Permutations avoiding 312 and another pattern, Chebyshev polynomials and longest increasing subsequences

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\textbf{ARTICLE INFO}

\textbf{Article history:}
Received 17 August 2018
Received in revised form 17 January 2020
Accepted 20 January 2020
Available online 24 January 2020

\textbf{MSC:}
05A05
05A15

\textbf{Keywords:}
Longest increasing subsequence problem
Pattern-avoiding permutations
Chebyshev polynomials
Generating functions

\textbf{ABSTRACT}

We study the longest increasing subsequence problem for random permutations avoiding the pattern 312 and another pattern τ under the uniform probability distribution. We determine the exact and asymptotic formulas for the average length of the longest increasing subsequences for such permutation classes specifically when the pattern τ is monotone increasing or decreasing, or any pattern of length four.

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https://doi.org/10.1016/j.aam.2020.102002
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1. Introduction

The study of longest increasing subsequences for uniformly random permutations is a wonderful example of a research program which begins with an easy-to-state question whose solution makes surprising connections with different branches of mathematics, and culminates with many astonishing results that have interesting applications in statistics, computer science, physics and biology, see [1,10–12]. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ be a permutation of $[n] := \{1, \ldots, n\}$. We denote by $L_n(\sigma)$ the length of a longest increasing subsequence in $\sigma$, that is,

$$L_n(\sigma) = \max\{k \in [n]: \text{there exist } 1 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ and } \sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}\}.$$  

Note that, in general, such a subsequence is not unique. Erdős-Szekeres theorem [15] states that every permutation of length $n \geq (r - 1)(s - 1) + 1$ contains either an increasing subsequence of length $r$ or a decreasing subsequence of length $s$. After this celebrated result, many researchers worked on the problem of determining the asymptotic behavior of $L_n$ on $S_n$, the set of all permutations of length $n$, under the uniform probability distribution [16,22,27,30,36]. The problem has been studied by several distinct methods from probability theory, random matrix theory, representation theory and statistical mechanics, see [1,4,21,29] and references therein. It is finally known that $E(L_n) \sim 2\sqrt{n}$ [22,31,36] and $n^{-1/6}(L_n - E(L_n))$ converges in distribution to the Tracy-Widom distribution as $n \to \infty$ [3,34]. For a thorough exposition of the subject, see the books [4,29].

We shall study the longest increasing subsequence problem for some pattern-avoiding permutation classes. First, we shall recall some definitions. For permutations $\tau \in S_k$ and $\sigma \in S_n$, we say that $\tau$ appears as a pattern in $\sigma$ if there is a subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k}$ of length $k$ in $\sigma$ which has the same relative order of $\tau$, that is, $\sigma_{i_s} < \sigma_{i_t}$ if and only if $\tau_s < \tau_t$ for all $1 \leq s, t \leq k$. For example, the permutation 312 appears as a pattern in 52341 because it has the subsequences 523 – , 52 – 4 – or 5 – 34 –. If $\tau$ does not appear as a pattern in $\sigma$, then $\sigma$ is called a $\tau$-avoiding permutation. We denote by $S_n(\tau)$ the set of permutations of length $n$ that avoid the pattern $\tau$. More generally, for a set $T$ of patterns, we use the notation $S_n(T) = \cap_{\tau \in T} S_n(\tau)$. Pattern-avoiding permutations have been studied from combinatorics perspectives for many years, for an introduction to the subject, see [8,35]. Recently probabilistic study of random pattern-avoiding permutations has also been initiated, and many interesting results have already appeared [5,17–20,23,24,26].

The longest increasing subsequence problem for the pattern-avoiding permutations was first studied for the patterns of length tree, that is, for $\tau \in S_3$ on $S_n(\tau)$ with uniform probability distribution in [14]. The case $S_n(\tau^1, \tau^2)$ with $\tau^1, \tau^2 \in S_3$ is studied for all possible cases in [24]. One of the corollaries of our main result, Theorem 2.2, covers the case $S_n(312, \tau)$ with either $\tau \in S_3(312)$ or $\tau \in S_4(312)$ and hence add some new results to this research program.
Note that for any $\sigma \in S_n$, we have
\[ L_n(\sigma) = L_n(\sigma^r) = L_n(\sigma^{-1}) \quad (1.1) \]
where the reverse, complement and inverse of $\sigma$ are defined as $\sigma^r_i = \sigma_{n+1-i}$, $\sigma_i^c = n+1 - \sigma_i$ and $\sigma_i^{-1} = j$ if and only if $\sigma_j = i$, respectively. These symmetries significantly reduce the number of cases needed to be studied.

In a different direction of research, the longest increasing subsequence problem has also been studied on $S_n$ under some non-uniform probability distributions such as Mallow distribution $[6,7,28]$ and in some other context such as colored permutations $[9]$, independent-identically distributed sequences, and random walks $[2,13]$.

The paper is organized as follows. We present our main result, Theorem 2.2, in Section 2 and as a first case apply it to $S_n(312, \tau)$ with $\tau \in S_3(312)$ which gives an alternative proof for some cases considered in $[24]$ through generating functions. In Section 3, we consider three specific longer patterns where $\tau$ is the monotone increasing/decreasing pattern or the pattern $(m-1)m(m-2)(m-3)\cdots321$. The last section presents the results for the case $S_n(312, \tau)$ with $\tau \in S_4(312)$.

For the rest of the paper, we only deal with random variables defined on sets $S_n(312, \tau)$ under the uniform probability distribution. That is, for any subset $A \subset S_n(312, \tau)$, $P^\tau(A) = \frac{|A|}{|S_n(312, \tau)|}$. The notation $E^\tau(X)$ denotes the expected value of a random variable $X$ on $S_n(312, \tau)$ under $P^\tau$. We denote the coefficient of $x^n$ in a generating function $G(x)$ by $[x^n]G(x)$. For two sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, we write $a_n \sim b_n$ if $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$.

