

## An extension of the Glauberman ZJ-theorem

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Let  $p$  be an odd prime and let  $J_o(X)$ ,  $J_r(X)$  and  $J_e(X)$  denote the three different versions of Thompson subgroups for a  $p$ -group  $X$ . In this paper, we first prove an extension of Glauberman's replacement theorem [G. Glauberman, A characteristic subgroup of a  $p$ -stable group, *Canad. J. Math.* **20** (1968) 1101–1135, Theorem 4.1]. Second, we prove the following: Let  $G$  be a  $p$ -stable group and  $P \in \text{Syl}_p(G)$ . Suppose that  $C_G(O_p(G)) \leq O_p(G)$ . If  $D$  is a strongly closed subgroup in  $P$ , then  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  are normal subgroups of  $G$ . Third, we show the following: Let  $G$  be a Qd( $p$ )-free group and  $P \in \text{Syl}_p(G)$ . If  $D$  is a strongly closed subgroup in  $P$ , then the normalizers of the subgroups  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  control strong  $G$ -fusion in  $P$ . We also prove a similar result for a  $p$ -stable and  $p$ -constrained group. Finally, we give a  $p$ -nilpotency criteria, which is an extension of Glauberman–Thompson  $p$ -nilpotency theorem.

*Keywords:* Controlling fusion; ZJ-theorem;  $p$ -stable groups.

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### 1. Introduction

Throughout the paper, all groups considered are finite. Let  $P$  be a  $p$ -group. For each abelian subgroup  $A$  of  $P$ , let  $m(A)$  be the rank of  $A$ , and let  $d_r(P)$  be the maximum of the numbers  $m(A)$ . Similarly,  $d_o(P)$  is defined to be the maximum of orders of abelian subgroups of  $P$  and  $d_e(P)$  is defined to be the maximum of orders of elementary abelian subgroups of  $P$ . Define

$$\mathcal{A}_o(P) = \{A \leq P \mid A \text{ is abelian and } |A| = d_o(P)\},$$

$$\mathcal{A}_r(P) = \{A \leq P \mid A \text{ is abelian and } m(A) = d_r(P)\}$$

and

$$\mathcal{A}_e(P) = \{A \leq P \mid A \text{ is elementary abelian and } |A| = d_e(P)\}.$$

Now we are ready to define three different versions of Thompson subgroup:  $J_r(P)$ ,  $J_o(P)$  and  $J_e(P)$  are subgroups of  $P$  generated by all members of  $\mathcal{A}_r(P)$ ,  $\mathcal{A}_o(P)$  and  $\mathcal{A}_e(P)$ , respectively. These definitions appear in [2] with the same notations.

Thompson proved his normal complement theorem according to  $J_r(P)$  in [13], which states: Let  $G$  be a group and  $P \in \text{Syl}_p(G)$ . If  $N_G(J_r(P))$  and  $C_G(Z(P))$  are both  $p$ -nilpotent and  $p$  is odd, then  $G$  is  $p$ -nilpotent. Later Thompson introduced “a replacement theorem” and a subgroup similar to  $J_o(P)$  in [14]. Glauberman generalized the replacement theorem of Thompson for odd primes (see [2, Theorem 4.1]) and worked with  $J_o(P)$  in [2] due to the compatibility of the replacement theorem with  $J_o(P)$ . We should note that Glauberman’s replacement theorem is one of the important ingredients of the proof of Glauberman ZJ-theorem [2, Theorem A].

**Definition ([3, p. 22]).** A group  $G$  is called  $p$ -stable if it satisfies the following condition: Whenever  $U$  is a  $p$ -subgroup of  $G$ ,  $g \in N_G(U)$  and  $[U, g, g] = 1$  then the coset  $gC_G(U)$  lies in  $O_p(N_G(U)/C_G(U))$ .

Now we are ready to state Glauberman ZJ-theorem.

**Theorem (Glauberman).** *Let  $p$  be an odd prime,  $G$  be a  $p$ -stable group, and  $P \in \text{Syl}_p(G)$ . Suppose that  $C_G(O_p(G)) \leq O_p(G)$ . Then  $Z(J_o(P))$  is a characteristic subgroup of  $G$ .*

There are many important consequences of the above theorem. A striking one is that  $N_G(Z(J_o(P)))$  controls strong  $G$ -fusion in  $P$  when  $G$  does not involve a subquotient isomorphic to  $\text{Qd}(p)$  (see [2, Theorem B]). Note that  $\text{Qd}(p)$  is defined to be a semidirect product of  $\mathbb{Z}_p \times \mathbb{Z}_p$  with  $\text{SL}(2, p)$  by the natural action of  $\text{SL}(2, p)$  on  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Another consequence of Glauberman ZJ-theorem is an improvement of Thompson normal complement theorem. This result says that if  $N_G(Z(J_o(P)))$  is  $p$ -nilpotent and  $p$  is odd, then  $G$  is  $p$ -nilpotent (see [2, Theorem D]).

There is still active research on the properties of Thompson’s subgroups. A current paper [12] is describing algorithms for determining  $J_e(P)$  and  $J_o(P)$ . We also refer to [12, 10] for more extensive discussions about literature and replacement theorems, which we do not state here. It deserves to be mentioned separately that Glauberman obtained remarkably more general versions of the Thompson replacement theorem in his later works (see [4, 5]). We should also note that even if [12, Theorem 1] is attributed to Thompson replacement theorem [13] in [12], it seems that the correct reference is Isaacs replacement theorem (see [9]).

In [1], the ZJ-theorem is given according to  $J_e(P)$  (see [1, Theorem 1.21, Definition 1.16]). Although it might be natural to think that Glauberman ZJ-theorem is also correct for “ $J_e(P)$  and  $J_r(P)$ ”, there is no reference verifying that. However, we should mention that Stellmacher showed that there exists a characteristic subgroup  $W$  of  $P$  such that  $\Omega(Z(P)) \leq W \leq \Omega(Z(J_e(P)))$  and  $W$  satisfies the conclusion of Glauberman’s theorem (see [11, Theorem 9.4.4]). The relations between Glauberman ZJ-theorem and spherical fibrations over classifying spaces are studied

in [15, Sec. 3]. Moreover, it is observed in [15] that  $J_e(P)$  is more useful to investigate these relations. This particular case shows that different Thompson subgroups may have distinct applications. This is one of our motivations to prove our extension of Glauberman ZJ-theorem (Theorem B) for all versions of Thompson’s subgroups.

One of the purposes of this paper is to generalize Glauberman replacement theorem (see [2, Theorem 4.1]), which was used in the proof of Glauberman ZJ-theorem. We also note that our replacement theorem is an extension of Isaacs replacement theorem (see [9]) when we consider odd primes. The following is the first main theorem of our paper.

**Theorem A.** *Let  $G$  be a  $p$ -group for an odd prime  $p$  and  $A \leq G$  be abelian. Suppose that  $B \leq G$  is of class at most 2 such that  $B' \leq A$ ,  $A \leq N_G(B)$  and  $B \not\leq N_G(A)$ . Then there exists an abelian subgroup  $A^*$  of  $G$  such that*

- (a)  $|A| = |A^*|$ ,
- (b)  $A \cap B < A^* \cap B$ ,
- (c)  $A^* \leq N_G(A) \cap A^G$ ,
- (d) *the exponent of  $A^*$  divides the exponent of  $A$ . Moreover,  $\text{rank}(A) \leq \text{rank}(A^*)$ .*

One of the main differences from [2, Theorem 4.1] is that we are not taking  $A$  to be of maximal order. By removing the order condition, we obtain more flexibility to apply the replacement theorem. Since our replacement theorem is easily applicable to all versions of Thompson subgroups and there is a gap in the literature whether the ZJ-theorem holds for other versions of Thompson subgroups, we shall prove our extensions of Glauberman ZJ-theorem for all different versions of Thompson subgroups.

**Definition 1.1.** Let  $G$  be a group,  $P \in \text{Syl}_p(G)$ , and  $D$  be a nonempty subset of  $P$ . We say that  $D$  is a **strongly closed set** in  $P$  (with respect to  $G$ ) if for all  $U \subseteq D$  and  $g \in G$  such that  $U^g \subseteq P$ , the containment  $U^g \subseteq D$  holds. In the case that  $D$  is a subgroup of  $P$ ,  $D$  is said to be a **strongly closed subgroup**.

Let  $K$  be a  $p$ -group. We write  $\Omega_i(K)$  to denote the subgroup  $\langle \{x \in K \mid x^{p^i} = 1\} \rangle$  of  $K$  for  $i \in \mathbb{Z}^+$  and we simply use  $\Omega(K)$  in place of  $\Omega_1(K)$ . Here is the second main theorem of the paper.

