < \theta_2 \left(\breve{d}^b_{2\kappa} + \frac{\breve{d}^b_{1\kappa}\breve{D}^b_{2\kappa}}{m_2} \right)$, $\exists \epsilon > 0$ s.t. a demand of size ϵ from the customers unique to the new-generation product cannot be met in period κ by using the recycled content.

(3) Period $\kappa + 1$: Recall from the proof of Theorem 1 that $\widehat{d}^n_{1(\kappa+1)} = \breve{d}^n_{1(\kappa+1)}$, $\widehat{s}_{2(\kappa+1)} = \breve{s}_{2(\kappa+1)} - AE\epsilon + \alpha\epsilon$, and $\breve{d}^b_{2(\kappa+1)} = \widehat{d}^b_{2(\kappa+1)} + A\epsilon$. Since $\breve{\nu}_{\kappa+1} > \theta_2 \left(\breve{d}^b_{2(\kappa+1)} + \frac{\breve{d}^b_{1(\kappa+1)}\breve{D}^b_{2(\kappa+1)}}{m_2} \right)$, $\exists \epsilon > 0$ s.t. $\widehat{\nu}_{\kappa+1} = \frac{(\gamma\theta_1 - \theta_2)\widehat{D}^b_{1\kappa}\widehat{d}^b_{2\kappa}}{m_2} = \frac{(\gamma\theta_1 - \theta_2)\breve{D}^b_{1\kappa}\breve{d}^b_{2\kappa}}{m_2} > \theta_2 \left(\breve{d}^b_{2(\kappa+1)} + \frac{\breve{d}^b_{1(\kappa+1)}\breve{D}^b_{2(\kappa+1)}}{m_2} \right) - \theta_2 A\epsilon + \theta_2\alpha\epsilon = \theta_2 \left(\widehat{d}^b_{2(\kappa+1)} + \frac{\widehat{d}^b_{1(\kappa+1)}\widehat{D}^b_{2(\kappa+1)}}{m_2} \right) + \theta_2\alpha\epsilon$.

(4) Period $\kappa+2$: Note that $\widehat{d}^b_{1t} = \breve{d}^b_{1t}$, $\forall t$. Recall from the proof of Theorem A.1 that $\widehat{d}^n_{1(\kappa+2)} = \breve{d}^n_{1(\kappa+2)} + \frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}$, $\breve{d}^b_{2(\kappa+2)} + A\epsilon \geq \widehat{d}^b_{2(\kappa+2)}$, $\widehat{d}^b_{2(\kappa+2)} \geq \breve{d}^b_{2(\kappa+2)}$, and $\widehat{s}_{2(\kappa+2)} = \breve{s}_{2(\kappa+2)} + \left(m_2 + \breve{D}^b_{1(\kappa+2)} \right) \left(\frac{\overline{a}A\epsilon}{m_2} + \frac{bq_2\epsilon(\alpha-1-A)}{m_2^2} + \frac{q_2 A\epsilon^2(\alpha-1-A)}{m_2^2} \right) - \frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}$. Since $\breve{d}^b_{2(\kappa+2)} + A\epsilon \geq \widehat{d}^b_{2(\kappa+2)}$ and $\breve{\nu}_{\kappa+2} > \theta_2 \left(\breve{d}^b_{2(\kappa+2)} + \frac{\breve{d}^b_{1(\kappa+2)}\breve{D}^b_{2(\kappa+2)}}{m_2} \right)$, $\exists \epsilon > 0$ s.t.

$$
\begin{aligned}
\widehat{\nu}_{\kappa+2} &= \widehat{\nu}_{\kappa+1} - \theta_2\widehat{d}^b_{2(\kappa+1)} - \frac{\theta_2\widehat{d}^b_{1(\kappa+1)}\widehat{D}^b_{2(\kappa+1)}}{m_2} - \theta_2\alpha\epsilon \\
&\quad + \frac{(\gamma\theta_1 - \theta_2)\,\widehat{D}^b_{1(\kappa+1)}\widehat{d}^b_{2(\kappa+1)}}{m_2} \\
&= \breve{\nu}_{\kappa+1} - \theta_2\breve{d}^b_{2(\kappa+1)} - \frac{\theta_2\breve{d}^b_{1(\kappa+1)}\breve{D}^b_{2(\kappa+1)}}{m_2} + \theta_2 A\epsilon - \theta_2\alpha\epsilon
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{(\gamma\theta_1-\theta_2)\,\breve{D}^b_{1(\kappa+1)}\breve{d}^b_{2(\kappa+1)}}{m_2}-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}\\
= \; & \breve{\nu}_{\kappa+2}+\theta_2 A\epsilon-\theta_2\alpha\epsilon-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}\\
> \; & \theta_2\left(\breve{d}^b_{2(\kappa+2)}+\frac{\breve{d}^b_{1(\kappa+2)}\breve{D}^b_{2(\kappa+2)}}{m_2}\right)+\theta_2 A\epsilon\\
= \; & \theta_2\left(\breve{d}^b_{2(\kappa+2)}+A\epsilon+\frac{\breve{d}^b_{1(\kappa+2)}\widehat{D}^b_{2(\kappa+2)}}{m_2}+\frac{A\epsilon\breve{d}^b_{1(\kappa+1)}}{m_2}\right)\\
\geq \; & \theta_2\left(\widehat{d}^b_{2(\kappa+2)}+\frac{\widehat{d}^b_{1(\kappa+2)}\widehat{D}^b_{2(\kappa+2)}}{m_2}\right).
\end{aligned}
$$

