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FIELD OF VALUES OF MATRIX POLYTOPES

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ABSTRACT

The tool of field of values (also known as the classical numerical range) is used to recover most results available in the literature and to obtain some new ones concerning Hurwitz and Schur stability of matrix polytopes. Some facts obtained by an application of the elementary properties of field of values are as follows. If the vertex matrices have polygonal field of values, then the matrix polytope is Hurwitz and Schur stable if and only if the vertex matrices are Hurwitz and Schur stable, respectively. If the polytope is nonnegative and the symmetric part of each vertex matrix is Schur stable, then the polytope is Schur stable. For polytopes with spectral vertex matrices, Schur stability of vertices is necessary and sufficient for the Schur stability of the polytope.

Key words: robust stability, structured perturbations, matrix polytopes, interval matrices, field of values, numerical range.

MATRİS POLİTOPLARININ DEĞERLER ALANI

ÖZ

Matris politoplarının Hurwitz ve Schur kararlılığı ile ilgili bilinen bir çok sonuç ve bazı yeni sonuçlar, nümerik kapsam olarak da bilinen, değerler alanı fikri kullanılarak elde edilmektedir. Bu şekilde ulaşılan bazı sonuçlar şunlardır: eğer politop, değerler alanı poligon olan köşe matrislerinden oluşmuşsa, köşe matrislerinin Hurwitz veya Schur kararlı olması, tüm politopun kararlılığı için gerek ve yeter şarttır. Eğer politop negatif olmayan matrislerden oluşuyorsa ve köşe matrislerinin simetrik kısımları Schur kararlı ise, tüm politop da kararlıdır. Eğer politopun köşe matrisleri spektral matrislerse, köşe matrislerinin Schur kararlılığı, tüm politopun Schur kararlılığı için gerek ve yeter şarttır.

Anahtar Kelimeler: gürbüz kararlılık, yapısal perturbasyonlar, matris politopları, aralık matrisi, değerler alanı, nümerik kapsam.

1. INTRODUCTION

One active area of research in stability robustness of linear time invariant systems is concerned with stability of matrix polytopes. Various structured real parametric uncertainties can be modeled by a family of matrices consisting of a convex hull of a finite number of known matrices yielding a matrix polytope. An interval matrix family consisting of matrices whose entries lie in given intervals are special types of matrix polytopes and it models a commonly encountered parametric uncertainty.

Results that allow the inference of the stability of the whole polytope from stability of a finite number of elements of the polytope are of interest. Deriving such results is known to be difficult and few results of sufficient generality exist. Apart from the obvious case where the vertices are all upper (lower) triangular, vertex results have been obtained only for polytopes with normal (in particular symmetric) vertex matrices, Wang (1991), Çevik (1995), Mansour (1988). It is also well known that a matrix polytope is Hurwitz stable if the symmetric part of every vertex matrix is negative definite, Jiang (1987), Mansour (1988). Concerning inter-

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val matrix families, Kharitonov theorem for interval polynomials yield a vertex result for interval matrices in companion form. A variety of vertex or test matrix results are available at the cost of rather severe assumptions on the family, Mori and Kokame (1987), Mansour (1988), Shi and Gao (1986), Soh (1990), Sezer and Siljak (1994). A fairly complete survey of existing vertex, edge, or face type of results for robust stability of matrix polytopes until 1994 can be found in Barmish (1994).

In this paper, we employ the concept of the field of values or the numerical range associated with a matrix to obtain conditions for the Hurwitz and Schur stability of matrix polytopes. The reader is referred to the book Horn and Johnson (1991) for an excellent exposure to various properties of the field of values and their applications. The field of values has been applide to robust stability problems earlier by Owens (1984). In Owens (1986) and Palazoglu and Khambanonda (1989), the merit of field of values in handling the phase information in structured multiplicative perturbations has been emphasized. classically, the field of values has been demonstrated to be an effective tool in giving estimates for the stability of numerical methods in boundary value problems, see e.g., Spijker (1993). The technique of *quadratic stability*, which has developed out of common Lyapunov function approaches to families of uncertain matrices, can also be applied to polytopes of matrices, and has strong links with the field of values approach taken here. The reader may refer to Khargonekar et al. (1990) and the references therein for more recent examples of the application of quadratic stability to uncertain systems. Some links with the field of values is clarified in Remarks 1 and 3 below.

In Section 2, we give a summary of those properties relevant to the stability of matrix polytopes. In Section 3, the field of values of the matrix polytope under consideration is examined. Sections 4 and 5 are devoted to the application of the concept of field of values to Hurwitz and Schur stability of matrix polytopes, respectively. The results reported here are based on the initial results of Saadaoui (1997).