**Theorem B.** *Let  $p$  be an odd prime,  $G$  be a  $p$ -stable group, and  $P \in \text{Syl}_p(G)$ . Suppose that  $C_G(O_p(G)) \leq O_p(G)$ . If  $D$  is a strongly closed subgroup in  $P$  then  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  are normal subgroups of  $G$ .*

We prove Theorem B by mainly following the original proof given by Glauberman and with the help of Theorem A. When we take  $D = P$ , we obtain that  $Z(J_o(P))$ ,  $\Omega(Z(J_r(P)))$  and  $\Omega(Z(J_e(P)))$  are characteristic subgroups of  $G$  under the hypothesis of Theorem B. Both  $Z(J_r(P))$  and  $Z(J_e(P))$  need an extra operation “ $\Omega$ ” and it does not seem quite possible to remove “ $\Omega$ ” by the method used here.

**Definition ([6, p. 268]).** A group  $G$  is called  $p$ -constrained if  $C_G(Y) \leq O_{p',p}(G)$  for a Sylow  $p$ -subgroup  $Y$  of  $O_{p',p}(G)$ .

**Theorem C.** Let  $p$  be an odd prime,  $G$  be a  $p$ -stable group, and  $P \in \text{Syl}_p(G)$ . Assume that  $N_G(U)$  is  $p$ -constrained for each nontrivial subgroup  $U$  of  $P$ . If  $D$  is a strongly closed subgroup in  $P$  then the normalizers of the subgroups  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  control strong  $G$ -fusion in  $P$ .

**Remark 1.2.** In [7], it is shown that if  $G$  is  $p$ -stable and  $p > 3$  then  $G$  is  $p$ -constrained and the proof uses the classification of finite simple groups (see [7, Proposition 2.3]). Note that a subgroup of a  $p$ -stable group is also  $p$ -stable (see Lemma 3.10), and so  $N_G(U)$  is  $p$ -stable. Hence, the assumption “ $N_G(U)$  is  $p$ -constrained for each nontrivial subgroup  $U$  of  $P$ ” is automatically satisfied when  $p > 3$  and  $G$  is a  $p$ -stable group.

**Theorem D.** Let  $p$  be an odd prime,  $G$  be a  $Qd(p)$ -free group, and  $P \in \text{Syl}_p(G)$ . If  $D$  is a strongly closed subgroup in  $P$  then the normalizers of the subgroups  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  control strong  $G$ -fusion in  $P$ .

**Remark 1.3.** In Theorem D, if we take  $D = P$ , then the proof of this special case follows by Theorem B and [3, Theorem 6.6]. However, the general case requires some extra work. We shall define the concept “the localization of a conjugacy functor” (see Definition 4.3) and deduce some of its properties (see Lemmas 4.4, 4.6 and 4.8). These are used in the proofs of Theorems C, D and E.

Lastly, we state an extension of Glauberman–Thompson  $p$ -nilpotency theorem.

**Theorem E.** Let  $p$  be an odd prime,  $G$  be a group and  $P \in \text{Syl}_p(G)$ . If  $D$  is a strongly closed subgroup in  $P$  then  $G$  is  $p$ -nilpotent if one among the normalizer of the subgroups  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  is  $p$ -nilpotent.

**Remark 1.4.** Let  $G$  be a group,  $P \in \text{Syl}_p(G)$  and  $D$  be a strongly closed subgroup in  $P$ . Assume that  $\Omega_i(D)$  is of exponent at most  $p^i$  for some  $i$ . Then it is routine to check that  $\Omega_i(D)$  is also strongly closed subgroup in  $P$ . In Theorems B, C, D and E if we write  $\Omega_i(D)$  in place of  $D$  (under the above assumption), we can obtain some variations of these theorems. We should note that the assumption is automatically satisfied when  $D$  is a regular  $p$ -group.

## 2. The Proof of Theorem A

We start with a lemma whose proof is extracted from the proof of Glauberman replacement theorem.

**Lemma 2.1 (Glauberman).** Let  $p$  be an odd prime and  $G$  be a  $p$ -group. Suppose that  $G = BA$  where  $B$  is a normal subgroup of  $G$  such that  $B' \leq Z(G)$  and  $A$  is an abelian subgroup of  $G$  such that  $[B, A, A, A] = 1$ . Then  $[b, A]$  is an abelian subgroup of  $G$  for each  $b \in B$ .

**Proof.** Let  $x, y \in A$ . Our aim is to show that  $[b, x]$  and  $[b, y]$  commute. Set  $u = [b, y]$ . If we apply the Hall–Witt identity to the triple  $(b, x^{-1}, u)$ , we obtain that

$$[b, x, u]^{x^{-1}} [x^{-1}, u^{-1}, b]^u [u, b^{-1}, x^{-1}]^b = 1.$$

Note that the above commutators of weight 3 lie in the center of  $G$  since  $B$  is normal in  $G$  and  $B' \leq Z(G)$ . Thus, we may remove conjugations in the above equation. Moreover,  $[u, b^{-1}, x^{-1}] = 1$  as  $[u, b^{-1}] \in B'$ . Thus, we obtain that  $[b, x, u][x^{-1}, u^{-1}, b] = 1$ , and so

$$[b, x, u] = [x^{-1}, u^{-1}, b]^{-1}.$$

Since  $[x^{-1}, u^{-1}, b] = [[x^{-1}, u^{-1}], b] \in Z(G)$ , we see that

$$[x^{-1}, u^{-1}, b]^{-1} = [[x^{-1}, u^{-1}], b]^{-1} = [[x^{-1}, u^{-1}]^{-1}, b] = [[u^{-1}, x^{-1}], b]$$

by [6, Lemma 2.2.5(ii)]. As a consequence, we get that  $[b, x, u] = [[u^{-1}, x^{-1}], b]$ . By inserting  $u = [b, y]$ , we obtain

$$[[b, x], [b, y]] = [[[b, y]^{-1}, x^{-1}], b].$$

Now set  $\overline{G} = G/B'$ . Then clearly  $\overline{B}$  is abelian. It follows that  $[\overline{B}, \overline{A}, \overline{A}] \leq Z(\overline{G})$  since  $G = AB$ ,  $[B, A, A, A] = 1$  and  $\overline{B}$  is abelian. Then we have

$$[[b, y]^{-1}, x^{-1}] \equiv [[b, y]^{-1}, x]^{-1} \equiv [[b, y], x] \pmod{B'}$$

by applying [6, Lemma 2.2.5(ii)] to  $\overline{G}$ . Since  $x$  and  $y$  commute and  $[\overline{b}, \overline{G}] \subseteq \overline{B}$  is abelian, we see that

$$[b, y, x] \equiv [b, x, y] \pmod{B'}$$

by [6, Lemma 2.2.5(i)].

Since  $B' \leq Z(G)$ , we obtain

$$[[b, x], [b, y]] = [[[b, y]^{-1}, x^{-1}], b] = [[[b, y], x], b] = [[b, x, y], b].$$

By symmetry, we also have that  $[[b, y], [b, x]] = [[b, x, y], b]$ . Then it follows that  $[[b, y], [b, x]] = [[b, y], [b, x]]^{-1}$ , and so  $[[b, x], [b, y]] = 1$  since  $G$  is of odd order.  $\square$

**Lemma 2.2.** *Let  $A$  be an abelian  $p$ -group and  $E$  be the largest elementary abelian subgroup of  $A$ . Then  $\text{rank}(E) = \text{rank}(A)$ .*

**Proof.** Consider the homomorphism  $\phi : A \rightarrow A$  by  $\phi(a) = a^p$  for each  $a \in A$ . Note that  $\phi(A) = \Phi(A)$  and  $E = \text{Ker}(\phi)$ , and so  $|A/\Phi(A)| = |E|$ . Since both  $E$  and  $A/\Phi(A)$  are elementary abelian groups of same order, we get  $\text{rank}(E) = \text{rank}(A/\Phi(A))$ . On the other hand,  $\text{rank}(A/\Phi(A)) = \text{rank}(A)$  and the result follows.  $\square$

**Proof of Theorem A.** We proceed by induction on the order of  $G$ . We can certainly assume that  $G = AB$ . Since  $A$  is not normal in  $G$ , there exists a maximal subgroup  $M$  of  $G$  such that  $A \leq M$ .

Clearly  $A$  normalizes  $M \cap B$  as both  $M$  and  $B$  are normal in  $G$ . Suppose that  $M \cap B$  does not normalize  $A$ . By induction applied to  $M$ , there exists a subgroup

$A^*$  of  $M$  such that  $A^*$  satisfies the conclusion of the theorem. Then  $A^*$  also satisfies (a), (c) and (d) in  $G$ . Moreover,  $A \cap (M \cap B) = A \cap B < A^* \cap B$ , and so  $G$  also satisfies the theorem. Hence, we can assume that  $M \cap B \leq N_G(A)$ . Note that  $M = M \cap AB = A(M \cap B)$ , and so  $M = N_G(A)$ .