(5) Periods $t>\kappa+2$: Recall from the proof of Theorem A.1 that $\widehat{D}^b_{2t}<\breve{D}^b_{2t}$, $\widehat{d}^n_{1t}\geq\breve{d}^n_{1t}$, $\widehat{d}^b_{2t}\geq\breve{d}^b_{2t}$, $\widehat{s}_{2t}\geq\breve{s}_{2t}-\left(\widehat{d}^n_{1t}-\breve{d}^n_{1t}\right)$, and $\widehat{d}^n_{1t}-\breve{d}^n_{1t}\leq\frac{A\epsilon\breve{d}^b_{1t}}{m_2}$, $\forall t\geq\kappa+3$. Note that $\breve{D}^b_{2(\kappa+1)}=\widehat{D}^b_{2(\kappa+1)}$ and $\widehat{d}^b_{2(\kappa+1)}=\breve{d}^b_{2(\kappa+1)}-A\epsilon$. Hence $\widehat{D}^b_{2(\kappa+4)}-\breve{D}^b_{2(\kappa+4)}=\widehat{D}^b_{2(\kappa+1)}+\breve{d}^b_{2(\kappa+1)}-A\epsilon+\widehat{d}^b_{2(\kappa+2)}+\widehat{d}^b_{2(\kappa+3)}-\breve{D}^b_{2(\kappa+1)}-\breve{d}^b_{2(\kappa+1)}-\breve{d}^b_{2(\kappa+2)}-\breve{d}^b_{2(\kappa+3)}=\widehat{d}^b_{2(\kappa+2)}+\widehat{d}^b_{2(\kappa+3)}-A\epsilon-\breve{d}^b_{2(\kappa+2)}-\breve{d}^b_{2(\kappa+3)}\leq 0$ and $\breve{d}^b_{2(\kappa+2)}+\breve{d}^b_{2(\kappa+3)}+A\epsilon\geq\widehat{d}^b_{2(\kappa+2)}+\widehat{d}^b_{2(\kappa+3)}$. Proceeding similarly, it can be shown that $\breve{d}^b_{2(\kappa+2)}+\cdots+\breve{d}^b_{2t}+A\epsilon\geq\widehat{d}^b_{2(\kappa+2)}+\cdots+\widehat{d}^b_{2t}$, $\forall t\geq\kappa+3$. Since $\breve{D}^b_{2(\kappa+2)}\geq\widehat{D}^b_{2(\kappa+2)}$, $\breve{D}^b_{2(\kappa+3)}\geq\widehat{D}^b_{2(\kappa+3)}$, $\widehat{d}^b_{2(\kappa+2)}\geq\breve{d}^b_{2(\kappa+2)}$, $\breve{d}^b_{2(\kappa+2)}+\breve{d}^b_{2(\kappa+3)}+A\epsilon\geq\widehat{d}^b_{2(\kappa+2)}+\widehat{d}^b_{2(\kappa+3)}$, and $\breve{\nu}_{\kappa+3}>\theta_2\left(\breve{d}^b_{2(\kappa+3)}+\frac{\breve{d}^b_{1(\kappa+3)}\breve{D}^b_{2(\kappa+3)}}{m_2}\right)$, $\exists\epsilon>0$ s.t.

$$
\begin{aligned}
\widehat{\nu}_{\kappa+3} = \; & \widehat{\nu}_{\kappa+2}-\theta_2\widehat{d}^b_{2(\kappa+2)}-\frac{\theta_2\widehat{d}^b_{1(\kappa+2)}\widehat{D}^b_{2(\kappa+2)}}{m_2}+\frac{(\gamma\theta_1-\theta_2)\,\widehat{D}^b_{1(\kappa+2)}\widehat{d}^b_{2(\kappa+2)}}{m_2}\\
= \; & \breve{\nu}_{\kappa+2}-\theta_2\alpha\epsilon-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}+\theta_2 A\epsilon-\theta_2\widehat{d}^b_{2(\kappa+2)}\\
& -\frac{\theta_2\breve{d}^b_{1(\kappa+2)}\widehat{D}^b_{2(\kappa+2)}}{m_2}+\frac{(\gamma\theta_1-\theta_2)\,\breve{D}^b_{1(\kappa+2)}\widehat{d}^b_{2(\kappa+2)}}{m_2}\\
= \; & \breve{\nu}_{\kappa+3}+\theta_2\breve{d}^b_{2(\kappa+2)}+\frac{\theta_2\breve{d}^b_{1(\kappa+2)}\breve{D}^b_{2(\kappa+2)}}{m_2}-\frac{(\gamma\theta_1-\theta_2)\,\breve{D}^b_{1(\kappa+2)}\breve{d}^b_{2(\kappa+2)}}{m_2}\\
& -\theta_2\alpha\epsilon-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}+\theta_2 A\epsilon-\theta_2\widehat{d}^b_{2(\kappa+2)}\\
& -\frac{\theta_2\breve{d}^b_{1(\kappa+2)}\widehat{D}^b_{2(\kappa+2)}}{m_2}+\frac{(\gamma\theta_1-\theta_2)\,\breve{D}^b_{1(\kappa+2)}\widehat{d}^b_{2(\kappa+2)}}{m_2}\\
= \; & \breve{\nu}_{\kappa+3}-\theta_2\alpha\epsilon-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}+\theta_2 A\epsilon+\theta_2\left(\breve{d}^b_{2(\kappa+2)}-\widehat{d}^b_{2(\kappa+2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta_2 \breve{d}^b_{1(\kappa+2)}\left(\breve{D}^b_{2(\kappa+2)}-\widehat{D}^b_{2(\kappa+2)}\right)}{m_2}+\frac{\left(\gamma\theta_1-\theta_2\right)\breve{D}^b_{1(\kappa+2)}\left(\widehat{d}^b_{2(\kappa+2)}-\breve{d}^b_{2(\kappa+2)}\right)}{m_2}\\
\geq\ & \breve{\nu}_{\kappa+3}-\theta_2\alpha\epsilon-\frac{\left(\gamma\theta_1-\theta_2\right)A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}+\theta_2 A\epsilon+\theta_2\left(\breve{d}^b_{2(\kappa+2)}-\widehat{d}^b_{2(\kappa+2)}\right)\\
>\ & \theta_2\left(\breve{d}^b_{2(\kappa+3)}+\frac{\breve{d}^b_{1(\kappa+3)}\breve{D}^b_{2(\kappa+3)}}{m_2}\right)+\theta_2 A\epsilon+\theta_2\left(\breve{d}^b_{2(\kappa+2)}-\widehat{d}^b_{2(\kappa+2)}\right)\\
=\ & \theta_2\left(\breve{d}^b_{2(\kappa+3)}+\breve{d}^b_{2(\kappa+2)}-\widehat{d}^b_{2(\kappa+2)}+A\epsilon+\frac{\breve{d}^b_{1(\kappa+3)}\breve{D}^b_{2(\kappa+3)}}{m_2}\right)\\
\geq\ & \theta_2\left(\widehat{d}^b_{2(\kappa+3)}+\frac{\widehat{d}^b_{1(\kappa+3)}\widehat{D}^b_{2(\kappa+3)}}{m_2}\right).
\end{aligned}
$$

