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The Paley graph conjecture and Diophantine m-tuples

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ABSTRACT

A Diophantine m-tuple with property $D(n)$, where n is a nonzero integer, is a set of m positive integers $\{a_1, ..., a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \leqslant i < j \leqslant m$. It is known that $M_n = \sup\{|\mathcal{S}| : \mathcal{S} \text{ is a } D(n) \text{ } m\text{-tuple}\}$ exists and is $O(\log |n|)$. In this paper, we show that the Paley graph conjecture implies that the upper bound can be improved to $\ll (\log |n|)^{\epsilon}$, for any $\epsilon > 0$.

1. Introduction

A Diophantine m-tuple with property $D(n)$, where n is a nonzero integer, is a set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \leqslant i < j \leqslant m$.

Diophantus found the quadruple $\{1, 33, 68, 105\}$ with property $D(256)$. The first $D(1)$-quadruple $\{1, 3, 8, 120\}$ was found by Fermat (cf. [9]). Baker and Davenport showed in 1969 (cf. [2]) that Fermat's set is the only extension of $\{1, 3, 8\}$ to a $D(1)$-quadruple,

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and thus it cannot be extended to a $D(1)$-quintuple. This result follows from a paper of Baker [1] published in 1968 on linear forms in logarithms of algebraic numbers, which is an effective version of Gelfond's theorem used for solutions of Diophantine equations in two unknowns and a reduction method introduced in [2].

Baker and Davenport's remarkable result was the first step towards the folklore conjecture which predicts that there are no $D(1)$-quintuples. In 2004, Dujella (cf. [13]) proved using similar methods to those of Baker and Davenport that there is no sextuple with property $D(1)$ and there are only finitely many effectively computable $D(1)$-quintuples. In 2018, in a very recent paper, He, Togbé and Ziegler (cf. [26]) have shown that there are no Diophantine quintuples, thereby settling this conjecture.

For $n \neq 1$, however, there are Diophantine quintuples and sextuples, such as the quintuple $\{1, 33, 105, 320, 18240\}$ with $n = 256$ (cf. [16]) and the sextuple $\{99, 315, 9920, 32768, 44460, 19534284\}$ with $n = 2985984$ (cf. [19]).

Thus, two related and important questions in the study of Diophantine m-tuples are (i) to determine, for a given n and m, the number of possible $D(n)$-m-tuples; and (ii) to estimate the quantity

$$M_n = \sup\{|\mathcal{S}| : \mathcal{S} \text{ is a } D(n) \text{ } m\text{-tuple}\}.$$

The first observation is that there is no infinite Diophantine m-tuples, for any $n \neq 0$, since the number of integral points on the elliptic curve

$$y^2 = (a_1x + n)(a_2x + n)(a_3x + n)$$

is finite, which follows from a celebrated theorem of Siegel (see, for example, [31]). Unfortunately, known bounds (cf. [30]) for the number of integral solutions depend on n, a_1, a_2, and a_3. On the other hand, as a result of a conjecture of Caporaso, Harris, and Mazur [6], the hyperelliptic curve of genus 2 given by

$$y^2 = (a_1x + n)(a_2x + n)(a_3x + n)(a_4x + n)(a_5x + n)$$

has a bounded number of integral points, independent of n and the coefficients $a_1, \ldots, a_5$. This would imply that $\sup_n M_n$ is bounded. This observation has been made by Dujella in [12] and the first result in this direction is due to Dujella and Luca [11], who proved that M_n is bounded by an absolute constant whenever $|n|$ is prime, and that, for every $\varepsilon > 0$, the set of positive integers n for which a $D(n)$ or $D(-n)$ Diophantine m-tuple exists with $m > (1 + \varepsilon) \log\log n$ is of asymptotic density zero.

A related elementary observation made independently by Brown [4], Gupta and Singh [21], and Mohanty and Ramasamy [28] is that $M_n \leqslant 3$ if $n \equiv 2 \pmod 4$. Indeed, being a square, $a_ia_j + n \equiv 0, 1 \pmod 4$. It follows that $a_i \not\equiv a_j \pmod 4$ and that at most one a_i can be even. On the other hand, if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then $M_n \geqslant 4$ (cf. [15]).

