OPTIMALITY BASED STRUCTURED CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

OPTIMALITY BASED STRUCTURED CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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This thesis proposes a complete procedure to obtain static output feedback (SOF) controllers for large scale discrete time linear time invariant (LTI) systems by considering two criteria: (1) use a small number of actuators and sensors, (2) calculate a SOF gain that minimizes a quadratic cost of the states and the input.

If the considered system is observable and stabilizable, the proposed procedure leads to a SOF gain which has a performance comparable to the linear quadratic regulator (LQR) problem in terms of the $\mathcal{H}_2$ norm of the closed loop system. When the system is not observable but detectable, only the observable part is considered.

Since the structure of input and output matrices for the LTI system have a significant importance for the success of the proposed algorithm, an optimal actuator/sensor placement problem is considered first. This problem is handled by taking the final goal of SOF stabilization into account. In order to formulate the actuator/sensor placement as an optimization problem, a method to calculate the generalized Gramians of unstable discrete time LTI systems is developed.

The results are demonstrated on a large scale flexible system and a biological network model.

Keywords: Static output feedback, fixed-order controller, optimal actuator/sensor placement, spatially distributed parameter systems, approximate dynamic programming.

ÖZET

DAĞITIK PARAMETRELİ SİSTEMLERİN ENİYİLİK ÖLÇÜTÜ ODAKLI YAPISAL KONTROLÜ

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Bu tez ayrık zamanlı lineer zamanda değişmez büyük ölçekli sistemlere statik sistem çıkışı geribeslemesi tasarlamak için bütünlüklü bir prosedür sunmaktadır. Önerilen metot iki kriteri gözetmektedir: (1) az sayıda aktüator ve sensör kullanılması, (2) statik sistem çıkışı geribeslemesi kullanarak ikinci dereceden bir maliyet fonksiyonun en aza indirilmesi.

Sistemin gözlenebilir ve kararlı hale getirilebilir olduğu durumda, tezde anlatılan prosedür izlenerek statik sistem çıkışı geribeslemesi kazancı hesaplanabilir. Bu geribesleme lineer kuadratik regülatör ile sağlanan kapalı döngü sistemin $\mathcal{H}_2$ performansı ile kıyaslanabilir iyilikte sonuçlar vermektedir. Sistemin gözlenebilir değil ama tespit edilebilir olduğu durumda ise, problem sistemin gözlenebilir kısmı üzerinden ele alınmıştır.

Sistemin giriş/çıkış matrislerinin yapısının, önerilen algoritmanın başarısına önemli etkisi olduğundan, tezde öncelikle aktüatörlerin ve sensörlerin eniyileme ile yerleştirmesi ele alınmıştır. Bu problem statik çıkış geribeslemesi gözetilerek ele alınmıştır. Kararsız sistemler için aktüatör/sensör yerleştirmeyi bir eniyileme problemi haline getirebilmek amacıyla, ayrık zamanlı zamanda değişmez sistemler için genelleştirilmiş Gramian hesaplama yöntemi geliştirilmiştir.

Sonuçlar bir büyük ölçekli esnek sistem ve bir biyolojik ağ sistemine uygulanmıştır.

Anahtar sözcükler: Statik çıkış geribeslemesi, sabit dereceli kontrolcü, eniyileme ile aktüator/sensör yerleştirilmesi, dağıtık parametreli sistemler, benzetimsel dinamik programlama.

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Chapter 1

Introduction

The practical applications of control theory require the control engineer to consider various design constraints, that can originate from the technical limitations, high cost of implementation, the difficulties in setting up and maintaining the system. An approach to overcome these issues would be deciding a controller structure beforehand by considering the essential constraints and then choosing an adequate one among all possible controllers with the given structure.

For example, the PID controller structure is a highly preferred one for process control applications in the industry. It is simple to implement and there is a vast literature on tuning a PID controller [1]. These facts make the PID controller a widely accepted control method in the industry. Another important example is the distributed control which is preferred for controlling network of systems having high dimensional state spaces models [2, 3]. This control scheme is an extension of the decentralized control [4], which is frequently used for systems modeling a large number of interconnected sub-systems [5, 4]. The distributed control assumes that each controller is responsible for a local part of the whole system and has any or limited communication with the surrounding subsystems. This approach is advantageous in terms of easier implementation and lower complexity [5]. The distributed control also assumes a controller structure defined locally for the subsystems and aims to achieve a control goal (a criterion for performance and robustness) subject to the structural constraints.

In contrast to the well developed full-order controller approaches (observer-state feedback, linear quadratic Gaussian (LQG)), there is not a straightforward solution for the structured control problems. There are several methods developed using linear matrix inequalities (LMI) derived from the Riccati equations for optimal control with additional matrix inequality constraints [6, 7, 8, 9], loop-shaping [10, 11] and non-linear programming [12]. Because of the non-convexity of problem, its solution is NP-hard [13, 14, 15], meaning that the numerical procedure for the solution is not easily tractable.

Choosing actuator and sensor locations can be considered as a first step of designing a control structure since the complexity and cost of a control task can be reduced by a proper placement of actuators and sensors. Different actuator/sensor configurations locate zeros of the system to different points on the complex plane which impose limitations on the closed loop performance. The actuator/sensor placement problem is a well developed subject with applications especially for the control of flexible structures [16, 17, 18]. Their main goal is to improve the robustness against disturbance and the closed loop performance. There are recent studies which extend the problem to complex networks [2, 19, 3]. It should also be noted that actuator and sensor placement decisions should be made by considering the closed loop performance limitations, otherwise this may lead to catastrophic results, as recently observed in crashes of Boeing 737 Max passenger airplanes [20].

The common approach is formulating the actuator and sensor placement problem in terms of improving *the degree of controllability* and *observability* of the system. More precisely, the higher controllability implies the ease of steering the system to a desired state. The higher observability implies the ease of observing the system's state from the measurements [21]. This "degree" in question is often quantified by using the norms of controllability and observability Gramians for linear time invariant (LTI) systems. The common method to choose better actuator/sensor locations is based on maximizing the norm of observability/controllability Gramians. The intuition behind this comes from the fact that greater norm of the controllability gramian means a larger influence of the input on the system states [21]. Similar logic applies for the observability Gramian. If the norm of the observability Gramian is larger, higher observation energy from states is obtained at the output.

This thesis first develops an optimal actuator/sensor selection algorithm for discrete time LTI systems based on maximization of the system Gramians. The second task is to calculate fixed order controllers with a special emphasis on the static output feedback (SOF) control. The proposed SOF calculation method is based on the solution of linear quadratic regulator (LQR) and minimizes a quadratic performance index.

In Chapter 4, the optimal sensor location problem is studied. The proposed method is based on [22, 3, 23, 24]. Their results are extended to the unstable discrete time LTI systems. For this regard, a method is developed to calculate the Gramians of unstable discrete time LTI systems in Chapter 3. It is shown that the output locations chosen by the proposed method can be steered to a desired value with a minimum amount of input energy.

In Chapter 5, the structured controller calculation algorithm is described. The solution is based on Approximate Dynamic Programming (ADP) and converges to a sub-optimal solution when a norm condition is satisfied. The norm condition requires to solve the structured control problem for an appropriate realization of the system. A method to calculate the similarity transformation for such a realization is also given in this chapter.

Lastly, the overall algorithm is explained and applied to two large scale example systems in Chapter 6, and conclusions are made in Chapter 7.

1.1 Notation Used in This Thesis

$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
j	Square root of -1
$\mathbb{R}^n$	The set of all n dimensional column vector with entries in $\mathbb{R}$
$\mathbb{R}^{m\times n}$	The set of all m by n matrices with entries in $\mathbb{R}$
I_n	n by n identity matrix
0_n	n by n zero matrix
$M > 0$	$v^T M v > 0,\ \forall\ v \neq 0$ where matrix M is symmetric and v is vector
$M \geq 0$	$v^T M v \geq 0,\ \forall\ v \neq 0$ where matrix M is symmetric and v is vector
$0_{m\times n}$	m by n zero matrix
$\rho(M)$	Spectral radius of M, $\max_i \lvert\lambda_i(M)\rvert$ where $\lambda_i(M)$ is the ith eigenvalue
$\sigma_{max}(M)$	Largest singular value of M
$\lVert v\rVert_p$	p-norm of the vector v
$\lVert M\rVert_p$	p-norm of the matrix M
M^H	Hermitian transpose of the matrix M
c^*	Conjugate of the complex number c
$tr(M)$	Trace of the square matrix M

$f_t(x)$	Function f explicitly depending on t
$G(z) := \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	Compact notation for the discrete time LTI system given by $G(z) = C\left(zI - A\right)^{-1} B + D$
$M^{\dagger}$	Pseudo-inverse of the matrix M
$null(M)$	Null space of the matrix M
$range(M)$	Range space of the matrix M

Chapter 2

Background and Problem Formulation

In this chapter, a brief background information is given about the problems considered in the thesis. After discussing the optimal actuator/sensor placement problem for discrete time LTI systems and pointing out the significance of the structured control problem, an abstract formulation of the main problem investigated is given.

2.1 A Brief Background on the Sensor/Actuator Placement Problem

Choosing the number of actuators/sensors and properly locating them are critical steps in control system design. This implementation step requires to consider many different criteria: physical constraints, the best achievable performance of the closed loop system, the implementation cost and the complexity of control application.

There is a vast literature on the optimal actuator/sensors selection problem. It is mainly investigated with emphasizing the improvement in the closed loop performance. This problem frequently finds application in the control of large scale flexible systems [25, 26, 17, 18, 16, 27]. In these studies, the common approach to the problem is to increase the degree of controllability and observability of these large scale LTI systems.

This thesis investigates the optimal actuator/sensor placement problem for discrete time LTI systems which are defined by the state space equations

$$\begin{aligned} x_{t+1} &= Ax_t + \tilde{B}\tilde{u}_t \\ \tilde{y}_t &= \tilde{C}x_t, \end{aligned}$$

where $x \in \mathbb{R}^n, \tilde{u} \in \mathbb{R}^{\tilde{m}}$, $\tilde{y} \in \mathbb{R}^{\tilde{r}}$, and appropriately sized matrices $A, \tilde{B}, \tilde{C}$ with real entries.

The degree of controllability indicates the ease of steering the system's state x to a desired point in the state space by the input $\tilde{u}$. The degree of observability can be interpreted as the strength of output signal $\tilde{y}$ produced by the state vector x. These degrees are often quantified by the norm of the system's controllability and observability Gramians. The larger norm means the larger degree of controllability/observability.

When the state matrix A is stable (i.e. $\rho(A) < 1$), the controllability ($\tilde{W}_c$) and observability ($\tilde{W}_o$) Gramians are defined as

$$\begin{aligned} W_c &= \sum_{t=0}^{\infty} A^t \tilde{B}\tilde{B}^T \left(A^T\right)^t \\ W_o &= \sum_{t=0}^{\infty} \left(A^T\right)^t \tilde{C}^T \tilde{C} A^t. \end{aligned}$$

Let us say that the input matrix $\tilde{B}$ has many columns which allows various possible actuator configurations. Similarly, many rows of the output matrix $\tilde{C}$ allow possible sensor configurations. Then, the optimal actuator/sensor selection is choosing the appropriate columns of $\tilde{B}$ and rows of $\tilde{C}$, to be used in the controller design.

Let us represent an optimal configuration by

$$
\begin{aligned}
B &= \tilde{B}\Gamma_b, \quad \Gamma_b = \mathrm{diag}\{[\gamma_1^b \; \gamma_2^b \; \cdots \; \gamma_{\tilde{m}}^b]\} \\
C &= \Gamma_c\tilde{C}, \quad \Gamma_c = \mathrm{diag}\{[\gamma_1^c \; \gamma_2^c \; \cdots \; \gamma_{\tilde{r}}^c]\},
\end{aligned}
$$

where $\Gamma_b,\ \Gamma_c$ are diagonal and $\gamma_i^b,\ \gamma_i^c \in \{0,1\}$ are binary variables used for deciding the inputs (actuators) and outputs (sensors) which are used. Assuming only m number of input and r number of output can be used, the optimal sets of γ_i^b and γ_i^c can be found as solutions of the mixed-integer problems

$$
\begin{aligned}
&\max_{\gamma_i^b} \|W_c\|, \quad \text{subject to} \;\; \sum_{i=1}^{\tilde{m}} \gamma_i^b = m \;\; \text{and} \;\; \gamma_i^b \in \{0,1\} \\
&\max_{\gamma_i^c} \|W_o\|, \quad \text{subject to} \;\; \sum_{i=1}^{\tilde{r}} \gamma_i^c = r \;\; \text{and} \;\; \gamma_i^c \in \{0,1\},
\end{aligned}
$$

where

$$
\begin{aligned}
W_c &= \sum_{t=0}^{\infty} A^t \tilde{B}\Gamma_b\tilde{B}^T \left(A^T\right)^t \\
W_o &= \sum_{t=0}^{\infty} \left(A^T\right)^t \tilde{C}^T\Gamma_c\tilde{C}A^t.
\end{aligned}
$$

Furthermore, the application areas can be extended to large network of systems which are used to model the power grids [19], social and biological networks [3, 2]. The methods used for deriving the degree of observability/controllability is extended to the graph theoretical approaches; [2] suggests that the system has a larger degree of controllability, if the density of connections between the nodes (state variables) of the system is greater (the system matrix has more non-zero entries).

This thesis broadens the ideas in [3, 28] to unstable discrete time systems by using generalized Gramians [29]. The generalized Gramians for discrete time LTI systems is described in Chapter 3. The optimization problem is solved in Chapter 4.

2.2 Structured Control and Static Output Feedback

Designing a full-order controller to stabilize a system with performance and robustness constraints has well known solutions. It can be achieved by designing an appropriate observer to estimate the system states and finding a stabilizing state feedback by considering problem's constraints. The resulting controller has the same degree as the controlled system itself, [30].

Assume, the considered system is the model of a flexible beam or a large network of subsystems connected to each other which can be modeled by a very large number of state variables. This leads to a high dimensional controller when the observer-state feedback structure is preferred. The control algorithm can be designed by working with a reduced order approximate model of the system. The common techniques for reduction are neglecting the insignificant terms in the transfer function [26] or balanced reduction of the system [31]. Another way of simplifying the control algorithm is designing a distributed controller that acts locally on a large network of small subsystems [32]. The latter is a type of structured control since is supposes a network structure that each controller acts locally.

A particular case of structured control is fixed-order controllers which requires the controller to have a transfer function with a given degree. It is known that this problem can be formulated as a bilinear matrix inequality (BMI) problem whose solution is NP-hard [13]. Meaning, there is no generally applicable efficient way of solving fixed-order controller problem.

The simplest achievable fixed-order controller is a static gain (zero order controller) which is called the static output feedback (SOF) controller. The SOF is considered as one of the fundamental problems in control theory [33, 15] and it is studied in several research papers [9, 34, 35, 36, 37, 38]. Not only finding a stabilizing SOF gain but also verifying that such a SOF gain exists is not straightforward. The reader can visit a recent survey [39] that covers many approaches for the solution of SOF.

A simple statement that defines the SOF problem is the following: given the system (A, B, C) where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{r\times n}$, find a matrix $K \in \mathbb{R}^{m\times r}$ that places all the eigenvalues of the closed loop state matrix

$$A_{cl} = A + BKC$$

inside the unit circle.

Furthermore, the fixed-order controller design problem can be reduced to the SOF problem by a proper augmentation of the state space matrices [40]. Let us define the fixed-order controller in the state space form

$$\begin{bmatrix} z_{t+1} \\ v_t \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} z_t \\ w_t \end{bmatrix} = K \begin{bmatrix} z_t \\ w_t \end{bmatrix},$$

where $z \in \mathbb{R}^{n_c}, w \in \mathbb{R}^p, v \in \mathbb{R}^m$ and a stabilizing K must be found. This is equivalent to the SOF problem for the augmented system

$$A' = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \quad B' = \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}.$$

The solution K must place the eigenvalues of the closed loop system matrix

$$A'_{cl} = A' + B'KC'$$

inside the unit circle.