Clearly  $M \cap B$  is a maximal subgroup of  $B$ . Then  $A$  acts trivially on  $B/(M \cap B)$ , and so  $[B, A] \leq M = N_G(A)$ . Thus, we see that  $[B, A, A] \leq A$  which yields  $[B, A, A, A] = 1$ . Moreover, we have that  $B' \leq Z(G)$  since  $B' \leq A$  and  $B' \leq Z(B)$ . It follows that  $[b, A]$  is abelian for any  $b \in B$  by Lemma 2.1.

Let  $b \in B \setminus M$ . Then  $A \neq A^b \trianglelefteq M$ . Set  $H = AA^b$  and  $Z = A \cap A^b$ . Then clearly  $H$  is a group and  $Z \leq Z(H)$ . On the other hand,  $H$  is of class at most 2 since  $H/Z$  is abelian. Note that the identity  $(xy)^n = x^n y^n [x, y]^{\frac{n(n-1)}{2}}$  holds for all  $x, y \in H$  as  $H$  is of odd order and  $[H, H] \leq Z(H)$  by [6, Lemma 2.2.2]. It follows that the exponent of  $H$  is the same as the exponent of  $A$ .

Next, we shall show that  $H \cap B$  is abelian. First we claim that  $H \cap B = (A \cap B)[b, A]$ . Clearly, we have  $[b, A] \subseteq H \cap B$  since  $H = AA^b$ . It follows that  $(A \cap B)[b, A] \subseteq H \cap B$  as  $A \cap B \leq H \cap B$ . Next, we obtain the reverse inequality. Let  $x \in H \cap B$ . Then  $x = ac^b$  for  $a, c \in A$  such that  $ac^b \in B$ . Since  $B \trianglelefteq G$ , we see that  $[c, b] \in B$ , and so  $ac \in B$  as  $ac[c, b] = ac^b \in B$ . It follows that  $ac \in A \cap B$  and  $x = ac[c, b] \in (A \cap B)[b, A]$ , which proves the equality  $H \cap B = (A \cap B)[b, A]$ . Since  $B' \leq A$ , we see that  $A \cap B \trianglelefteq B$ . Then  $A \cap B = A^b \cap B$  and hence  $A \cap B = Z \cap B$ . In particular, we see that  $A \cap B \leq Z \leq Z(H)$ . It follows that  $H \cap B = (A \cap B)[b, A]$  is abelian since  $[b, A]$  is an abelian subgroup of  $H$  and  $(A \cap B) \leq Z(H)$ .

Set  $A^* = (H \cap B)Z$ . Note that  $A^*$  is abelian as  $H \cap B$  is abelian and  $Z \leq Z(H)$ . Now we shall show that  $A^*$  is the desired subgroup. Clearly, the exponent of  $A^*$  divides the exponent of  $H$ , which we showed is equal to the exponent of  $A$ , and so the first part of (d) follows. Note that  $A < H$  and  $H = H \cap AB = A(H \cap B)$ , and so  $H \cap B > A \cap B$ . It follows that  $A^* \cap B \geq H \cap B > A \cap B$ , which shows (b). On the other hand,

$$A^* \leq H = AA^b \leq M \cap A^G = N_G(A) \cap A^G,$$

which shows (c). It remains to prove (a) and the second part of (d). Since  $A^* = (H \cap B)Z$ , we have

$$|A^*| = \frac{|H \cap B||Z|}{|Z \cap B|} = \frac{|H \cap B||Z|}{|A \cap B|}.$$

As  $H = AA^b = A(H \cap B)$ , we obtain that

$$\frac{|AA^b|}{|A^b|} = \frac{|A(H \cap B)|}{|A|} = \frac{|H \cap B|}{|A \cap B|}.$$

On the other hand, we see that

$$\frac{|AA^b|}{|A^b|} = \frac{|A|}{|A \cap A^b|} = \frac{|A|}{|Z|}.$$

Thus, we get the equality  $|A| = |A^*|$  as desired.

Let  $E$  be the largest elementary abelian subgroup of  $A$ . We shall observe that  $E$  and  $A$  enjoy some similar properties. Note that  $E \trianglelefteq M = N_G(A)$  since  $E$  is a characteristic subgroup of  $A$ . Hence,  $EE^b$  is a group. Set  $H_1 = EE^b$ ,  $Z_1 = E \cap E^b$  and  $E^* = (H_1 \cap B)Z_1$ . First observe that  $Z_1 \leq Z(H_1)$ , and so  $H_1$  is of class at most 2. It follows that the exponent of  $E^*$  is  $p$  since  $H_1$  is of odd order. Thus,  $E^*$  is elementary abelian as  $E^* \leq A^*$  and  $A^*$  is abelian. Note also that  $E \cap B = E \cap (A \cap B)$ , and so  $E \cap B$  is characteristic in  $A \cap B$ . Then we see that  $E \cap B \trianglelefteq B$  as  $A \cap B \trianglelefteq B$ . This also yields that  $E \cap B = (E \cap B)^b = E^b \cap B$ , and hence  $E \cap B = Z_1 \cap B$ . Finally, observe that  $H_1 = EE^b = EE^b \cap EB = E(H_1 \cap B)$ . Now we can show that  $|E| = |E^*|$  by using the same method used for showing that  $|A| = |A^*|$ . Then we see that  $\text{rank}(A) = \text{rank}(E) = \text{rank}(E^*) \leq \text{rank}(A^*)$  by Lemma 2.2.  $\square$

### 3. The Proof of Theorem B

**Lemma 3.1.** *Let  $P$  be a  $p$ -group and  $R$  be a subgroup of  $P$ . Then if there exists  $A \in \mathcal{A}_x(P)$  such that  $A \leq R$  then  $J_x(R) \leq J_x(P)$  for  $x \in \{o, r, e\}$ . Moreover,  $J_x(P) = J_x(R)$  if and only if  $J_x(P) \leq R$  for  $x \in \{o, r, e\}$ .*

**Proof.** Let  $A \subseteq R$  for some  $A \in \mathcal{A}_x(P)$ . Note that  $\mathcal{A}_x(R) \subseteq \mathcal{A}_x(P)$  by the definition of  $\mathcal{A}_x(P)$  for each  $x \in \{o, r, e\}$ , and so  $J_x(R) \leq J_x(P)$  in that case.

Next observe that  $J_x(P) \leq R$  if and only if  $\mathcal{A}_x(P) = \mathcal{A}_x(R)$ . Then the second part follows.  $\square$

**Lemma 3.2 ([6, Theorem 8.1.3]).** *Let  $G$  be a  $p$ -stable group such that  $C_G(O_p(G)) \leq O_p(G)$ . If  $P \in \text{Syl}_p(G)$  and  $A$  is an abelian normal subgroup of  $P$  then  $A \leq O_p(G)$ .*

**Proof.** Since  $O_p(G)$  normalizes  $A$ , we see that  $[O_p(G), A, A] = 1$ . Write  $C = C_G(O_p(G))$ . Then we have  $AC/C \leq O_p(G/C)$ . Note that  $O_p(G/C) = O_p(G)/C$  since  $C \leq O_p(G)$ . It follows that  $A \leq O_p(G)$ .  $\square$

**Lemma 3.3.** *Let  $G$  be a group and  $P \in \text{Syl}_p(G)$ . Suppose that  $D$  is a strongly closed subset in  $P$ . If  $N \trianglelefteq G$  and  $D \cap N$  is nonempty then  $D \cap N$  is also a strongly closed subset in  $P$ . Moreover,  $G = N_G(D \cap N)N$ .*

**Proof.** Write  $D^* = D \cap N$ . Let  $U \subseteq D^*$  and  $g \in G$  such that  $U^g \subseteq P$ . It follows that  $U^g \subseteq D$  as  $U \subseteq D$  and  $D$  is strongly closed in  $G$ . Since  $N \trianglelefteq G$ , we see that  $U^g \subseteq N$  which yields that  $U^g \subseteq N \cap D = D^*$  which shows the first part.

Let  $Q = P \cap N$ . Then we see that  $Q \in \text{Syl}_p(N)$ , and so  $G = N_G(Q)N$  by the Frattini argument. Thus, it is enough to show that  $N_G(Q) \leq N_G(D^*)$ . Let  $x \in N_G(Q)$ . Then  $D^{*x} \subseteq Q \leq P$ . Since  $D^*$  is strongly closed in  $P$ , we see that  $D^{*x} = D^*$ . It follows that  $x \in N_G(D^*)$ , as desired.  $\square$

**Lemma 3.4.** *Let  $P$  be a  $p$ -group,  $p$  be odd, and let  $B, N \trianglelefteq P$ . Suppose that  $B$  is of class at most 2 and  $B' \leq A$  for all  $A \in \mathcal{A}_x(N)$ . Then there exists  $A \in \mathcal{A}_x(N)$  such that  $B$  normalizes  $A$  while  $x \in \{o, r, e\}$ .*

**Proof.** Choose  $A \in \mathcal{A}_x(N)$  such that  $A \cap B$  has the maximum possible order. Assume that  $B$  does not normalize  $A$ . Then there exists an abelian subgroup  $A^* \leq P$  such that  $|A^*| = |A|$ ,  $A^* \leq A^P \cap N_P(A)$ ,  $A \cap B < A^* \cap B$ , the exponent of  $A^*$  divides that of  $A$  and  $\text{rank}(A) \leq \text{rank}(A^*)$  by Theorem A. Since  $A \leq N \trianglelefteq P$ , we see that  $A^* \leq A^P \leq N$ .