2. General results

Note that if $\tau \notin S_k(312)$, then $S_n(312, \tau) = S_n(312)$ for all $n \geq 1$. For any $\tau \in S_k(312)$ with $k \geq 2$, we define the generating function
\[ F_\tau(x, q) = \sum_{n \geq 0} \sum_{\sigma \in S_n(312, \tau)} x^n q^{L_n(\sigma)} \quad (2.1) \]
with $F_1(x, q) \equiv 1$.

We will use the following facts repeatedly in our proofs:
\[ E^\tau(L_n) = \frac{1}{s_n} [x^n] \frac{\partial}{\partial q} F_\tau(x, q) \bigg|_{q=1} \quad \text{and} \]
\[ E^\tau(L_n^2) = \frac{1}{s_n} [x^n] \left( \frac{\partial^2}{\partial q^2} F_\tau(x, q) \bigg|_{q=1} + \frac{\partial}{\partial q} F_\tau(x, q) \bigg|_{q=1} \right) \]
where $s_n = |S_n(312, \tau)|$. Note also that $s_n = [x^n]F_\tau(x, 1)$.

For any sequence $w = w_1w_2 \cdots w_m$ of $m$-distinct integers, we define the corresponding reduced form to be the unique permutation $v = v_1v_2 \cdots v_m$ where $v_i = \ell$ if the $w_i$ is the
\( \ell \)-th smallest term in \( w \). For example, the reduced form of 253 is 132. For any sequence \( w \), we define \( F_w(x, q) \) to be \( F_v(x, q) \) where \( v \) is the reduced form of \( w \).

In order to determine \( F_\tau(x, q) \) explicitly, we shall introduce some notations. Let \( w^1, w^2 \) be two sequences of integers, we write \( w^1 < w^2 \) or \( w^2 > w^1 \) if \( w^1_i < w^2_j \) for all possible \( i, j \). Recall that for any permutation \( \tau = \tau_1 \cdots \tau_k \), \( \tau_i \) is called a right-to-left minimum if \( \tau_i < \tau_j \) for all \( j > i \). Note that, by definition, the last entry \( \tau_k \) is a right-to-left minimum.

Let \( \tau \in S_k(312) \) and let \( m_0 = 1 < m_1 < \ldots < m_r \) be the right-to-left minima of \( \tau \) written from left to right. Then \( \tau \) can be represented as

\[
\tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r,
\]

where \( m_0 < \tau^{(0)} < m_1 < \tau^{(1)} < \cdots < m_r < \tau^{(r)} \), and \( \tau^{(j)} \) (may possibly be empty) avoids 312 for each \( j = 0, 1, \ldots, r \). In this case we call this representation the normal form of \( \tau \). For instance, if \( \tau = 214365 \), then the normal form of \( \tau \) is \( \tau^{(0)} 1 \tau^{(1)} 3 \tau^{(2)} 5 \) with \( \tau^{(0)} = 2, \tau^{(1)} = 4 \) and \( \tau^{(2)} = 6 \).

Assume that \( \tau \in S_k(312) \) is given in its normal form, that is, \( \tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r \). We use \( \Theta^{(j)} \) to denote \( \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(j)} m_j \), which we call the \( j \)th prefix of \( \tau \). We use \( \Theta^{<j>} \) to denote the reduced form of \( \tau^{(j)} m_j \tau^{(j+1)} m_{j+1} \cdots \tau^{(r)} m_r \), which we call the \( j \)th suffix of \( \tau \). We set \( \Theta^{(-1)} = \emptyset \).

The following lemma plays a key role in the proof of Theorem 2.2. For the sake of the reader, we provide a proof for it which is indeed very similar to the main result of [25].

**Lemma 2.1.** Assume that \( \tau \in S_k(312) \) is given in its normal form, \( \tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r \). Then \( \sigma = \sigma' 1 \sigma'' \) avoids both 312 and \( \tau \) if and only if there exists \( i, 0 \leq i \leq r \), such that \( \sigma' \) avoids \( \Theta^{(i)} \) and contains \( \Theta^{(i-1)} \), while \( \sigma'' \) avoids \( \Theta^{<i>} \).

**Proof.** We denote the set of all permutations, including the empty permutation, that avoid both 312 and \( \tau \) by \( T_\tau \), and that avoid both 312 and \( \tau \) and contain \( \tau' \) by \( T_{\tau, \tau'} \). Let \( \sigma \) be any nonempty permutation in \( T_\tau \). We can write \( \sigma \) as \( \sigma = \sigma' 1 \sigma'' \). Note that \( \sigma \) avoids 312 if and only if \( \sigma' < \sigma'' \) and both \( \sigma' \) and \( \sigma'' \) avoid 312. Note also that \( \Theta^{(j+1)} \) is a prefix of \( \Theta^{(j+1)} \) and \( \Theta^{<j+1>} \) is a suffix of \( \Theta^{<j>} \) for all \( j = 0, 1, \ldots, r - 1 \), and \( \Theta^{(r)} = \Theta^{<0>} = \tau \).

We have

\[
T_\tau = T_{\Theta^{(0)}} \cup T_{\tau; \Theta^{(0)}},
\]

\[
T_{\tau; \Theta^{(s)}} = T_{\Theta^{(s+1)}; \Theta^{(s)}} \cup T_{\tau; \Theta^{(s+1)}}, \quad s = 0, 1, 2, \ldots, r - 1,
\]

with \( T_{\tau; \Theta^{(r)}} = T_{\tau; \tau} = \emptyset \). Therefore,

\[
T_\tau = T_{\Theta^{(0)}} \cup T_{\Theta^{(1)}; \Theta^{(0)}} \cup T_{\Theta^{(2)}; \Theta^{(1)}} \cup \cdots \cup T_{\Theta^{(r)}; \Theta^{(r-1)}}.
\]

Thus, since \( \sigma \in T_\tau \) we have that \( \sigma' 1 \in T_\tau \), so there exists \( i, 0 \leq i \leq r \), such that \( \sigma' 1 \in T_{\Theta^{(i)}; \Theta^{(i-1)}} \). But then we must have \( \sigma'' \in T_{\Theta^{<i>}} \). Hence, \( \sigma \in T_\tau \) implies that
(\ast) there exists \(i, 0 \leq i \leq r\), such that \(\sigma' \in T_{\Theta^{(i)}:\Theta^{(i-1)}}\) and \(\sigma'' \in T_{\Theta^{<i>}}\).