2.2.1 Dynamic Programming

In this thesis, the proposed SOF calculation algorithm is based on dynamic programming (DP). The DP deals with optimization problems containing dynamic equality constraints. A general formulation can be given by

$$\min_{u_t} \quad J = c_{t_f}(x_{t_f}) + \sum_{t=0}^{t_f-1} c_t(x_t, u_t) \tag{2.1}$$

$$\text{subject to} \quad x_{t+1} = f_t(x_t, u_t), \tag{2.2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, $c_t(x_t, u_t) \in \mathbb{R}$ is the cost which must be minimized with respect to u_t, $c_{t_f}(x_{t_f})$ is the final cost at time step t_f and (2.2) is the constraint that the solution must satisfy.

At a time step t_0 the optimal solution $u^\star_{t_0}$ is not isolated from future steps $t > t_0$ and effects the future costs. Because of this fact, the optimal solution must regard the future costs, $c_t(x_t, u_t)$ for $t > t_0$. A decision made at step t_0 does not effect the past steps $t < t_0$. As a consequence of this, dynamic programming carries out the optimization task backwards starting from a final step t_f.

At the final step t_f, the cost is given by $J_{t_f}(x_{t_f}) = c_{t_f}(x_{t_f})$. Then the optimal input for step $t_f - 1$ can be found from the solution of

$$
\begin{aligned}
J_{t_f-1}(x_{t_f-1}, u_{t_f-1}) &= \min_{u_{t_f-1}} \left[c_{t_f-1}(x_{t_f-1}, u_{t_f-1}) + J_{t_f}\left(x_{t_f}\right)\right] \\
J_{t_f-1}(x_{t_f-1}, u_{t_f-1}) &= \min_{u_{t_f-1}} \left[c_{t_f-1}(x_{t_f-1}, u_{t_f-1}) + J_{t_f}\left(f_{t_f-1}(x_{t_f-1}, u_{t_f-1})\right)\right] \\
&= \min_{u_{t_f-1}} \left[c_{t_f-1}(x_{t_f-1}, u_{t_f-1}) + c_{t_f}\left(f_{t_f-1}(x_{t_f-1}, u_{t_f-1})\right)\right],
\end{aligned}
$$

which can be iterated backwards in time

$$
J_{t-1}(x_{t-1}, u_{t-1}) = \min_{u_{t-1}} \left[c_{t-1}(x_{t-1}, u_{t-1}) + J_t\left(f_{t-1}(x_{t-1}, u_{t-1})\right)\right].
$$

The iterations accumulates to reach a total cost

$$
J_0(x_0, u_0) = \min_{u_0} \left(c_0(x_0, u_0) + J_1\left(f_0(x_0, u_0)\right)\right).
$$

The optimal control interpretation of DP for LTI systems is used to solve the linear quadratic control (LQR) problem. In this case the cost function is

$$
J = \frac{1}{2} x_{t_f}^T S_{t_f} x_{t_f} + \frac{1}{2} \sum_{t=0}^{t_f-1} x_t^T Q x_t + u_t^T R u_t,
$$

with the constraint

$$
x_{t+1} = A x_t + B u_t,
$$

where symmetric matrices $S_{t_f} \geq 0, Q \geq 0$ and $R > 0$ are given. The solution is given by [41, Chapter 4]

$$u_t^\star = -(R + B^T S_{t+1} B)^{-1} B^T S_{t+1} A x_t \tag{2.3}$$

$$= F_t x_t, \tag{2.4}$$

where

$$S_t = Q + A^T \left(S_{t+1} - S_{t+1} B (R + B^T S_{t+1})^{-1} B^T S_{t+1} \right) A,$$

which is called the Riccati difference equation. As $t \searrow -\infty$, Riccati difference equation converges to a unique, positive definite S that satisfies the discrete time Riccati equation

$$S = Q + A^T \left(S - SB(R + B^T S)^{-1} B^T S \right) A,$$

provided that (A, B) is controllable and (Q, A) is observable [41, Chapter 4].

In this thesis, the proposed method uses a projected version of the optimal input $u_t^\star$ in (2.4) onto the range space of C^T.

2.2.2 Projection Matrices

In the previous section, the optimal input $u_t^\star$ is defined as $u_t^\star = F_t x_t$ where F_t is a state feedback gain. For the SOF case, a gain matrix K_t must be found which generates a sub-optimal u_t given by $\hat{u}_t = K_t y_t = K_t C x_t$. Such a K_t can be found by solving the least squares problem

$$K_t = F_t C^T (CC^T)^{-1} = F_t C^\dagger,$$

where C has full row rank and $\dagger$ denotes the pseudo-inverse. An other representation can be

$$\hat{F}_t = F_t C^T (CC^T)^{-1} C = F_t \Pi_c, \tag{2.5}$$

where Π_c is an orthogonal projection matrix onto the $range(C^T)$ and $\hat{F}$ is the sub-optimal state feedback.

Definition 2.1. *(Projection matrix)* Π *is a projection matrix if* $\Pi = \Pi^2$.

Definition 2.2. *(Orthogonal projection matrix) If* Π *is a projection matrix and symmetric then* Π *is an orthogonal projection matrix.*

An orthogonal projection matrix Π satisfies:

- $\Pi = \Pi^T = \Pi^2$
- $\Pi = U\Lambda U^T$ where U is orthogonal and

$$\Lambda = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

With these observations, the eigenvalue decomposition of Π_c in (2.5) is $\Pi_c = U_c \Lambda_c U_c^T$ where

$$\Lambda_c = \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix},$$

r and n are row and column counts of C.

2.3 An Abstract Formulation of the Main Problem

This thesis develops a SOF stabilization procedure that minimizes a quadratic cost function of the states and input by using a small number of actuators/sensors for discrete time LTI systems. A high level abstract formulation can be given by

Problem 1. *(Main problem) Given* $Q = Q^T \geq 0, R = R^T > 0$ *and constants* $\beta_c > 0, \beta_b > 0$,

$$\begin{aligned}
\min_{\gamma_i^c, \gamma_j^b, K} \quad & \beta_c \sum_{i=1}^{\tilde{r}} \gamma_i^c + \beta_b \sum_{j=1}^{\tilde{m}} \gamma_j^b + \sum_{t=0}^{\infty} x_t^T Q x_t + u_t^T R u_t \qquad (2.6) \\
\textit{subject to} \quad & x_{t+1} = A x_t + B u_t \\
& y_t = C x_t \\
& u_t = K y_t \\
& \gamma_i^c, \gamma_j^b \in \{0, 1\},
\end{aligned}$$

where $C = \Gamma_c \tilde{C}, B = \tilde{B} \Gamma_b$. *Matrices* $\tilde{C}, \tilde{B}$ *contain all possible sensor and actuator configurations and*

$$\begin{aligned}
\Gamma_c &= diag\{[\gamma_1^c \ \gamma_2^c \ \cdots \ \gamma_{\tilde{r}}^c]\} \\
\Gamma_b &= diag\{[\gamma_1^b \ \gamma_2^b \ \cdots \ \gamma_{\tilde{m}}^b]\}.
\end{aligned}$$

The weights of first two terms in (2.6) are adjusted by the positive scalar values β_c and β_b. Large β_c puts large penalty on the number of sensors used and similarly, β_b penalizes the number of actuators used.

Finding a solution for the whole problem can not be done directly. For this reason, the problem is separated into two parts. Firstly, an optimal actuator/sensor placement procedure is developed to decide γ_i^c and γ_j^b values. Then, the last term in (2.6) is solved by a dynamic programming based algorithm, to determine K.

Chapter 3

Generalized Gramians

The observability and controllability attributes of discrete time LTI systems are generally examined by means of their Gramians. For the system given by the state space equations

$$x_{t+1} = Ax_t + Bu_t \tag{3.1}$$

$$y_t = Cx_t, \tag{3.2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^r$ and A, B, C are real matrices of appropriate sizes and assuming $\rho(A) < 1$, the controllability Gramian W_o is defined as

$$W_c = \sum_{t=0}^{\infty} A^t BB^T \left(A^T\right)^t, \tag{3.3}$$

where $W_c \geq 0$ is also the solution of discrete time Lyapunov equation

$$W_c = BB^T + AW_cA^T.$$

Similarly, the observability Gramian W_o is defined as

$$W_o = \sum_{t=0}^{\infty} \left(A^T\right)^t C^T C A^t, \tag{3.4}$$

where $W_o \geq 0$ is the solution of discrete time Lyapunov equation

$$W_o = C^T C + A^T W_o A.$$

The infinite sums in (3.4) and (3.3) imply that all eigenvalues of matrix A must be inside the unit circle for convergence. More precisely, the Gramians (3.3) and (3.4) are defined for stable discrete time LTI systems which has a state matrix A having spectral radius $\rho(A) < 1$.

The Gramians in (3.4) and (3.3) can be equivalently written in frequency domain by using Plancherel's theorem [42, Chapter 4]

$$W_c = \frac{1}{2\pi}\int_0^{2\pi} \left(e^{j\omega}I - A\right)^{-1} BB^T \left(e^{-j\omega}I - A^T\right)^{-1} d\omega, \tag{3.5}$$

$$W_o = \frac{1}{2\pi}\int_0^{2\pi} \left(e^{-j\omega}I - A^T\right)^{-1} C^T C \left(e^{j\omega}I - A\right)^{-1} d\omega \tag{3.6}$$

where $\left(e^{j\omega}I - A\right)^{-1} B$ and $C\left(e^{j\omega}I - A\right)^{-1}$ are the discrete time Fourier transforms (DTFT) of $A^t B$ and CA^t respectively.

A similar formulation based on the frequency domain formula in (3.5) and (3.6) can be developed for unstable discrete time LTI systems. In [29], the Gramians of unstable system is called the generalized Gramians. In this chapter, a discrete time counterpart of the formulation in [29] is developed to calculate the generalized Gramians for discrete time LTI systems.

Now, consider the unstable discrete time LTI system $G(z)$ in the state space form

$$G(z) := \left[\begin{array}{c|c} A & B \\ \hline I_n & 0_{n\times m} \end{array}\right],$$

with the right coprime factorization $G(z) = N(z)M(z)^{-1}$ where $G(z)$ has no poles on the unit circle, $M(z)$ is an inner transfer function and $N(z)$ is stable. In [43, Chapter 21], a realization of the coprime factorized form is given by $G = NM^{-1}$, where

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A + BF & B\tilde{R}^{-1/2} \\ \hline F & \tilde{R}^{-1/2} \\ I_n & 0_{n\times m} \end{array}\right],$$

with

$$\tilde{R} = I_m + B^T S B \tag{3.7}$$

$$F = -\tilde{R}^{-1} B^T S A \tag{3.8}$$

and $S = S^T \geq 0$ is the solution of the Riccati equation

$$S = A^T S (I_n + B B^T S)^{-1} A. \tag{3.9}$$

Using Matrix Inversion Lemma, the Riccati equation (3.9) can be equivalently written as [44, Chapter 12]

$$S = A^T (S - SB(I_m + B^T S B)^{-1} B^T S) A, \tag{3.10}$$

where S is a stabilizing solution that places all eigenvalues of $A + BF$ inside the unit circle.

The transfer function $G(z)$ in coprime factorized form is given by the product

$$G(z) = N(z) M(z)^{-1},$$

where

$$N(z) := \left[\begin{array}{c|c} A + BF & \tilde{R}^{-1/2} \\ \hline I_n & 0_{n \times m} \end{array}\right], \quad \text{and} \quad M(z) := \left[\begin{array}{c|c} A + BF & \tilde{R}^{-1/2} \\ \hline F & \tilde{R}^{-1/2} \end{array}\right].$$

Similar to (3.5), the frequency domain representation of the generalized controllability Gramian of $G(z)$ can be written by

$$\begin{aligned} W_c &= \frac{1}{2\pi} \int_0^{2\pi} G(z) G(z)^H d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} N(z) M(z)^{-1} \left(M(z)^{-1}\right)^H N(z)^H d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} N(z) N(z)^H d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(e^{j\omega} I - (A + BF)\right) B \tilde{R}^{-1} B^T \left(e^{-j\omega} I - (A + BF)^T\right), \end{aligned}$$

since $M(z)$ is inner, $M(z)^{-1} \left(M(z)^H\right)^{-1} = I$. Then, the generalized controllability Gramian W_c associated to the pair (A, B) is solution of the Lyapunov equation

$$W_c = B \tilde{R}^{-1} B^T + (A + BF) W_c (A + BF)^T. \tag{3.11}$$

The generalized observability Gramian W_o associated to (C, A) can be calculated similarly as the transpose of the generalized controllability Gramian for the transposed system matrices (A^T, C^T).

Chapter 4

Optimal Actuator and Sensor Placement

In this thesis, the sensor placement algorithm is developed with a focus on the SOF stabilization which is solved by using an approximate solution of the well-known LQR problem which is a type of $\mathcal{H}_2$ synthesis when the system's output matrix C is considered to be the identity matrix.

Definition 4.1. *$\mathcal{H}_2$ norm of the discrete time LTI systems: For the stable discrete time LTI system $G(z) = C\left(zI - A\right)^{-1} B$ defined by state space matrices (A, B, C), the $\mathcal{H}_2$ norm $\|G(z)\|_{\mathcal{H}_2}$ is given by [42, Chapter 5]*

$$\|G(z)\|_{\mathcal{H}_2} = \sqrt{tr(CW_cC^T)},$$

where W_c is the controllability Gramian of $G(z)$.

Define a matrix $\tilde{C} \in \mathbb{R}^{\tilde{m}\times n}$ which contains a large set of possible sensor configurations (for example, $\tilde{C} = C = I$ when all states can be directly measured) and $m < \tilde{m}$ sensor configurations must be selected among $\tilde{m}$ possibilities. In the proposed method, sensor selection problem is equivalent to selecting the "best" rows of $\tilde{C}$ by multiplying a diagonal matrix Γ_c with diagonal entries $\gamma_i^c \in \{0, 1\}$

to obtain

$$C = \Gamma_c \tilde{C},$$

and removing the zero rows.

Let us say, $\tilde{\mathcal{Y}}$ and $\mathcal{Y}$ denote the $\mathcal{H}_2$ norms of the systems given by $(A, B, \tilde{C})$ and (A, B, C) respectively. The proposed method aims to choose γ_i^c values such that $\mathcal{Y}$ becomes as close to $\tilde{\mathcal{Y}}$ as possible. By this way, the LQR based SOF calculation algorithm described in Chapter 5 efforts to suppress a larger energy which is closer to the one in state feedback. It leads to better results in terms of the closed loop performance, as demonstrated by the examples given in Chapter 6.

4.1 Optimal Actuator and Sensor Placement for Stable Systems

Definition 4.2. *(Output controllability) For the system given by (A, B, C), if an input sequence u_t can be found that steers the system output from y_0 to a desired y_f in finite time, it can be said that (A, B, C) is output controllable, [3].*

Output controllability can be formalized by a matrix rank condition:

$$\text{If } \ rank(C) = rank\left(C\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix}\right), \tag{4.1}$$

the system is output controllable. Similarly, the output controllability gramian can be defined as [3, 23, 19],

$$Y_c = CW_cC^T, \tag{4.2}$$

where W_c is the controllability gramian. The norm of Y_c gives a measure about ease of steering the output of the system to a desired value if the system is output controllable. Output controllability condition (4.1) implies non-singularity of the output controllability Gramian Y_c [3].

