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Corrigendum

Corrigendum to "Representations of ∗-semigroups associated to invariant kernels with values adjointable operators" [Linear Algebra Appl. 486 (2015) 361–388]

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We correct a lemma by adding the assumption that the ordered ∗-space is Archimedean and show by counter-examples and examples that this is needed.

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Recall that, see e.g. [2], a complex vector space Z is called an *ordered $*$-space*, if:

(a1) Z has an *involution* $*$, that is, a map $Z \ni z \mapsto z^* \in Z$ that is *conjugate linear* ($(sx+ty)^* = \bar{s}x^* + \bar{t}y^*$ for all $s, t \in \mathbb{C}$ and all $x, y \in Z$) and *involutive* ($(z^*)^* = z$ for all $z \in Z$).

(a2) In Z there is a *cone* Z^+ ($sx + ty \in Z^+$ for all numbers $s, t \geq 0$ and all $x, y \in Z^+$), that is *strict* ($Z^+ \cap -Z^+ = \{0\}$), and consisting of *selfadjoint elements* only ($z^* = z$ for all $z \in Z^+$). This cone is used to define a *partial order* on the real vector space of all selfadjoint elements in Z: $z_1 \geq z_2$ if $z_1 - z_2 \in Z^+$.

An ordered $*$-space is called an *Archimedean ordered $*$-space* if its cone Z_+ has the *Archimedean property*, that is,

(a3) If $y \in Z$ is selfadjoint and $z \in Z_+$ having the property $y + rz \geq 0$ for all $r \in (0, +\infty)$, it follows that $y \in Z_+$.

Remark 1.1. Letting Z be an ordered $*$-space with its specified cone Z_+, the following assertions are equivalent:

(i) Z is Archimedean.
(ii) If $y = y^* \in Z$ and $x \in Z_+$ is such that $ny + x \in Z_+$ for all $n \in \mathbb{N}$ then $y \in Z_+$.
(iii) If $y = y^* \in Z$ and $x \in Z_+$ is such that $y + r_n x \in Z_+$ for all $n \in \mathbb{N}$ and some sequence $(r_n)_n$ with all positive elements and $r_n \to 0$ then $y \in Z_+$.

The complex vector space Z is called a *topologically ordered $*$-space* if it is an ordered $*$-space, that is, axioms (a1) and (a2) hold and, in addition:

(a4) Z is a *Hausdorff locally convex space*.
(a5) The cone Z_+ is closed with respect to this topology.
(a6) There exists a collection of seminorms $\{p_j\}_{j \in \mathcal{J}}$ defining the topology of Z such that, for any $j \in \mathcal{J}$, p_j is *increasing*, in the sense that, $0 \leq x \leq y$ implies $p_j(x) \leq p_j(y)$.

Remark 1.2. Any topologically ordered $*$-space is Archimedean. Indeed, if $y = y^* \in Z$ and $x \in Z_+$ is such that $y + rx \geq 0$ for all $r > 0$ then $y = \lim_{r \to 0+}(y + rx) \in Z_+$.

As a consequence of the previous remark, all ordered $*$-spaces from Examples 1.2 in [2] are Archimedean since they are topologically ordered $*$-spaces, as seen in Examples 1.2 in [3]. In the following we give an example of a non-Archimedean ordered $*$-space, inspired from [5].

Example 1.3. Consider the ordered $*$-space $\mathcal{F}_{\mathbb{N}}$ of all complex sequences indexed on $\mathbb{N}$ with finite support, endowed with the $*$ operation of taking complex conjugate of all entries, and the cone $\mathcal{F}_{\mathbb{N}}^{+}$ consisting of all nonzero selfadjoint elements which have their last nonzero entry positive, plus the sequence with all entries 0. It is easy to see that $(\mathcal{F}_{\mathbb{N}}, \mathcal{F}_{\mathbb{N}}^{+})$ is non-Archimedean: for instance, consider $y = (1, -1, 0, 0, \ldots)$ which is selfadjoint and, letting $x = (0, 0, 1, 0, 0, \ldots) \in \mathcal{F}_{\mathbb{N}}^{+}$ we observe that the sum $y + rx \in \mathcal{F}_{\mathbb{N}}^{+}$ for all $r > 0$, however, $y \notin \mathcal{F}_{\mathbb{N}}^{+}$.

Given an ordered $*$-space Z with its cone Z_+, an element $e \in Z_+$ is called an *order unit* if for any $x = x^* \in Z$ there exists $r > 0$ such that $re \geq x$. An order unit $e \in Z$ is called *Archimedean* if whenever $x = x^* \in Z$ is such that $re + x \in Z_+$ for all numbers $r > 0$ it follows that $x \in Z_+$, cf. [1].

Remark 1.4. Let Z be an ordered $*$-space with order unit e. Then Z is Archimedean if and only if the order unit e is Archimedean. Clearly, if Z is Archimedean then the order unit e is Archimedean. Conversely, assume that the order unit e is Archimedean, let $y = y^* \in Z$ and $x \in Z_+$ be such that $y + rx \in Z_+$ for all real numbers $r > 0$. Let $t > 0$ be such that $te \geq x$ hence $y + rte \geq y + rx \geq 0$, hence $y + rte \geq 0$, equivalently, $se + y \geq 0$ for all real numbers $s > 0$. Since e is Archimedean it follows that $y \geq 0$.