We claim that  $A^* \in \mathcal{A}_x(N)$  for  $x \in \{o, r, e\}$ . In the case that  $x = o$ , the claim is obviously true as  $|A^*| = |A|$ . Let  $x = e$ . Since the exponent of  $A^*$  divides the exponent of  $A$  and  $A$  is elementary abelian, we see that  $A^*$  is also elementary abelian. That yields that  $A^* \in \mathcal{A}_e(N)$  as  $|A^*| = |A|$ . Now suppose that  $x = r$ . We see that  $\text{rank}(A^*) = \text{rank}(A)$  as  $\text{rank}(A^*) \geq \text{rank}(A)$  and the rank of  $A$  is the maximum possible in  $N$ . Then we get that  $A^* \in \mathcal{A}_r(N)$ . So,  $A^* \in \mathcal{A}_x(N)$  for  $x \in \{o, r, e\}$ , contradicting the maximality of  $|A \cap B|$ . Thus,  $B$  normalizes  $A$  as desired. □

**Notation 3.5.** Let  $P$  be a  $p$ -group. We denote the following subgroups  $Z(J_o(P))$ ,  $\Omega(Z(J_r(P)))$  and  $\Omega(Z(J_e(P)))$  of  $P$  by  $Z_o(P)$ ,  $Z_r(P)$  and  $Z_e(P)$ , respectively.

**Lemma 3.6.** *Let  $P$  be a  $p$ -group. Then  $Z_x(P) \leq A$  for all  $A \in \mathcal{A}_x(P)$  while  $x \in \{o, r, e\}$ .*

**Proof.** Let  $Z = Z_x(P)$ . Since  $Z \leq Z(J_x(P))$  and  $A \leq J_x(P)$ , we see that  $ZA$  is an abelian subgroup of  $P$ . If  $x = o$ , then  $ZA \leq A$  as  $A$  is a maximal abelian subgroup of  $P$ . Thus,  $Z \leq A$ . In the case that  $x = r$ , we again obtain that  $Z \leq A$  as  $A$  has the maximum possible rank in  $P$  and  $Z = \Omega(Z(J_r(P)))$  is elementary abelian. Now let  $x = e$ . We see that  $ZA$  is elementary abelian as both  $Z$  and  $A$  are elementary abelian. So,  $Z \leq A$  as  $A$  is a maximal elementary abelian subgroup of  $P$ . □

**Theorem 3.7.** *Let  $p$  be an odd prime,  $G$  be a  $p$ -stable group, and  $P \in \text{Syl}_p(G)$ . Let  $D$  be a strongly closed subset in  $P$  and  $B$  be a normal  $p$ -subgroup of  $G$ . Write  $K = \langle D \rangle$ . If all members of  $\mathcal{A}_x(K)$  are included in  $D$  then  $Z_x(K) \cap B \trianglelefteq G$  while  $x \in \{o, r, e\}$ .*

**Proof.** Fix  $x \in \{o, r, e\}$ . Write  $J(U) = J_x(U)$  for any  $p$ -subgroup  $U$  of  $P$  and set  $Z = Z_x(K)$ . We can clearly assume that  $B \neq 1$ . Let  $G$  be a counter example, and choose  $B$  to be the smallest possible normal  $p$ -subgroup contradicting the theorem. As  $D$  is strongly closed in  $P$ , it is a normal subset of  $P$ . It then follows that  $\langle D \rangle = K \trianglelefteq P$ , and so  $Z \trianglelefteq P$ . In particular,  $B$  normalizes  $Z$ .

Set  $B_1 = (Z \cap B)^G$ . Clearly  $B_1 \leq B$ . Suppose that  $B_1 < B$ . By our choice of  $B$ , we get  $Z \cap B_1 \trianglelefteq G$ . Since  $Z \cap B \leq B_1$ , we have  $Z \cap B \leq Z \cap B_1 \leq Z \cap B$ , and hence  $Z \cap B = Z \cap B_1$ . This contradiction shows that  $B = B_1 = (Z \cap B)^G$ .



Clearly  $B' < B$ , and hence  $Z \cap B' \trianglelefteq G$  by our choice of  $B$ . Since  $Z$  and  $B$  normalize each other,  $[Z \cap B, B] \leq Z \cap B'$ . Since  $B$  and  $Z \cap B'$  are both normal subgroups of  $G$ , we obtain  $[(Z \cap B)^g, B] \leq Z \cap B'$  for all  $g \in G$ . This yields  $[(Z \cap B)^G, B] = [B, B] = B' \leq Z \cap B'$ . In particular, we have  $B' \leq Z$ , and so  $[Z \cap B, B'] = 1$ . It follows that  $[B, B'] = 1$  as  $B = (Z \cap B)^G$ . As a consequence, we see that  $B$  is of class at most 2. Note that  $Z \leq A$  for all  $A \in \mathcal{A}_x(K)$  by Lemma 3.6. In particular,  $B' \leq A$  for all  $A \in \mathcal{A}_x(K)$ .

Let  $N$  be the largest normal subgroup of  $G$  that normalizes  $Z \cap B$ . Set  $D^* = D \cap N$ . Note that all members of  $\mathcal{A}_x(K)$  are included in  $D$  by the hypothesis, and so the identity element lies in  $D$ . Thus,  $D^*$  is nonempty. Write  $K^* = \langle D^* \rangle$ . We see that  $G = N_G(D^*)N$  by Lemma 3.3, and so  $G = N_G(K^*)N$ . It follows that  $G = N_G(J(K^*))N$  since  $J(K^*)$  is a characteristic subgroup of  $K^*$ . Suppose that  $J(K) \leq K^*$ . Then we see that  $J(K) = J(K^*)$ , and hence  $Z \cap B$  is normalized by  $N_G(J(K^*))$ . It follows that  $Z \cap B \trianglelefteq G$ . Thus, we may assume that  $J(K) \not\leq K^*$ .

There exists  $A \in \mathcal{A}_x(K)$  such that  $B$  normalizes  $A$  by Lemma 3.4. Hence,  $[B, A, A] = 1$  as  $[B, A] \leq A$ . Since  $G$  is  $p$ -stable and  $B \trianglelefteq G$ , we have that  $AC/C \leq O_p(G/C)$ , where  $C = C_G(B)$ . Note that  $C$  normalizes  $Z \cap B$ , and so  $C \leq N$  by the choice of  $N$ . It follows that  $AN/N \leq O_p(G/N)$ . Now we claim that  $O_p(G/N) = 1$ . Let  $L \trianglelefteq G$  such that  $L/N = O_p(G/N)$ . Then  $L = (L \cap P)N$ , and hence  $L$  normalizes  $Z \cap B$  as both  $N$  and  $L \cap P$  normalize  $Z \cap B$ . The maximality of  $N$  forces that  $N = L$ , which yields that  $O_p(G/N) = 1$ . Thus,  $A \leq N$ . Note that  $A \subseteq D$  by hypothesis, and so  $A \subseteq N \cap D = D^* \subseteq K^*$ .

We see that  $Z \leq A \leq J(K^*)$ , and so we have  $J(K^*) \leq J(K)$  and  $Z \leq Z(J(K^*))$ . It follows that  $Z \cap B \leq Z \leq Z_x(K^*)$ . Set  $U = Z_x(K^*)$ . Then we see that  $G = NN_G(U)$  since  $G = NN_G(K^*)$  and  $U$  is characteristic in  $K^*$ . As  $N$  normalizes  $Z \cap B$ , each distinct conjugate of  $Z \cap B$  comes via an element of  $N_G(U)$ . Thus,  $B = (Z \cap B)^G = (Z \cap B)^{N_G(U)} \leq U$ .