Proposition 4.1. *The output controllability Gramian Y_c is positive definite if the discrete time LTI system given by (A, B, C) is output controllable.*

Proof. Output controllability indicates that the matrix

$$\mathcal{O}_c = C \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix},$$

has full row rank. Then, the matrix $\mathcal{O}_c \mathcal{O}_c^T$ is positive definite. The output controllability Gramian is given by

$$\begin{aligned} Y_c &= \sum_{t=0}^{\infty} CA^t BB^T \left(A^T\right)^t C^T \\ &= \mathcal{O}_c \mathcal{O}_c^T + \sum_{t=n}^{\infty} CA^t BB^T \left(A^T\right)^t C^T > 0, \end{aligned}$$

is positive definite. □

The next lemma describes the relation between the required input energy to steer the output and the norm of output controllability Gramian.

Lemma 4.1. *Define the finite horizon output controllability Gramian $Y_c(t)$ by*

$$\begin{aligned} Y_c(t) &= \sum_{\tau=0}^{t-1} CA^\tau BB^T \left(A^T\right)^\tau C^T \\ &= \sum_{\tau=0}^{t-1} CA^{t-\tau-1} BB^T \left(A^T\right)^{t-\tau-1} C^T. \end{aligned}$$

For the stable discrete time LTI system defined by (A, B, C), the minimum energy required to steer the output y_t from $y_0 = 0$ to $y_{t_f} = y_f$ is given by

$$J = y_f^T Y_c(t_f)^{-1} y_f, \tag{4.3}$$

where J is also the solution of the optimal control problem

$$\min_{u_t} \quad J = \frac{1}{2} \sum_{t=0}^{t_f-1} u_t^T u_t \tag{4.4}$$

$$s.t. \quad x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t \tag{4.5}$$

$$y_f = y_{t_f} = Cx_{t_f}, \tag{4.6}$$

where $y_{t_f} = y_f$ is a terminal condition.

Proof. The Hamiltonian of the system is given by

$$H(x, \lambda, u) = \sum_{t=0}^{t_f-1} H_t + \nu^T \left(y_f - Cx_{t_f}\right)$$

where

$$H_t(x_t, \lambda_{t+1}, u_t) = \frac{1}{2}u_t^T u_t + \lambda_{t+1}^T \left(Ax_t + Bu_t\right),$$

$\lambda \in \mathbb{R}^n$, $\nu \in \mathbb{R}^r$ are Lagrange multipliers and ν comes from the terminal condition. The optimal solution must satisfy the constraints [45, Chapter 4]

$$x_{t+1} = \frac{\partial H_t}{\partial \lambda_{t+1}} = Ax_t + Bu_t, \quad \text{for} \quad t < t_f - 1 \tag{4.7}$$

$$\lambda_t = \frac{\partial H_t}{\partial x_t} = A^T \lambda_{t+1}, \quad \text{for} \quad t < t_f - 1 \tag{4.8}$$

$$0 = \frac{\partial H_t}{\partial u_t} = u_t + B^T \lambda_{t+1} \tag{4.9}$$

with the terminal condition

$$\frac{\partial H}{\partial x_{t_f}} = \lambda_{t_f} = C^T \nu, \tag{4.10}$$

From (4.9), the optimal solution is

$$u_t^\star = -B^T \lambda_{t+1}.$$

By using (4.8) and (4.10), one can write

$$\lambda_t = \left(A^T\right)^{t_f - t} \lambda_{t_f} = \left(A^T\right)^{t_f - t} C^T \nu \tag{4.11}$$

$$u_t^\star = -B^T \lambda_{t+1} = -B^T \left(A^T\right)^{t_f - t - 1} C^T \nu. \tag{4.12}$$

The output response of the system for the optimal input (4.12) is given by

$$y_t = CA^t x_0 + \sum_{\tau=0}^{t-1} CA^{t-\tau-1} Bu_\tau^\star \tag{4.13}$$

$$= CA^t x_0 - \sum_{\tau=0}^{t-1} CA^{t-\tau-1} BB^T \left(A^T\right)^{t-\tau-1} C^T \nu. \tag{4.14}$$

Note that, the term $\sum_{\tau=0}^{t-1} CA^{t-\tau-1} BB^T \left(A^T\right)^{t-\tau-1} C^T$ is equal to the finite horizon output controllability Gramian $Y_c(t)$. Then,

$$y_t = CA^t x_0 - Y_c(t)\nu.$$

Choose $x_0 = 0$ for simplicity, then one can write

$$y_{t_f} = -Y_c(t_f)\nu \tag{4.15}$$

$$\nu = -Y_c(t_f)^{-1} y_f. \tag{4.16}$$

Substitute (4.16) in (4.12)

$$u_t^\star = B^T \left(A^T\right)^{t_f-t-1} C^T Y_c(t_f)^{-1} y_f.$$

Then, the optimal solution $J^\star$ is

$$\begin{aligned}
J^\star &= \sum_{t=0}^{t_f-1} \left(u_t^\star\right)^T u_t^\star \\
&= y_f^T Y_c(t_f)^{-1} \left(\sum_{t=0}^{t_f-1} C A^{t_f-t-1} B B^T \left(A^T\right)^{t_f-t-1} C^T \right) Y_c(t_f)^{-1} y_f \\
&= y_f^T Y_c(t_f)^{-1} y_f.
\end{aligned}$$

□

Remark 4.1. *As $t_f \to \infty$, $Y_c(t_f) \to Y_c$ where Y_c is the output controllability Gramian.*

Minimization of J in (4.3) with respect to the output controllability matrix Y can be equivalently written as an LMI problem by introducing an additional variable f [46]

$$\begin{aligned}
&\min_{Y_c} \quad f \\
&\text{subject to} \quad \begin{bmatrix} Y_c & y_f \\ y_f^T & f \end{bmatrix} \geq 0, \quad Y_c > 0.
\end{aligned} \tag{4.17}$$

If the norm $\|Y_c\|$ is larger, a smaller f can be found that satisfies the inequality constraint for any given y_f. This formulation does not contain the constraints imposed by system dynamics which increase the complexity of the problem. An appropriate way to highly reduce the complexity can be achieved by considering maximization of the trace of Y_c which found several applications in the literature [47, 19, 23, 24]. This approach is more suitable for calculating the SOF gain by an approximate solution of the LQR problem.

Let us revisit the definitions in Section 2.1, where the matrices $\tilde{B} \in \mathbb{R}^{n\times\tilde{m}}$ and $\tilde{C} \in \mathbb{R}^{\tilde{r}\times n}$ contains all possible input and output configurations. The diagonal matrices Γ_b and Γ_c with binary diagonal entries are use to decide which columns of $\tilde{B}$ and rows of $\tilde{C}$ are selected as inputs and outputs. Then, a choice of input/output matrices can be represented by

$$B = \tilde{B}\Gamma_b, \quad C = \Gamma_c\tilde{C}. \tag{4.18}$$

Assuming the state matrix A is diagonalizable by the similarity transformation T_d, the output controllability Gramian for the transformed system ($\Sigma = T_d A T_d^{-1}, \bar{B} = T_d\tilde{B}, \bar{C} = \tilde{C}T_d^{-1}$) is

$$Y_c = \sum_{t=0}^{\infty} \Gamma_c\tilde{C}A^t\tilde{B}\Gamma_b\tilde{B}^T\left(A^T\right)^t\tilde{C}^H\Gamma_c, \tag{4.19}$$

$$= \sum_{t=0}^{\infty} \Gamma_c\bar{C}\Sigma^t\bar{B}\Gamma_b\bar{B}^H\left(\Sigma^H\right)^t\bar{C}^H\Gamma_c, \tag{4.20}$$

where $\Sigma \in \mathbb{C}^{n\times n}$ is the diagonal matrix of eigenvalues $\{\lambda_i : i = 1, \cdots, n\}$ of A, $\bar{B}$ and $\bar{C}$ are complex matrices. Define

$$\bar{C} = \begin{bmatrix} \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{b}_1^H \\ \bar{b}_2^H \\ \vdots \\ \bar{b}_n \end{bmatrix},$$

where $\bar{c}_i$ and $\bar{b}_i$ are complex column vector. Then,

$$\bar{C}\Sigma^t\bar{B}\bar{B}^H\left(\Sigma^H\right)^t C^H = \left(\sum_{i=1}^{n}\lambda_i^t\Gamma_c\bar{c}_i\bar{b}_i^H\Gamma_b\right)\left(\sum_{j=1}^{n}\left(\lambda_j^*\right)^t\Gamma_b\bar{b}_j\bar{c}_j^H\Gamma_c\right) \tag{4.21}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\lambda_i\lambda_j^*\right)^t\Gamma_c\bar{c}_i\bar{b}_i^H\Gamma_b\bar{b}_j\bar{c}_j^H\Gamma_c. \tag{4.22}$$

Substitute (4.22) into (4.20)

$$Y_c = \sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\lambda_i\lambda_j^*\right)^t\Gamma_c\bar{c}_i\bar{b}_i^H\Gamma_b\bar{b}_j\bar{c}_j^H\Gamma_c. \tag{4.23}$$

Since A is a stable matrix, $\left|\lambda_i \lambda_j^*\right| < 1$ for all i, j. The sum over time index t can be eliminated by defining

$$
\begin{aligned}
\Lambda_{ij} &= \sum_{t=0}^{\infty} \left(\lambda_i \lambda_j^*\right)^t \\
&= \frac{1}{1 - \lambda_i \lambda_j^*}.
\end{aligned}
$$

The trace of Y_c in (4.23) is given by

$$
\mathrm{tr}(Y_c) = \sum_{k=1}^{\tilde{r}} [Y_c]_{kk},
$$

where $\tilde{r}$ is the number of rows in $\bar{C}$ and $[Y_c]_{kk}$ is the entry on the kth row and kth column of Y_c. Then,

$$
[Y_c]_{kk} = \sum_{i=1}^{n} \sum_{j=1}^{n} \Lambda_{ij} \gamma_k^c \bar{c}_{ik} \left(\bar{b}_i^H \Gamma_b \bar{b}_j\right) \gamma_k^c \bar{c}_{jk}^*,
$$

where $\bar{c}_{ik}$ is the kth element in vector $\bar{c}_i$. Therefore,

$$
\begin{aligned}
\mathrm{tr}(Y_c) &= \sum_{k=1}^{\tilde{r}} \gamma_k^c \sum_{i=1}^{n} \sum_{j=1}^{n} \Lambda_{ij} c_{ik} c_{jk}^* \sum_{l=1}^{\tilde{m}} \gamma_l^b b_{il}^* b_{jl} && (4.24) \\
&= \sum_{k=1}^{\tilde{r}} \gamma_k^c \sum_{l=1}^{\tilde{m}} \gamma_l^b \sum_{i=1}^{n} \sum_{j=1}^{n} \Lambda_{ij} c_{ik} c_{jk}^* b_{il}^* b_{jl},
\end{aligned}
$$

where $\tilde{m}$ is the number of columns in $\bar{B}$, $\bar{b}_{il}$ is the lth element of the vector $\bar{c}_i$ and $\left(\gamma_k^c\right)^2 = \gamma_k^c$, $\left(\gamma_k^b\right)^2 = \gamma_k^b$ since $\gamma_k^c, \gamma_k^b \in \{0, 1\}$. When the sums over i and j is evaluated, the last equation can be brought into a simpler form

$$
\begin{aligned}
\mathrm{tr}(Y_c) &= \sum_{k=1}^{\tilde{r}} \gamma_k^c \sum_{l=1}^{\tilde{m}} \gamma_l^b H_{kl}, \quad H_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} \Lambda_{ij} c_{ik} c_{jk}^* b_{il} b_{jl}^* && (4.25) \\
&= \underline{\gamma}_c^T H \underline{\gamma}_b, && (4.26)
\end{aligned}
$$

where $H \in \mathbb{R}^{\tilde{r} \times \tilde{m}}$, $H_{kl} \geq 0$ for all k, l, $\underline{\gamma}_c = [\gamma_1^c \ \cdots \ \gamma_{\tilde{r}}^c]^T$ and $\underline{\gamma}_b = [\gamma_1^b \ \cdots \ \gamma_{\tilde{m}}^b]^T$.

Problem 2.

$$
\begin{aligned}
&\max_{\underline{\gamma}_c, \underline{\gamma}_b} \ \underline{\gamma}_c^T H \underline{\gamma}_b && (4.27) \\
&\textit{subject to} \ \sum_{k=1}^{\tilde{r}} \gamma_k^c = r \quad \textit{and} \quad \sum_{l=1}^{\tilde{m}} \gamma_l^b = m && (4.28) \\
&\gamma_k^c, \gamma_l^b \in \{0, 1\}, && (4.29)
\end{aligned}
$$

where H is defined in (4.25).

The Problem 2 is a quadratic mixed-integer problem with linear equality constraints. When input or output configuration is fixed and problem is solved only for $\underline{\gamma}_c$ or $\underline{\gamma}_b$, optimal solution can simply be obtained by ordering the entries in vectors, $H\underline{\gamma}_b$ or $\underline{\gamma}_c^T H$ and picking the indices of the first r or m largest entries. Nevertheless, obtaining a simultaneous solution with respect to both $\underline{\gamma}_c$ and $\underline{\gamma}_b$ requires to use a mixed-integer program solver. The SCIP Optimization Suite [48] is used to obtain the results presented in this thesis.

4.2 Optimal Sensor and Actuator Placement for Unstable Systems

The optimization problem given in the previous section can be applied to unstable discrete time LTI systems by using generalized Gramians. The results of Lemma 4.1 are also valid for the unstable case with slight modifications.

Lemma 4.2. *For the unstable discrete time LTI system defined by* (A, B, C), *define*

$$\begin{aligned} Y_c(t) = C \sum_{\tau=0}^{t-1} (A+BF)^{t-\tau-1} B(I+B^T SB)^{-1} B^T \\ \times \left((A+BF)^T\right)^{t-\tau-1} C^T, \end{aligned}$$

where

$$\begin{aligned} S &= A^T(S - SB(I_m + B^T SB)^{-1} B^T S)A, \\ F &= -(I + B^T SB)^{-1} B^T SA \end{aligned}$$

and $A + BF$ is stable. Then, the minimum energy required to steer the output $y_0 = 0$ to a desired final value $y_{t_f} = y_f$ is given by

$$J = \frac{1}{2} y_f^T Y_c(t_f)^{-1} y_f, \tag{4.30}$$

where J is also the solution of

$$\min_{u_t} \quad J = \frac{1}{2}x_{t_f}^T S x_{t_f} + \frac{1}{2}\sum_{t=0}^{t_f-1} u_t^T u_t \tag{4.31}$$

$$s.t. \quad x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t \tag{4.32}$$

$$y_f = y_{t_f} = Cx_{t_f}. \tag{4.33}$$

Proof. Proof is based on the solution of optimal state feedback problem with fixed terminal condition in [45, Chapter 4]. Start by defining quadratic cost

$$\min_{u_t} \quad J = \frac{1}{2}x_{t_f}^T S_{t_f} x_{t_f} + \frac{1}{2}\sum_{t=0}^{t_f-1} u_t^T u_t, \tag{4.34}$$

with terminal condition $y_f = y_{t_f} = Cx_{t_f}$. Which can be equivalently written with Lagrange multipliers λ_t and ν

$$\begin{aligned} J = &\frac{1}{2}x_{t_f}^T S_{t_f} x_{t_f} + \left(y_f - Cx_{t_f}\right)^T \nu \\ &+ \sum_{t=0}^{t_f} \frac{1}{2}u_t^T u_t + \lambda_{t+1}\left(Ax_t + Bu_t - x_{t+1}\right). \end{aligned}$$