Notation: The field of real and complex numbers are denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. If $c \in \mathbf{C}$, then $\bar{c}$ denotes the complex conjugate of c, Re(c) the real part, Im(c) the imaginary part, and $|c|$ the magnitude of c. The angle or phase θ of a complex number $c = |c|\, e^{j\theta}$ is denoted by $\angle c$. Given a matrix $A = [a_{ij}] \in \mathbf{C}^{n\times m}$, A' denotes the transpose of A, A^* denotes the complex conjugate transpose of A, and $|A|$ denotes the matrix $[|a_{ij}|]$. A nonnegative matrix, such as $|A|$, is a real matrix with each entry nonnegative; $A \geq 0$ denotes that A is (real and) nonnegative. When n = m, $\sigma(A)$ stands for the set of eigenvalues of A called the spectrum of A. For the notation, terminology, and for various unproved elementary facts concerning vector norms and induced matrix norms (or operator norms) used in this paper, we refer the reader to Noble and Daniel (1977).

The set of points in the open left half complex plane and the open unit disk are denoted by $\mathbf{C}_-$ and $\mathbf{D}_1$, respectively. A polynomial $p(s)$ with real or complex coefficients is said to be Hurwitz (Schur) stable if all its roots lie in $\mathbf{C}_-$ ($\mathbf{D}_1$). A square matrix $A \in \mathbf{C}^{n\times n}$ is said to be Hurwitz (Schur) stable if its characteristic polynomial is Hurwitz (Schur) stable, which is equivalent to $\sigma(A) \subseteq \mathbf{C}_-$ ($\sigma(A) \subseteq \mathbf{D}_1$). Given a matrix family A we say that A is (robustly) Hurwitz (Schur) stable if all its members are Hurwitz (Schur) stable.

If S is a set, then conv(S) denotes the convex hull of S which is the smallest convex set containing S. Alternatively, conv(S) is the set of all convex combinations of any finite number of elements of S. The reader is referred to Rockafellar (1970) for the algebra and the properties of convex sets.

2. ELEMENTARY PROPERTIES OF THE FIELD OF VALUES

This section contains the definition, a summary of the properties of field of values, and its computation. For a more in-depth discussion and for the proofs Horn and Johnson (1991) can be consulted.

The field of values of $A \in \mathbf{C}^{n\times n}$ is

$$F(A) = \{x^* Ax : x \in \mathbf{C}^n ,\ x^*x = 1\} .$$

Thus, F(A) is the range in the complex plane of the continuous function $x \to x^* Ax$ with the unit Euclidean ball $\{x \in \mathbf{C}^n : x^*x = 1\}$ as its domain. Alternatively, F(.) can be viewed as a function from $\mathbf{C}^{n\times n}$ to the complex plane like the spectrum $\sigma(.)$. By considering the unit eigenvectors associated with each eigenvalue of A, it immediately follows that

$$\sigma(A) \subseteq F(A). \qquad (1)$$

A fundamental property of F(A), known as the Toeplitz-Hausdorff theorem, is that it is a (compact and) convex subset of the complex plane. Any information on the location and the shape of this convex set can be used to bound the eigenvalues. For matrices of size 2, the field of values is always an ellipse (possibly degenerate) with eigenvalues at the foci. When the size of the matrix is larger than 2 however, a variety of shapes are possible in general. The field of values of real matrices are symmetrically located with respect to the real axis.

A useful measure of the size of F(A) is the radius of the smallest disc centered at the origin of the complex plane that contains F(A). This is the *numerical radius* of $A \in C^{nxn}$ defined by

$$r(A) := \max \{ |z| : z \in F(A)\} .$$

Since the spectral radius $\rho(A) = \max \{ |\lambda| : \lambda \in \sigma(A)\}$ is the radius of the smallest disc centered at the origin in the complex plane that includes all eigenvalues of A, (1) gives

$$\rho(A) \le r(A) \quad (2)$$

for any $A \in C^{nxn}$.

The field of values is invariant under unitary similarity transformations, by an easy consequence of its definition. For all $A \in C^{nxn}$ and unitary $U \in C^{nxn}$,

$$F(U^* AU) = F(U^{-1} AU) = F(A).$$

Moreover, if $V \in C^{nxk}$ with $k \le n$ is such that $V^*V = I$, then

$$F(V^* AV) \subset F(A). \quad (3)$$

If A is a normal matrix (i.e., $A^*A = AA^*$), then it is unitarily similar to a diagonal matrix having its eigenvalues as diagonal entries. The field of values of a diagonal matrix, on the other hand, can easily be seen to be a polygon in the complex plane having the diagonal elements at its vertices. By unitary similarity invariance of F(A), it follows that if $A \in C^{nxn}$ is normal, then

$$F(A) = \mathrm{conv}(\sigma(A)) = \left\{ \sum_{i=1}^{n} \alpha_i\lambda_i : \alpha_i \ge 0 \, , \sum_{i=1}^{n} \alpha_i = 1 \, , \lambda_i \in \sigma(A) \right\} \quad (4)$$

In particular, if Q is Hermitian, then F(Q) is the interval $[\lambda_{min}(Q), \lambda_{max}(Q)]$, where $\lambda_{min}(Q)$, $\lambda_{max}(Q)$ denote the minimum and maximum eigenvalues of Q.