Note that in the case \(i = 0\), \(\sigma'\) avoids \(\Theta^{(0)}\), while \(\sigma''\) avoids \(\Theta^{<0>}\), and we defined \(\Theta^{(-1)} = \emptyset\). Then clearly \(\sigma'\) contains \(\Theta^{(-1)}\).

On the other hand, let both \(\sigma'\) and \(\sigma''\) avoid 312 and satisfy the condition (\ast), that is, there exists \(i, 0 \leq i \leq r\), such that \(\sigma' \in T_{\Theta^{(i)}:\Theta^{(i-1)}}\) and \(\sigma'' \in T_{\Theta^{<i>}}\). Thus, both \(\sigma'\) and \(\sigma''\) avoid \(\tau\). If \(\sigma = \sigma'1\sigma''\) contains \(\tau\), then \(\sigma'\) contains \(\Theta^{(j-1)}\) and \(\sigma''\) contains \(\Theta^{<j>}\), for some \(j\), \(0 \leq j \leq r\). If we choose \(j\) to be maximal (it exists since \(\sigma'\) avoids \(\Theta^{(r)} = \tau\) and \(\Theta^{(j)}\) is a prefix of \(\Theta^{(j+1)}\)), then we see that \(\sigma'\) avoids \(\Theta^{(j)}\) and contains \(\Theta^{(j-1)}\) while \(\sigma''\) contains \(\Theta^{<j>}\), which contradicts (\ast). Thus the condition (\ast) implies that \(\sigma \in T_{\tau}\). This completes the proof. \(\Box\)

Note that in the above lemma \(\sigma'\) avoids \(\Theta^{(i)}\) and contains \(\Theta^{(<i>}\) if and only if \(\sigma'\) avoids \(\Theta^{(i)}\) and contains \(\Theta^{(i-1)}\), for all \(2 \leq i \leq r\).

Our main result gives a functional equation for the generating function.

**Theorem 2.2.** Let \(\tau \in S_k(312)\) be given in its normal form \(\tau^{(0)}m_0\tau^{(1)}m_1 \cdots \tau^{(r)}m_r\), with \(k \geq 2\).

- If \(\tau^{(0)} = \emptyset\), then

\[
F_\tau(x,q) = 1 + xq + x(F_\tau(x,q) - 1) + xq(F_{\Theta^{<1>}}(x,q) - 1) \\
+ x \sum_{j=1}^{r} (F_{\Theta^{(j)}}(x,q) - F_{\Theta^{(j-1)}}(x,q))(F_{\Theta^{<j>}}(x,q) - 1); \\
\]

- if \(\tau^{(0)} \neq \emptyset\), then

\[
F_\tau(x,q) = 1 + xq + x(F_{\tau^{(0)}}(x,q) - 1)\delta_{r=0} + x(F_\tau(x,q) - 1)\delta_{r\geq 1} + xq(F_\tau(x,q) - 1) \\\n+ x \sum_{j=2}^{r} (F_{\Theta^{(j)}}(x,q) - F_{\Theta^{(j-1)}}(x,q))(F_{\Theta^{<j>}}(x,q) - 1) \\
+ x(F_{\Theta^{(1)}}(x,q) - F_{\tau^{(0)}}(x,q))(F_{\Theta^{<1>}}(x,q) - 1)\delta_{r\geq 1} \\\n+ x(F_{\tau^{(0)}}(x,q) - 1)(F_\tau(x,q) - 1),
\]

where we define \(F_{\emptyset}(x,q) = 0\), and \(\delta_\chi\) denotes 1 if the condition \(\chi\) holds, and 0 otherwise.

**Proof.** Note that the contributions of the empty permutation and the permutation of length 1 to the generating function are 1 and \(xq\), respectively. Henceforth, we assume that \(\tau\) has at least two entries. Let \(\sigma = \sigma'1\sigma''\) be any nonempty permutation which avoids both 312 and \(\tau\).
Case $\tau^{(0)} = \emptyset$: Since $\tau$ has at least two entries and $\tau^{(0)} = \emptyset$, we have that $r \geq 1$. If $\sigma = \sigma'$ with $\sigma' \neq \emptyset$, then we have the contribution of $x(F_\tau(x, q) - 1)$. If $\sigma = 1\sigma''$ with $\sigma'' \neq \emptyset$, then we have the contribution of $xq(F_{\Theta^{<1>}}(x, q) - 1)$. As a last case, we need to consider the permutations in the form of $\sigma = \sigma'1\sigma''$ with $\sigma', \sigma'' \neq \emptyset$. By Lemma 2.1, the contribution of this case is given by

$$x \sum_{j=2}^{r} (F_{\Theta^{(j)}}(x, q) - F_{\Theta^{(j-1)}}(x, q))(F_{\Theta^{<j>}}(x, q) - 1) + x(F_{\Theta^{(1)}}(x, q) - 1)(F_{\Theta^{<1>}}(x, q) - 1)$$

$$= x \sum_{j=1}^{r} (F_{\Theta^{(j)}}(x, q) - F_{\Theta^{(j-1)}}(x, q))(F_{\Theta^{<j>}}(x, q) - 1),$$

where in last equality we used that $\Theta^{(0)} = 1$ and $F_{\Theta^{(0)}}(x, q) = 1$. By summing over all the contributions, we complete the proof.