The Hamiltonian is given by

$$H_t = \frac{1}{2}u_t^T u_t + \lambda_{t+1}^T\left(Ax_t + Bu_t\right),$$

where the optimal solution satisfies

$$\begin{aligned} x_{t+1} &= \frac{\partial H_t}{\partial \lambda_{t+1}} = Ax_t + Bu_t \\ \lambda_t &= \frac{\partial H_t}{\partial x_t} = A^T \lambda_{t+1}, \end{aligned}$$

with u_t is given by

$$u_t = -B^T \lambda_{t+1}. \tag{4.35}$$

By considering the fixed terminal condition $y_{t_f} = y_f$, guess a solution given by

$$\lambda_t = S_t x_t + V_t \nu, \tag{4.36}$$

where S_t and V_t are [45]

$$S_t = A^T S_{t+1}(A + BF_t), \quad \text{for a given } S_{t_f}, \tag{4.37}$$

$$V_t = (A + BF_t)^T V_{t+1}, \quad V_{t_f} = C^T, \tag{4.38}$$

where $F_t = -(I + B^T S_{t+1} B) B^T S_{t+1} A$. If V_t is iterated backwards, we obtain

$$V_t = \Phi(t_f, t)^T C^T,$$

where

$$\begin{aligned}\Phi(t, \tau) = (A + BF_{t-1})(A + BF_{t-2}) \cdots (A + BF_\tau),\\ \text{for } \tau < t.\end{aligned}$$

Optimal u_t (4.35) can be written as

$$\begin{aligned}u_t &= -B^T(S_{t+1}x_{t+1} + V_{t+1}\nu)\\ &= -B^T\left(S_{t+1}(Ax_t + Bu_t) + V_{t+1}\nu\right)\\ &= -(I + B^T S_{t+1} B)^{-1} B^T S_{t+1} A\\ &- (I + B^T S_{t+1} B)^{-1} B^T V_{t+1}\nu\\ &= F_t x_t + r_t = w_t + r_t.\end{aligned}$$

The boundary condition ν can be found from the system's output response at time t_f for the input $u_t = w_t + r_t$.

$$\begin{aligned}x_{t_f} &= \Phi(t_f, 0)x_0 + \sum_{\tau=0}^{t_f-1} \Phi(t_f, \tau + 1)Br_\tau\\ &= \Phi(t_f, 0)x_0 - \sum_{\tau=0}^{t_f-1} \Phi(t_f, \tau + 1)\\ &\times B(I + B^T S_{\tau+1} B)^{-1} B^T V_{t+1}\nu\\ &= \Phi(t_f, 0)x_0 - \sum_{\tau=0}^{t_f-1} \Phi(t_f, \tau + 1)\\ &\times B(I + B^T S_{\tau+1} B)^{-1} B^T \Phi(t_f, \tau + 1)^T C^T \nu\\ y_{t_f} &= C\Phi(t_f, 0)x_0 - Y_c(t_f)\nu,\end{aligned}$$

where

$$Y_c(t_f) = C \sum_{\tau=0}^{t_f-1} \Phi(t_f, \tau+1)$$
$$\times B(I + B^T S_{\tau+1} B)^{-1} B^T \Phi(t_f, \tau+1)^T C^T.$$

That leads to

$$\nu = Y_c(t_f)^{-1} \left(C\Phi(t_f, 0)x_0 - y_{t_f} \right).$$

For simplicity, let us choose $S_{t_f} = S$ of (4.37) and $x_0 = 0$. Then,

$$\begin{aligned}
&S_t = S \quad \forall\, t \leq t_f \\
&F_t = F = -(I + B^T SB)^{-1} B^T SA \\
&\Phi(t, \tau+1) = (A + BF)^{t-\tau-1} = A_{cl}^{t-\tau-1} \\
&u_t = Fx_t + r_t,
\end{aligned}$$

where

$$r_t = -(I + B^T SB)^{-1} B^T \left(A_{cl}^T \right)^{t_f-\tau-1} C^T Y_c(t_f)^{-1} y_f.$$

Substitute u_t into the cost function (4.34)

$$\begin{aligned}
J &= \frac{1}{2} x_{t_f}^T S_{t_f} x_{t_f} + \frac{1}{2} \sum_{t=0}^{t_f-1} J_t \\
J_t &= w_t^T w_t + 2r_t^T w_t + r_t^T r_t.
\end{aligned}$$

From (4.35)

$$w_t = -B^T (Sx_{t+1} + V_{t+1}\nu) - r_t.$$

Then

$$\begin{aligned}
r_t^T w_t &= -r_t^T B^T Sx_{t+1} - r_t^T B^T V_{t+1} v - r_t^T r_t \\
&= -r_t^T B^T Sx_{t+1} + r_t^T (I + B^T SB) r_t - r_t^T r_t,
\end{aligned}$$

is obtained. By adding and subtracting $x_{t+1}^T S x_{t+1}$, J_t can be simplified:

$$J_t = w_t^T w_t - 2r_t^T B^T S x_{t+1} + 2r_t^T (I + B^T SB) r_t \tag{4.39}$$

$$- 2r_t^T r_t + r_t^T r_t - x_{t+1}^T S x_{t+1} + x_{t+1}^T S x_{t+1} \tag{4.40}$$

$$= x_t^T F^T F x_t - x_{t+1}^T S x_{t+1} + x_{t+1}^T S x_{t+1} \tag{4.41}$$

$$- 2r_t^T B^T S x_{t+1} + r_t^T B^T S B r_t + r_t^T (I + B^T SB) r_t \tag{4.42}$$

$$= x_t^T F^T F x_t + (x_{t+1} - Br_t)^T S (x_{t+1} - Br_t) \tag{4.43}$$

$$- x_{t+1}^T S x_{t+1} + r_t^T (I + B^T SB) r_t \tag{4.44}$$

$$= x_t^T (F^T F + A_{cl}^T S A_{cl}) x_t - x_{t+1}^T S x_{t+1} \tag{4.45}$$

$$+ r_t^T (I + B^T SB) r_t \tag{4.46}$$

$$= x_t^T S x_t - x_{t+1}^T S x_{t+1} + r_t^T (I + B^T SB) r_t \tag{4.47}$$

by using the equality $S = F^T F + A_{cl}^T S A_{cl}$ which is another representation of the DARE in (3.10). Finally, substitute J_t into J and by using (4.39)

$$J = \frac{1}{2} x_{t_f}^T S x_{t_f} + \frac{1}{2} \sum_{t=0}^{t_f - 1} J_t \tag{4.48}$$

$$= \frac{1}{2} x_{t_f}^T S x_{t_f} + \frac{1}{2} \sum_{t=0}^{t_f - 1} (x_t S x_t - x_{t+1} S x_{t+1}) \tag{4.49}$$

$$+ \frac{1}{2} \sum_{t=0}^{t_f} r_t^T (I + B^T SB) r_t \tag{4.50}$$

$$= \frac{1}{2} x_{t_f}^T S x_{t_f} + \frac{1}{2} \sum_{t=0}^{t_f - 1} \left(x_t^T S x_t - x_{t+1}^T S x_{t+1} \right) \tag{4.51}$$

$$+ \frac{1}{2} y_f^T Y_c(t_f)^{-1} y_f \tag{4.52}$$

$$= \frac{1}{2} x_0^T S x_0 + \frac{1}{2} y_f^T Y_c(t_f)^{-1} y_f = \frac{1}{2} y_f^T Y_c(t_f)^{-1} y_f. \tag{4.53}$$

□

Remark 4.2. *Note that,*

$$\begin{aligned} Y_c(t) &= C \sum_{\tau=0}^{t-1} (A+BF)^{t-\tau-1} B(I+B^T SB)^{-1} B^T \\ &\times \left((A+BF)^T\right)^{t-\tau-1} C^T \\ &= C \sum_{\tau=0}^{t-1} (A+BF)^{\tau} B(I+B^T SB)^{-1} B^T \\ &\times \left((A+BF)^T\right)^{\tau} C^T \\ &= C W_c(t) C^T, \end{aligned}$$

where $W_c(t)$ is the generalized controllability gramian in (3.11) *and $W_c(t) \to W_c$ as $t \to \infty$, where W_c is the generalized controllability Gramian of the discrete time unstable LTI system given by (A, B, C).*

The optimization problem 2 can be also used similarly for the unstable case by using the generalized output controllability Gramian.

4.3 Summary of the Results

In this chapter, a procedure for simultaneous optimal placement of actuator and sensors is described. Assuming that there are large sets of all possible actuator ($\tilde{B}$) and sensor ($\tilde{C}$) configurations, to control and observe the states of discrete time LTI system, The proposed method aims to choose a subset (B, C) of possible configurations that maximizes the $\mathcal{H}_2$ norm of the system given by (A, B, C).

The $\mathcal{H}_2$ norm of a system can be obtained by the trace of output controllability Gramian $Y_c = CW_cC^T$ where W_c is the controllability Gramian. Since the Gramians are defined for stable systems, it is extended to unstable systems by using the generalized Gramians. By using the generalized output controllability Gramians the optimal actuator/sensor places are obtained from the result of a quadratic mixed-integer problem.

The overall algorithm can be described as:

Algorithm 1. *(Actuator/sensor placement) Given m and r which denote the number of input and output will be used;*

1. *If the system is stable solve the Lyapunov equation*

$$W_c = \tilde{B}\tilde{B}^T + AW_cA^T$$

2. *Else solve*

$$W_c = \tilde{B}\tilde{R}^{-1}\tilde{B}^T + (A + \tilde{B}F)W_c(A + \tilde{B}F)^T,$$

   where F and $\tilde{R}$ are given in (3.8) *and* (3.7) *to obtain the controllability Gramian W_c.*

3. *Calculate H of* (4.26) *by using $Y_c = \tilde{C}W_c\tilde{C}^T$.*

4. *Solve Problem 2 to obtain optimal γ_i^c and γ_j^b values.*

An example will be given in Chapter 6.

Chapter 5

Structured Control Design

This chapter presents a framework for calculating fixed-order controllers which expands the method developed in our paper [49]. The fixed-order controller problem is transformed into a SOF problem and a step-by-step procedure is developed to obtain the SOF gain which minimizes a quadratic performance index.

The proposed algorithm utilizes an approximate solution of dynamic programming related to the LQR problem for discrete time LTI systems. The approximate solution requires and appropriate realization of the system. Dynamic programming based method leads to a modified version of the Riccati difference equations. The iterations converge for a realization while diverging for another one. It is shown that the projected state matrix $A\Pi_{\bar{c}}$ onto the null space of C must be stable for a steady state solution of the modified Riccati equation, but it is not sufficient for convergence. Although the convergence of the modified Riccati difference equation is not easily tractable, it is observed that iterations converge if the largest singular value of $A\Pi_{\bar{c}}$ is less than one.

It is assumed that (C, A) is observable and (A, B) is stabilizable for the systems considered. However the problem is solved for the observable part, when the observability condition is not satisfied.

5.1 Static Output Feedback Design

For the static output feedback stabilization problem, the goal is to calculate a gain matrix K that generates the stabilizing feedback input $u_t = Ky_t$ where y_t is the output of the system considered. More precisely, the SOF gain K places all eigenvalues of the closed-loop system matrix $A_{cl} = A + BKC$ inside the unit circle.

Calculating the SOF gain is known to be an NP-hard problem, i.e. it is difficult to find a computationally efficient algorithm for its solution in complete generality [13, 33, 15, 14]. It is also one of the fundamental questions in the control theory literature and studied in many research papers [37, 35, 38, 34, 9, 36]. Furthermore, dynamic control problems, in which the controller is a dynamical system instead of a static gain, can be reduced to a SOF stabilization problem by an appropriate augmentation of the system matrices.

There are several different approaches in the literature for solving SOF problem. In [50], the SOF is handled as a pole placement problem and the author concludes with a result that strengthens the NP-hardness assertion. On the other hand, [34] proposes a closed form solution for the discrete time SOF stabilization if the system satisfies some restrictive constraints. There are many approaches formulating SOF as a Linear Matrix Inequalities (LMI) problem derived from Lyapunov equations related to closed loop stability which has well-known solution fo the regular state feedback stabilization problem. The LMI formulations of the SOF introduces additional inequality and equality constraints eliminates convexity [38, 51, 6, 7, 8]. Some approaches obtain the SOF gain by iterative solutions of the Riccati equations related to the LQR problem [36, 9]. A recent survey on the SOF, [39], gives a broad overview of all the available approaches in the literature.

In the proposed method, the SOF gain K is a projected solution of dynamic programming for the discrete time LTI systems. The proposed solution is analogous to approximate dynamic programming (ADP) approach since the policy iteration step is approximated by a least squares solution [52].

5.1.1 Dynamic Programming Based Solution of the State Feedback Problem

Definition of the finite horizon discrete time LQR problem aims to minimize a quadratic cost J given by

$$\min_{u_t} \quad J = x_{t_f}^T Q x_{t_f} + \sum_{t=0}^{t_f-1} x_t^T Q x_t + u_t^T R u_t \tag{5.1}$$

subject to the system dynamics

$$x_{t+1} = Ax_t + Bu_t.$$

Starting from a final cost S_{t_f}, dynamic programming algorithm can be written backwards in time as [41, Chapter 4]

$$\begin{aligned} J_t &= x_t^T Q x_t + u_t^T R u_t + x_{t+1}^T S_{t+1} x_{t+1} &(5.2)\\ &= x_t^T Q x_t + u_t^T R u_t + (Ax_t + Bu_t)^T S_{t+1} (Ax_t + Bu_t), &(5.3) \end{aligned}$$

where J_t is the cost at time t which can be minimized by

$$u_t = -(R + B^T S_{t+1} B)^{-1} B^T S_{t+1} A x_t = F_t x_t.$$

When u_t is substituted into (5.3),

$$J_t = x_t^T \left(Q + F_t^T R F_t + (A + BF_t)^T S_{t+1} (A + BF_t)\right) x_t = x_t^T S_t x_t$$

$$S_t = Q + F_t^T R F_t + (A + BF_t)^T S_{t+1} (A + BF_t), \tag{5.4}$$

is obtained where (5.4) is the value iteration step by using the optimal policy F_t.

Proposition 5.1. *[44, Chapter 17] If (A, B) is stabilizable and (Q, A) is detectable, $R = R^T > 0$, $Q = Q^T \geq 0$ and starting from $S_{t_f} = Q$, as $t \to -\infty$, S_t converges to a symmetric and non-negative definite solution $S_t = S$ that satisfies the discrete time algebraic Riccati equation (DARE)*

$$S = Q + F^T R F + (A + BF)^T S (A + BF),$$

where

$$F = -(R + B^T SB)^{-1} B^T SA, \tag{5.5}$$

is a stabilizing state feedback gain.

Remark 5.1. *[41, Chapter 4] If the stabilizability and detectability assumptions are replaced by controllability and observability conditions, S_t converges to a positive definite solution $S > 0$.*

5.1.2 Approximate Dynamic Programming Based Solution of the SOF problem

For the SOF case, feedback gain matrix must have a structure in the form of $F = KC$, where C is the output matrix and has full row rank. If a SOF gain K and symmetric $S \geq 0$ can be found that satisfies the Lyapunov equation

$$S = Q + C^T K^T RKC + (A + BKC)^T S(A + BKC),$$

then it can be said that K is a stabilizing SOF gain.

Problem 3.

$$\min_K \quad \sum_{t=0}^{\infty} x_t^T Q x_t + u_t^T R u_t \tag{5.6}$$

$$\textit{subject to} \quad x_{t+1} = Ax_t + Bu_t \tag{5.7}$$

$$y_t = Cx_t, \;\; u_t = Ky_t. \tag{5.8}$$

An optimal F in the structure $F = KC$ may not be achievable, but at each iteration of (5.4) a sub-optimal K_t can be found by solving the following least squares problem

$$\begin{aligned} &K_t C = F_t \\ &K_t = F_t C^T (CC^T)^{-1} = F_t C^\dagger, \end{aligned}$$

where C is full row rank and $\dagger$ denotes the pseudo-inverse. Now, K_tC can be substituted into (5.4) instead of F_t. If S_t converges to a $S_t = S$ by using the sub-optimal least squares solution, it can be said that $K = -(R+B^TSB)^{-1}B^TSAC^\dagger$ is a stabilizing SOF gain.