Given $A \in C^{nxn}$, let H(A) and S(A) denote the Hermitian and the skew-Hermitian parts of A, respectively, i. e.,

$$H(A) := \frac{A + A^*}{2} \, , \; S(A) = \frac{A - A^*}{2}$$

For any $x \in C^n$ such that $x^*x = 1$, we have $x^*H(A)x = \mathrm{Re}(x^*Ax)$ and $x^*S(A)x = j\,\mathrm{Im}(x^*Ax)$, by a straightforward computation. It follows that For $A \in C^{nxn}$ with Hermitian part H(A) and skew-Hermitian part S(A).

$$F(H(A)) = \mathrm{Re}(F(A)) := \{\mathrm{Re}(z) : z \in F(A)\} \, , \quad (5)$$

$$F(S(A)) = j\,\mathrm{Im}(F(A)) := \{j\,\mathrm{Im}(z) : z \in F(A)\}. \quad (6)$$

Using the facts that $F(S(A)) = jF(-jS(A))$ and $-jS(A)$ is Hermitian, we obtain

$$F(H(A)) = [\lambda_{min}(H(A)), \lambda_{max}(H(A))] \, , \quad (7)$$

$$F(S(A)) = [j\,\lambda_{min}(-jS(A)) \, , j\,\lambda_{max}(-jS(A))].$$

The properties (5) and (6) thus yield a rectangular region containing F(A) with vertical sides going through the smallest and the largest eigenvalues of

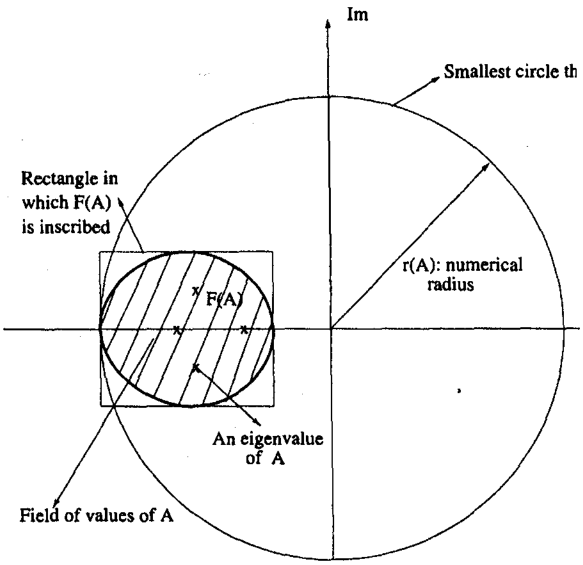

Figure 1. The field of values of a matrix A.

H(A) and with horizontal sides going through the smallest and the largest eigenvalues of -j S(A). The two regions, one circular and one rectangular, in which F(A) is inscribed are shown in Figure 1 for a real matrix A.

A simple bound on the numerical radius is easily obtained on noting that

$$r(A) = \max_{\|x\|_2=1} |x^* Ax| \le \max_{\|x\|_2=1} \|Ax\|_2 \, \|x\|_2 = \|A\|_2.$$

Hence, for any $A \in C^{nxn}$,

$$r(A) \le \|A\|_2 \, , \quad (8)$$

namely, the numerical radius is not larger than the largest singular value. In the case of l_1 and l_∞ induced norms, a similar inequality to (8) is not possible. However, it can be shown that (see Corollary 1.5.4 in Horn and Johnson (1991))

$$r(A) \le \frac{1}{2} \left(\|A\|_1 + \|A\|_\infty\right) . \quad (9)$$

For nonnegative matrices, better bounds on the numerical radius are possible. Recall that if a real A is nonnegative, then the spectral radius ρ(A) is an eigenvalue of A. If A is nonnegative, then so is H(A). By (7), it follows that $r(H(A)) = \rho(H(A))$. On the other hand, for any $x \in C^n$ and nonnegative $A = [a_{ij}]$, we have

$$|x^* Ax| = \left|\sum_i \sum_j a_{ij}\bar{x}_i x_j\right| \le \sum_i \sum_j a_{ij} |x_i| \, |x_j|$$

so that $r(A) \le \max \{x' Ax : x \in \mathbf{R}^n \, , x_i \ge 0 \, , x'x = 1\} = \max \{x' H(A)x : x \in \mathbf{R}^n \, , x_i \ge 0 \, , x'x = 1\} = \rho(H(A))$. Moreover, by property (5), it is easily seen that $r(H(A)) \le r(A)$. We thus arrive at the folloving property of the field of values of nonnegative matrices. If $A \in \mathbf{R}^{nxn}$ is nonnegative, then