Case $\tau^{(0)} \neq \emptyset$: If $\sigma = \sigma'$ with $\sigma' \neq \emptyset$, then we have the contribution of $x(F_\tau(x, q) - 1)$ when $r \geq 1$, and $x(F_\tau(x, q) - 1)$ when $r = 0$. If $\sigma = 1\sigma''$ with $\sigma'' \neq \emptyset$, then we have the contribution of $xq(F_\tau(x, q) - 1)$. As a last case, we need to consider the permutations in the form of $\sigma = \sigma'1\sigma''$ with $\sigma', \sigma'' \neq \emptyset$. By Lemma 2.1, the contribution of this case is given by

$$x \sum_{j=2}^{r} (F_{\Theta^{(j)}}(x, q) - F_{\Theta^{(j-1)}}(x, q))(F_{\Theta^{<j>}}(x, q) - 1)$$

$$+ x(F_{\Theta^{(1)}}(x, q) - F_{\tau^{(0)}}(x, q))(F_{\Theta^{<1>}}(x, q) - 1)\delta_{r \geq 1}$$

$$+ x(F_{\tau^{(0)}}(x, q) - 1)(F_\tau(x, q) - 1).$$

By summing over all the contributions, we complete the proof. □

We can also deduce the rationality of the generating function $F_\tau(x, q)$ for any nonempty pattern $\tau$ by using the induction on $k$ with the observations in the proof of Theorem 2.2 and $F_1(x, q) = 1$.

**Corollary 2.3.** For any $k \geq 1$ and $\tau \in S_k(312)$, the generating function $F_\tau(x, q)$ is a rational function in $x$ and $q$.

Note that $F_1(x, q) = 1$ (the only permutation that avoids the pattern 1 is the empty permutation). Theorem 2.2 with $\tau = 21$ gives

$$F_{21}(x, q) = 1 + xq + x(F_1(x, q) - 1) + xq(F_{21}(x, q) - 1) + x(F_1(x, q) - 1)(F_{21}(x, q) - 1),$$

where $F_1(x, q) = 1$. Thus, $F_{21}(x, q) = \frac{1}{1-xq}$. Theorem 2.2 with $\tau = 12$ gives

$$F_{12}(x, q) = 1 + xq + x(F_{12}(x, q) - 1) + xq(F_1(x, q) - 1) + x(F_1(x, q) - 1)(F_{12}(x, q) - 1).$$
Thus, $F_{12}(x, q) = \frac{1 + xq - x}{1-x} = 1 + \frac{xq}{1-x}$. We summarize these results in the following corollary for future references.

**Corollary 2.4.** For $\tau \in S_2$, the generating functions are given by

$$F_{21}(x, q) = \frac{1}{1-xq} \text{ and } F_{12}(x, q) = 1 + \frac{xq}{1-x}.$$  

Next we will consider the application of Theorem 2.2 to the patterns of length three, that is, $\tau \in S_3(312)$. In each case we will also use Corollary 2.4 and $F_1(x, q) = 1$.

**Pattern $\tau = 123$.** We have $\Theta^{(0)} = 1$, $\Theta^{(1)} = 12$, $\Theta^{(2)} = 123$, and $\Theta^{<0>} = 123$, $\Theta^{<1>} = 12$, $\Theta^{<2>} = 1$. Thus,

$$F_{123}(x, q) = 1 + xq + x(F_{123}(x, q) - 1) + xq(F_{12}(x, q) - 1) + x(F_{12}(x, q) - 1)(F_{12}(x, q) - 1),$$

which gives $F_{123}(x, q) = 1 + xq/(1 - x) + x^2q^2/(1-x)^3$.

**Pattern $\tau = 132$.** We have $\Theta^{(0)} = 1$, $\Theta^{(1)} = 132$, and $\Theta^{<0>} = 132$, $\Theta^{<1>} = 21$. Thus,

$$F_{132}(x, q) = 1 + xq + x(F_{132}(x, q) - 1) + xq(F_{21}(x, q) - 1) + x(F_{132}(x, q) - 1)(F_{21}(x, q) - 1),$$

which gives $F_{132}(x, q) = \frac{1-x}{1-x-xq}$.

**Pattern $\tau = 213$.** We have $\Theta^{(0)} = 21$, $\Theta^{(1)} = 213$, and $\Theta^{<0>} = 213$, $\Theta^{<1>} = 1$. Thus,

$$F_{213}(x, q) = 1 + xq + x(F_{213}(x, q) - 1) + xq(F_{213}(x, q) - 1),$$

which gives $F_{213}(x, q) = \frac{1-x}{1-x-xq}$.

**Pattern $\tau = 231$.** We have $\Theta^{(0)} = \Theta^{<0>} = 231$. Thus,

$$F_{231}(x, q) = 1 + xq + x(F_{12}(x, q) - 1) + xq(F_{231}(x, q) - 1) + x(F_{12}(x, q) - 1)(F_{231}(x, q) - 1),$$

which gives $F_{231}(x, q) = \frac{1-x}{1-x-xq}$.

**Pattern $\tau = 321$.** We have $\Theta^{(0)} = \Theta^{<0>} = 321$. Thus,

$$F_{321}(x, q) = 1 + xq + x(F_{21}(x, q) - 1) + xq(F_{321}(x, q) - 1) + x(F_{21}(x, q) - 1)(F_{321}(x, q) - 1),$$

which gives $F_{321}(x, q) = \frac{1-xq}{(1-xq)^2-xq}$. Hence, we can state the following result.

**Corollary 2.5.** For $\tau \in S_3(312)$, the generating functions are given by

$$F_{123}(x, q) = 1 + \frac{xq}{1-x} + \frac{x^2q^2}{(1-x)^3},$$

$$F_{132}(x, q) = F_{213}(x, q) = F_{231}(x, q) = \frac{1-x}{1-x-xq},$$

$$F_{321}(x, q) = \frac{1-xq}{(1-xq)^2-xq}.$$
\[
F_{321}(x, q) = \frac{1 - xq}{(1 - xq)^2 - x^2}. 
\]

The results in Corollary 2.5 indeed extend the relevant results of Simion and Schmidt [32] for the permutations avoiding two patterns of length three. Here we find the generating functions for the number of permutations \( \sigma \in S_n(312, \tau) \) with \( \tau \in S_3(312) \) according to the length of the longest increasing subsequence in \( \sigma \).

Our next result considers a specific type of pattern in which the last entry is 1.