Lemma 5.1. *Given an observable and controllable realization (A, B, C), where C has full row rank and the matrix $Q = \beta C^TC$ for $\beta > 0$. A necessary condition for the existence of a stabilizing SOF gain K that satisfies*

$$S = Q + C^TK^TRKC + (A + BKC)^TS(A + BKC), \tag{5.9}$$

for a symmetric $S > 0$ is such that the projected system matrix $A\Pi_{\bar{c}}$ must be stable where $\Pi_{\bar{c}} = I_n - C^T(CC^T)^{-1}C$ is the orthogonal projection on null(C).

Proof. Eigenvalue decomposition of the orthogonal projection matrix $\Pi_c = C^T(CC^T)^{-1}C$ is in the form of $\Pi_c = U_c\Lambda_cU_c^T$, where U_c is unitary and $\Lambda_c = \text{diag}(I_r, 0_{n-r})$ where 0_{n-r} is an $(n-r)\times(n-r)$ matrix of zeros. Use U_c as a similarity transformation to obtain $\tilde{A} = U_c^TAU_c, \tilde{B} = U_c^TB, \tilde{C} = CU_c, \tilde{Q} = \gamma\tilde{C}^T\tilde{C}$ and $\tilde{S} = U_c^TSU_c$ where $\tilde{C}$ is now in the form of $\tilde{C} = \begin{bmatrix}\tilde{C}_1, & 0\end{bmatrix}$ and the corresponding projection matrix on the null space of C is $\Lambda_{\bar{c}} = I_n - \Lambda_c$. Project $\tilde{S}$ by multiplying $\Lambda_{\bar{c}}$ from both sides to obtain

$$\begin{aligned}
\Lambda_{\bar{c}}\tilde{S}\Lambda_{\bar{c}} &= \Lambda_{\bar{c}}\tilde{A}^T\tilde{S}\tilde{A}\Lambda_{\bar{c}} \\
\tilde{S} &\geq \Lambda_{\bar{c}}\tilde{A}^T\tilde{S}\tilde{A}\Lambda_{\bar{c}} \\
\begin{bmatrix}\tilde{S}_1 & \tilde{S}_3 \\ \tilde{S}_3^T & \tilde{S}_2\end{bmatrix} &\geq \begin{bmatrix}0 & 0 \\ \tilde{A}_{12}^T & \tilde{A}_{22}^T\end{bmatrix}\begin{bmatrix}\tilde{S}_1 & \tilde{S}_3 \\ \tilde{S}_3^T & \tilde{S}_2\end{bmatrix}\begin{bmatrix}0 & \tilde{A}_{12} \\ 0 & \tilde{A}_{22}\end{bmatrix}
\end{aligned}$$

$\tilde{S}$ can be decomposed to

$$\begin{aligned}
\tilde{S} &= \begin{bmatrix}\tilde{S}_1 & \tilde{S}_3 \\ \tilde{S}_3^T & \tilde{S}_2\end{bmatrix} \\
&= \begin{bmatrix}I_r & 0 \\ \tilde{S}_3^T\tilde{S}_1^{-1} & I_{n-r}\end{bmatrix}\begin{bmatrix}\tilde{S}_1 & 0 \\ 0 & \bar{S}_2\end{bmatrix}\begin{bmatrix}I_r & \tilde{S}_1^{-1}\tilde{S}_3 \\ 0 & I_{n-r}\end{bmatrix} \\
&= U_p^T\bar{S}U_p,
\end{aligned}$$

where $\tilde{S}_1$ and $\bar{S}_2 = \tilde{S}_2 - \tilde{S}_3^T \tilde{S}_1^{-1} \tilde{S}_3$ are positive definite. Then,

$$\begin{aligned}
\bar{S} &\geq \left(U_p^T\right)^{-1} \Lambda_{\bar{c}} \tilde{A}^T U_p^T \bar{S} U_p \tilde{A} \Lambda_{\bar{c}} U_p^{-1} \\
\begin{bmatrix} \tilde{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix} &\geq \begin{bmatrix} 0 & 0 \\ \tilde{A}_{12}^T + \tilde{A}_{22}^T \tilde{S}_3^T \tilde{S}_1^{-1} & \tilde{A}_{22}^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} \tilde{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix} \begin{bmatrix} 0 & \tilde{A}_{12} + \tilde{S}_1^{-1} \tilde{S}_3 \tilde{A}_{22} \\ 0 & \tilde{A}_{22} \end{bmatrix},
\end{aligned}$$

which leads to

$$\begin{aligned}
\bar{S}_2 &\geq \tilde{A}_{22}^T \bar{S}_2 \tilde{A}_{22} \\
&\quad + (\tilde{A}_{12} + \tilde{S}_1^{-1} \tilde{S}_3 \tilde{A}_{22})^T \tilde{S}_1 (\tilde{A}_{12} + \tilde{S}_1^{-1} \tilde{S}_3 \tilde{A}_{22}) \\
&\quad > \tilde{A}_{22}^T \bar{S}_2 \tilde{A}_{22}, \\
0 &> \tilde{A}_{22}^T \bar{S}_2 \tilde{A}_{22} - \bar{S}_2.
\end{aligned}$$

The final inequality imposes stability of $\tilde{A}_{22}$ meaning that $\tilde{A}\Lambda_{\bar{c}}$ is stable. Hence, the proof can be concluded by saying $A\Pi_{\bar{c}}$ must be stable, since the similarity transformation U_c is unitary. □

Lemma 5.1 may be satisfied for a realization of the system while being unsatisfied for an other. Moreover, the convergence of S_t highly depends on the realization. In the next section, a method to obtain an adequate realization for the SOF algorithm is given.

5.2 An Adequate Realization for Static Output Feedback Calculation

Lemma 5.1 indicates that the projected system matrix $A\Pi_{\bar{c}}$ must be stable to find a SOF gain with the proposed dynamic programming based method. In [49], adequate results are obtained by using the balanced form of the considered system. Nevertheless, the balanced form does not guarantee the stability condition in Lemma 5.1.

In this section, a similarity transformation for the linear discrete time LTI system defined by ($A_0 \in \mathbb{R}^{n\times\, n}, B_0 \in \mathbb{R}^{n\times\, n}, C_0 \in \mathbb{R}^{m\times\, n}$) investigated, that makes projected version ($A\Pi_{\bar{c}}$) of the transformed system matrix A stable to satisfy the condition in Lemma 5.1. The symmetric matrix $\Pi_{\bar{c}}$ is an orthogonal projection to the null space of $C \in \mathbb{R}^{r\times n}$ and given by $\Pi_{\bar{c}} = I - C^T \left(CC^T\right)^{-1} C$. Problem is formulated as an LQR problem for the transposed system (A^T, C^T) and given cost function weights $Q_0 > 0$ and $R = I_r$.

Lemma 5.2. *Let us say S is the positive definite solution of the Riccati equation*

$$S = Q_0 + A_0 \left(S - SC_0^T \left(I_r + C_0 S C_0^T\right)^{-1} C_0 S\right) A_0^T, \tag{5.10}$$

where $Q_0 = Q_0^T > 0$, (A_0^T, C_0^T) is controllable, (Q_0, A_0^T) is observable and C_0 has full row rank.

If the system is transformed by $\sqrt{S}$ to obtain $(A = \sqrt{S}^{-1} A_0 \sqrt{S}, C = C_0\sqrt{S})$, the projected system matrix $A\Pi_{\bar{c}}$ has the largest singular value $\sigma_{max}(A\Pi_{\bar{c}}) < 1$ where $\Pi_{\bar{c}} = I_n - C^T \left(CC^T\right)^{-1} C$ is the projection onto null(C).

Proof. After the similarity transformation $\sqrt{S}$ is applied, the Riccati equation (5.10) becomes

$$I_n = \sqrt{S}^{-1} Q_0 \sqrt{S}^{-1} + A\left(I_n - C^T \left(I_r + CC^T\right)^{-1} C\right) A^T. \tag{5.11}$$

Given that C has full row rank and using the matrix inversion lemma

$$\left(I_r + CC^T\right)^{-1} = \left(CC^T\right)^{-1} - C\left(I_n + \left(C^T C\right)\right)^{-1} C^T$$

is obtained. Substitute (5.12) into (5.11)

$$\begin{aligned} I_n &= Q + A\left(I_n - C^T \left(CC^T\right)^{-1} C - C^T C \left(I_n + \left(C^T C\right)\right)^{-1} C^T C\right) A^T \\ &= Q + A\Pi_{\bar{c}} A^T + A\left(C^T C \left(I_n + \left(C^T C\right)\right)^{-1} C^T C\right) A^T \\ &= Q + Q_1 + A\Pi_{\bar{c}}\Pi_{\bar{c}} A^T \end{aligned}$$

where $Q_1 \geq 0$. This lets us write

$$I > A\Pi_{\bar{c}}\Pi_{\bar{c}} A^T.$$

Thus, $\sigma_{max}(A\Pi_{\bar{c}}) < 1$. $\square$

Remark 5.2. *The eigenvalues of $A\Pi_{\bar{c}}$ is upper bounded by $\sigma_{max}(A\Pi_{\bar{c}})$ [53, Chapter 5], which implies the stability of $A\Pi_{\bar{c}}$.*

5.3 Fixed-order Controller Design

In this section, a controller structure is assumed in the form of

$$\begin{bmatrix} z_{t+1} \\ v_t \end{bmatrix} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \begin{bmatrix} z_t \\ w_t \end{bmatrix}, \tag{5.12}$$

where $z \in \mathbb{R}^{n_c}$ is the state vector, $w \in \mathbb{R}^{m_c}, v \in \mathbb{R}^{r_c}$ are input and output vectors of the controller and A_c, B_c, C_c are appropriately sized matrices with real entries.

If the state space matrices of the system to be controlled is augmented in a way given by

$$A' = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \quad B' = \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}, \tag{5.13}$$

the fixed order controller problem can be formulated as: find a SOF gain K'

$$K' = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right], \tag{5.14}$$

which stabilizes the closed loop system matrix $A'_{cl} = A' + B'K'C'$. If such a K' can be found, the closed loop system shown in Figure 5.1 is stable.

Nevertheless, the A_c, B_c, C_c in (5.14) is out of the range space of the augmented system (5.13) when the proposed SOF calculation algorithm is used. Dynamic programming iterations leads to a result in which A_c, B_c, C_c are zero. In order to overcome this, a C_c and D_c in K' is fixed and problem is solved only for A_c and B_c. In this case, for a given C_c and D_c the augmented state space matrices become

$$A' = \begin{bmatrix} A + BD_cC & BC_c \\ 0 & 0_{n_c} \end{bmatrix}, \quad B' = \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}, . \tag{5.15}$$

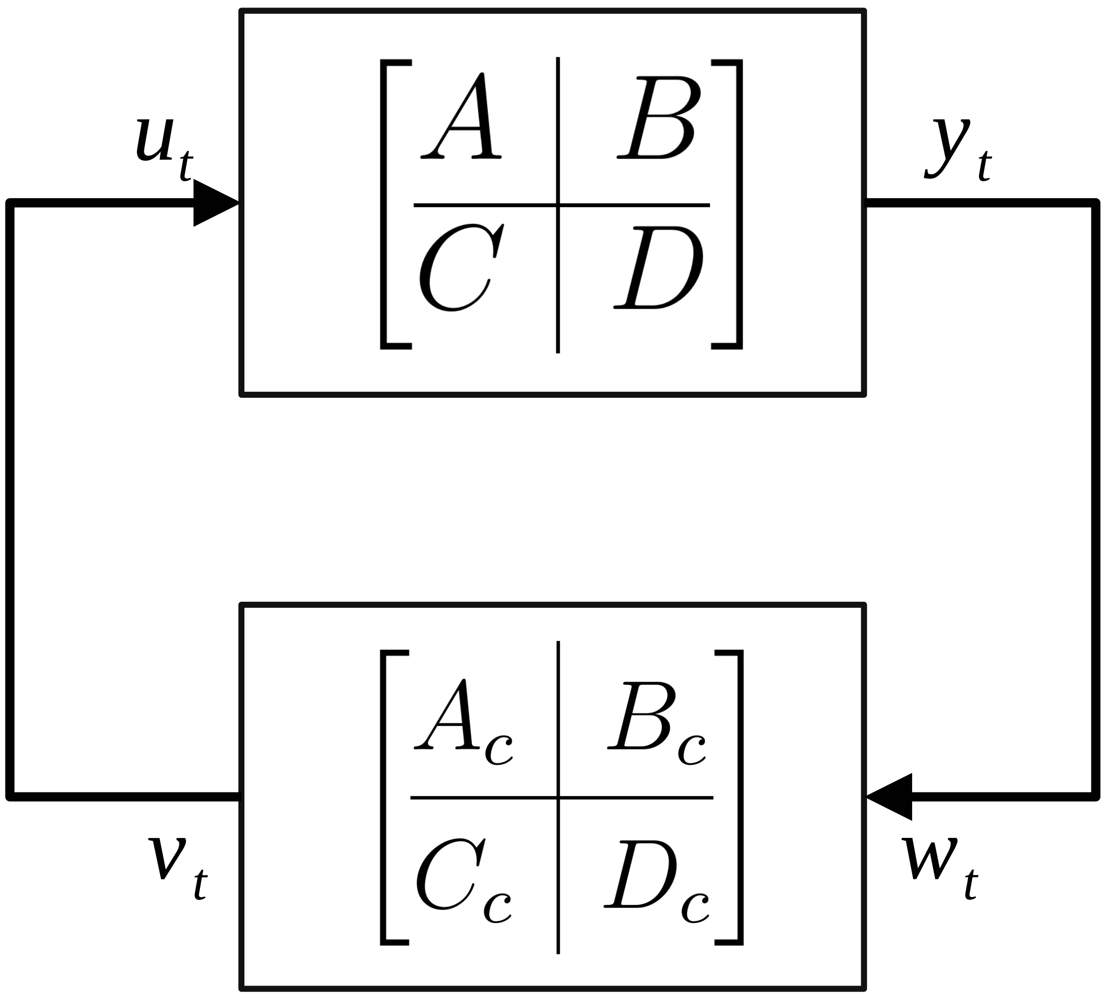

Figure 5.1: The closed loop system diagram with the fixed-order controller.

The problem is to find a K' given by

$$K' = \begin{bmatrix} A_c & B_c \end{bmatrix},$$

that stabilizes $A'_{cl} = A' + B'K'C'$. Then approximate dynamic programming approach described earlier can be applied to solve this problem. An example is given in Section 5.4.3.