$$r(A) = r(H(A)) = \rho(H(A)). \quad (10)$$

Given $A = [a_{ij}] \in \mathbf{C}^{n\times n}$, let $|A| := [|a_{ij}|]$. Clearly, for any $x \in \mathbf{C}^n$ and any $A = [a_{ij}]$, we have

$$|x^* AX| = \left|\sum_i \sum_j a_{ij}\bar{x}_i x_j\right| \le \sum_i \sum_j |a_{ij}|\,|x_i|\,|x_j|$$

so that $r(A) \le r(|A|)$. By property 10, we get the following bounds on the numerical radius of any matrix $A \in \mathbf{C}^{n\times n}$

$$\rho(A) \le r(A) \le r(|A|) = \rho(H(A)). \quad (11)$$

Although our main concern is to utilize the field of values as a theoretical tool, a comment on the numerical or graphical computation is in order. One method of computation of F(A) is based on the fact that

$$F(A) = \bigcap_{0 \le \theta < 2\pi} H_\theta ,\ H_\theta :$$
$$= \text{the half-plane } e^{-j\theta}\{z : \mathrm{Re}(z) \le \lambda_{max}(H(e^{-j\theta} A))\}.$$

Upon choosing a finite number of angular mesh points $\{\theta_1, \dots, \theta_k\}$, the convex set $F_k(A) := \cap_{i=1}^{k} H_{\theta_i}$ (outer) approximates F(A) and converges to F(A) as $k \to \infty$. Alternatively, an inner approximation or a combination of the two are also possible, Horn and Johnson (1991), Owens (1984), Palazoglu and Khambanonda (1989).

3. FIELD OF VALUES OF MATRIX POLYTOPES

We now turn to our main objective of examining the stability of a *matrix polytope*

$$\mathcal{A} = \left\{A = \sum_{i=1}^{N} \alpha_i E_i : E_i \in \mathbf{C}^{n\times n}, \alpha_i \ge 0, \sum_{i=1}^{n} \alpha_i = 1\right\}, \quad (12)$$

The matrices E_i, $i = 1, \dots, N$ are called *vertex matrices* since

$$\mathcal{A} = \mathrm{conv}\{E_1, \dots, E_N\}.$$

If the vertex matrices are real, then the whole polytope is real and we denote the *real matrix polytope* by A_r. An important subfamily of A_r is the *interval matrix family*

$$\mathcal{A}_{int} = \left\{A = [a_{ij}] : \underline{a}_{ij} \le a_{ij} \le \bar{a}_{ij},\ \underline{a}_{ij}, \bar{a}_{ij} \in \mathbf{R}, i,j = 1,\dots,n\right\}$$

sometimes specified by alternative notations

$$\mathcal{A}_{int} = \left[\left[\underline{a}_{ij}, \bar{a}_{ij}\right]\right] \text{ or } \mathcal{A}_{int} =$$
$$\left\{A \in \mathbf{R}^{n\times n} : \underline{A} \le A \le \bar{A},\ \underline{A}, \bar{A} \in \mathbf{R}^{n\times n}\right\}.$$

Upon choosing the vertex matrices as

$$E_v = [e_{ij}] : e_{ij} \in \left\{\underline{a}_{ij}, \bar{a}_{ij}\right\}.\ i,j = 1,2,\dots,n, \quad (13)$$
$$v = 1,2,\dots,2^{n^2},$$

it is clear that an interval family is a matrix polytope, Also observe that any real matrix polytope can be *imbedded* in an interval matrix family upon choosing $\underline{a}_{ij}$ and $\bar{a}_{ij}$ to be, respectively, the minimum and maximum among the ij-th entries of $E_1, \dots, E_N$.

For a general matrix polytope (12), by the definition of F(A), we easily obtain the inclusion

$$F(A) = F\left(\sum_{i=1}^{N} \alpha_i E_i\right) \subseteq \sum_{i=1}^{N} \alpha_i F(E_i) \subseteq \mathrm{conv}\left(F(E_1) \cup \dots \cup F(E_N)\right) \quad (14)$$

for any $A = \sum_{i=1}^{N} \alpha_i E_i \in \mathcal{A}$ so that

$$\cup\{F(A) : A \in \mathcal{A}\} \subseteq \mathrm{conv}\,(F(E_1) \cup \dots \cup F(E_N)). \quad (15)$$

The reverse inclusion holds if the left hand side is convex, e.g, if $E_i = e_i I$ for $e_i \in \mathbf{C}$, $i = 1, \dots, N$. Even when E_i is normal for each $i = 1, \dots, N$, the reverse inclusion in (15) may fail to hold as the following example shows.