Proceeding similarly, for all $t \geq \kappa+3$, it can be shown that $\widehat{\nu}_t > \theta_2\left(\widehat{d}^b_{2t}+\frac{\widehat{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\right)$ and

$$
\begin{aligned}
\widehat{\nu}_t\ =\ & \breve{\nu}_t-\theta_2\alpha\epsilon-\frac{\left(\gamma\theta_1-\theta_2\right)A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}+\theta_2 A\epsilon+\theta_2\sum_{i=1}^{t-\kappa-2}\frac{\widehat{d}^b_{1(t-i)}\left(\breve{D}^b_{2(t-i)}-\widehat{D}^b_{2(t-i)}\right)}{m_2}\\
&+\theta_2\sum_{i=1}^{t-\kappa-2}\left(\breve{d}^b_{2(t-i)}-\widehat{d}^b_{2(t-i)}\right)+\left(\gamma\theta_1-\theta_2\right)\sum_{i=1}^{t-\kappa-2}\frac{\widehat{D}^b_{1(t-i)}\left(\widehat{d}^b_{2(t-i)}-\breve{d}^b_{2(t-i)}\right)}{m_2}.
\end{aligned}
$$

In order to prove part (a) of Theorem A.2, suppose that $r_1-c_1 \geq r_2-c_2$ and $\gamma\theta_1 \geq \theta_2$. We will show that sales plan (i) is more profitable than sales plan (ii). Combining our observations in (1) – (5):

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}\\
&+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right)\right\}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&\quad+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\widehat{s}_{2(\kappa+1)}-\frac{\theta_2\breve{D}^b_{1(\kappa+1)}\widehat{d}^b_{2(\kappa+1)}}{m_2}\right]\\
&\quad+p_r\left[\widehat{s}_{2(\kappa+2)}-\frac{\theta_2\breve{D}^b_{1(\kappa+2)}\widehat{d}^b_{2(\kappa+2)}}{m_2}+\theta_2\sum_{t=\kappa+3}^{T}\left(\widehat{s}_{2t}-\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right)+\theta_2\sum_{t=\kappa+3}^{T}\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+p_r\left[\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}+\frac{\theta_2\breve{D}^b_{1(\kappa+1)}\widehat{d}^b_{2(\kappa+1)}}{m_2}+\frac{\theta_2\breve{D}^b_{1(\kappa+2)}\widehat{d}^b_{2(\kappa+2)}}{m_2}\right]\\
=&\,(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
\geq&\,(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}^n_{1t}+\widehat{d}^n_{1\kappa}+\widehat{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^n_{1t}\right]\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
=&\,(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}-\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\right]\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)+\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
=&\,(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\widehat{D}^b_{2t}}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
\geq&\,(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\widehat{d}^n_{1(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}^n_{1t}-\breve{d}^n_{1t})\right)\right]\\
\geq{}&(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\breve{d}^n_{1(\kappa+2)}+\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)-\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right]\\
&+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\theta_2\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}^b_{1t}}{m_2}\right]\\
&+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
={}&(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}^n_{1t}+\breve{d}^n_{1\kappa}+\breve{d}^n_{1(\kappa+1)}+\breve{d}^n_{1(\kappa+2)}+\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}+\sum_{t=\kappa+3}^{T}\breve{d}^b_{1t}\right]\\
&-[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+3}^{T}\frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}+\theta_2p_r\left[\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\frac{A\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right]\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{d}^b_{2t}\left(1+\frac{\breve{D}^b_{1t}}{m_2}\right)-\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right]\\
&+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon\right]\\
&+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
={}&(r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)\right\}\\
&+\theta_2p_r\sum_{t=\tau}^{T}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}+[(r_1-c_1)-(r_2-c_2)]\left(\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right)+(r_2-c_2)\left(-\epsilon-AE\epsilon+\alpha\epsilon\right)\\
&+(r_2-c_2)\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&+\theta_2p_r\left(\alpha\epsilon-AE\epsilon\right)+\theta_2p_r\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&-\frac{\theta_2p_rA\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}.
\end{aligned}
$$

In order to show that

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}\\
&+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\right)\right\}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\\
&\geq(r_1-c_1)\sum_{t=1}^{T}\breve{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}\\
&+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}_{1t}^{b}\breve{d}_{2t}^{b}}{m_2}\right)\right\}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\breve{D}_{1t}^{b}\breve{d}_{2t}^{b}}{m_2},
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&\frac{(r_1-c_1)\,A\breve{d}_{1(\kappa+2)}^{b}}{m_2}+(r_2-c_2)\left(\left(m_2+\breve{D}_{1(\kappa+2)}^{b}\right)\left(\frac{\overline{a}A}{m_2}+\frac{\underline{b}q_2(\alpha-1-A)}{m_2^2}\right)\right)\\
&+(r_2-c_2)\left((\alpha-1-AE)-\frac{A\breve{d}_{1(\kappa+2)}^{b}}{m_2}\right)-\frac{\theta_2 p_r A\left(\breve{D}_{1(T+1)}^{b}-\breve{D}_{1(\kappa+2)}^{b}\right)}{m_2}\\
&+\theta_2 p_r\left(\left(m_2+\breve{D}_{1(\kappa+2)}^{b}\right)\left(\frac{\overline{a}A}{m_2}+\frac{\underline{b}q_2(\alpha-1-A)}{m_2^2}\right)+(\alpha-AE)\right)>0.
\end{aligned}
$$

The above inequality holds as we assume

$$
\begin{aligned}
\alpha \;>\; & 1+A+\frac{\frac{A\breve{D}_{1(\kappa+1)}^{b}}{m_2}-\frac{(r_1-c_1)-(r_2-c_2)}{\theta_2 p_r+r_2-c_2}\left(\frac{A\breve{d}_{1(\kappa+2)}^{b}}{m_2}\right)}{1+AC-ABC}\\
&-\frac{\frac{\theta_2 p_r}{\theta_2 p_r+r_2-c_2}\left(1-\frac{A\left(\breve{D}_{1(T+1)}^{b}-\breve{D}_{1(\kappa+2)}^{b}\right)}{m_2}\right)+ABC\,(1+A)}{1+AC-ABC}.
\end{aligned}
$$