5.4 An Application

5.4.1 Aircraft Model

The proposed SOF calculation method is applied on a discretized model of the lateral-directional command augmentation system of an F-16 aircraft linearized around its nominal operating conditions [36, 54]. The continuous time state space realization $(A_{cont}, B_{cont}, C_{cont})$ is given by

$$A_{cont} = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 & 0.003 & 0.0008 & 0 \\ 0 & 0 & 1 & 0.0037 & 0 & 0 & 0 \\ -30.6492 & 0 & -3.6784 & 0.6646 & -0.7333 & 0.1315 & 0 \\ 8.5396 & 0 & -0.0254 & -0.4764 & -0.0319 & -0.0620 & 0 \\ 0 & 0 & 0 & 0 & -20.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.2 & 0 \\ 0 & 0 & 0 & 57.2958 & 0 & 0 & -1 \end{bmatrix}$$

$$B_{cont} = \begin{bmatrix} 0 & 0 & 0 & 0 & 20.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20.2 & 0 \end{bmatrix}^T,$$

$$C_{cont} = \begin{bmatrix} 0 & 0 & 0 & 57.2958 & 0 & 0 & -1 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 & 0 \\ 57.2958 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 57.2958 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$x_c = \begin{bmatrix} \beta & \phi & p & r & \delta_a & \delta_r & x_w \end{bmatrix}$$

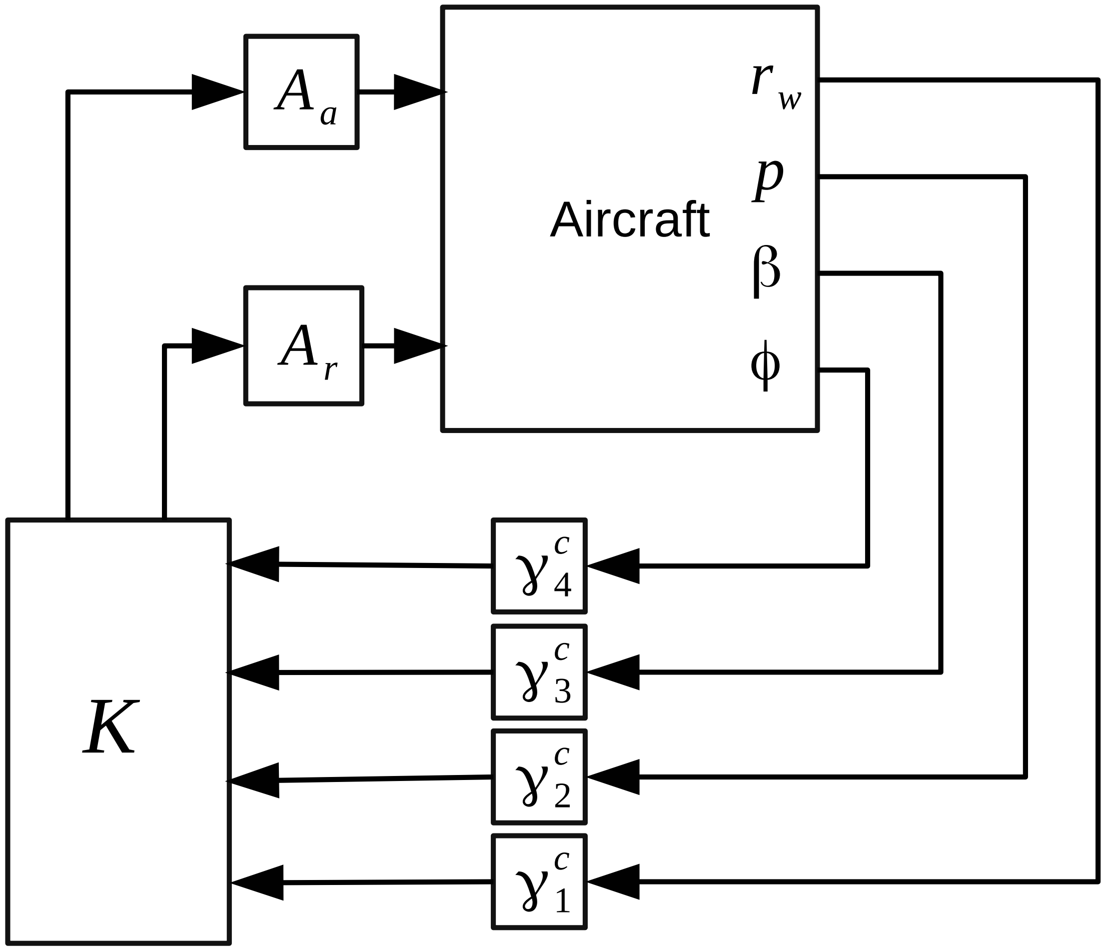

Figure 5.2: In the aircraft model, A_a and A_r are aileron and rudder actuators. $\gamma_i^c \in \{0, 1\}$ determines that corresponding output is used for feedback.

where x_c is the state vector and the states are side-slip angle β, bank angle ϕ, roll rate p, yaw rate r (see Figure 5.2). The variables δ_a and δ_r come from the aileron and rudder actuator models. The washout filter state is denoted by x_w. They constitute a stable system with the given state matrix A_{cont} [54].

The model is discretized by zero order hold (ZOH) method with the sampling period $h = 0.01$ sec. The discrete time model is

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= \tilde{C}x_t \end{aligned}$$

where $A = e^{hA_{cont}}$, $B = \left(\int_0^h e^{A_c\tau} d\tau\right) B_{cont}$ and $C = C_{cont}$.

Firstly, an adequate realization (A, B, C) is obtained by solving (5.10) for $Q_0 = I_n$. In the SOF calculation step, the cost function weights are chosen as $Q = \alpha C^T C$ and $R = I_r$. The results found for different values of α are shown in Table 5.1. As α increases, the effort to suppress the energy of the output increases which leads to a decrease in $\mathcal{H}_2$ norm of the closed loop system.

Furthermore, the SOF gain is calculated by using less than 4 available outputs after solving the sensor placement problem for less than 4 outputs. The resulting optimal γ_i^c values are given in Table 5.2. The most significant output is the bank angle followed by the roll rate and yaw rate. The SOF gains found for different sensor configurations is shown in Table 5.3.

α	K	$\|T(z)\|_{\mathcal{H}_2}$
1	$\begin{bmatrix} 0.2748 & 0.6781 & -1.5333 & 0.8157 \\ 0.8208 & -0.1383 & 0.3504 & -0.1500 \end{bmatrix}$	0.5554
10	$\begin{bmatrix} 0.7017 & 1.5227 & -2.8858 & 1.7705 \\ 2.8201 & -0.3464 & -0.5328 & -0.3237 \end{bmatrix}$	0.4140
100	$\begin{bmatrix} 1.6191 & 2.1110 & -4.5079 & 2.4563 \\ 7.9177 & -0.6299 & -5.1806 & -0.4623 \end{bmatrix}$	0.3796
1000	$\begin{bmatrix} 3.1779 & 2.1826 & -6.2642 & 2.5919 \\ 17.2117 & -1.0146 & -14.6734 & -0.5555 \end{bmatrix}$	0.3751

Table 5.1: The SOF gains and $\mathcal{H}_2$ norm of the closed-loop system matrix for different α values.

r	yaw (r_w)	roll (p)	$\underline{\gamma^c}$ side-slip (β)	bank (ϕ)
1	[0	0	0	1]
2	[0	1	0	1]
3	[1	1	0	1]

Table 5.2: Optimal γ_i^c values found by the algorithm given in Chapter 4.

r	K	$\rho(A+BKC)$
1	$\begin{bmatrix} 0.8218 \\ 0.4305 \end{bmatrix}$	0.9950
2	$\begin{bmatrix} 0.6887 & 0.8145 \\ -0.1406 & -0.1860 \end{bmatrix}$	0.9953
3	$\begin{bmatrix} 0.2031 & 0.6921 & 0.8131 \\ 0.7516 & -0.1499 & -0.1984 \end{bmatrix}$	0.9897
4	$\begin{bmatrix} 0.2748 & 0.6781 & -1.5333 & 0.8157 \\ 0.8208 & -0.1383 & 0.3504 & -0.1500 \end{bmatrix}$	0.9885

Table 5.3: The SOF gains obtained when the sensor configurations in Table 5.2 are used and Q is the identity matrix.

5.4.2 Aircraft Model with Actuator Failure

In the continuous time model, the rudder actuator is modeled as $A_r(s) = 20.2/(20.2 + s)$ [54, Chapter 4]. In this example, the rudder actuator is assumed to have failed and its model is replaced by $\Sigma_r(s) = 1/(s + \epsilon)$ to approximate a stuck actuator integrating ($\epsilon = -0.01$ is chosen to avoid imaginary axis poles in the system). Stabilization by using a minimum number of outputs is investigated for this approximately unstable aircraft model.

The output controllability gramian maximization problem is solved for different m (see Table 5.4). When a rudder failure occurs, sensing the side-slip angle is now preferred to the yaw rate. The proposed method can find a stabilizing gain for all m values. Results are given in Table 5.5.

		$\underline{\gamma}^c$		
r	yaw (r_w)	roll (p)	side-slip (β)	bank (ϕ)
1	[0	0	0	1]
2	[0	1	0	1]
3	[0	1	1	1]

Table 5.4: Optimal γ_i^c values found by the algorithm given in Chapter 4 and using the generalized Gramians for unstable systems.

r	K
1	$\begin{bmatrix} 0.020692 \\ 0.004343 \end{bmatrix}$
2	$\begin{bmatrix} 0.003579 & 0.013430 \\ 0.014666 & 0.002685 \end{bmatrix}$
3	$\begin{bmatrix} 0.004374 & -0.011155 & 0.013707 \\ 0.004374 & -0.011155 & 0.013707 \end{bmatrix}$
4	$\begin{bmatrix} 3.803015 & -0.033977 & -0.826673 & 0.067775 \\ 2.279953 - 0.065955 & 0.678481 & -0.093710 & \end{bmatrix}$

Table 5.5: The SOF gains obtained when the sensor configurations in Table 5.4 are used and Q is the identity matrix.

5.4.3 Fixed-order Controller for the Aircraft Model with Actuator Failure

For the fixed order controller case, the controller order is chosen $n_c = 2$. Since the system model has 2 input and 4 output, m_c must be 4 and $r_c = 2$. Lastly, fixed part of the controller's state space matrices is chosen to be $C_c = I_2$, $D_c = 0$. The SOF problem is solved for the augmented system given in (5.15) for $Q = I_7$ and $R = I_2$. The resulting controller is

$$A_c = \begin{bmatrix} -0.1350 & -0.2534 \\ -0.2558 & -0.7480 \end{bmatrix},\ B_c = \begin{bmatrix} 1.1459 & -0.0122 & -0.4214 & 0.0323 \\ 2.3547 & -0.0503 & -0.0773 & -0.0118 \end{bmatrix},$$

with the controller poles at -0.043025 and -0.839919. The given 2nd order controller improves the $\mathcal{H}_2$ norm of the closed loop system. The $\mathcal{H}_2$ norm with the SOF gain given in Table 5.5 for 4 outputs is 4.8204. The 2nd order controller results to an $\mathcal{H}_2$ norm of 4.1491.

5.5 Summary of the Results

The proposed algorithm yields promising results, nevertheless the convergence of the proposed dynamic programming based algorithm is not easily tractable.

It is shown that the stability of $A\Pi_{\bar{c}}$ is a necessary condition for the existence of a SOF gain. Furthermore, it is observed from the simulations that algorithm converges when $\sigma_{max}(A\Pi_{\bar{c}}) < 1$. As previously mentioned, there must be a non-negative definite symmetric S that satisfies the equality

$$S = Q + C^T K^T RKC + (A + BKC)^T S(A + BKC), \tag{5.16}$$

where

$$K = -(R + B^T SB)^{-1} B^T SAC^{\dagger},$$

which makes $A + BKC$ stable. Substituting the sub-optimal state feedback

$$\hat{F} = KC = -(R + B^T SB)^{-1} B^T SA\Pi_c = F\Pi_c$$

into (5.16) we obtain

$$
\begin{aligned}
S &= Q + \Pi_c F^T R F \Pi_c + A^T S A + A^T S B F \Pi_c + \Pi_c F^T B^T S A \\
&= Q + A^T S A + \Pi_c F \left(R + B^T S B\right) F \Pi_c + A^T S B F \Pi_c + \Pi_c F^T B^T S A \\
&= Q + A^T S A + \Pi_c A^T S B \left(R + B^T S B\right)^{-1} B^T S A \Pi_c \\
&- A S B \left(R + B^T S B\right)^{-1} B^T S A \Pi_c - \Pi_c A S B \left(R + B^T S B\right)^{-1} B^T S A
\end{aligned}
$$

By using $\Pi_c = I_n - \Pi_{\bar{c}}$, a modified version of the Riccati equation for discrete time systems is obtained,

$$
\begin{aligned}
S &= Q + A^T S A - A^T S B \left(R + B^T S B\right)^{-1} B^T S A \\
&+ \Pi_{\bar{c}} A^T S B \left(R + B^T S B\right)^{-1} B^T S A \Pi_{\bar{c}} \\
&= Q_1 + A^T S A - A S B \left(R + B^T S B\right)^{-1} B^T S A,
\end{aligned}
$$

where $Q_1 = Q + \Pi_{\bar{c}} A^T S B \left(R + B^T S B\right)^{-1} B^T S A \Pi_{\bar{c}}$.

Assume that the second term in Q_1 is already known, the total cost J of the LQR problem with the cost function

$$
J = x_{t_f}^T Q_1 x_{t_f} + \sum_{t=0}^{t_f-1} x_t^T Q_1 x_t + u_t^T R u_t
$$

is given by

$$
\begin{aligned}
J &= x_{t_f}^T Q_1 x_{t_f} + \sum_{t=0}^{t_f-1} x_t^T (Q_1 + F^T R F) x_t \\
&= x_{t_f}^T Q_1 x_{t_f} + \sum_{t=0}^{t_f-1} x_t^T \left(S - (A + BF)^T S (A + BF)\right) x_t \\
&= x_{t_f}^T Q_1 x_{t_f} + \sum_{t=0}^{t_f-1} x_t^T S x_t - x_{t+1}^T S x_{t+1} \\
&= x_0^T S x_0 + x_{t_f}^T (Q_1 - S) x_{t_f},
\end{aligned}
$$

when $u_t = -\left(R + B^T S B\right)^{-1} B^T S A x_t = F x_t$. The proposed method in this thesis uses a projected version of the state feedback $\hat{u}_t = F \Pi_c x_t = \hat{F} x_t$. Similarly, the total cost $\hat{J}$ for the input $\hat{u}_t$ can be written as

$$
\begin{aligned}
\hat{J} &= x_{t_f}^T Q x_{t_f} + \sum_{t=0}^{t_f-1} x_t^T Q x_t + \hat{u}_t^T R \hat{u}_t \\
&= x_0^T S x_0 + x_{t_f}^T (Q - S) x_{t_f}.
\end{aligned}
$$

Sensor count	$\underline{\gamma}_c$	$\rho(A+BKC)$
$m=3$	**[0 1 1 1]**	0.9996
	[1 1 1 0]	0.9999
	[1 1 0 1]	0.9999
	[1 0 1 1]	0.9999

Table 5.6: Comparison of the spatial radius of the closed loop system obtained by non-optimal placement of sensors for the aircraft model. Bold γ_i^c shows the optimal value. Cost weights are chosen as $Q = I_n$ and $R = I_m$.

Since the closed loop system is stable for both cases, $J \approx \hat{J}$ for an arbitrarily large t_f. Eventually, it can be said that the proposed algorithm leads to a similar quadratic cost as the LQR problem with a larger weight (Q_1) on the system's states.

Additionally, efficiency of the proposed method for sensor placement can be demonstrated by solving the SOF problem for non-optimal sensor sets. It is observed that the proposed SOF calculation converges slower to a worse minimum when other possible $\underline{\gamma}^c$ configurations are used. The results for non-optimal sensor placement combinations for the unstable aircraft model are given in Table 5.6.

The step-by-step approach to calculate stabilizing SOF for a discrete time LTI system is as follows:

Algorithm 2. *(SOF design)*

1. *Create a dictionary of actuator/sensor places for different number of actuators and sensors by following the steps described in Section 4.3.*
2. *By considering the control application limitations and the cost function in Problem 1 choose an appropriate pair of* (B, C) *from the dictionary*
3. *Calculate an adequate similarity transformation by using the procedure described in Section 5.2 with a* $Q_0 > 0$.

4. *By using the realization from Step 3, choose quadratic cost function weights Q and R of* (5.1) *and calculate the iterations described in Section 5.1*

5. *If iterations converge, the algorithm yields a stabilizing SOF gain K.*

6. *Else, increase the number of actuators or sensors by considering the cost function in Problem 1 and go to Step 2.*

Chapter 6

Applications

6.1 Simply supported flexible beam

This example is the model of a simply supported beam [17] with the continuous time state space representation

$$\dot{x} = A_{cont}x + \tilde{B}_{cont}u$$
$$y = \tilde{C}_{cont}x,$$

where

$$A_{cont} = \text{diag}A_i, \quad A_i = \begin{bmatrix} -2\zeta_i\omega_i & -\omega_i \\ \omega_i & 0 \end{bmatrix}$$
$$\omega_i = \left(\frac{i\pi}{l}\right)^2 \sqrt{\frac{EI}{\rho A}}, \quad i = 1, \cdots, N,$$

where l is the beam length, EI is the bending stiffness, ρ is the density of material, A is the cross sectional area and ω_i, ζ_i represents ith natural frequency and the corresponding damping. The most accurate model has an infinite number of natural modes but higher order modes can not be excited in practice. The higher order natural modes $i > N$, for some large N can be neglected.