Example 1. Let $n = N = 2$ and

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Both vertex matrices are normal so that $F(E_1)$ and $F(E_2)$ are intervals $[-1,1]$ and $[-j,+j]$, respectively. The convex hull of $F(E_1) \cup F(E_2)$ is the region bounded by a square with vertices at $\pm 1, \pm j$. Any $A \in \mathcal{A}$ is of the form

$$A = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}, \quad a \in [-1, 1].$$

The field of values of A is an ellipse with center at the origin, foci at the eigenvalues, and major and minor axes of lengths $1 \pm |a|$. Since A is real, the major axis is parallel to either the real or imaginary axis. At $a = 0$, the ellipse degenerates to a circle of radius $\frac{1}{2}$ with center at the origin. For $a < 0$, the major axis is parallel to the imaginary axis and for $a > 0$, the major axis is parallel to the real axis. The ellipse in rectangular coordinates $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$ has then the equation.

$$\frac{x^2}{(1+a)^2} + \frac{y^2}{(1-a)^2} = \frac{1}{4}, \quad 0 < |a| < 1.$$

It is easy to see that this ellipse has no intersection with the lines $\pm x + \pm y = 1$ for any $a \in (-1, 0) \cup (0, 1)$. Infinitely many points in $\mathrm{conv}\{F(E_1) \cup F(E_2)\}$ do not belong to F(A) for any $A \in \mathcal{A}$. The situation is illustrated in Figure 2.

Since each $F(E_i)$ is convex, the disk of radius $\max_i\{r(E_i)\}$ contains the right hand side in (15). This yields the following inequality for the numerical radii:

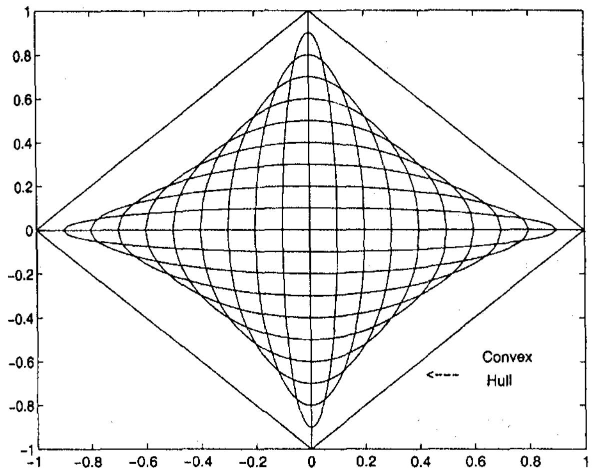

Figure 2. The field of values of the polytope of Example 1.

$$r(A) \le \max_i \{r(E_i)\}\ ,\ \forall A \in \mathcal{A}. \qquad (16)$$

Furthermore, by (5) and (15),

$$Re(F(A)) \subseteq conv(F(H(E_1)) \cup \dots \cup F(H(E_N)))$$

and since each $F(H(E_i))$ is an interval on the real axis, we have that

$$\max\{Re(z) : z \in F(A)\} \le \max_i \{\beta : \beta \in F(H(E_i))\}\ ,\ \forall A \in A. \qquad (17)$$

We thus obtain the following central result.

Theorem 1. A matrix polytope $\mathcal{A} = conv\{E_1, \dots, E_N\}$ is

(i) Hurwitz stable if $\max\{\beta: \beta \in F(H(E_i))\} < 0$ for each $i = 1, \dots, N$,

(ii) Schur stable if $r(E_i) < 1$ for each $i = 1, \dots, N$.

Proof. (i) If $\max\{\beta : \beta \in F(H(E_i))\} < 0$ for each $i = 1,\dots,N$, then by (17), $Re(F(A))$ is contained in the negative real axis and hence, by (5), $F(A) \subseteq \mathbf{C}_-$ for any $A \in \mathcal{A}$. The spectral containment property (1) gives that $\mathcal{A}$ is Hurwitz stable. (ii) If $r(E_i) < 1$ for each $i = 1, \dots, N$, then by (16), $r(A) < 1$ so that $F(A) \subseteq D_1$ for every $A \in \mathcal{A}$. By (1), we have that $\mathcal{A}$ is Schur stable.

4. HURWITZ STABILITY OF MATRIX POLYTOPES

An immediate consequence of Theorem 1.i is the following.

Fact 1. Consider $\mathcal{A} = conv\{E_1, \dots, E_n\}$.

(i) If $H(E_i), i = 1, \dots, N$ are Hurwitz stable, then $\mathcal{A}$ is Hurwitz stable

(ii) If $E^*_i \in \mathcal{A}$ for $i = 1, \dots, N$, then $\mathcal{A}$ is Hurwitz stable if and only if $H(E_i), i = 1, \dots, N$ are Hurwitz stable.

(iii) If each E_i is Hermitian, then $\mathcal{A}$ is Hurwitz stable if and only if $E_i, i = 1, \dots, N$ are Hurwitz stable.