**Corollary 2.6.** Assume \( \tau = \rho 1 \in S_k(312) \) and \( k \geq 2 \). Then \( F_\tau(x, q) = \frac{1}{1 - xq - x(F_\rho(x, q) - 1)} \).

Moreover,

\[
\frac{\partial}{\partial q} F_\tau(x, q) \bigg|_{q=1} = x F_\tau^2(x, 1) \left( 1 + \frac{\partial}{\partial q} F_\rho(x, q) \bigg|_{q=1} \right). 
\]

**Proof.** By Theorem 2.2 with \( \tau = \rho 1 \ (r = 0, m_0 = 1 \) and \( \tau^{(0)} = \rho \), we have

\[
F_\tau(x, q) = 1 + xq + x(F_\rho(x, q) - 1) + xq(F_\tau(x, q) - 1) + x(F_\rho(x, q) - 1)(F_\tau(x, q) - 1),
\]

which implies

\[
F_\tau(x, q) = \frac{1}{1 - xq - x(F_\rho(x, q) - 1)}. 
\]

In particular, \( F_\tau(x, 1) = \frac{1}{1 - x F_\rho(x, 1)} \), as shown in [25]. Moreover, by differentiating \( F_\tau(x, q) \) with respect to \( q \) and evaluating at \( q = 1 \), we obtain

\[
\frac{\partial}{\partial q} F_\tau(x, q) \bigg|_{q=1} = \frac{x \left( 1 + \frac{\partial}{\partial q} F_\rho(x, q) \bigg|_{q=1} \right)}{(1 - x F_\rho(x, 1))^2} = x F_\tau^2(x, 1) \left( 1 + \frac{\partial}{\partial q} F_\rho(x, q) \bigg|_{q=1} \right),
\]

which completes the proof. \( \square \)

By Corollary 2.4 and 2.5, we recover the relevant results in [24].

**Theorem 2.7.** For all \( n \geq 1 \), we have

\[
E_{123}(L_n) = \frac{2(n^2 - n + 1)}{n^2 - n + 2}, \quad E_{123}(L_n^2) = \frac{2(2n^2 - 2n + 1)}{n^2 - n + 2},
\]

\[
E_{132}(L_n) = E_{213}(L_n) = E_{231}(L_n) = \frac{n + 1}{2}, \quad E_{132}(L_n^2) = E_{213}(L_n^2) = E_{231}(L_n^2) = \frac{n(n + 3)}{4},
\]

\[
E_{321}(L_n) = \frac{3n}{4}, \quad E_{321}(L_n^2) = \frac{n(9n + 1)}{16}. 
\]
3. Special cases of longer patterns

The main result of this paper, Theorem 2.2, can be used to obtain general results for several longer patterns. In this subsection, as an example, we apply it to the following three specific patterns $12 \cdots m$, $m(m-1) \cdots 21$ and $(m-1)m(m-2) \cdots 21$.

Recall that the Chebyshev polynomials of the second kind are defined by $U_j(\cos \theta) = \frac{\sin((j+1)\theta)}{\sin \theta}$. It is well known that these polynomials satisfy

$$U_0(t) = 1, U_1(t) = 2t, \text{ and } U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t) \text{ for all integers } m, \quad (3.1)$$

and

$$U_n(t) = 2^n \prod_{j=1}^{n} \left(t - \cos \left(\frac{j\pi}{n+1}\right)\right). \quad (3.2)$$

3.1. Monotone increasing pattern $\tau = 12 \cdots m$

In this subsection, we study the pattern $\tau = 12 \cdots m$. By Corollaries 2.4 and 2.5, we see that $F_1(x, q) = 1$, $F_{12}(x, q) = 1 + \frac{xq}{1-x}$ and $F_{123}(x, q) = 1 + \frac{xq}{1-x} + \frac{x^2q^2}{(1-x)^3}$. By Theorem 2.2 with $\tau = 12 \cdots m$, we have

$$F_{12\cdots m}(x, q) = 1 + xq + x(F_{12\cdots m}(x, q) - 1) + xq(F_{12\cdots(m-1)}(x, q) - 1)$$

$$+ x \sum_{j=2}^{m} (F_{12\cdots j}(x, q) - F_{12\cdots(j-1)}(x, q))(F_{12\cdots(m-j+1)}(x, q) - 1),$$

which is equivalent to

$$F_{12\cdots m}(x, q) = 1 + xqF_{12\cdots(m-1)}(x, q)$$

$$+ x \sum_{j=2}^{m} (F_{12\cdots j}(x, q) - F_{12\cdots(j-1)}(x, q))F_{12\cdots(m-j+1)}(x, q).$$

Define $G(x, q, v) = \sum_{m \geq 1} F_{12\cdots m}(x, q)v^m$. Then, by multiplying the above recurrence by $v^m$ and summing over $m$, we obtain

$$\sum_{m \geq 1} F_{12\cdots m}(x, q)v^m = \sum_{m \geq 1} v^m + xqv \sum_{m \geq 1} v^{m-1}F_{12\cdots(m-1)}(x, q)$$

$$+ x \sum_{m \geq 2} v^m \sum_{j=2}^{m} (F_{12\cdots j}(x, q) - F_{12\cdots(j-1)}(x, q))F_{12\cdots(m-j+1)}(x, q),$$

which implies
\( G(x, q, v) = \frac{v}{1-v} + xqvG(x, q, v) + \frac{x}{v} (G(x, q, v) - v)G(x, q, v) - x(G(x, q, v))^2. \)

Thus, \( G(x, q, v) \) satisfies
\[
\frac{v}{1-v} + (1 + x - qvx)G(x, q, v) - \frac{x(1-v)}{v} G^2(x, q, v) = 0.
\]

By solving the above equation for \( G(x, q, v) \) we obtain
\[
G(x, q, v) = \frac{(1 + x - qxv - \sqrt{1 + x - qxv})^2 - 4xv}{2x(1-v)}.
\]

Then
\[
G(x, q, \frac{v(1-x)^2}{q}) = \frac{1 - \frac{v(1-x)^2}{q}}{v(1-x)^2} - 1 = \frac{1 - v(1-x) - \sqrt{1 - 2v(1+x) + v^2(1-x)^2}}{2xv(1-v)}
\]

which, by definition of Narayana numbers (see Sequence A001263 in [33]), leads to
\[
G(x, q, \frac{v(1-x)^2}{q}) = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^{k-1} v^n.
\]

Therefore, by replacing \( v \) by \( xqv/(1-x)^2 \), we have
\[
G(x, q, v) = \frac{1 - qvx}{1-x} + \sum_{n \geq 1} \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \frac{x^{n+k-1} q^n v^n}{(1-x)^{2n+1}}.
\]

which implies
\[
G(x, q, v) = \frac{v}{1-v} \left( 1 + \frac{qvx}{1-x} + \sum_{n \geq 1} \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \frac{q^{n+k} x^{n+k} v^{n+1}}{(1-x)^{2n+1}} \right).
\]

By finding the coefficient of \( v^m \), we obtain the following result.