In order to prove part (b) of Theorem A.2, suppose that $r_1-c_1\leq r_2-c_2$ and $\gamma\theta_1\geq\theta_2$. We will show that sales plan (i) is more profitable than sales plan (ii). With similar arguments to those used in the proof of part (a):

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\left[\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\right)\right\}+\theta_2\sum_{t=\tau}^{T}\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\right]\\
&=(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\widehat{d}_{1t}^{n}+\widehat{d}_{1\kappa}^{n}+\widehat{d}_{1(\kappa+1)}^{n}+\widehat{d}_{1(\kappa+2)}^{n}+\sum_{t=\kappa+3}^{T}\widehat{d}_{1t}^{n}\right]\\
&\quad+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+p_r\left[\sum_{t=\tau}^{\kappa}\widehat{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}_{1t}^b\widehat{d}_{2t}^b}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\widehat{s}_{2t}\right]\\
\geq\ &(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}_{1t}^n+\breve{d}_{1\kappa}^n+\breve{d}_{1(\kappa+1)}^n+\widehat{d}_{1(\kappa+2)}^n+\sum_{t=\kappa+3}^{T}\left(\breve{d}_{1t}^n+(\widehat{d}_{1t}^n-\breve{d}_{1t}^n)\right)\right]\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\widehat{s}_{2(\kappa+2)}+\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}_{1t}^n-\breve{d}_{1t}^n)\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}_{1t}^b\breve{d}_{2t}^b}{m_2}+\theta_2\widehat{s}_{2(\kappa+1)}+\theta_2\widehat{s}_{2(\kappa+2)}+\theta_2\sum_{t=\kappa+3}^{T}\left(\breve{s}_{2t}-(\widehat{d}_{1t}^n-\breve{d}_{1t}^n)\right)\right]\\
\geq\ &(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}_{1t}^n+\breve{d}_{1\kappa}^n+\breve{d}_{1(\kappa+1)}^n+\breve{d}_{1(\kappa+2)}^n+\sum_{t=\kappa+3}^{T}\breve{d}_{1t}^n\right]\\
&+[(r_1-c_1)-(r_2-c_2)]\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}_{1t}^b}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]-\theta_2p_r\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}_{1t}^b}{m_2}\\
&+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}_{1(\kappa+2)}^b\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}_{1t}^b\breve{d}_{2t}^b}{m_2}+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\theta_2\alpha\epsilon+\theta_2\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]\\
&+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}_{1(\kappa+2)}^b\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
=\ &(r_1-c_1)\left[\sum_{t=1}^{\kappa-1}\breve{d}_{1t}^n+\breve{d}_{1\kappa}^n+\breve{d}_{1(\kappa+1)}^n+\breve{d}_{1(\kappa+2)}^n+\sum_{t=\kappa+3}^{T}\breve{d}_{1t}^n\right]\\
&+[(r_1-c_1)-(r_2-c_2)]\frac{A\epsilon\left(\breve{D}_{1(T+1)}^b-\breve{D}_{1(\kappa+2)}^b\right)}{m_2}\\
&+(r_2-c_2)\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-AE\epsilon+\alpha\epsilon+\sum_{t=\kappa+3}^{T}\breve{s}_{2t}\right]\\
&+(r_2-c_2)\left[\breve{s}_{2(\kappa+2)}+\left(m_2+\breve{D}_{1(\kappa+2)}^b\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]\\
&+p_r\left[\sum_{t=\tau}^{\kappa}\breve{\nu}_t+\theta_2\sum_{t=\tau}^{\kappa}\frac{\breve{D}_{1t}^b\breve{d}_{2t}^b}{m_2}+\theta_2\breve{s}_{2(\kappa+1)}-\theta_2AE\epsilon+\alpha\epsilon\right]\\
&+\theta_2p_r\left[\sum_{t=\kappa+3}^{T}\breve{s}_{2t}-\frac{A\epsilon\left(\breve{D}_{1(T+1)}^b-\breve{D}_{1(\kappa+2)}^b\right)}{m_2}\right]\\
&+p_r\left[\theta_2\breve{s}_{2(\kappa+2)}+\theta_2\left(m_2+\breve{D}_{1(\kappa+2)}^b\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (r_1 - c_1) \sum_{t=1}^{T} \breve{d}_{1t}^n + (r_2 - c_2) \sum_{t=\tau}^{T} \breve{s}_{2t} + p_r \sum_{t=\tau}^{T} \min \left\{ \breve{\nu}_t, \theta_2 \left(\breve{s}_{2t} - \frac{\breve{D}_{1t}^b \breve{d}_{2t}^b}{m_2} \right) \right\} \\
&\quad + \theta_2 p_r \sum_{t=\tau}^{T} \frac{\breve{D}_{1t}^b \breve{d}_{2t}^b}{m_2} + \theta_2 p_r \left(\alpha\epsilon - AE\epsilon \right) \\
&\quad + (r_2 - c_2) \left(m_2 + \breve{D}_{1(\kappa+2)}^b \right) \left(\frac{\overline{a} A \epsilon}{m_2} + \frac{\underline{b} q_2 \epsilon (\alpha - 1 - A)}{m_2^2} + \frac{q_2 A \epsilon^2 (\alpha - 1 - A)}{m_2^2} \right) \\
&\quad + \theta_2 p_r \left(m_2 + \breve{D}_{1(\kappa+2)}^b \right) \left(\frac{\overline{a} A \epsilon}{m_2} + \frac{\underline{b} q_2 \epsilon (\alpha - 1 - A)}{m_2^2} + \frac{q_2 A \epsilon^2 (\alpha - 1 - A)}{m_2^2} \right) \\
&\quad + \frac{\left((r_1 - c_1) - (r_2 - c_2) - \theta_2 p_r \right) A \epsilon \left(\breve{D}_{1(T+1)}^b - \breve{D}_{1(\kappa+2)}^b \right)}{m_2} + (r_2 - c_2) \left(-\epsilon - AE\epsilon + \alpha\epsilon \right).
\end{aligned}
$$