Since  $J(K) \not\leq K^*$ , some members of  $\mathcal{A}_x(K)$  do not lie in  $K^*$ . Among such members, choose  $A_1 \in \mathcal{A}_x(K)$  such that  $A_1 \cap B$  has the maximum possible order. Note that  $B$  does not normalize  $A_1$ , since otherwise this forces  $A_1 \leq K^*$  as in previous paragraphs. Then there exists  $A^* \leq P$  such that  $|A^*| = |A_1|$ ,  $A^* \leq A_1^P \cap N_P(A_1)$ ,  $A_1 \cap B < A^* \cap B$ , the exponent of  $A^*$  divides that of  $A_1$  and  $\text{rank}(A_1) \leq \text{rank}(A^*)$  by Theorem A. Note that  $A^* \leq K$  as  $A_1^P \leq K \trianglelefteq P$ . The order, rank and the exponent of  $A^*$  force that  $A^* \in \mathcal{A}_x(K)$  for  $x \in \{o, r, e\}$ . It follows that  $A^* \leq K^*$  due to the choice of  $A_1$ , and so  $A^* \in \mathcal{A}_x(K^*)$ . We see that  $B \leq U = Z_x(K^*) \leq A^*$  by Lemma 3.6. It follows that  $B \leq A^* \leq N_P(A_1)$ , which is the final contradiction. □

When we work with  $J_o(K)$ , we do not need to use  $\Omega$  operation due to the fact that  $Z(J_o(K)) \leq A$  for all  $A \in \mathcal{A}_o(K)$ . However, this does not need to be satisfied for  $Z(J_e(K))$  and  $Z(J_r(K))$ . This difference causes the use of  $\Omega$  operation necessary for  $Z(J_e(K))$  and  $Z(J_r(K))$ .

**Proof of Theorem B.** As in our hypothesis, let  $p$  be an odd prime,  $G$  be a  $p$ -stable group such that  $C_G(O_p(G)) \leq O_p(G)$  and  $D$  be a strongly closed subgroup in  $P$ . Since all these subgroups  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  are abelian normal subgroups of  $G$ , we see that they must be included in  $O_p(G)$  by Lemma 3.2. Note that  $D$  is also a strongly closed subset in  $P$  and satisfies the hypothesis of Theorem 3.7. Then the results follow from Theorem 3.7, applied with  $B = O_p(G)$ . □

As an application of Theorem 3.7, we prove the following theorem, which we shall need in the next section.

**Theorem 3.8.** *Let  $p$  be an odd prime,  $G$  be a  $p$ -stable and  $p$ -constrained group, and  $P \in \text{Syl}_p(G)$ . Let  $D$  be a strongly closed subset in  $P$ . Write  $K = \langle D \rangle$ . If all members of  $\mathcal{A}_x(K)$  are included in  $D$ , then the normalizer of  $Z_x(K)$  controls strong  $G$ -fusion in  $P$  while  $x \in \{o, r, e\}$ .*

We need the following lemma in the proof of Theorem 3.8.

**Lemma 3.9 ([2, Lemma 7.2]).** *If  $G$  is a  $p$ -stable group, then  $G/O_{p'}(G)$  is also  $p$ -stable.*

Since the  $p$ -stability definition we used here is not same with that of [2] and [2, Lemma 7.2] has also the extra assumption that  $O_p(G) \neq 1$ , it is appropriate to give a proof of this lemma here.

**Proof.** Write  $N = O_{p'}(G)$  and  $\overline{G} = G/N$ . Let  $V$  be  $p$ -subgroup of  $\overline{G}$ . Then there exists a  $p$ -subgroup  $U$  of  $G$  such that  $\overline{U} = V$ .

Let  $\overline{x} \in N_{\overline{G}}(\overline{U})$  such that  $[\overline{U}, \overline{x}, \overline{x}] = \overline{1}$ . Clearly, we can write  $\overline{x} = \overline{x}_1 \overline{x}_2$  such that  $\overline{x}_1$  is a  $p$ -element,  $\overline{x}_2$  is a  $p'$ -element and  $[\overline{x}_1, \overline{x}_2] = \overline{1}$  for some  $x_1, x_2 \in G$ . It follows that  $[\overline{U}, \overline{x}_i, \overline{x}_i] = \overline{1}$  for  $i = 1, 2$ . Then we see that  $\overline{x}_2 \in C_{\overline{G}}(\overline{U})$  by [8, Lemma 4.29]. Thus, it is enough to show that  $\overline{x}_1 \in O_p(N_{\overline{G}}(\overline{U})/C_{\overline{G}}(\overline{U}))$  to finish the proof.

Since  $\overline{x}_1$  is a  $p$ -element of  $\overline{G}$ ,  $x_1 = sn$  where  $n \in N$  and  $s$  is a  $p$ -element of  $G$ , which yields that  $\overline{x}_1 = \overline{s}$ . Then we see that  $[UN, s, s] \in N$  and  $s \in N_G(UN)$  by the previous paragraph. Note that  $U \in \text{Syl}_p(UN)$  and  $|\text{Syl}_p(UN)|$  is a  $p'$ -number. Consider the action of  $\langle s \rangle$  on  $\text{Syl}_p(UN)$ . Then we observe that  $s$  normalizes  $U^n$  for some  $n \in N$ . Thus, we get that  $[U^n, s, s] \leq U^n \cap N = 1$ . Note that  $\overline{U} = \overline{U^n}$ , and so we take  $U^n = U$  without loss of generality.

Let  $K \leq N_G(U)$  such that  $K/C_G(U) = O_p(N_G(U)/C_G(U))$ . Thus we observe that  $s \in K$  as  $G$  is  $p$ -stable. Note that  $N_{\overline{G}}(\overline{U}) = N_G(U)$  and  $C_{\overline{G}}(\overline{U}) = C_G(U)$  by [8, Lemma 7.7]. Hence, we see that  $\overline{x}_1 = \overline{s} \in \overline{K}$  and  $\overline{K}/C_{\overline{G}}(\overline{U}) \leq O_p(N_G(U)/C_G(U)) = O_p(N_{\overline{G}}(\overline{U})/C_{\overline{G}}(\overline{U}))$ , which completes the proof. □

**Proof of Theorem 3.8.** Write  $\overline{G} = G/O_{p'}(G)$ . Then  $\overline{G}$  is  $p$ -stable by Lemma 3.9. Since  $G$  is  $p$ -constrained, we have  $C_{\overline{G}}(O_p(\overline{G})) \leq O_p(\overline{G})$  by [6, Theorem 1.1(ii)]. Note that  $Z_x(\overline{K}) \leq O_p(\overline{G})$  by Lemma 3.2 for  $x \in \{o, r, e\}$ . We see that  $\overline{G}$  satisfies the

hypotheses of Theorem 3.7 as  $\overline{P}$  is isomorphic to  $P$  and  $\overline{D}$  is the desired strongly closed set in  $\overline{P}$ . It follows that  $Z_x(\overline{K}) \trianglelefteq \overline{G}$  by Theorem 3.7. Thus, we get  $G = O_{p'}(G)N_G(Z_x(K))$  as  $\overline{Z_x(K)} = Z_x(\overline{K}) \trianglelefteq \overline{G}$ . Hence,  $N_G(Z_x(K))$  controls strong  $G$ -fusion in  $P$  by [2, Lemma 7.1] for  $x \in \{o, r, e\}$ .  $\square$

We shall use the following fact in the proof of Theorem C.

**Lemma 3.10.** *A subgroup of a  $p$ -stable group is  $p$ -stable.*

**Proof.** Let  $G$  be a  $p$ -stable group and  $H \leq G$ . Assume that  $[U, h, h] = 1$  where  $U$  is a  $p$ -subgroup of  $H$  and  $h \in H$ . Now write  $N = N_G(U)$  and  $C = C_G(U)$ . Let  $O$  be the full inverse image of  $O_p(N/C)$  in  $N$ . Clearly,  $O \trianglelefteq N$  and  $O/C$  is a  $p$ -group. Since  $G$  is  $p$ -stable, we have  $h \in O$ , and so  $h \in O \cap H \leq N_H(U)$ . We see that  $O \cap H \trianglelefteq N_H(U)$  as  $N_H(U)$  normalizes both  $O$  and  $H$ . Moreover,  $O \cap H/C \cap H = O \cap H/C_H(U)$  is a  $p$ -group as  $O/C$  is a  $p$ -group. It follows that  $O \cap H/C_H(U)$  is a normal  $p$ -subgroup of  $N_H(U)/C_H(U)$ , and so we obtain  $hC_H(U) \in O \cap H/C_H(U) \leq O_p(N_H(U)/C_H(U))$ , that is,  $H$  is  $p$ -stable.  $\square$

#### 4. The Proofs of Theorems C, D and E

**Lemma 4.1.** *Let  $P \in \text{Syl}_p(G)$  and  $D$  be a strongly closed subset in  $P$ . Let  $H \leq G$ ,  $N \trianglelefteq G$  and  $g \in G$  such that  $P^g \cap H \in \text{Syl}_p(H)$ . Then*

- (a)  $D^g \cap H$  is strongly closed in  $P^g \cap H$  with respect to  $H$  if  $D^g \cap H$  is nonempty.
- (b) If  $y \in G$  such that  $P^y \cap H \in \text{Syl}_p(H)$ , then  $D^g \cap H$  and  $D^y \cap H$  are conjugate by an element of  $H$ .
- (c)  $DN/N$  is strongly closed in  $PN/N$  with respect to  $G/N$ .