Examples 1.5. (1) The ordered $*$-space $\mathcal{F}_{\mathbb{N}}$ from Example 1.3 does not have any order unit. Indeed, due to the way the cone $\mathcal{F}_{\mathbb{N}}^{+}$ is defined, such an order unit cannot have a finite support and hence does not belong to $\mathcal{F}_{\mathbb{N}}$.

(2) Let $Z = \mathbb{C}^3$, canonically embedded in $\mathcal{F}_{\mathbb{N}}$ and let $Z_+ = Z \cap \mathcal{F}_{\mathbb{N}}^{+}$. Then Z is not Archimedean but, for example, $e = (0, 0, 1)$ is an order unit.

Also, recall that given a complex linear space $\mathcal{E}$ and an ordered $*$-space space Z, a *Z-gramian*, also called a *Z-valued inner product*, is a mapping $\mathcal{E} \times \mathcal{E} \ni (x, y) \mapsto [x, y] \in Z$ subject to the following properties:

(ve1) $[x, x] \geq 0$ for all $x \in \mathcal{E}$, and $[x, x] = 0$ if and only if $x = 0$.
(ve2) $[x, y] = [y, x]^*$ for all $x, y \in \mathcal{E}$.
(ve3) $[x, \alpha y_1 + \beta y_2] = \alpha[x, y_1] + \beta[x, y_2]$ for all $\alpha, \beta \in \mathbb{C}$ and all $x_1, x_2 \in \mathcal{E}$.

If the axiom (ve1) is replaced by the weaker axiom

(ve1)$'$ $[x, x] \geq 0$ for all $x \in \mathcal{E}$,

then we call $[\cdot, \cdot]$ a *positive semidefinite Z-gramian*.

A complex linear space $\mathcal{E}$ onto which a Z-gramian $[\cdot, \cdot]$ is specified, for a certain ordered $*$-space Z, is called a *VE-space* (Vector Euclidean space) over Z, cf. [6].

Our correction is that, Lemma 1.4 in [2] must be stated for Archimedean ordered $*$-spaces instead of ordered $*$-spaces. This error has its origin in a wrong attribution, more precisely, the lemma was proven in [6] for the case of a topologically ordered $*$-space Z and, see Remark 1.2, this implies that Z is Archimedean.

Lemma 1.6. *Let Z be an Archimedean ordered $*$-space, $\mathcal{E}$ a complex vector space and $[\cdot,\cdot]\colon \mathcal{E}\times\mathcal{E}\to Z$ a positive semidefinite Z-gramian. If $f\in\mathcal{E}$ is such that $[f,f]=0$, then $[f,f']=[f',f]=0$ for all $f'\in\mathcal{E}$.*

Proof. For arbitrary $\lambda\in\mathbb{C}$ we have

$$0\leq [f+\lambda f', f+\lambda f'] = \lambda[f',f]+\overline{\lambda}[f,f']+|\lambda|^2[f',f']. \tag{1.1}$$

We claim that

$$\lambda[f',f]+\overline{\lambda}[f,f']=0. \tag{1.2}$$

To see this, let $x=|\lambda|^2[f',f']\in Z_+$ and $y=\lambda[f',f]+\overline{\lambda}[f,f']$. Clearly, $y=y^*$ and then, by (1.1), by changing λ to $r\lambda$ for arbitrary real number $r>0$ it follows that $y+rx\geq 0$. Since Z is Archimedean, it follows that $y\geq 0$ and hence (1.2) is proven.

Then, we take first $\lambda=1$ and then $\lambda=\mathrm{i}$ in (1.2) and get

$$[f',f]+[f,f']=0 \text{ and } [f',f]-[f,f']=0,$$

hence $[f,f']=0$. □

In the following we provide an example showing the necessity of the extra condition of Archimedean property in Lemma 1.6.

Example 1.7. Consider the ordered $*$-space Z as in Example 1.5.(2). Now consider the vector space $\mathcal{E}=\mathbb{C}^3$ with the positive semidefinite Z-gramian $[\cdot,\cdot]\colon\mathcal{E}\times\mathcal{E}\to Z$, given by

$$[x,y]:=(x_1\overline{y_1}-(x_1+x_3)\overline{(y_1+y_3)}, x_2\overline{y_2}-(x_2+x_3)\overline{(y_2+y_3)}, x_3\overline{y_3}), \tag{1.3}$$

where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$. This Z-gramian is positive semidefinite but not positive definite, i.e. there is a nonzero $x\in\mathcal{E}$ such that $[x,x]=0$, e.g. take $x=(1,0,0)$. On the other hand, for $y=(0,0,1)$ we have $[x,y]=(-1,0,0)$, hence the conclusion in Lemma 1.6 fails.

Lemma 1.6 is used essentially in the constructions of linearizations and reproducing kernel spaces in [2] and hence throughout it the ordered $*$-space Z should be assumed to be Archimedean.

This lemma is used as well in the algebraic dilation theorems for weakly positive semidefinte kernels in [4]. Consequently, in [4], Theorem 3.4, our main algebraic dilation theorem, which uses Lemma 1.6, must be stated for an Archimedean ordered $*$-space Z.

Lemma 3.1(2) and 3.1(3), Theorem 3.10 and Theorem 4.3, must also be stated for an Archimedean ordered $*$-space Z. On the other hand, clauses in Theorem 4.3 concerning a topologically ordered $*$-space and all the other theorems remain the same.

Declaration of competing interest

We declare that we have no conflict of interest.

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