More generally, Dujella proved in [14] that $M_n \leqslant 31$ for $|n| \leqslant 400$ and

$$M_n < C \log |n|,$$

where $C = 15.476$ if $|n| > 400$ and $C = 9.078$ if $n > 10^{100}$. This is done by estimating separately the quantities

$$A_n = \sup\{|\mathcal{S} \cap [|n|^3, \infty)| : \mathcal{S} \text{ has } D(n)\}$$
$$B_n = \sup\{|\mathcal{S} \cap [n^2, |n|^3)| : \mathcal{S} \text{ has } D(n)\}$$
$$C_n = \sup\{|\mathcal{S} \cap [1, n^2]| : \mathcal{S} \text{ has } D(n)\}.$$

He proved that $A_n \leqslant 21$, $B_n < 0.6071 \log |n| + 2.152$, and $C_n < 11.006 \log |n|$ for $|n| > 400$. If $|n| > 10^{100}$, he showed that $C_n < 8.37 \log |n|$ and the final result is derived by combining all of these estimates. The most significant contribution comes from C_n and is obtained by using Gallagher's sieve inequality together with an estimate on double Dirichlet character sums due to Vinogradov. Improving these results, Murty and Becker [3] have recently shown that for any n,

$$M_n \leqslant 2.6071 \log |n| + O\left(\frac{\log |n|}{(\log \log |n|)^2}\right).$$

Our purpose in this manuscript is to relate the estimate of M_n for all sufficiently large n to the Paley graph conjecture (stated below) and show how one can improve the known estimates. The basic idea is to use this conjecture together with Gallagher's sieve inequality to handle B_n and C_n simultaneously which will lead to Theorem 1. We first recall the Paley Graph Conjecture.

Conjecture 1 *(Paley graph conjecture). Let $\varepsilon > 0$ be a real number, $S, T \subseteq \mathbb{F}_p$ for an odd prime p with $|S|, |T| > p^{\varepsilon}$, and χ any nontrivial multiplicative character modulo p. Then, there is some number $\delta = \delta(\varepsilon)$ for which the inequality*

$$\left| \sum_{a \in S, b \in T} \chi(a+b) \right| \leqslant p^{-\delta} |S||T| \tag{1}$$

holds for primes larger than some constant $C(\varepsilon)$.

The related estimate from Murty and Becker [3] yields

$$\left| \sum_{a \in S, b \in T} \chi(ab+n) \right| \leqslant \sqrt{p|S||T|},$$

where $p \nmid n$ is an odd prime and not both sets S and T contain 0.

In general, a positive answer for this conjecture is known only in the case $|S| > p^{1/2+\varepsilon}$, and $|T| > p^{\varepsilon}$. However, if the sets S and T have a certain structure, there are nontrivial estimates that can be obtained under weaker constraints on their size (see [7,18,27,32,33]) using recent advances in additive combinatorics. We quote from [32] why this conjecture is called the Paley graph conjecture below.

> A Paley graph is a graph $G(V, E)$ with vertex set $V = \mathbb{F}_p$ and edge set E such that $(a, b) \in E$ if and only if $a - b$ is a quadratic residue modulo p. For this graph to be undirected, it is also necessary that $p \equiv 1 \bmod 4$. Under this assumption, setting $S = -T$ in the conjecture and taking the Legendre symbol for the multiplicative character χ, we obtain the following remarkable statement: the clique number of the Paley graph and its independence number increase slower than p^{ε} for any positive ε.

On the other hand, Graham and Ringrose [20] proved that for infinitely many primes p, the least quadratic non-residue q is at least $c(\log p)(\log\log\log p)$ for some constant $c > 0$. Obviously, for these primes, (1) does not hold for $S = T = \{1, 2, \ldots, q/2\}$. See also [29] for a recent result of Mrazović relating Paley graphs to the result of Graham and Ringrose.

Our main theorem is:

Theorem 1. *If the Paley graph conjecture holds for some $\varepsilon \in (0, 1)$, then*

$$M_n \ll_{\varepsilon} (\log|n|)^{\frac{\varepsilon}{1-\varepsilon}} \left(1 + O\left(\frac{1}{(\log\log|n|)^2}\right)\right).$$

Remark 1. In particular, if for any $\varepsilon > 0$, the Paley graph conjecture holds, then

$$M_n \ll_{\varepsilon} (\log|n|)^{\varepsilon}.$$

We refer the reader to several other related papers that apply results from graph theory to certain problems concerning Diophantine m-tuples such as [23,24,5,10,25].

2. Preliminaries and proof of Theorem 1

We collect here some results needed to prove our main theorem.

Lemma 1 *(cf. [17, Thm 4.2]). For $x \geqslant 2$,*

$$|\theta(x) - x| < 3.965\frac{x}{\log^2 x},$$

where $\theta(x) = \sum_{p \leqslant x} \log p$. Also, for $x \geqslant 1$, $\theta(x) < 2x$.