Proof. (i) If $H(E_i)$ is Hurwitz stable, then by (7), $F(H(E_i))$ is contained in the negative real axis. The result follows by Theorem 1.i. (ii) If $E^*_i \in \mathcal{A}$, then $H(E_i)$ which is a convex combination of E_i and E^*_i is also in $\mathcal{A}$. Stability of $\mathcal{A}$ hence implies stability of $H(E_i)$ for $i = 1, \dots, N$. The converse follows by (i). (iii) This is a special case of (ii).

Remark 1. By elementary properties of the field of values, we thus obtained the results noted by Jiang (1987), Mansour (1988), Soh (1990), Shi and Gao (1986) and Çevik (1995). As noted by Mansour (1988), Fact 1.i is also a very simple consequence of Lyapunov's theorem. If $H(E_i)$ is Hurwitz stable, then $Q_i := -(E_i + E^*_i)$ is negative definite and $P = I$ is a positive definite solution of the Lyapunov equation

$$E^*_i P + PE_i = -Q_i. \qquad (18)$$

The same P is also a positive definite solution of $A^*P + PA = -Q$ for any A which is a convex combination of vertex matrices, where Q is the same convex combination of positive definite $Q_i, i = 1, \dots, N$. Fact 1.i follows by Lyapunov's theorem. The reason why this result follows equally easily by the two approaches may be explained by the strong links, Horn and Johnson (1991), that exist between the field of values and the Lyapunov's theorem. To emphasize this point, suppose P is a positive definite solution of (18) for some positive definite Q_i for all $i = 1, \dots, N$. Let $P^{\frac{1}{2}}$ be the unique positive definite square root of P. Then, by (18), we have $P^{\frac{1}{2}}E_iP^{-\frac{1}{2}} + \left(P^{\frac{1}{2}}E_iP^{-\frac{1}{2}}\right)^* = -P^{-\frac{1}{2}}Q_iP^{-\frac{1}{2}}$ and hence $H\left(P^{\frac{1}{2}}E_iP^{-\frac{1}{2}}\right)$ is negative definite, or equivalently, Hurwitz stable. By projection property (5), $F\left(P^{\frac{1}{2}}E_iP^{-\frac{1}{2}}\right) \subseteq \mathbf{C}_-$. Now, given $A \in \mathcal{A}$, by (14), we have $F\left(P^{\frac{1}{2}}AP^{-\frac{1}{2}}\right) \subseteq \mathbf{C}_-$ and spectral containment (1) yields that $\sigma(A) = \sigma\left(P^{\frac{1}{2}}AP^{-\frac{1}{2}}\right) \subseteq \mathbf{C}_-$ This shows, using field of values, that $\mathcal{A}$ is Hurwitz stable whenever the Lyapunov equations for vertex matrices admit a simultaneous solution P.

Remark 2. Fact 1.ii has an interesting application to interval matrix families. A particular case in which the assumption in Fact 1.ii is fulfilled is for the interval matrix family

$$\mathcal{A}_{int} = [[\underline{a}_{ij}, \bar{a}_{ij}]],$$

with the additional property

$$\underline{a}_{ij} = \underline{a}_{ji}\ ,\ \bar{a}_{ij} = \bar{a}_{ji}\ ,\ i, j = 1, \dots, n. \qquad (19)$$

Such a family is Hurwitz stable if and only if the symmetric parts of the vertex matrices are Hurwitz stable. This result implies but is different from the result of Shi and Gao (1986) or Soh (1990) who show that if $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$ for all $A = [a_{ji}] \in A_{int}$, then the resulting symmetric interval matrix family is stable if and only if (a subset of) the vertex matrices are stable. A similar result was established by Rhon (1994) who additionally showed that only a portion of the vertex matrices is sufficient to infer Hurwitz stability of the family.

Hurwitz stability of the Hermitian part of any matrix is sufficient to conclude the Hurwitz stability of the matrix by (5). The converse is not generally true but turns out to be true for normal matrices and a bit more.

Fact 2. *Suppose* $A \in C^{nxn}$ *satisfies*

$$F(A) = \text{conv}\,(\sigma(A)). \qquad (20)$$

Then, A is Hurwitz stable if and only if H(A) is Hurwitz stable.

Proof. If (20) holds for A, then as conv(σ(A)) is a polygon with vertices consisting of some (or all) eigenvalues of A, we have

$$\max\{\text{Re}\, z : z \in F(A)\} = \max\{\text{Re}(\lambda_i) : \lambda_i \in \sigma(A)\}$$

and the result follows.

By (4), a normal matrix satisfies (20). If $n \le 4$, then any matrix satisfying (20) is normal. If $n > 4$, then there are matrices satisfying (20) which are not normal. Such matrices are characterized by Theorem 1.6.8 of Horn and Johnson (1991) : E satisfies (20) if and only if either E is normal or E is unitarily similar to a matrix of the form

$$\begin{bmatrix} \tilde{E} & 0 \\ 0 & \hat{E} \end{bmatrix}, \ \tilde{E} \text{ is normal and } F(\hat{E}) \subseteq F(\tilde{E}) \qquad (21)$$

The following fact recovers the result by Wang (1991) concerning polytopes with normal vertex matrices.