**Proof.** (a) Let  $U \subseteq D^g \cap H$  and  $h \in H$  such that  $U^h \subseteq P^g \cap H$ . Since  $U \subseteq D^g$  and  $U^h \subseteq P^g$ , we see that  $U^h \subseteq D^g$  as  $D^g$  is strongly closed in  $P^g$  with respect to  $G$ . Thus,  $U^h \subseteq D^g \cap H$  as  $U^h \subseteq H$ .

(b) Clearly, there exists  $h \in H$  such that  $(P^g \cap H)^h = P^y \cap H$ . Thus, we have  $(D^g \cap H)^h \subseteq P^y$ . Since  $D^y$  is strongly closed in  $P^y$ , we get that  $(D^g \cap H)^h \subseteq D^y$ , and so  $(D^g \cap H)^h \subseteq D^y \cap H$ . By a symmetric argument, we can reach that  $(D^y \cap H)^{h^{-1}} \subseteq D^g \cap H$ , and so we get  $(D^g \cap H)^h = D^y \cap H$ .

(c) Let  $X \subseteq DN/N$  and suppose that  $(X)^y \subseteq PN/N$  for some  $y \in G$ . By an easy argument, we can find  $V \subseteq D$  such that  $X = VN/N$ .

Then we see that  $VN \subseteq DN$  and  $(VN)^y = V^yN \subseteq PN$ . We need to show that  $V^yN \subseteq DN$ . Note that  $\langle V^y \rangle = \langle V \rangle^y$  is a  $p$ -subgroup of  $PN$ . Since  $P \in \text{Syl}_p(PN)$ , there exists  $x \in PN$  such that  $V^y \subseteq P^x$ . Thus, we obtain that  $V^y \subseteq D^x$  as  $D^x$  is strongly closed in  $P^x$  and  $V^y \subseteq D^x$ . It follows that  $V^yN \subseteq D^xN$ . Write  $x = mn$  for  $m \in P$  and  $n \in N$ . Note that  $D^x = D^{mn} = D^n$  as  $D$  is a normal set in  $P$ . It follows that  $D^xN = D^nN = DN$ . Consequently,  $V^yN \subseteq DN$  as desired.  $\square$

Let  $\mathcal{L}_p(G)$  be the set of all  $p$ -subgroups of  $G$ . A map  $W : \mathcal{L}_p(G) \rightarrow \mathcal{L}_p(G)$  is called a **conjugacy functor** if the followings hold for each  $U \in \mathcal{L}_p(G)$ :

- (i)  $W(U) \leq U$ ,
- (ii)  $W(U) \neq 1$  unless  $U = 1$ , and
- (iii)  $W(U)^g = W(U^g)$  for all  $g \in G$ .

A **section of  $G$**  is a quotient group  $H/K$  where  $K \trianglelefteq H \leq G$ . Let  $\mathcal{L}_p^*(G)$  be the set of all sections of  $G$  that are  $p$ -groups. A map  $W : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  is called a **section conjugacy functor** if the followings hold for each  $H/K \in \mathcal{L}_p^*(G)$ :

- (i)  $W(H/K) \leq H/K$ ,
- (ii)  $W(H/K) \neq 1$  unless  $H/K = 1$ , and
- (iii)  $W(H/K)^g = W(H^g/K^g)$  for all  $g \in G$ .
- (iv) Suppose that  $N \trianglelefteq H$ ,  $N \leq K$  and  $K/N$  is a  $p'$ -group. Let  $P/N$  be a Sylow  $p$ -subgroup of  $H/N$  and set  $W(P/N) = L/N$ . Then  $W(H/K) = LK/K$ .

For more information about section conjugacy functors and their properties, we refer to [3]. Note that a sufficient condition for (iii) and (iv) is the following: whenever  $Q, R \in \mathcal{L}_p^*(G)$  and  $\phi : Q \rightarrow R$  is an isomorphism,  $\phi(W(Q)) = W(R)$ . Thus, the operations like  $ZJ_x, \Omega ZJ_x$  and  $J_x$  are section conjugacy functors for  $x \in \{o, r, e\}$ .

**Remark 4.2.** Let  $G$  be a group,  $K \trianglelefteq H \leq G$  and  $W : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  be a section conjugacy functor. The restriction of  $W$  on  $\mathcal{L}_p(H/K)$  is a function satisfying (i), (ii) and (iii), and so  $W$  is also a conjugacy functor on  $\mathcal{L}_p(H/K)$ . Now let  $K = 1$ . We identify this trivial quotient  $H/1$  with  $H$ , and so instead of saying that  $W$  is a conjugacy functor on  $\mathcal{L}_p(H/1)$ , we simply say that  $W$  is a conjugacy functor on  $\mathcal{L}_p(H)$ . Thus,  $W$  is also a conjugacy functor on  $\mathcal{L}_p(G)$  in particular.

**Definition 4.3.** Let  $P \in \text{Syl}_p(G)$  and  $D$  be a strongly closed subset in  $P$ . Let  $W : \mathcal{L}_p(G) \rightarrow \mathcal{L}_p(G)$  be a conjugacy functor. We define the **localization of a conjugacy functor  $W$**  on  $D$  as a function  $W_D : \mathcal{L}_p(G) \rightarrow \mathcal{L}_p(G)$  with the following settings: For each  $p$ -subgroup  $U$  of  $P$ , set

$$W_D(U) = \begin{cases} W(\langle U \cap D \rangle) & \text{if } \langle U \cap D \rangle \neq 1 \\ W(U) & \text{if } \langle U \cap D \rangle = 1 \end{cases}$$

and for all  $V \in \mathcal{L}_p(G)$  and  $x \in G$  such that  $V^x \leq P$  set  $W_D(V) = (W_D(V^x))^{x^{-1}}$ .

**Lemma 4.4.** Let  $P \in \text{Syl}_p(G)$  and  $D$  be a strongly closed subset in  $P$ . Let  $W : \mathcal{L}_p(G) \rightarrow \mathcal{L}_p(G)$  be a conjugacy functor. Then the localization of  $W$  on  $D$ , denoted by  $W_D$ , is a conjugacy functor. Moreover, the equality  $W_D = W_{D^s}$  holds for all  $s \in G$ .

**Proof.** Since  $W$  is a conjugacy functor, it is easy to see that  $W_D(U) \leq U$  and  $W_D(U) \neq 1$  unless  $U = 1$  for each  $U \in \mathcal{L}_p(G)$  by our settings.

Now we need to show that  $W_D(U)^g = W_D(U^g)$  for all  $g \in G$  and  $U \in \mathcal{L}_p(G)$ , and indeed  $W_D$  is well defined. Suppose that  $U, U^g \leq P$  for some  $g \in G$ . We first show that  $W_D(U)^g = W_D(U^g)$  for this special case. Note that  $(U \cap D)^g \subseteq U^g \leq P$ , and so  $(U \cap D)^g \subseteq U^g \cap D$  as  $D$  is strongly closed in  $P$ . On the other hand,  $(U^g \cap D)^{g^{-1}} \subseteq U \leq P$ , and so  $(U^g \cap D)^{g^{-1}} \subseteq U \cap D$  as  $D$  is strongly closed in  $P$ . By showing the reverse inequality, we obtain that  $(U \cap D)^g = U^g \cap D$ . If  $\langle U \cap D \rangle = 1$  then  $\langle U^g \cap D \rangle = \langle U \cap D \rangle^g = 1$ , and so we have  $W_D(U)^g = W(U)^g = W(U^g) = W_D(U^g)$ . The second equality holds as  $W$  is a conjugacy functor. Assume  $\langle U \cap D \rangle \neq 1$ . Then we have  $\langle U^g \cap D \rangle \neq 1$  as  $U^g \cap D = (U \cap D)^g$ . It follows that  $W_D(U)^g = W(\langle U \cap D \rangle)^g = W(\langle U \cap D \rangle^g) = W(\langle U^g \cap D \rangle) = W_D(U^g)$ .

Let  $V \in \mathcal{L}_p(G)$  and  $x, y \in G$  such that  $V^x, V^y \leq P$ . Then by setting  $U = V^x$  and  $g = x^{-1}y$ , we have  $U^g = V^y$  and  $W_D(U)^g = W_D(U^g)$  by the previous paragraph. It follows that  $W_D(V^x)^{x^{-1}y} = W_D(V^y)$ . Then  $W_D(V^x)^{x^{-1}} = W_D(V^y)^{y^{-1}}$ , and so  $W_D$  is well defined. Let  $z \in G$  and set  $t = z^{-1}x$ . Then  $(V^z)^t = V^x \leq P$ . Thus,

$$W_D(V^z) = W_D(V^{zt})^{t^{-1}} = W_D(V^x)^{x^{-1}z} = (W_D(V^x)^{x^{-1}})^z = W_D(V)^z$$

which shows that  $W_D$  is a conjugacy functor.