Lemma 2. *For any* $\alpha \in (0,1)$,

$$\sum_{p \leqslant Q} \frac{\log p}{p^{\alpha}} = \frac{Q^{1-\alpha}}{1-\alpha}\left(1 + O\Big(\frac{1}{\log^2 Q}\Big)\right)$$

with an effectively computable implied constant. Furthermore,

$$\sum_{p \leqslant Q} \frac{\log p}{p^{\varepsilon}} < \frac{2Q^{1-\varepsilon}}{1-\varepsilon}. \tag{2}$$

Proof. Both results follow from Lemma 1 and partial integration. □

Lemma 3 *(cf. [17, Thm 5.1, Lemma 5.10]). For* $x > 1$,

$$\pi(x) \leqslant \frac{x}{\log x}\Big(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{7.59}{\log^3 x}\Big). \tag{3}$$

Furthermore, for $k \geqslant 4$,

$$p_k \leqslant k(\log k + \log\log k + 1) < 2k \log k, \tag{4}$$

where p_k *denotes the* k*th prime.*

Lemma 4 *(cf. [22, Thm 11]). For* $n \geqslant 3$,

$$\omega(n) \leqslant 1.38402 \frac{\log n}{\log\log n} \tag{5}$$

with equality for $n = p_1 p_2 \cdots p_9$, *where* $\omega(n)$ *is the number of distinct prime divisors of* n.

The following is an inequality which is a result of Gallagher's Sieve Inequality.

Lemma 5. *Let* $\mathcal{S}$ *be a subset of* $\{1, 2, \ldots, N\}$ *for some positive integer* N. *For any* $1 < Q \leqslant N$,

$$|\mathcal{S}| \leqslant \frac{\sum\limits_{p \leqslant Q} \log p - \log N}{\sum\limits_{p \leqslant Q} \frac{\log p}{|\mathcal{S}_p|} - \log N},$$

where $\mathcal{S}_p = \mathcal{S} \bmod p$, *provided the denominator is positive.*

Proof of Theorem 1. As mentioned in the introduction, $A_n \leqslant 21$. Thus, it is enough to take a $D(n)$-m-tuple $\mathcal{S}$ lying inside $[1, N]$, where $N = |n|^3$.

Assume Conjecture 1 holds for some $\varepsilon > 0$ at least for the quadratic character given by the Legendre's symbol $\left(\frac{\cdot}{p}\right)$. Then, there exists some $\delta = \delta(\varepsilon) > 0$ such that (1) holds for $p > C(\varepsilon)$ for some constant $C(\varepsilon) > 0$. Increasing $C(\varepsilon)$ if necessary we can make sure that the inequality

$$p^{\varepsilon}(1 - p^{-\delta}) \geqslant 3 \tag{6}$$

also holds for $p > C(\varepsilon)$. Note that $p^{\varepsilon} > 3$ for these primes. If $N \leqslant C(\varepsilon)$, we get $|\mathcal{S}| \leqslant C(\varepsilon)$. Otherwise, assume that N is large enough so that we can take a prime $p \nmid n$ with $C(\varepsilon) < p \leqslant Q < N$, where Q is to be chosen later.

For $i = \pm 1$, let S_i denote the elements a of $\mathcal{S}_p$ for which $\left(\frac{a}{p}\right) = i$. Thus, we have

$$|\mathcal{S}_p| \leqslant |S_1| + |S_{-1}| + 1,$$

with equality when $0 \in \mathcal{S}_p$. Since $p \nmid n$, for each $a \in S_i$, there is at most one $b_0 \in S_i$ such that $p \mid ab_0 + n$, and for $b \in S_i \setminus \{b_0, a\}$, $\left(\frac{ab+1}{p}\right) = 1$. Finally, it may happen that $\left(\frac{a^2+1}{p}\right) = -1$. Thus, assuming $|S_i| > p^{\varepsilon}$ (so that $|S_i| > 3$ as well) would result in

$$\begin{aligned} 0 < |S_i|(|S_i| - 3) &\leqslant \sum_{a,b \in S_i} \left(\frac{ab+n}{p}\right) = \left| \sum_{a,b \in S_i} \left(\frac{b + na^{-1}}{p}\right) \right| \\ &= \left| \sum_{\substack{b \in S_i \\ a \in nS_i^{-1}}} \left(\frac{b+a}{p}\right) \right| \leqslant p^{-\delta(\varepsilon)} |S_i|^2, \end{aligned}$$

implying

$$p^{\varepsilon} < |S_i| \leqslant \frac{3}{1 - p^{-\delta}},$$

which contradicts (6). Thus, we must have $|S_i| \leqslant p^{\varepsilon}$ for $C(\varepsilon) < p \leqslant Q$ with $p \nmid n$.