Fact 3. Suppose a matrix polytope $\mathcal{A}$ = conv $\{E_1, \dots, E_N\}$ is such that each E_i is either normal or unitarily similar to a matrix of (21). Then, the following are equivalent:

(i) $\mathcal{A}$ is Hurwitz stable,

(ii) $H(E_i)$, i=1,...,N are Hurwitz stable,

(iii) E_i, i= 1,...,N are Hurwitz stable.

Proof. By the hypothesis, vertex matrices satisfy

$$F(E_i) = \text{conv}\,(\sigma(E_i)),\ i=1,\dots,N.$$

By Fact 2, (i) and (ii) are equivalent. The fact that (i) implies (iii) is obvious. Finally, (ii) implies (i) by Fact 1.i.

Fact 3 provides a strict extension of the result in Wang (1991). By (3), the hypothesis of Fact 3 is satisfied whenever each vertex matrix E_i is unitarily similar (or equal) to

$$\begin{bmatrix} \tilde{E}_i & 0 \\ 0 & P_i^* \tilde{E}_i P_i \end{bmatrix}, \ \tilde{E}_i \text{ is normal,}$$

for some $P_i \in C^{nxk}$ such that $P^*_i P_i = I$. Note that even though $\tilde{E}_i$ is normal, $P_i^* \tilde{E}_i P_i$ *may not be normal* unless $n = k$.

5. SCHUR STABILITY OF MATRIX POLYTOPES

We now examine the robust Schur stability of a general matrix polytope (12) using Theorem 1.ii.

A matrix $A \in C^{nxn}$ is called *spectral* if

$$\rho(A) = r(A). \qquad (22)$$

Note that, by (2), $\rho(A) \le r(A)$ for *any* A. In view of (4), normal matrices or matrices for which (20) hold are spectral. The converse is true only in the case n = 2. For instance, the 3 x 3 matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (23)$$

satisfies (22) but not (20), see Figure 3.

Fact 4. *Let* $\mathcal{A}$ *be such that every vertex matrix* E_i *is spectral. Then, the following are equivalent:*

(i) $\mathcal{A}$ is *Schur stable,*

(ii) $r(E_i) < 1$ for i = 1, ... , N,

(iii) E_i, i = 1, ... , N *are Schur stable.*

Proof. Obviously (i) implies (iii). Since a matrix is Schur stable if and only if its spectral radius is

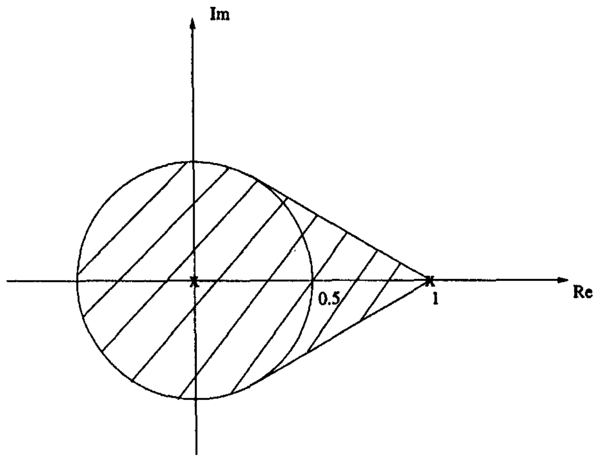

Figure 3. The field of values of (23).

less than 1, (ii) is equivalent to (iii) by the hypothesis $r(E_i) = p(E_i)$, i = 1 , ... , N. Finally, (ii) implies (i) by Theorem 1.ii and by (2) which is true for any matrix A.

A corollary of Fact 4 is that a polytope with normal (or Hermitian) vertex matrices is Schur stable if and only if the vertices are Schur stable, a result of Wang (1991). among the many characterizations for spectral matrices, the following can be cited, Horn and Johnson (1991), pp. 61-62: A matrix A is spectral if and only if A is unitarily similar to a matrix of the form

$$r(A)\begin{bmatrix} U & 0 \\ 0 & B \end{bmatrix}, \text{U is unitaray}, \rho(B) < 1, r(B) \leq 1.$$

The norm bounds (8) and (9) together with Theorem 1.ii yield that if

$$\min\left\{\|E_i\|_2, \frac{1}{2}(\|E_i\|_1 + \|E_i\|_\infty)\right\} < 1, \quad (24)$$

for every i = 1 , ... , N, then $\mathcal{A}$ is Schur stable if and only if the vertex matrices are Schur stable. However, this result is weaker than what can be obtained by directly using the spectral property of induced norms. Since every induced matrix norm ||.|| is spectrally dominant, i.e., $\rho(.) \leq ||.||$ and satisfies the triangle inequality, it immediately follows that if

$$\min\{\|E_i\|_2, (\|E_i\|_1 + \|E_i\|_\infty)\} < 1,$$

for all i = 1, ... , N, then $\mathcal{A}$ is Schur stable, a result noted by Mori and Kokame (1987) for interval matrix families.