Since  $D^s$  is strongly closed in  $P^s$ ,  $W_{D^s}$  is a conjugacy functor for  $s \in G$  by the first part. Let  $U \leq P$ . Note that  $U^s \leq P^s$ . Assume  $\langle U^s \cap D^s \rangle = 1$ . Then  $W_{D^s}(U) = W_{D^s}(U^s)^{s^{-1}} = W(U^s)^{s^{-1}} = W(U) = W_D(U)$ . The last equality hold as we have  $\langle U \cap D \rangle = 1$  in that case. Assume  $\langle U^s \cap D^s \rangle \neq 1$ . Then  $\langle U \cap D \rangle = \langle U^s \cap D^s \rangle^{s^{-1}} \neq 1$ . It follows that  $W_{D^s}(U) = W_{D^s}(U^s)^{s^{-1}} = W(\langle U^s \cap D^s \rangle)^{s^{-1}} = W(\langle U \cap D \rangle)^{s^{-1}} = W(\langle U \cap D \rangle) = W_D(U)$ . Thus, these functions agree on the subgroups of  $P$ . Let  $V \in \mathcal{L}_p(G)$ . Then  $V = U^t$  for some  $U \leq P$  and  $t \in G$ . Then we have  $W_D(V) = W_D(U^t) = W_D(U)^t = W_{D^s}(U)^t = W_{D^s}(U^t) = W_{D^s}(V)$  by using the fact that both functions are conjugacy functors.  $\square$

**Remark 4.5.** Although a strongly closed set is nonempty according to Definition 1.1, if we take  $D = \emptyset$  in Definition 4.3, we get  $W_\emptyset(U) = W(U)$ . Thus, we set  $W_\emptyset = W$  for any conjugacy functor  $W$ .

**Lemma 4.6.** *Let  $P \in \text{Syl}_p(G)$  and  $D$  be a strongly closed subset in  $P$ . Let  $K \trianglelefteq H \leq G$ ,  $N \trianglelefteq G$  and  $g \in G$  such that  $P^g \cap H \in \text{Syl}_p(H)$ . Assume  $W : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  is a section conjugacy functor. Then the followings hold:*

- (a)  $W_{D^g \cap H} : \mathcal{L}_p(H) \rightarrow \mathcal{L}_p(H)$  is a conjugacy functor. Moreover,  $W_{D^g \cap H}$  is equal to the restriction of  $W_D$  to  $\mathcal{L}_p(H)$ .
- (b)  $W_{DN/N} : \mathcal{L}_p(G/N) \rightarrow \mathcal{L}_p(G/N)$  is a conjugacy functor.
- (c)  $W_{(D^g \cap H)K/K} : \mathcal{L}_p(H/K) \rightarrow \mathcal{L}_p(H/K)$  is a conjugacy functor.

**Proof.** (a) By taking the restrictions of  $W$  to  $\mathcal{L}_p(H)$ , we obtain a conjugacy functor  $W : \mathcal{L}_p(H) \rightarrow \mathcal{L}_p(H)$ (see Remark 4.2). Note that  $D^g \cap H$  is strongly closed in  $H \cap P^g$  with respect to  $H$  if  $D^g \cap H$  is nonempty by Lemma 4.1(a). Then  $W_{D^g \cap H} :$

$\mathcal{L}_p(H) \rightarrow \mathcal{L}_p(H)$  is a conjugacy functor by Lemma 4.4 and Remark 4.5. Let  $U \in \mathcal{L}_p(H)$ . Then it is easy to see that  $W_{D^g}(U) = W_{D^g \cap H}(U)$  by their definitions, and so we get  $W_D(U) = W_{D^g}(U) = W_{D^g \cap H}(U)$  by Lemma 4.4. Thus, the map  $W_{D^g \cap H}$  is equal to the restriction of  $W_D$  to  $\mathcal{L}_p(H)$ .

Part (b) follows by Lemma 4.1(c) and Lemma 4.4. Part (c) also follows in a similar fashion. □

**Remark 4.7.** It should be noted that we only need  $W$  to be a conjugacy functor to establish Lemma 4.6(a).

**Lemma 4.8.** *Let  $P \in \text{Syl}_p(G)$ ,  $D$  be a strongly closed subset in  $P$  and  $W : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  be a section conjugacy functor. For each  $H/K \in \mathcal{L}_p^*(G)$ , pick  $g \in G$  such that  $P^g \cap H \in \text{Syl}_p(H)$ . We define  $W_D^* : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  by setting  $W_D^*(H/K) = W_{(D^g \cap H)K/K}(H/K)$  for each  $H/K \in \mathcal{L}_p^*(G)$ . Then*

$$W_D^*(H/K) = \begin{cases} W(\langle D^g \cap H \rangle K/K), & \text{if } D^g \cap H \not\subseteq K. \\ W(H/K), & \text{if } D^g \cap H \subseteq K. \end{cases}$$

Moreover,  $W_D^*$  is a section conjugacy functor.

**Proof.** First, we show that  $W_D^*$  is well defined. Pick  $y \in G$  such that  $P^y \cap H \in \text{Syl}_p(H)$ . We claim that  $W_{(D^g \cap H)K/K}(H/K) = W_{(D^y \cap H)K/K}(H/K)$ . We see that  $D^g \cap H$  is conjugate to  $D^y \cap H$  by an element of  $H$  by Lemma 4.1(b), and so  $(D^g \cap H)K/K$  and  $(D^y \cap H)K/K$  are conjugate in  $H/K$ . It follows that  $W_{(D^g \cap H)K/K} = W_{(D^y \cap H)K/K}$  by Lemma 4.4. Hence,  $W_D^*$  is well defined.

Suppose that  $D^g \cap H \subseteq K$ . Then  $H/K \cap (D^g \cap H)K/K = K/K$ , and so  $W_{(D^g \cap H)K/K}(H/K) = W(H/K)$ . If  $D^g \cap H \not\subseteq K$  then  $H/K \cap (D^g \cap H)K/K \neq K/K$ , and so  $W_{(D^g \cap H)K/K}(H/K) = W(\langle D^g \cap H \rangle K/K)$  by its definition, which shows the first part.

Note that  $W_D^*(H/K) \leq H/K$  and  $W_D^*(H/K) \neq 1$  unless  $H/K = 1$  by Lemma 4.6(c). Now we need to show that (iii) and (iv) in the definition of a section conjugacy functor hold.

Pick  $x \in G$ . Since  $(D^g \cap H)K/K$  is a strongly subset in  $(P^g \cap H)K/K$ ,  $(D^g \cap H)^x K^x / K^x$  is a strongly closed subset in  $(P^g \cap H)^x K^x / K^x$ . Moreover,  $D^g \cap H \subseteq K$  if and only if  $D^{g^x} \cap H^x \subseteq K^x$ . Thus, if  $W_D^*(H/K) = W(H/K)$ , then  $W_D^*(H^x / K^x) = W(H^x / K^x)$ . It follows that

$$W_D^*(H^x / K^x) = W(H^x / K^x) = W(H/K)^x = W_D^*(H/K)^x.$$

The second equality holds as  $W$  is a section conjugacy functor.

If  $W_D^*(H/K) = W(\langle D^g \cap H \rangle K/K)$  then

$$W_D^*(H^x / K^x) = W(\langle D^{g^x} \cap H^x \rangle K^x / K^x) = W(\langle \langle D^g \cap H \rangle K/K \rangle^x) = W_D^*(H/K)^x.$$

The last equality holds as  $W$  is a section conjugacy functor. Thus, we see that (iii) is satisfied.

Let  $N \trianglelefteq H$  such that  $N \leq K$  and  $K/N$  is a  $p'$ -group. Let  $X/N$  be a Sylow  $p$ -subgroup of  $H/N$ . We need to show that if  $W_D^*(X/N) = L/N$  then  $W_D^*(H/K) = LK/K$ . Pick  $h \in H$  such that  $(X/N)^h \supseteq (D^g \cap H)N/N$ . By part (iii), we have  $W_D^*(X/N)^h = L^h/N^h = L^h/N$ . If we could show that  $W_D^*(H/K) = L^hK/K$ , we can conclude that

$$W_D^*(H/K) = W_D^*((H/K)^{h^{-1}}) = W_D^*(H/K)^{h^{-1}} = (L^hK/K)^{h^{-1}} = LK/L$$

by part (iii). Thus, we see that it is enough to show the claim for  $(X/N)^h$ , and so we may simply assume that  $(D^g \cap H)N/N \subseteq X/N$ .