We conclude that $|\mathcal{S}_p| \leqslant 1 + 2p^{\varepsilon}$. Take $\gamma = 2 + C(\varepsilon)^{-\varepsilon}$. Then, $|\mathcal{S}_p| < \gamma p^{\varepsilon}$ for the primes in question. Therefore,

$$\begin{aligned} \sum_{p \leqslant Q} \frac{\gamma \log p}{|\mathcal{S}_p|} &> \sum_{\substack{C(\varepsilon) < p \leqslant Q \\ p \nmid n}} \frac{\log p}{p^{\varepsilon}} \\ &\geqslant \sum_{p \leqslant Q} \frac{\log p}{p^{\varepsilon}} - \sum_{p \leqslant C(\varepsilon)} \frac{\log p}{p^{\varepsilon}} - \sum_{p \mid n} \frac{\log p}{p^{\varepsilon}}. \end{aligned}$$

If n has no prime divisor exceeding $e^{1/\varepsilon}$, which is the only critical point of $x^{-\varepsilon} \log x$ on $[2, \infty)$, then using the inequality (3) it follows that

$$\sum_{p|n} \frac{\log p}{p^{\varepsilon}} \leqslant \frac{\pi(e^{1/\varepsilon})}{e\varepsilon} \leqslant (1+11\varepsilon)e^{1/\varepsilon-1}.$$

Otherwise, using (2) we obtain

$$\sum_{p|n} \frac{\log p}{p^{\varepsilon}} \leqslant \sum_{p\leqslant p_{\omega(n)}} \frac{\log p}{p^{\varepsilon}} < \frac{2p_{\omega(n)}^{1-\varepsilon}}{1-\varepsilon}.$$

Using the inequalities (4) and (5) to estimate the last term, and combining this with the previous estimate above we derive that

$$\sum_{p|n} \frac{\log p}{p^{\varepsilon}} \ll_{\varepsilon} (\log|n|)^{1-\varepsilon}.$$

Using Lemma 2 again yields

$$\sum_{p\leqslant Q} \frac{\gamma \log p}{|\mathcal{S}_p|} > Q^{1-\varepsilon}\left(1-\frac{c_1}{\log^2 Q}\right) - c_2(\log N)^{1-\varepsilon}$$

for some positive constants c_1 and c_2 depending on ε. Since we need the sum on the left larger than $\gamma \log N$ to be able to use Lemma 5, we choose $Q = (\lambda^{-1}\log N)^{1/(1-\varepsilon)}$ for some $\lambda < 1$. Combining the estimates above and using Lemmas 2 and 5 we obtain

$$\begin{aligned} |\mathcal{S}| &\leqslant \gamma \frac{Q(1+O(1/\log^2 Q)) - \log N}{Q^{1-\varepsilon}\left(1-\frac{c_1}{\log^2 Q}\right) - c_2(\log N)^{1-\varepsilon} - \log N} \\ &\leqslant \frac{\gamma}{(1-\lambda)\lambda^{\frac{\varepsilon}{1-\varepsilon}}} \frac{(\log N)^{\frac{\varepsilon}{1-\varepsilon}}(1+O(1/(\log\log N)^2))}{1-\frac{c_4}{(\log\log N)^2}}. \end{aligned}$$

Choosing $\lambda = \varepsilon$ minimizes the coefficient above and we obtain

$$|\mathcal{S}| \leqslant \frac{2C(\varepsilon)^{\varepsilon}+1}{C(\varepsilon)^{\varepsilon}(1-\varepsilon)}\left(\frac{3}{\varepsilon}\right)^{\frac{\varepsilon}{1-\varepsilon}} (\log|n|)^{\frac{\varepsilon}{1-\varepsilon}}\left(1+O\Big(\frac{1}{(\log\log|n|)^2}\Big)\right). \quad \square$$

3. Concluding remarks

The Paley graph conjecture is not only important in graph theory but has important consequences in computer science as explained in [8]. There is some progress towards this conjecture in the emerging field of additive combinatorics as evident in the paper of [7]. However, here, one needs to have some information on the additive structure of our sets S and T. Our main idea in the paper is to demonstrate the intimate connection between this important conjecture and the problem of Diophantine m-tuples, which was hitherto unknown.

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