In order to show that

$$
\begin{aligned}
&(r_1 - c_1) \sum_{t=1}^{T} \widehat{d}_{1t}^n + (r_2 - c_2) \sum_{t=\tau}^{T} \widehat{s}_{2t} \\
&+ p_r \left[\sum_{t=\tau}^{T} \min \left\{ \widehat{\nu}_t, \theta_2 \left(\widehat{s}_{2t} - \frac{\widehat{D}_{1t}^b \widehat{d}_{2t}^b}{m_2} \right) \right\} + \theta_2 \sum_{t=\tau}^{T} \frac{\widehat{D}_{1t}^b \widehat{d}_{2t}^b}{m_2} \right] \\
&\geq (r_1 - c_1) \sum_{t=1}^{T} \breve{d}_{1t}^n + (r_2 - c_2) \sum_{t=\tau}^{T} \breve{s}_{2t} \\
&+ p_r \left[\sum_{t=\tau}^{T} \min \left\{ \breve{\nu}_t, \theta_2 \left(\breve{s}_{2t} - \frac{\breve{D}_{1t}^b \breve{d}_{2t}^b}{m_2} \right) \right\} + \theta_2 \sum_{t=\tau}^{T} \frac{\breve{D}_{1t}^b \breve{d}_{2t}^b}{m_2} \right],
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&\frac{\left((r_1 - c_1) - (r_2 - c_2) \right) A \left(\breve{D}_{1(T+1)}^b - \breve{D}_{1(\kappa+2)}^b \right)}{m_2} - \frac{\theta_2 p_r A \left(\breve{D}_{1(T+1)}^b - \breve{D}_{1(\kappa+2)}^b \right)}{m_2} \\
&+ (r_2 - c_2) \left(\left(m_2 + \breve{D}_{1(\kappa+2)}^b \right) \left(\frac{\overline{a} A}{m_2} + \frac{\underline{b} q_2 (\alpha - 1 - A)}{m_2^2} \right) + (\alpha - 1 - AE) \right) \\
&+ \theta_2 p_r \left(\left(m_2 + \breve{D}_{1(\kappa+2)}^b \right) \left(\frac{\overline{a} A}{m_2} + \frac{\underline{b} q_2 (\alpha - 1 - A)}{m_2^2} \right) + (\alpha - AE) \right) > 0.
\end{aligned}
$$

The above inequality holds as we assume

$$
\begin{aligned}
\alpha \quad > \quad & 1 + A + \frac{\frac{A \breve{D}_{1(\kappa+1)}^b}{m_2} + \frac{\theta_2 p_r + (r_2 - c_2) - (r_1 - c_1)}{\theta_2 p_r + r_2 - c_2} \left(\frac{A \left(\breve{D}_{1(T+1)}^b - \breve{D}_{1(\kappa+2)}^b \right)}{m_2} \right)}{1 + AC - ABC} \\
& - \frac{\frac{\theta_2 p_r}{\theta_2 p_r + r_2 - c_2} + ABC(1 + A)}{1 + AC - ABC}.
\end{aligned}
$$

(c) Suppose that the conditions in part (a) or (b) hold and T is sufficiently large. Thus the partial-fulfillment policy is optimal. Also, suppose that the partial-fulfillment policy is initiated in period $\kappa' > \kappa$ at optimality. We use the tilde $\tilde{}$ to denote the variables of this sales plan. Note that

$\widetilde{D}^n_{2(T+1)} \cong \breve{D}^n_{2(T+1)} \cong m_1 + m_2$, and $\widetilde{D}^n_{2t} = \breve{D}^n_{2t}$, $\forall t \leq \kappa' + 1$. Proposition 1 implies that $\sum_{t=\kappa'+1}^{T} \widetilde{D}^b_{1t}\widetilde{d}^b_{2t} > \sum_{t=\kappa'+1}^{T} \breve{D}^b_{1t}\breve{d}^b_{2t}$. Hence $\sum_{t=\kappa'+1}^{T} \left(\widetilde{d}^n_{2t} - \frac{\widetilde{D}^b_{1t}\widetilde{d}^b_{2t}}{m_2}\right) < \sum_{t=\kappa'+1}^{T} \left(\breve{d}^n_{2t} - \frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)$. For the customers who have not bought the early-generation product, the total amount of recycled material used in manufacturing equals $\sum_{t=1}^{\kappa} \breve{\nu}_t + \sum_{t=\kappa+1}^{T} \theta_2 \left(\breve{d}^n_{2t} - \frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)$ under the immediate-fulfillment policy, while it cannot exceed $\sum_{t=1}^{\kappa} \widetilde{\nu}_t + \sum_{t=\kappa+1}^{T} \left(\widetilde{d}^n_{2t} - \frac{\widetilde{D}^b_{1t}\widetilde{d}^b_{2t}}{m_2}\right) = \sum_{t=1}^{\kappa} \breve{\nu}_t + \sum_{t=\kappa+1}^{\kappa'} \theta_2 \left(\breve{d}^n_{2t} - \frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right) + \sum_{t=\kappa'+1}^{T} \theta_2 \left(\widetilde{d}^n_{2t} - \frac{\widetilde{D}^b_{1t}\widetilde{d}^b_{2t}}{m_2}\right)$ under the partial-fulfillment policy. Since $\sum_{t=\kappa'+1}^{T} \theta_2 \left(\widetilde{d}^n_{2t} - \frac{\widetilde{D}^b_{1t}\widetilde{d}^b_{2t}}{m_2}\right) < \sum_{t=\kappa'+1}^{T} \theta_2 \left(\breve{d}^n_{2t} - \frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)$, the total amount of recycled material used in manufacturing for the customers who have not bought the early-generation product is lower under the partial-fulfillment policy. $\square$

Theorem A.3. *Suppose that the consumers' timing of end-of-life product returns coincides with their timing of new-generation product purchases and the recyclable materials extracted from the early-generation product returns are available for salvaging at the end of the selling horizon.*

(a) Suppose that $r_1 - c_1 \geq r_2 - c_2$ and $\gamma\theta_1 \geq \theta_2$. For optimality of partial demand fulfillment in some period over the T-period selling horizon, it is sufficient that $\exists \kappa \in \{\tau, .., T-2\}$ s.t. $\breve{\nu}_t < \theta_2 \left(\breve{d}^b_{2t} + \frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\right)$ for $\tau \leq t \leq \kappa$, $\breve{\nu}_t > \theta_2 \left(\breve{d}^b_{2t} + \frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\right)$ for $t > \kappa$,

$$\alpha > \frac{(\theta_2 p_r + r_2 - c_2)\left(\frac{A\breve{D}^b_{1(\kappa+1)}}{m_2} + A + AC(1+A) - 2ABC(1+A) + r_2 - c_2\right)}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s} + \frac{\frac{\theta_2 p_r A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2} + \frac{(\gamma\theta_1 - \theta_2)p_s A\breve{D}^b_{1(\kappa+1)}}{m_2} - \frac{((r_1-c_1)-(r_2-c_2))A\breve{d}^b_{1(\kappa+2)}}{m_2}}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s},$$

and $\alpha > \frac{(1+A)(1-2B)}{1-B}$, where $A = q_2 - \frac{q_2\breve{D}^b_{2(\kappa+1)}}{m_2}$, $B = p_2 + \frac{q_2\breve{D}^b_{2(\kappa+1)}}{m_2}$, and $C = 1 + \frac{\breve{D}^b_{1(\kappa+2)}}{m_2}$.