Clearly  $\langle D^g \cap H \rangle$  is a  $p$ -group. Since  $K/N$  is a  $p'$ -group, we see that  $D^g \cap H \subseteq K$  if and only if  $D^g \cap H \subseteq N$ . Thus, if  $W_D^*(H/K) = W(H/K)$  then  $W_D^*(X/N) = W(X/N)$ . It follows that  $W_D^*(H/K) = LK/K$  as  $W$  is a section conjugacy functor.

Assume that  $D^g \cap H \not\subseteq K$ . Then  $W_D^*(H/K) = W(\langle D^g \cap H \rangle K/K)$  and  $W_D^*(X/N) = W(\langle D^g \cap H \rangle N/N) = L/N$ . Now write  $H^* = \langle D^g \cap H \rangle K$  and  $P^* = \langle D^g \cap H \rangle N$ . Observe that  $P^*/N \in \text{Syl}_p(H^*/N)$  and recall  $K/N$  is a  $p'$ -group. Since  $W$  is a section conjugacy functor and  $W(P^*/N) = L/N$ , we get  $W(H^*/K) = LK/K$ . Then the result follows.  $\square$

**Proof of Theorem C.** Let  $p$  be an odd prime,  $G$  be a  $p$ -stable group and  $P \in \text{Syl}_p(G)$ . Suppose that  $D$  is a strongly closed subgroup in  $P$ . Let  $H$  be a  $p$ -constrained subgroup of  $G$  and  $g \in G$  such that  $P^g \cap H \in \text{Syl}_p(H)$ . We see that  $H$  is  $p$ -stable by Lemma 3.10.

Let  $W \in \{ZJ_o, \Omega ZJ_e, \Omega ZJ_r\}$ . It follows that  $W_{D^g \cap H}$  is a conjugacy functor by Lemma 4.6(a). Note that  $W_{D^g \cap H}(P^g \cap H) \in \{W(D^g \cap H), W(P^g \cap H)\}$ , where the former case occurs if  $D^g \cap H \neq 1$  and the later case occurs if  $D^g \cap H = 1$ . Thus  $N_H(W_{D^g \cap H}(P^g \cap H))$  controls strong  $H$ -fusion in  $P^g \cap H$  by Theorem 3.8 in both cases. Since  $W_{D^g \cap H}(P^g \cap H) = W_D(P^g \cap H)$  by the second part of Lemma 4.6(a), we have that  $N_H(W_D(P^g \cap H))$  controls strong  $H$ -fusion in  $P^g \cap H$ .

Now assume that  $N_G(U)$  is  $p$ -constrained for each nontrivial subgroup  $U$  of  $P$ . Fix  $1 \neq U \leq P$ . Let  $S \in \text{Syl}_p(N_G(U))$ . By the arguments in the previous paragraph, we see that the normalizer of  $W_D(S)$  in  $N_G(U)$  controls strong  $N_G(U)$ -fusion in  $S$ , and so we obtain that  $N_G(W_D(P))$  control strong  $G$ -fusion in  $P$  by [3, Theorem 5.5(i)]. It follows that the normalizers of the subgroups  $Z(J_o(D))$ ,  $\Omega(Z(J_r(D)))$  and  $\Omega(Z(J_e(D)))$  control strong  $G$ -fusion in  $P$ .  $\square$

**Lemma 4.9.** *Let  $p$  be an odd prime,  $G$  be a group, and  $P \in \text{Syl}_p(G)$ . Suppose that  $D$  is a strongly closed subgroup in  $P$ . Let  $G^*$  be a section of  $G$  such that  $G^*$  is  $p$ -stable and  $C_{G^*}(O_p(G^*)) \leq O_p(G^*)$ . If  $S \in \text{Syl}_p(G^*)$ , then  $W_D^*(S) \trianglelefteq G^*$  for each  $W \in \{ZJ_o, \Omega ZJ_e, \Omega ZJ_r\}$ .*

**Proof.** Note that  $D$  is also a strongly closed set in  $P$ . We assume the notation of Lemma 4.8. Let  $W \in \{ZJ_o, \Omega ZJ_e, \Omega ZJ_r\}$ . Then clearly  $W$  is a section conjugacy

functor. It follows that  $W_D^* : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  is a section conjugacy functor by Lemma 4.8. Let  $G^* = X/K$  be a section of  $G$  such that

$$C_{G^*}(O_p(G^*)) \leq O_p(G^*).$$

Let  $S = H/K \in \text{Syl}_p(G^*)$ . Then we see that  $W_D^*(S) = W(S)$  if  $D^g \cap H \subseteq K$ . In this case,  $W(S) = Z(J_o(S)), \Omega(Z(J_e(S)))$  or  $\Omega(Z(J_r(S)))$  which are normal subgroups of  $G^*$  by Theorem B. If  $D^g \cap H \not\subseteq K$  then  $(D^g \cap H)K/K$  is a strongly closed subgroup in  $S = H/K$  with respect to  $G^*$  by Lemma 4.1. Write  $D^* = (D^g \cap H)K/K$ . Then we have

$$W_D^*(S) = W(D^*) = Z(J_o(D^*)), \Omega(Z(J_e(D^*))), \text{ or } \Omega(Z(J_r(D^*)))$$

which are normal subgroups of  $G^*$  by Theorem B. Thus, we see that  $W_D^*(S) \trianglelefteq G^*$  for all cases. □

Now we are ready to prove Theorems D and E.

**Proof of Theorem D.** Let  $p$  be an odd prime,  $G$  be a Qd( $p$ )-free group, and  $P \in \text{Syl}_p(G)$  as in our hypothesis. Since  $G$  does not involve a section isomorphic to Qd( $p$ ), every section of  $G$  is  $p$ -stable by [3, Proposition 14.7]. Now let  $W \in \{ZJ_o, \Omega ZJ_e, \Omega ZJ_r\}$ . Then we have that  $W_D^* : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  is a section conjugacy functor by Lemma 4.8. Let  $G^*$  be a section of  $G$  such that  $C_{G^*}(O_p(G^*)) \leq O_p(G^*)$  and let  $S \in \text{Syl}_p(G^*)$ . Then we see that  $W_D^*(S) \trianglelefteq G^*$  by Lemma 4.9. It follows that  $N_G(W_D^*(P))$  controls strong  $G$ -fusion in  $P$  by [3, Theorem 6.6]. We see that  $W_D^*(P) = Z(J_o(D)), \Omega(Z(J_e(D))),$  or  $\Omega(Z(J_r(D)))$  according to choice of  $W$ , which completes the proof. □

**Proof of Theorem E.** Let  $W \in \{ZJ_o, \Omega ZJ_e, \Omega ZJ_r\}$ . Then  $W_D^* : \mathcal{L}_p^*(G) \rightarrow \mathcal{L}_p^*(G)$  is a section conjugacy functor by Lemma 4.8. Let  $G^*$  be a section of  $G$  such that  $C_{G^*}(O_p(G^*)) \leq O_p(G^*)$  and  $G^*/O_p(G^*)$  is  $p$ -nilpotent. Suppose also that  $S^* \in \text{Syl}_p(G^*)$  is a maximal subgroup of  $G^*$ . Let  $H$  be the normal Hall  $p'$ -subgroup of  $G^*/O_p(G^*)$ . Write  $S = S^*/O_p(G^*)$ . Then  $S$  is also maximal in  $G^*/O_p(G^*)$  and  $S$  acts on  $H$  via coprime automorphisms. If  $1 < U \leq H$  is  $S$ -invariant then  $SU = G^*/O_p(G^*)$  by the maximality of  $S$ . Since  $SH = G^*/O_p(G^*)$  and  $S \cap H = 1$ , we see that  $U = H$ . Thus, there is no proper nontrivial  $S$ -invariant subgroup of  $H$ . On the other hand, we may choose an  $S$ -invariant Sylow subgroup of  $H$  by [8, Theorem 3.23(a)]. This forces  $H$  to be a  $q$ -group for some prime  $q$ , and so  $H' < H$ . It follows that  $H$  is abelian due to the fact that  $H'$  is  $S$ -invariant.

Let  $H^*$  be a Hall  $p'$ -subgroup of  $G^*$ . Then we see that  $H^*O_p(G^*)/(O_p(G^*)) \cong H^*$ . Thus, we observe that Hall  $p'$ -subgroups of  $G^*$  are also abelian. Since  $p$  is odd, we see that a Sylow 2-subgroup of  $G^*$  is abelian. This yields that  $G^*$  does not involve a section isomorphic to  $SL(2, p)$ , and so every section of  $G^*$  is  $p$ -stable by [3, Proposition 14.7]. Then, we obtain that  $W_D^*(S^*) \trianglelefteq G^*$  by Lemma 4.9. It follows that  $G$  is  $p$ -nilpotent by [3, Theorem 8.7]. □



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