It is assumed that external forces can be applied from the points

$\{d_1^b, d_2^b, \cdots, d_{\tilde{m}}^b\}$ which constitutes the input matrix

$$B_i = \begin{bmatrix} \Phi_i(d_1^b) & \Phi_i(d_2^b) & \cdots & \Phi_i(d_{\tilde{m}}^b) \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \tilde{B}_{cont} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix},$$

where

$$\Phi_i(d) = \sqrt{\frac{2}{l}} \sin \frac{i\pi d}{l},$$

and d is the position variable.

In this example, the output is considered to be the displacement of distinct points ($\{d_1^c, d_2^c, \cdots, d_{\tilde{r}}^c\}$) on the beam. It leads to the output matrix

$$C_i = \begin{bmatrix} 0 & \Phi_i(d_1^c) \\ 0 & \Phi_i(d_2^c) \\ \vdots & \\ 0 & \Phi_i(d_{\tilde{r}}^c) \end{bmatrix}, \quad \tilde{C}_{cont} = \begin{bmatrix} C_1 & C_2 & \cdots C_N \end{bmatrix}.$$

The list of parameters used are given in Table 6.1. In the examples, 35 actuators and sensors sites are assigned at equidistant points between 0 and l (Figure 6.1) ($\tilde{m} = \tilde{r} = 35$). First 20 modes are considered in the truncated model ($N = 20$). The continuous time model is discretized with the sampling rate $h = 2\pi/(30 \cdot \omega_{20})$ *sec*, so the sampling frequency is 30 times of the largest natural frequency.

6.1.1 Case 1

In this case, the problem is solved for only one actuator and sensor. The parameters used for Algorithm 1 are $Q_0 = I_n + C^T C$ and $R_0 = 1$. Algorithm 1 finds the optimal actuator/sensor locations at 18th actuator site which corresponds to the middle of beam. Then, the SOF calculated with the cost function weights $Q = I_n$ and $R = 1$. The results in Table 6.2 compare the energy of impulse responses of the states for the open-loop case and closed-loop case. The results reveal that an optimal placing actuator/sensor lets us calculate a SOF gain with higher performance.

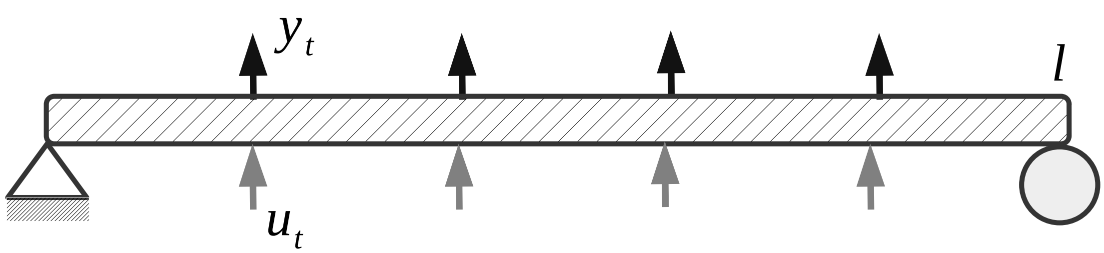

Figure 6.1: Simply supported beam with equidistantly placed actuator sensor sites.

Parameter	Value
EI	$40\ N\dot{m}^2$
ρA	$252.9\ kg/m$
l	$1\ m$
$\zeta_i\ \forall\ i$	0.01
$\tilde{m}$	35
$\tilde{r}$	35

Table 6.1: The list of parameters used in the examples of this section.

Actuator/sensor sites	K	Closed-loop/open-loop $\mathcal{H}_2$ norms	Suppression
18 / 18	-0.3701	0.0280 / 0.0641	0.4370
10 / 18	-0.1428	0.0434 / 0.0564	0.7696
20 / 18	-0.2897	0.0323 / 0.0636	0.5082
30 / 18	-0.1044	0.0424 / 0.0450	0.9435
18 / 10	-0.1642	0.0413 / 0.0641	0.6444
18 / 20	-0.2957	0.0306 / 0.0641	0.4773
18 / 30	-0.1615	0.0469 / 0.0641	0.7323

Table 6.2: The first row shows the optimal solution. It is shown that the optimal actuator/sensor position results to better energy suppression in terms of $\mathcal{H}_2$ norm.

6.1.2 Case 2

In this case, the actuator/sensor placement problem is solved for more than one sites. Table 6.3 shows the optimal actuator/sensor locations and corresponding SOF gains found by the proposed algorithms.

Then, the problem is solved for the fixed-order controller case. The system matrices C_c and D_c of the controller are chosen as

$$C = I_m, \quad D = K,$$

where K is the corresponding SOF gain given in Table 6.3. The fixed-order controllers are given in Tables 6.4, 6.5 and 6.6. The comparison between the closed-loop $\mathcal{H}_2$ norms of the SOF case and fixed order controller cases are given in Table 6.7.

Actuator/sensor sites	K
18 / 18	-0.3701
17, 18 / 17, 18	$\begin{bmatrix} -0.30036 & -0.01488 \\ -0.09779 & -0.22356 \end{bmatrix}$
17, 18, 19 / 17, 18, 19	$\begin{bmatrix} -0.40239 & 0.19354 & -0.21881 \\ -0.28314 & 0.11948 & -0.28314 \\ -0.21881 & 0.19354 & -0.40239 \end{bmatrix}$

Table 6.3: The optimal actuator/sensor places and the SOF gains obtained for different number of actuators/sensors.

System matrices	
A_c	-0.000341
B_c	-0.139772
C_c	1
D_c	-0.3701

Table 6.4: The fixed-order controller obtained with the given C_c and D_c when the actuator/sensor are at 18th site.

System matrices	
$A_c =$	$\begin{bmatrix} -0.000285 & -0.000214 \\ -0.000214 & -0.000262 \end{bmatrix}$
$B_c =$	$\begin{bmatrix} -0.129460 & 0.014017 \\ -0.029450 & -0.087680 \end{bmatrix}$
$C_c =$	I_2
$D_c =$	$\begin{bmatrix} -0.300360 & -0.014880 \\ -0.097790 & -0.223560 \end{bmatrix}$

Table 6.5: The fixed-order controller obtained with the given C_c and D_c when the actuators/sensors are at 17 and 18th sites.

System matrices
$$A_c = \begin{bmatrix} -0.000220 & -0.000148 & -0.000040 \\ -0.000148 & -0.000193 & -0.000148 \\ -0.000040 & -0.000148 & -0.000220 \end{bmatrix}$$
$$B_c = \begin{bmatrix} -0.123957 & 0.036629 & -0.034476 \\ -0.063124 & -0.000107 & -0.063124 \\ -0.034476 & 0.036629 & -0.123957 \end{bmatrix}$$
$$C_c = I_3$$
$$D_c = \begin{bmatrix} -0.402390 & 0.193540 & -0.218810 \\ -0.283140 & 0.119480 & -0.283140 \\ -0.218810 & 0.193540 & -0.402390 \end{bmatrix}$$

Table 6.6: The fixed-order controller obtained with the given C_c and D_c when actuators/sensors are at $17, 18$ and 19th sites.

Actuator/sensor sites	Open-loop	SOF	Fixed-order
18 / 18	0.0641	0.0280	0.0257
$17, 18$ / $17, 18$	0.1257	0.0448	0.0408
$17, 18, 19$ / $17, 18, 19$	0.1878	0.0542	0.0502

Table 6.7: Comparisons of the closed-loop $\mathcal{H}_2$ norms for the SOF and fixed-order controllers given in Tables 6.3, 6.4, 6.5 and 6.6.

6.1.3 Case 3

In this example, the flexible beam model is destabilized by assigning negative damping coefficients to some natural modes. Each negative damping leads to a pair of unstable poles. Firstly, damping of the first natural mode is changed to -0.1 from its previous value 0.01. Then, damping of the 10th mode is also set to -0.1. The actuator/sensor locations found for these two configurations are given in Table 6.8. The unstable system is stabilized by the SOF and fixed-order controller. The SOF gains obtained are given in Table 6.9. The fixed-order controller problem is also solved for the same actuator/sensor configurations and the results are given in Tables 6.10 and 6.11. Lastly, Table 6.12 compares the closed loop performances of the SOF and fixed-order controllers. The results show that the performance can be further increased by adding a dynamic part to the controller.

Unstable modes	Damping	Unstable-poles	Actuator/sensor locations
1st	-0.1	$1.00005 \pm 0.00052j$	$9, 27$ / $9, 27$
1st & 10th	-0.1, -0.1	$1.00389 \pm 0.05235j$	$8, 9, 27, 28$ / $8, 9, 27, 28$

Table 6.8: Negative damping creates a complex pair of unstable poles. The last column shows the optimal actuator/sensor locations.

Actuator/sensor sites	K
$9, 27$ / $9, 27$	$\begin{bmatrix} -0.45070 & -0.17235 \\ -0.17235 & 0.45070 \end{bmatrix}$
$8, 9, 27, 28$ / $8, 9, 27, 28$	$\begin{bmatrix} -4.4543 & 1.1853 & -1.4409 & 4.3052 \\ 9.1862 & -17.5270 & 17.1252 & -9.2079 \\ -9.2079 & 17.1252 & -17.5270 & 9.1862 \\ 4.3052 & -1.4409 & 1.1853 & -4.4543 \end{bmatrix}$

Table 6.9: The optimal actuator/sensor places and the SOF gains obtained for different number of actuators/sensors.

System matrices
$A_c = \begin{bmatrix} -0.00033058 & -0.00004969 \\ -0.00004969 & -0.00033058 \end{bmatrix}$
$B_c = \begin{bmatrix} -0.130826 & -0.022540 \\ -0.022540 & -0.130826 \end{bmatrix}$
$C_c = I_2$
$D_c = \begin{bmatrix} -0.45070 & -0.17235 \\ -0.17235 & 0.45070 \end{bmatrix}$

Table 6.10: The fixed-order controller obtained with the given C_c and D_c when actuators/sensors are at $17, 18$ and 19th sites.

System matrices

$$A_c = \begin{bmatrix} -0.000142 & -0.000117 & -0.000100 & -0.000111 \\ -0.000116 & -0.000159 & -0.000125 & -0.000100 \\ -0.000100 & -0.000125 & -0.000159 & -0.000116 \\ -0.000111 & -0.000100 & -0.000117 & -0.000142 \end{bmatrix}$$

$$B_c = \begin{bmatrix} -0.036489 & -0.024617 & -0.033858 & -0.019593 \\ 0.000438 & -0.063560 & -0.054455 & -0.000776 \\ -0.000776 & -0.054455 & -0.063560 & 0.000438 \\ -0.019593 & -0.033858 & -0.024617 & -0.036489 \end{bmatrix}$$

$$C_c = I_3$$

$$D_c = \begin{bmatrix} -4.4543 & 1.1853 & -1.4409 & 4.3052 \\ 9.1862 & -17.5270 & 17.1252 & -9.2079 \\ -9.2079 & 17.1252 & -17.5270 & 9.1862 \\ 4.3052 & -1.4409 & 1.1853 & -4.4543 \end{bmatrix}$$

Table 6.11: The fixed-order controller obtained with the given C_c and D_c when actuators/sensors are at $17, 18$ and 19th sites.

Actuator/sensor sites	SOF	Fixed-order
$9, 27$ / $9, 27$	0.04762	0.04175
$8, 9, 27, 28$ / $8, 9, 27, 28$	0.05637	0.05013

Table 6.12: The comparison of the closed loop $\mathcal{H}_2$ norm of the unstable flexible beam for the controllers given in Tables 6.10 and 6.11.

6.2 Biological Network

This example considers a linearized version of the Partial Differential Equation (PDE) that models aggregation of cellular slime molds [55]. The slime mold cells are distributed on 1 dimensional spatial region according to the function $a(d, \tau)$, where d and τ are position and time variables. and they produce cAMP chemical which attracts the slime mold cells and leads to the aggregation of cells. The cAMP distribution is given by $c(d, \tau)$ and its concentration decreases with respect to a decay rate. The aggregation is modeled by the PDEs

$$\begin{aligned}\frac{\partial a(d,\tau)}{\partial \tau} &= \mu \frac{\partial^2 a(d,\tau)}{\partial d^2} - \chi \bar{a} \frac{\partial^2 c(d,\tau)}{\partial d^2} \\ \frac{\partial c(d,\tau)}{\partial \tau} &= D \frac{\partial^2 c(d,\tau)}{\partial d^2} + f(d)a(d,\tau) - k(d)c(d,\tau),\end{aligned}$$

where μ determines the cell mobility, χ is the chemotactic coefficient, D is the diffusion rate of cAMP, $f(d)$ and $k(d)$ are cAMP generation and decay rates. The PDE is spatially discretized with the sampling period Δd. The discretized version is given by

$$\begin{aligned}\frac{a(i,\tau)}{d\tau} &= \mu \frac{a(i+1,\tau) - 2a(i,\tau) + a(i-1,\tau)}{\Delta d^2} \\ &\quad - \chi \bar{a} \frac{c(i+1,\tau) - 2c(i,\tau) + c(i-1,\tau)}{\Delta d^2} \\ \frac{c(i,\tau)}{d\tau} &= D \frac{c(i+1,\tau) - 2c(i,\tau) + c(i-1,\tau)}{\Delta d^2} \\ &\quad + f(i)a(i,\tau) - k(i)c(i,\tau),\end{aligned}$$

where $a(i, \tau) = a(i\Delta d, \tau)$, $c(i, \tau) = c(i\Delta d, \tau)$. In the state space form,

$$\begin{aligned}\frac{dx_c}{d\tau}(i) &= \begin{bmatrix} -2\mu/\Delta d^2 & 2\chi\bar{a}/\Delta d^2 \\ f(i) & -2D/\Delta d^2 - k(i) \end{bmatrix} x_c(i) \\ &\quad + \begin{bmatrix} \mu/\Delta d^2 & -\chi\bar{a}/\Delta d^2 \\ 0 & D/\Delta d^2 \end{bmatrix} x_c(i-1) \\ &\quad + \begin{bmatrix} \mu/\Delta d^2 & -\chi\bar{a}/\Delta d^2 \\ 0 & D/\Delta d^2 \end{bmatrix} x_c(i+1) \\ \dot{x}_c(i) &= A_i x(i) + M_{i,i-1} x_c(i-1) + M_{i,i+1} x_c(i+1), \\ &\quad \text{for} \quad i = 1, \cdots, N,\end{aligned}$$

where $x_c(i) = \begin{bmatrix} a(i,\tau) & c(i,\tau) \end{bmatrix}^T$. It is assumed that for some $i \in \mathbb{Y}$ the concentration of slime molds can be sensed and for some $i \in \mathbb{U}$, the cAMP concentration can be modified externally. In particular, the input and output matrices for each subsystem i are

$$B_i = \begin{cases} [0 \;\; 1]^T & \text{if } \; i \in \mathbb{U} \\ [0 \;\; 0]^T & \text{otherwise} \end{cases}$$

$$C_i = \begin{cases} [1 \;\; 0] & \text{if } \; i \in \mathbb{Y} \\ [0 \;\; 0] & \text{otherwise} \end{cases}.$$

State vectors $x(i)$ of the subsystems can be combined in a large scale system with sparse A_{cont}, B_{cont} and C_{cont} matrices. The subsystems are connected to each other to create a ring shaped structure (Figure 6.2b). Nonzero structure of the system matrix A_{cont} can be found in Figure 6.3 when $N = 17$.