(b) Suppose that $r_1 - c_1 \leq r_2 - c_2$ and $\gamma\theta_1 \geq \theta_2$. For optimality of partial demand fulfillment in some period over the T-period selling horizon, it is sufficient that $\exists \kappa \in \{\tau, .., T-2\}$ s.t. $\breve{\nu}_t < \theta_2 \left(\breve{d}^b_{2t} + \frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\right)$ for $\tau \leq t \leq \kappa$, $\breve{\nu}_t > \theta_2 \left(\breve{d}^b_{2t} + \frac{\breve{d}^b_{1t}\breve{D}^b_{2t}}{m_2}\right)$

for $t > \kappa$,

$$\alpha > \frac{(\theta_2 p_r + r_2 - c_2)\left(\frac{A\left(\breve{D}^b_{1(T+1)} + \breve{D}^b_{1(\kappa+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2} - 2ABC(1+A)\right)}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s}$$
$$+ \frac{A + AC(1+A) + r_2 - c_2 + \frac{(\gamma\theta_1 - \theta_2) p_s A \breve{D}^b_{1(\kappa+1)}}{m_2} - \frac{(r_1 - c_1) A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2}}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s},$$

and $\alpha > \frac{(1+A)(1-2B)}{1-B}$, *where* $A = q_2 - \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$, $B = p_2 + \frac{q_2 \breve{D}^b_{2(\kappa+1)}}{m_2}$, *and* $C = 1 + \frac{\breve{D}^b_{1(\kappa+2)}}{m_2}$.

Proof of Theorem A.3. We consider sales plans (i) and (ii) in the proof of Theorem A.1. We again show that sales plan (i) is more profitable than sales plan (ii). Recall from the proof of Theorem A.2 that

$$\begin{aligned}
\widehat{\nu}_t &= \breve{\nu}_t - \theta_2 \alpha\epsilon - \frac{(\gamma\theta_1 - \theta_2) A\epsilon \breve{D}^b_{1(\kappa+1)}}{m_2} + \theta_2 A\epsilon + \theta_2 \sum_{i=1}^{t-\kappa-2}\left(\breve{d}^b_{2(t-i)} - \widehat{d}^b_{2(t-i)}\right) \\
&\quad + \theta_2 \sum_{i=1}^{t-\kappa-2} \frac{\widehat{d}^b_{1(t-i)}\left(\breve{D}^b_{2(t-i)} - \widehat{D}^b_{2(t-i)}\right)}{m_2} + (\gamma\theta_1 - \theta_2) \sum_{i=1}^{t-\kappa-2} \frac{\widehat{D}^b_{1(t-i)}\left(\widehat{d}^b_{2(t-i)} - \breve{d}^b_{2(t-i)}\right)}{m_2}
\end{aligned}$$

for $t \geq \kappa + 2$. Recall also from the proof of Theorem A.1 that $\widehat{d}^b_{2t} \geq \breve{d}^b_{2t}$ and $\breve{D}^b_{2t} \geq \widehat{D}^b_{2t}$, $\forall t \geq \kappa + 2$, and from the proof of Theorem A.2 that $\breve{d}^b_{2(\kappa+2)} + \cdots + \breve{d}^b_{2t} + A\epsilon \geq \widehat{d}^b_{2(\kappa+2)} + \cdots + \widehat{d}^b_{2t}$, $\forall t \geq \kappa + 3$. Hence:

$$\begin{aligned}
& p_s\left[\widehat{\nu}_T - \theta_2\left(\widehat{s}_{2T} - \frac{\widehat{D}^b_{1T}\widehat{d}^b_{2T}}{m_2}\right)\right] \\
&= p_s\left[\breve{\nu}_T - \theta_2\left(\breve{s}_{2T} - \frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right) - \theta_2\alpha\epsilon - \frac{(\gamma\theta_1 - \theta_2) A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2} + \theta_2 A\epsilon\right] \\
&\quad + p_s\left[\theta_2 \sum_{t=1}^{T-\kappa-2}\left(\breve{d}^b_{2(T-t)} - \widehat{d}^b_{2(T-t)}\right) + \theta_2 \sum_{t=1}^{T-\kappa-2} \frac{\widehat{d}^b_{1(T-t)}\left(\breve{D}^b_{2(T-t)} - \widehat{D}^b_{2(T-t)}\right)}{m_2}\right] \\
&\quad + p_s\left[(\gamma\theta_1 - \theta_2) \sum_{t=1}^{T-\kappa-2} \frac{\widehat{D}^b_{1(T-t)}\left(\widehat{d}^b_{2(T-t)} - \breve{d}^b_{2(T-t)}\right)}{m_2}\right] \\
&\quad + p_s\left[\theta_2\left(\breve{s}_{2T} - \frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right) - \theta_2\left(\widehat{s}_{2T} - \frac{\widehat{D}^b_{1T}\widehat{d}^b_{2T}}{m_2}\right)\right] \\
&= p_s\left[\breve{\nu}_T - \theta_2\left(\breve{s}_{2T} - \frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right) - \theta_2\alpha\epsilon - \frac{(\gamma\theta_1 - \theta_2) A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}\right]
\end{aligned}$$

$$
\begin{aligned}
& +p_s\left[\theta_2\left(\sum_{t=0}^{T-\kappa-2}\left(\breve{d}^b_{2(T-t)}-\widehat{d}^b_{2(T-t)}\right)+A\epsilon\right)\right] \\
& +p_s\left[\theta_2\sum_{t=0}^{T-\kappa-2}\frac{\widehat{d}^b_{1(T-t)}\left(\breve{D}^b_{2(T-t)}-\widehat{D}^b_{2(T-t)}\right)}{m_2}\right] \\
& +p_s\left[(\gamma\theta_1-\theta_2)\sum_{t=1}^{T-\kappa-2}\frac{\widehat{D}^b_{1(T-t)}\left(\widehat{d}^b_{2(T-t)}-\breve{d}^b_{2(T-t)}\right)}{m_2}\right] \\
\geq \quad & p_s\left[\breve{\nu}_T-\theta_2\left(\breve{s}_{2T}-\frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right)-\theta_2\alpha\epsilon-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}\right].
\end{aligned}
$$