**Corollary 3.1.** For all \( m \geq 1 \),
\[
F_{12\ldots m}(x, q) = 1 + \frac{qx}{1-x} + \sum_{j=2}^{m-1} \left( \frac{q^j x^j}{(1-x)^{2j-1}} + \sum_{k=1}^{j-1} \frac{1}{j-1} \binom{j-1}{k} \binom{j-1}{k-1} x^{k-1} \right).
\]

By Corollary 3.1, we see that the generating function \( F_{12\ldots m}(x,1) \) has a pole at \( x = 1 \) of order \( 2m - 3 \). Thus,
\[ [x^n] F_{12 \ldots m}(x, 1) \sim \frac{n^{2m-4}}{(2m-4)!} (m-2) \sum_{k=1}^{m-2} \binom{m-2}{k} \binom{m-2}{k-1}, \]

which, by definition of Narayana numbers, implies that

\[ [x^n] F_{12 \ldots m}(x, 1) \sim \frac{n^{2m-4}}{(2m-4)!} c_{m-2}, \]

where \( c_n = \frac{1}{n+1} \binom{2n}{n} \) is the \( n \)th Catalan number.

Also, by Corollary 3.1, we see that the generating function \( \frac{\partial}{\partial q} F_{12 \ldots m}(x, q) \big|_{q=1} \) has a pole at \( x = 1 \) of order \( 2m - 3 \). Thus,

\[ [x^n] \frac{\partial}{\partial q} F_{12 \ldots m}(x, q) \big|_{q=1} \sim \frac{(m-1)n^{2m-4}}{(2m-4)!} (m-2) \sum_{k=1}^{m-2} \binom{m-2}{k} \binom{m-2}{k-1} \]

\[ = \frac{(m-1)n^{2m-4}}{(2m-4)!} c_{m-2}. \]

Hence, we can state the following result.

**Theorem 3.2.** Let \( m \geq 1 \). When \( n \to \infty \), we have

\( E_{12 \ldots m}(L_n) \sim m - 1 \).

3.2. Monotone decreasing pattern \( \tau = m(m-1) \ldots 21 \)

In this subsection, we study the pattern \( m = m(m-1) \ldots 21 \).

**Corollary 3.3.** Let \( m = m(m-1) \ldots 21 \). Then

\[ F_m(x, q) = \frac{U_{m-2}(t) - \sqrt{x} U_{m-3}(t)}{\sqrt{x}(U_{m-1}(t) - \sqrt{x} U_{m-2}(t))}, \]

where \( t = \frac{1+x-xq}{2\sqrt{x}} \).

**Proof.** The proof is given by induction on \( m \). Clearly, \( F_1(x, q) = 1 \) and \( F_{21}(x, q) = \frac{1}{1-xq} \), so the claim holds for \( m = 1, 2 \). Assume that the claim holds for \( 1, 2, \ldots, m \) and let us prove it for \( m+1 \). Since \( m+1 = (m+1)1 \), then Corollary 2.6 gives \( F_{m+1}(x, q) = \frac{1}{1-xq-x(F_m(x, q)-1)} \). Thus by induction assumption, we obtain

\[ F_{m+1}(x, q) = \frac{1}{1-xq-x(F_m(x, q)-1)} \]

\[ = \frac{\sqrt{x}(U_{m-1}(t) - \sqrt{x} U_{m-2}(t))}{x(2tU_{m-1}(t) - U_{m-2}(t)) - x \sqrt{x}(2tU_{m-2}(t) - U_{m-3}(t))}. \]
where we used the fact (3.1) and \(2t \sqrt{x} = 1 + x - x^q\). \(\square\)

By Corollary 3.3 with \(q = 1\) and (3.1), we have \(F_m(x, 1) = \frac{U_{m-1}(\frac{1}{x^q})}{\sqrt{x}U_m(\frac{1}{2\sqrt{x}})}\), as shown in [25]. Moreover, by Corollary 2.6, we have

\[
\frac{\partial}{\partial q} F_m(x, q) \bigg|_{q=1} = xF_m^2(x, 1) \left(1 + \frac{\partial}{\partial q} F_{m-1}(x, q) \bigg|_{q=1}\right)
\]

with \(\frac{\partial}{\partial q} F_1(x, q) \bigg|_{q=1} = 0\). Thus, by induction on \(m\), we can state the following result.