Continuous time model is discretized by the sampling rate h. Complete parameter list used in the examples are given in Table 6.13. In some subsystems, it is assumed that the slime molds produces cAMP with a higher rate f' which causes unstability (Table 6.13). The circular arrangement places poles on the imaginary axis. The symmetry is broken by scaling $M_{N,1}$ by a factor of 0.99 to eliminate the poles on the imaginary axis.

In this case N is chosen to be 31. It is assumed that there is one anomalous subsystem which produces the cAMP at a greater rate. The goal is to place one actuator and one sensor by Algorithm 1.

When the 10th subsystem is anomalous, the system matrix A has an unstable pole at 1.0002. The proposed actuator/sensor placement algorithm locates both actuator and sensor at 10th subsystem. The proposed SOF calculation algorithm finds the stabilizing SOF gain

$$K = -0.4297,$$

when Q_0 of Algorithm 2 is chosen as $Q_0 = I_n + 0.1C^TC$ and Q and R of Algorithm 1 are $Q = 0.1I_n$ and $R = 1$. The resulting K leads to the spectral radius of the closed loop system matrix $\rho(A + BKC) = 0.9999$. The actuator/sensor placements for different configurations are given in Table 6.14 and corresponding stabilizing SOF gains are given in Table 6.15.

Furthermore, the fixed-order controllers are calculated by setting $C = I_m$ and $D = K$ where K is the corresponding SOF gain in Table 6.15. The state space matrices of the controllers are given in Tables 6.16–6.19. The $\mathcal{H}_2$ norms of the closed loop systems for the SOF and fixed-order controller cases are compared in Table 6.20.

Parameter	Value
Δd	1×10^{-4}
μ	1×10^{-7}
χ	4×10^{-4}
$\bar{a}$	1.6×10^{-3}
D	3×10^{-8}
h	5×10^{-3}
f	0.3
f'	0.65
k	2

Table 6.13: The list of parameters used in the examples of this section.

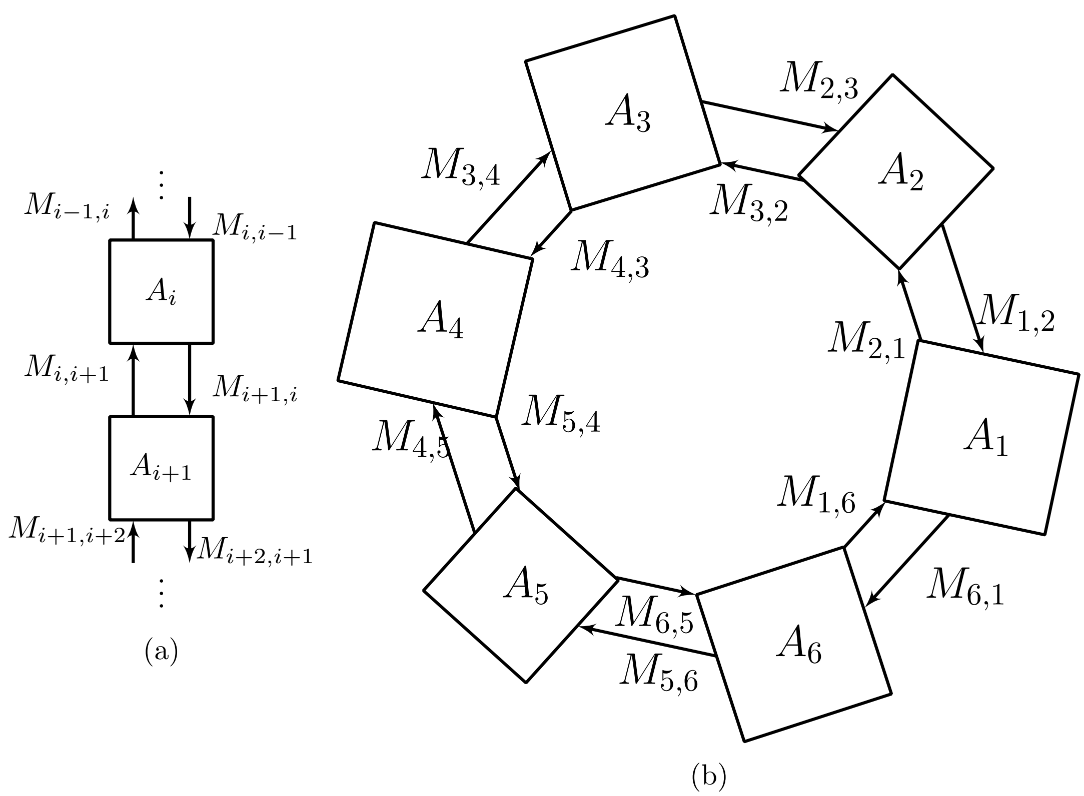

Figure 6.2: Example 5: (6.2a) Interconnection of the subsystems at discrete positions. (6.2b) The circular interconnection of subsystems.

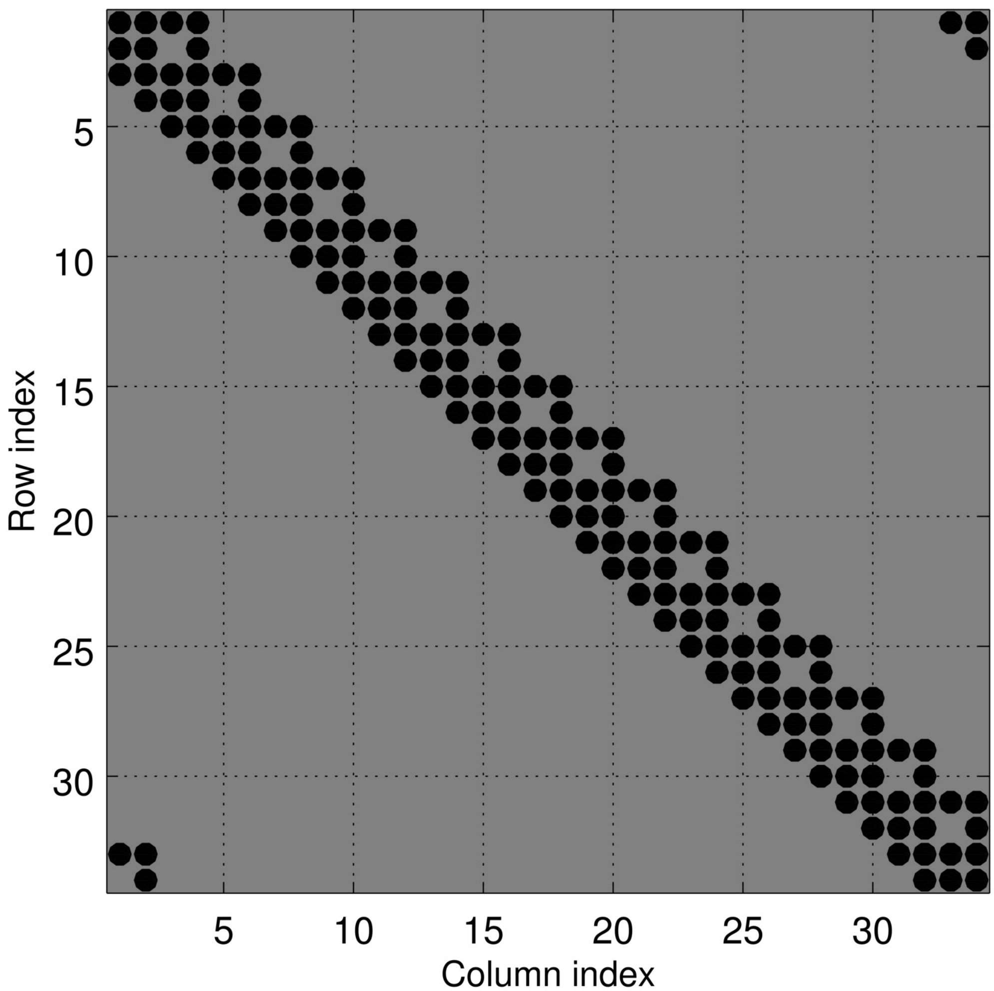

Figure 6.3: Example 5: The nonzero structure of the system matrix A_{cont} for the ring shaped structure where $N = 17$.

Anomalous subsystems	Unstable poles	Optimal actuator/sensor locations
10	1.0002	10 / 10
23	1.0002	23 / 23
$8, 19$	$1.00002, 1.00032$	$8, 19$ / $8, 19$
$4, 11$	$1.00009, 1.00031$	$4, 11$ / $4, 11$

Table 6.14: The optimal actuator/sensor locations obtained for different cases when $Q_0 = I_n + 0.1C^TC$.

Anomalous subsystems	K
10	-0.4297
23	-0.4295
$8, 19$	$\begin{bmatrix} -0.4243 & -0.0117 \\ -0.0115 & -0.4243 \end{bmatrix}$
$4, 11$	$\begin{bmatrix} -0.4181 & -0.0213 \\ -0.0217 & -0.4188 \end{bmatrix}$

Table 6.15: The SOF gains obtained for different cases when $Q = 0.1I_n$ and $R = I_m$.

	System matrices
$A_c =$	-0.00151
$B_c =$	-0.17708
$C_c =$	1
$D_c =$	-0.42965

Table 6.16: The fixed-order controller obtained with the given C_c and D_c when the anomaly is at 10th subsystem.

	System matrices
$A_c =$	-0.00151
$B_c =$	-0.17695
$C_c =$	1
$D_c =$	-0.42949

Table 6.17: The fixed-order controller obtained with the given C_c and D_c when the anomaly is at 23th subsystem.

System matrices

$$A_c = \begin{bmatrix} -0.001514 & 0.000081 \\ 0.000081 & -0.001514 \end{bmatrix}$$

$$B_c = \begin{bmatrix} -0.176552 & -0.005800 \\ -0.005899 & -0.176632 \end{bmatrix}$$

$$C_c = I_2$$

$$D_c = \begin{bmatrix} -0.4243 & -0.0117 \\ -0.0115 & -0.4243 \end{bmatrix}$$

Table 6.18: The fixed-order controller obtained with the given C_c and D_c when the anomaly is at 8th and 19th subsystems.

System matrices

$$A_c = \begin{bmatrix} -0.001536 & 0.000162 \\ 0.000162 & -0.001532 \end{bmatrix}$$

$$B_c = \begin{bmatrix} -0.172239 & -0.003813 \\ -0.004337 & -0.172563 \end{bmatrix}$$

$$C_c = I_2$$

$$D_c = \begin{bmatrix} -0.4181 & -0.0213 \\ -0.0217 & -0.4188 \end{bmatrix}$$

Table 6.19: The fixed-order controller obtained with the given C_c and D_c when the anomaly is at 4th and 11th subsystems.

Anomalous subsystems	SOF	Fixed-order
10	0.0928	0.0827
23	0.0928	0.0827
$8, 19$	0.1325	0.1176
$4, 11$	0.1342	0.1188

Table 6.20: The comparison of closed loop $\mathcal{H}_2$ norms for the controllers given in Tables 6.15–6.19.

Chapter 7

Conclusions

In this thesis, a framework is developed to calculate stabilizing SOF gains by using a small number of actuators and sensors. Furthermore, the SOF gain minimizes a quadratic cost function of the states and inputs which leads to a cost that is comparable to the state feedback regulator.

The problem is separated into two parts. Firstly, an optimal actuator/sensor placement algorithm is systematized in Chapter 4. In the literature, the optimal actuator/sensor placement problem is often investigated in terms of the degree of controllability/observability of the LTI system. In other words, the actuator and sensor configuration must be chosen in a way that maximizes the controllability and observability Gramians. A similar approach is developed that aims to maximize the $\mathcal{H}_2$ norm of the considered system. Moreover, this approach is extended to unstable systems by using the generalized Gramians. A coprime factorization based approach to calculate the generalized Gramians for unstable discrete time LTI systems is described in Chapter 3.

The first problem handled is the optimal actuator/sensor placement which is beneficial for the SOF calculation. The proposed solution of the SOF problem is similar to the LQR by dynamic programming but the obtained state feedback gain is projected onto the range space of the output. By this, the intuition behind the proposed sensor placement method is to choose a set of output that leads to $\mathcal{H}_2$ norm which is as close as possible the state feedback case where the output matrix $C = I_n$. Similar logic applies to the actuator placement problem.

The $\mathcal{H}_2$ norm maximization based solution allows us to simply formulate the problem as a quadratic mixed-integer problem. If the input or output matrices are fixed and only sensor or actuator placement is considered, the problem can be further simplified.

It is also shown that $\mathcal{H}_2$ norm maximization is equivalent to minimizing the required input to steer the output to some desired value. This result is also extended to unstable discrete time LTI systems by using the generalized Gramians.

After the input and output matrices are obtained, a full realization of the system is available. In Chapter 5, the structured control problem is investigated with a special emphasis on the SOF. A solution is proposed using dynamic programming which tries to minimize a quadratic cost of the states and input. Dynamic programming based solution leads to a modified version of the Riccati difference equation. Since a steady state solution is searched for, the modified Riccati difference equation must be convergent. It is known that the Riccati difference equation for the LQR problem converges if (A, B) is controllable and (Q, A) is observable. However, it is not true for the modified version.

It is shown that the stability of projected system matrix $A\Pi_{\bar{c}}$ is a necessary condition to obtain a stabilizing steady state solution. The observations from the simulations reveal that the modified Riccati difference equation converges if the largest singular value of $A\Pi_{\bar{c}}$ is less than one. On the other hand, the singular value condition is directly satisfied after a similarity transformation that diagonalizes $A\Pi_{\bar{c}}$ when $A\Pi_{\bar{c}}$ is stable.

The projected matrix $A\Pi_{\bar{c}}$ can be stable for a realization while being unstable for another. A procedure to obtain a similarity transformation that makes the largest singular value of $A\Pi_{\bar{c}}$ less than 1 is also given in Chapter 5.

Finally, the proposed methods are applied to two different spatially distributed systems. The first example investigates the optimal actuator/sensor placement and the SOF design for a simply supported flexible beam. The results of actuator/sensor placement algorithm coincides with the results in [17, 18]. The proposed actuator/sensor placement method collocates pairs of actuators and sensors which is expected by intuition. Because, the transfer functions associated to collocated actuators and sensor are typically minimum phase and easier to control than non-minimum phase systems.

Non-minimum phase systems have zeros in the right half plane. Depending on the location of right half plane poles, such systems may not even be stabilizable by a SOF gain (which is the simplest stable controller). The parity interlacing property (PIP) is a necessary condition for the existence of SOF. When the PIP is satisfied, we know that there exists a stable stabilizing controller but its order may be very high [56].

Furthermore the proposed SOF design algorithm gives promising results in terms of $\mathcal{H}_2$ performance. It is also shown by how much the performance can be further improved by increasing the order of the controller.

The second example is a linearized version of a morphogenetic model which are used to represent biological pattern formations. The spatial distribution is assumed to have a circular shape [57]. It is assumed that some points on the circular spatial region create an anomaly which causes instability in the whole system. As expected, the proposed actuator/sensor placement algorithm locates actuators and sensors on these problematic points and stabilizing SOF gains are obtained by the proposed SOF design algorithm.

As demonstrated by the examples, the proposed numeric method for SOF calculation works well when the largest singular value of the projected state matrix, $A\Pi_{\bar{c}}$, is less than one. It is shown that obtained SOF controllers give results as efficient as the state feedback case in terms of the closed loop system's $\mathcal{H}_2$-norm. The applications of proposed SOF calculation method is also used for fixed-order controller design.

Future studies can extend the application areas further to the simultaneous stabilization of multiple systems. This can be useful in controlling switched systems and linear parameter varying (LPV) systems.

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