In order to prove part (a) of Theorem A.3, recall from the proof of Theorem A.2 that

$$
\begin{aligned}
& (r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+p_r\left[\theta_2\sum_{t=\tau}^{T}\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}+\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right)\right\}\right] \\
& \geq (r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+\theta_2p_r\sum_{t=\tau}^{T}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2} \\
& \quad +p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)\right\}+(r_2-c_2)\left(-\epsilon-AE\epsilon+\alpha\epsilon\right)+\theta_2p_r\left(\alpha\epsilon-AE\epsilon\right) \\
& \quad +\left[(r_1-c_1)-(r_2-c_2)\right]\left(\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right)-\frac{\theta_2p_rA\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2} \\
& \quad +(r_2-c_2)\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right) \\
& \quad +\theta_2p_r\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right).
\end{aligned}
$$

The sum of the above two inequalities implies that

$$
\begin{aligned}
& (r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+\theta_2p_r\sum_{t=\tau}^{T}\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2} \\
& \quad +p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right)\right\}+p_s\left[\widehat{\nu}_T-\theta_2\left(\widehat{s}_{2T}-\frac{\widehat{D}^b_{1T}\widehat{d}^b_{2T}}{m_2}\right)\right] \\
& \geq (r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+\theta_2p_r\sum_{t=\tau}^{T}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2} \\
& \quad +p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+p_s\left[\breve{\nu}_T-\theta_2\left(\breve{s}_{2T}-\frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right)-\theta_2\alpha\epsilon-\frac{(\gamma\theta_1-\theta_2)\,A\epsilon\breve{D}^b_{1(\kappa+1)}}{m_2}\right]\\
&+[(r_1-c_1)-(r_2-c_2)]\left(\frac{A\epsilon\breve{d}^b_{1(\kappa+2)}}{m_2}\right)-\frac{\theta_2p_rA\epsilon\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\\
&+(r_2-c_2)\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&+(r_2-c_2)\left(-\epsilon-AE\epsilon+\alpha\epsilon\right)+\theta_2p_r\left(\alpha\epsilon-AE\epsilon\right)\\
&+\theta_2p_r\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right).
\end{aligned}
$$

In order to show that

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+\theta_2p_r\sum_{t=\tau}^{T}\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\\
&\quad+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right)\right\}+p_s\left[\widehat{\nu}_T-\theta_2\left(\widehat{s}_{2T}-\frac{\widehat{D}^b_{1T}\widehat{d}^b_{2T}}{m_2}\right)\right]\\
&\geq(r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+\theta_2p_r\sum_{t=\tau}^{T}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\\
&\quad+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)\right\}+p_s\left[\breve{\nu}_T-\theta_2\left(\breve{s}_{2T}-\frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right)\right],
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&[(r_1-c_1)-(r_2-c_2)]\left(\frac{A\breve{d}^b_{1(\kappa+2)}}{m_2}\right)+(r_2-c_2)\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A}{m_2}+\frac{\underline{b}q_2(\alpha-1-A)}{m_2^2}\right)\\
&+\theta_2p_r\left[\left(m_2+\breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A}{m_2}+\frac{\underline{b}q_2(\alpha-1-A)}{m_2^2}\right)+(\alpha-AE)-\frac{A\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right]\\
&(r_2-c_2)(\alpha-AE-1)-p_s\left[\theta_2\alpha+\frac{(\gamma\theta_1-\theta_2)\,A\breve{D}^b_{1(\kappa+1)}}{m_2}\right]>0.
\end{aligned}
$$

The above inequality holds as we assume

$$
\begin{aligned}
\alpha\quad>\quad&\frac{(\theta_2p_r+r_2-c_2)\left(\frac{A\breve{D}^b_{1(\kappa+1)}}{m_2}+A+AC(1+A)-2ABC(1+A)\right)+\frac{\theta_2p_rA\left(\breve{D}^b_{1(T+1)}-\breve{D}^b_{1(\kappa+2)}\right)}{m_2}}{(\theta_2p_r+r_2-c_2)(1+AC-ABC)-\theta_2p_s}\\
&+\frac{r_2-c_2+\frac{(\gamma\theta_1-\theta_2)p_sA\breve{D}^b_{1(\kappa+1)}}{m_2}-\frac{((r_1-c_1)-(r_2-c_2))A\breve{d}^b_{1(\kappa+2)}}{m_2}}{(\theta_2p_r+r_2-c_2)(1+AC-ABC)-\theta_2p_s}.
\end{aligned}
$$

In order to prove part (b) of Theorem A.3, recall from the proof of Theorem A.2

that

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\\
&\quad+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\right)\right\}\\
&\geq(r_1-c_1)\sum_{t=1}^{T}\breve{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\breve{D}_{1t}^{b}\breve{d}_{2t}^{b}}{m_2}\\
&\quad+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}_{1t}^{b}\breve{d}_{2t}^{b}}{m_2}\right)\right\}+\theta_2 p_r\left(\alpha\epsilon-AE\epsilon\right)\\
&\quad+(r_2-c_2)\left(\left(m_2+\breve{D}_{1(\kappa+2)}^{b}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\right)\\
&\quad+\theta_2 p_r\left(m_2+\breve{D}_{1(\kappa+2)}^{b}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&\quad+\frac{\left((r_1-c_1)-(r_2-c_2)-\theta_2 p_r\right)A\epsilon\left(\breve{D}_{1(T+1)}^{b}-\breve{D}_{1(\kappa+2)}^{b}\right)}{m_2}+(r_2-c_2)\left(-\epsilon-AE\epsilon+\alpha\epsilon\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
p_s\left[\widehat{\nu}_T-\theta_2\left(\widehat{s}_{2T}-\frac{\widehat{D}_{1T}^{b}\widehat{d}_{2T}^{b}}{m_2}\right)\right] \quad\geq\quad & p_s\left[\breve{\nu}_T-\theta_2\left(\breve{s}_{2T}-\frac{\breve{D}_{1T}^{b}\breve{d}_{2T}^{b}}{m_2}\right)\right]\\
&-p_s\left[\theta_2\alpha\epsilon+\frac{(\gamma\theta_1-\theta_2)A\epsilon\breve{D}_{1(\kappa+1)}^{b}}{m_2}\right],
\end{aligned}
$$