**Corollary 3.4.** Let \(m = m(m-1) \cdots 21\). Then

\[
\frac{\partial}{\partial q} F_m(x, q) \bigg|_{q=1} = \frac{1}{U_m^2(\frac{1}{2\sqrt{x}})} \sum_{j=1}^{m-1} U_j^2 \left(\frac{1}{2\sqrt{x}}\right).
\]

Since the smallest pole of \(1/U_n(x)\) is \(\cos\left(\frac{\pi}{n+1}\right)\), it follows, by Corollary 3.4, that the coefficient of \(x^n\) in the generating function \(\frac{\partial}{\partial q} F_m(x, q) \bigg|_{q=1}\) is given by,

\[
[x^n] \frac{\partial}{\partial q} F_m(x, q) \bigg|_{q=1} \sim \alpha_m n \left(4 \cos^2\left(\frac{\pi}{m+1}\right)\right)^n \text{ as } n \to \infty.
\]

Let \(v_0 = \frac{1}{4 \cos^2\left(\frac{\pi}{m+1}\right)}\). The constant \(\alpha_m\) can be computed explicitly as

\[
\alpha_m = \lim_{x \to v_0} \frac{\left(1 - 4x \cos^2\left(\frac{\pi}{m+1}\right)\right)^2}{U_m^2(\frac{1}{2\sqrt{x}})} \sum_{j=1}^{m-1} U_j^2 \left(\frac{1}{2\sqrt{x}}\right)
\]

\[
= \sum_{j=1}^{m-1} U_j^2 \left(\cos\left(\frac{\pi}{m+1}\right)\right) \left(4 \cos^2\left(\frac{\pi}{m+1}\right)\right)^m \prod_{j=2}^{m-1} \left(1 - \frac{\cos\left(\frac{j\pi}{m+1}\right)}{\cos\left(\frac{\pi}{m+1}\right)}\right)^2
\]

\[
= \sum_{j=1}^{m-1} U_j^2 \left(\cos\left(\frac{\pi}{m+1}\right)\right) \left(4 \cos^4\left(\frac{\pi}{m+1}\right)\right)^{m-1} \left(\cos\left(\frac{\pi}{m+1}\right) - \cos\left(\frac{j\pi}{m+1}\right)\right)^2.
\]

Moreover, the coefficient of \(x^n\) in the generating function \(F_m(x, 1) = \frac{U_{m-1}(\frac{1}{x^q})}{\sqrt{2}U_m(\frac{1}{2\sqrt{x}})}\) is given by
\[ [x^n] F_m(x, 1) \sim \tilde{\alpha}_m \left( 4 \cos^2 \left( \frac{\pi}{m+1} \right) \right)^n \text{ as } n \to \infty, \]  

where

\[
\tilde{\alpha}_m = \lim_{x \to 0} \frac{1 - 4x \cos^2 \left( \frac{\pi}{m+1} \right)}{\sqrt{x} U_m \left( \frac{1}{2\sqrt{x}} \right)} U_{m-1} \left( \frac{1}{2\sqrt{x}} \right) \cos \left( \frac{\pi}{m+1} \right)
\]

\[= 2^{m-1} \cos^{m-1} \left( \frac{\pi}{m+1} \right) \prod_{j=2}^{m-1} \left( 1 - \cos \left( \frac{j\pi}{m+1} \right) \right) \frac{U_{m-1} \left( \cos \left( \frac{\pi}{m+1} \right) \right)}{U_{m-1} \left( \frac{\pi}{m+1} \right)} \prod_{j=2}^{m-1} \left( \cos \left( \frac{\pi}{m+1} \right) - \cos \left( \frac{j\pi}{m+1} \right) \right).
\]

Thus we have \( E_m^m(L_n) \sim \tilde{\alpha}_m n \). By substituting expressions of \( \alpha_m \) and \( \tilde{\alpha}_m \), it leads to the following result.

**Theorem 3.5.** Let \( m \geq 1 \). When \( n \to \infty \), we have

\[ E_m^m(L_n) \sim \frac{\sum_{j=1}^{m-1} U_j^2 \left( \cos \left( \frac{\pi}{m+1} \right) \right)}{2^{m+1} \cos^3 \left( \frac{\pi}{m+1} \right) U_{m-1} \left( \cos \left( \frac{\pi}{m+1} \right) \right) \prod_{j=2}^{m-1} \left( \cos \left( \frac{\pi}{m+1} \right) - \cos \left( \frac{j\pi}{m+1} \right) \right)} n. \]

For example, Theorem 3.5 for \( m = 3 \) gives that \( E_3^{321}(L_n) \sim \frac{3n}{4} \) as shown in [24], and for \( m = 4 \), we have \( E_4^{321}(L_n) \sim \left( 2 - \frac{3}{\sqrt{5}} \right) n. \)

**3.3. Pattern \( \tau = (m - 1)m(m - 2)(m - 3) \cdots 321 \)**

In this subsection, we study the pattern \( \check{m} = (m - 1)m(m - 2)(m - 3) \cdots 321 \).

**Corollary 3.6.** Let \( \check{m} = (m - 1)m(m - 2)(m - 3) \cdots 321 \) with \( m \geq 3 \). Then

\[ F_{\check{m}}(x, q) = \frac{(1 - x) U_{m-3}(t) - \sqrt{x}(1 - x + qx) U_{m-4}(t)}{\sqrt{x}(1 - x) U_{m-2}(t) - \sqrt{x}(1 - x + qx) U_{m-3}(t)}, \]

where \( t = \frac{1 + x - qx}{2\sqrt{x}} \).

**Proof.** We proceed the proof by induction on \( m \). By Example \( \tau = 231 \) in Section 2, we have \( F_3(x, q) = \frac{1 - x}{1 - x - qx} \), so the result holds for \( m = 3 \). Assume that the result holds for \( m - 1 \) and let us prove for \( m \). By Corollary 2.6, we have

\[ F_{\check{m}}(x, q) = \frac{1}{1 + x - qx - x F_{\check{m}-1}(x, q)}. \]
Thus by induction hypothesis, we have

\[ F_m(x, q) = \frac{1}{2\sqrt{x} - \sqrt{x}((1-x)U_{m-4}(t) - \sqrt{x}(1-x+q)U_{m-4}(t))} \]

\[ = \frac{(1-x)U_{m-3}(t) - \sqrt{x}(1-x+q)U_{m-4}(t)}{(1-x)(2tU_{m-3}(t) - U_{m-4}(t)) - \sqrt{x}(1-x+q)(2tU_{m-4}(t) - U_{m-5}(t))}, \]

which, by (3.1), implies

\[ F_m(x, q) = \frac{(1-x)U_{m-3}(t) - \sqrt{x}(1-x+q)U_{m-4}(t)}{\sqrt{x}((1-x)U_{m-2}(t) - \sqrt{x}(1-x+q)U_{m-3}(t))}, \]

which completes the proof. □

By Corollary 3.6 with \( q = 1 \) and (3.1), we have

\[ F_m(x, 1) = \frac{U_{m-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_m\left(\frac{1}{2\sqrt{x}}\right)}. \]

By induction on \( m \), we can state the following result.

**Corollary 3.7.** Let \( m = (m-1)m(m-2)(m-3) \cdots 321 \) with \( m \geq 4 \). Then

\[ \frac{\partial}{\partial q} F_m(x, q) \bigg|_{q=1} = \frac{1}{U_m^2\left(\frac{1}{2\sqrt{x}}\right)} \left( U_2\left(\frac{1}{2\sqrt{x}}\right) + \sum_{j=2}^{m-1} U_j^2\left(\frac{1}{2\sqrt{x}}\right) \right). \]

By similar arguments as in the proof of Theorem 3.5, we obtain the following result.

**Theorem 3.8.** Let \( m \geq 4 \). As \( n \to \infty \), we have

\[ E^m(L_n) \sim \frac{U_2\left(\cos\left(\frac{\pi}{m+1}\right)\right)}{2^{m+1}\cos^3\left(\frac{\pi}{m+1}\right)U_m-1\left(\cos\left(\frac{\pi}{m+1}\right)\right)} \prod_{j=2}^{m-1} \left( \cos\left(\frac{\pi}{m+1}\right) - \cos\left(\frac{j\pi}{m+1}\right) \right) n. \]

4. The case \( S_n(312, \tau) \) where \( \tau \in S_4(312) \)

In this section, we present the results for random permutations from \( S_n(312, \tau) \) where \( \tau \in S_4(312) \). A summary of the results for all \( \tau \in S_4(312) \) is given in Table 1. We present the details only for the two patterns, \( \tau = 1234 \) and \( \tau = 1243 \). Since the computations for other cases are very similar, the details are omitted.
Table 1
A summary of the results for $S_n(312, \tau)$ with $\tau \in S_4(312)$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$F_\tau(x, q)$</th>
<th>$E^\tau(L_n)$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234</td>
<td>$1 + \frac{xq}{1-x} + \frac{x^2q^2}{(1-x)^2} + \frac{x^3(1+x)q^3}{(1-x)^3}$</td>
<td>$\frac{3(n^4-4n^3+9n^2-6n+4)}{n^4-4n^3+11n^2-8n+12} \to 3$</td>
<td>Example 4.1</td>
</tr>
<tr>
<td>1243,1324</td>
<td>$1 + \frac{xq(q^2(2x-1)+(1-x)^2)}{(1-x-qx)^2(1-x)}$</td>
<td>$\frac{2^{n-3}(n^2-n+4)}{(n-1)2^{n-3}+1} \sim \frac{n}{2}$</td>
<td>Example 4.2</td>
</tr>
<tr>
<td>2134</td>
<td>$1 + \frac{xq(1-x)}{(1-x)^2-qx}$</td>
<td>$\sim \frac{n}{\sqrt{3}}$</td>
<td>Theorem 2.2</td>
</tr>
<tr>
<td>2143,3214</td>
<td>$\frac{1-x-qx}{(1-qx)^2-x}$</td>
<td>$\sim \frac{n}{\sqrt{3}}$</td>
<td>Theorem 2.2</td>
</tr>
<tr>
<td>2431,3241</td>
<td>$\frac{(1-x)^3}{(1-x)^3-xq(1-x)^2-xq^2}$</td>
<td>$\sim \frac{3(-5a^2+22a-9)n}{31} \Rightarrow a \approx 2.4657\cdots, a^3-4a^2+5a-3=0$</td>
<td>Theorem 2.2</td>
</tr>
<tr>
<td>3421,1432</td>
<td>$\frac{(1-x)^3}{(1-x)^3-xq(1-x)^2-xq^2}$</td>
<td>$\sim \frac{3(-5a^2+22a-9)n}{31} \Rightarrow a \approx 2.4657\cdots, a^3-4a^2+5a-3=0$</td>
<td>Theorem 2.2</td>
</tr>
</tbody>
</table>

Example 4.1. By Theorem 2.2 with $\tau = 1234$, we have

$$F_{1234}(x, q) = 1 + xqF_{123}(x, q) + x(F_{12}(x, q) - F_1(x, q))F_{123}(x, q) + x(F_{123}(x, q) - F_{12}(x, q))F_{12}(x, q) + x(F_{1234}(x, q) - F_{123}(x, q))F_1(x, q).$$

By Corollaries 2.4 and 2.5, we have

$$F_{1234}(x, q) = 1 + \frac{xq}{1-x} + \frac{x^2q^2}{(1-x)^3} + \frac{x^3(1+x)q^3}{(1-x)^5},$$

which agrees with Theorem 3.1 with $m = 4$. Therefore, we have

$$E_{1234}(L_n) = \frac{3(n^4-4n^3+9n^2-6n+4)}{n^4-4n^3+11n^2-8n+12}$$

and

$$E_{1234}(L_n) = \frac{3(3n^4-12n^3+23n^2-14n+4)}{n^4-4n^3+11n^2-8n+12}.$$

Example 4.2. By Theorem 2.2 with $\tau = 1243$, we have

$$F_{1243}(x, q) = 1 + xqF_{132}(x, q) + x(F_{1243}(x, q) - 1) + x(F_{12}(x, q) - 1)(F_{132}(x, q) - 1) + x(F_{1243}(x, q) - F_{12}(x, q))(F_{21}(x, q) - 1),$$

which, by Corollaries 2.4 and 2.5, leads to

$$F_{1243}(x, q) = 1 + \frac{xq(2x-1) + (1-x)^2}{(1-x-qx)^2(1-x)}.$$ 

Thus we have
\[ E^{1243}(L_n) = \frac{2^{n-3}(n^2 - n + 4)}{(n-1)2^{n^2} + 1} \]

and

\[ E^{1243}(L_n^2) = \frac{2^{n-4}(n^3 + 5n + 2)}{(n-1)2^{n^2} + 1}. \]

References