the total profits under sales plans (i) and (ii) obey

$$
\begin{aligned}
&(r_1-c_1)\sum_{t=1}^{T}\widehat{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\\
&\quad+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t,\theta_2\left(\widehat{s}_{2t}-\frac{\widehat{D}_{1t}^{b}\widehat{d}_{2t}^{b}}{m_2}\right)\right\}+p_s\left[\widehat{\nu}_T-\theta_2\left(\widehat{s}_{2T}-\frac{\widehat{D}_{1T}^{b}\widehat{d}_{2T}^{b}}{m_2}\right)\right]\\
&\geq(r_1-c_1)\sum_{t=1}^{T}\breve{d}_{1t}^{n}+(r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t}+\theta_2 p_r\sum_{t=\tau}^{T}\frac{\breve{D}_{1t}^{b}\breve{d}_{2t}^{b}}{m_2}-(r_2-c_2)\left(\epsilon-AE\epsilon+\alpha\epsilon\right)\\
&\quad+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t,\theta_2\left(\breve{s}_{2t}-\frac{\breve{D}_{1t}^{b}\breve{d}_{2t}^{b}}{m_2}\right)\right\}+p_s\left[\breve{\nu}_T-\theta_2\left(\breve{s}_{2T}-\frac{\breve{D}_{1T}^{b}\breve{d}_{2T}^{b}}{m_2}\right)\right]\\
&\quad+(r_2-c_2)\left(m_2+\breve{D}_{1(\kappa+2)}^{b}\right)\left(\frac{\overline{a}A\epsilon}{m_2}+\frac{bq_2\epsilon(\alpha-1-A)}{m_2^2}+\frac{q_2A\epsilon^2(\alpha-1-A)}{m_2^2}\right)\\
&\quad+\theta_2 p_r\left(\alpha\epsilon-AE\epsilon\right)-p_s\left[\theta_2\alpha\epsilon+\frac{(\gamma\theta_1-\theta_2)A\epsilon\breve{D}_{1(\kappa+1)}^{b}}{m_2}\right]
\end{aligned}
$$

$$+\theta_2 p_r \left(m_2 + \breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A\epsilon}{m_2} + \frac{\underline{b}q_2\epsilon(\alpha-1-A)}{m_2^2} + \frac{q_2 A\epsilon^2(\alpha-1-A)}{m_2^2}\right)$$

$$+\frac{((r_1-c_1)-(r_2-c_2)-\theta_2 p_r)\, A\epsilon\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2}.$$

In order to show that

$$(r_1-c_1)\sum_{t=1}^{T}\widehat{d}^n_{1t} + (r_2-c_2)\sum_{t=\tau}^{T}\widehat{s}_{2t} + \theta_2 p_r \sum_{t=\tau}^{T}\frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}$$

$$+p_r\sum_{t=\tau}^{T}\min\left\{\widehat{\nu}_t, \theta_2\left(\widehat{s}_{2t} - \frac{\widehat{D}^b_{1t}\widehat{d}^b_{2t}}{m_2}\right)\right\} + p_s\left[\widehat{\nu}_T - \theta_2\left(\widehat{s}_{2T} - \frac{\widehat{D}^b_{1T}\widehat{d}^b_{2T}}{m_2}\right)\right]$$

$$\geq (r_1-c_1)\sum_{t=1}^{T}\breve{d}^n_{1t} + (r_2-c_2)\sum_{t=\tau}^{T}\breve{s}_{2t} + \theta_2 p_r \sum_{t=\tau}^{T}\frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}$$

$$+p_r\sum_{t=\tau}^{T}\min\left\{\breve{\nu}_t, \theta_2\left(\breve{s}_{2t} - \frac{\breve{D}^b_{1t}\breve{d}^b_{2t}}{m_2}\right)\right\} + p_s\left[\breve{\nu}_T - \theta_2\left(\breve{s}_{2T} - \frac{\breve{D}^b_{1T}\breve{d}^b_{2T}}{m_2}\right)\right],$$

it suffices to show that

$$\frac{((r_1-c_1)-(r_2-c_2))\, A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2} + (r_2-c_2)(\alpha - AE - 1)$$

$$+\theta_2 p_r\left[\left(m_2 + \breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A}{m_2} + \frac{\underline{b}q_2(\alpha-1-A)}{m_2^2}\right) + (\alpha - AE)\right]$$

$$-\theta_2 p_r\left[\frac{A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2}\right] - p_s\left[\theta_2\alpha + \frac{(\gamma\theta_1-\theta_2)\, A\breve{D}^b_{1(\kappa+1)}}{m_2}\right]$$

$$+(r_2-c_2)\left(m_2 + \breve{D}^b_{1(\kappa+2)}\right)\left(\frac{\overline{a}A}{m_2} + \frac{\underline{b}q_2(\alpha-1-A)}{m_2^2}\right) > 0.$$

The above inequality holds as we assume

$$\alpha > \frac{(\theta_2 p_r + r_2 - c_2)\left(\frac{A\left(\breve{D}^b_{1(T+1)} + \breve{D}^b_{1(\kappa+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2} + A\right)}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s}$$

$$+\frac{(\theta_2 p_r + r_2 - c_2)(AC(1+A) - 2ABC(1+A))}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s}$$

$$+\frac{r_2 - c_2 + \frac{(\gamma\theta_1-\theta_2)p_s A\breve{D}^b_{1(\kappa+1)}}{m_2} - \frac{(r_1-c_1)A\left(\breve{D}^b_{1(T+1)} - \breve{D}^b_{1(\kappa+2)}\right)}{m_2}}{(\theta_2 p_r + r_2 - c_2)(1 + AC - ABC) - \theta_2 p_s}.$$

$\square$