Remark 3. A contact with the discrete-time version of the Lyapunov's theorem is possible. Suppose that for every vertex matrix E_i , the discrete-time Lyapunov equation

$$E^*_i P E_i - P = -Q_i \quad (25)$$

has a positive definite (common) solution P for some positive definite Q_i, i = 1 , ..., N. Let $P^{\frac{1}{2}}$ be the unique positive definite square root of P and note that $G_i := P^{\frac{1}{2}} E_i P^{-\frac{1}{2}}$ is such that $G^*_i G_i - I$ is negative definite for i = 1,..., N. It follows that $|\lambda_{max}(G_i^* G_i)| = \|G_i\|_2^2 < 1$ which by (8) gives $r(G_i) < 1$. Now, given $A \in \mathcal{A}$, we have by (16) that $r\left(P^{\frac{1}{2}} A P^{-\frac{1}{2}}\right) < 1$ and hence $\sigma(A) = \sigma\left(P^{\frac{1}{2}} A P^{-\frac{1}{2}}\right)$ is contained in $\mathbf{D}_1$. With field of values arguments, we have thus recovered the fact noted by Mansour (1988) that if there is a simultaneous solution P of (25), then te matrix polytope $\mathcal{A}$ is Schur stable

Let us now consider a nonnegative matrix polytope

$\mathcal{A}_{nn} :=$ conv $\{E_1, ..., E_N\}$, $E_i \in \mathbf{R}^{n\times n}$, $E_i \geq 0$, i = 1, ...,N.

Fact 5. A *nonnegative matrix polytope* $\mathcal{A}_{nn}$ is Schur stable if either of the following hold:

(i) $H(E_i)$, i = 1,..., N are *Schur stable*,

(ii) $B = [b_{ij}]$ *is Schur stable, where,* b_{ij} *is the maximum among all ij-th entries of* E_k, k = 1, ..., N.

(iii) $C = [c_{ij}]$ *is Schur stable, where,* b_{ij} *is the maximum among all ij-th entries of* $H(E_k)$ = 1,..., N.

Proof. (i) If $h(E_i)$, i = 1,..., N are Schur stable then $r(h(E_i)) < 1$ and by (10) $r(E_i) < 1$, i = 1,..., N. The result follows by Theorem 1.ii and by (2). (ii) Note nonnegative matrices, $\rho(A) \leq \rho(B) < 1$ for all $A \in$ A. (iii) Note thet $0 \leq H(E_i) \leq C$ for all i = 1,..., N. Schur stability of C thus implies the Schur satibilty of $H(E_i)$ for i = 1,..., N. The result follows by (i).

Remark 4. The above proof of Fact 5. ii uses a basic property of nonnegative matrices and does not resort to field of values. One useful consequence of Fact 5. ii is for nonnegative interval matrice, obtained in Shafai et al. (1991), Sezer and Siljak (1994) by other means. Consider the nonnegative inteval matrix family

$$\mathcal{A}_{int\text{-}nn} := \{A \in \mathbf{R}^{n\times n} : 0 \leq A \leq A \leq \tilde{A}\}.$$

Using Fact 5. ii, $\mathcal{A}_{int\text{-}nn}$ is Schur stable if and onl if $B = \tilde{A}$ is. Applying Fact 5. iii to $\mathcal{A}_{int\text{-}nn}$, we see that if $H(\tilde{A})$ is Schur stable, then $\mathcal{A}_{int\text{-}nn}$ is olso Schur stable. This however is a consequence of the italicized statement, since by (10) $\rho(A) \leq \rho(H(A))$ for any $A \geq 0$.

There is no implication in general between conditions (ii) and (iii) of Fact 5.

Example 2. Consider

$$E_1 = \begin{bmatrix} 0.5 & 0.6 \\ 0.3 & 0.5 \end{bmatrix}, E_2 = \begin{bmatrix} 0.5 & b \\ c & 0.5 \end{bmatrix}.$$

If $b \leq 0.6$ and $c \geq 0.3$, then we have

$$B = \begin{bmatrix} 0.5 & 0.6 \\ c & 0.5 \end{bmatrix}, C = \begin{bmatrix} 0.5 & d \\ d & 0.5 \end{bmatrix},$$

where d = max {0.45, 0.5 (b + c)}. For b = 0.1, c = 0.7, the matrix C is Schur but not B. On the other hand, for b = 0.6 , c = 0.4, the matrix B is Schur stable but not C.

The enquality (11) gives a slight generalization of Fact 1.i.

Fact 6. A *matrix polytope* $\mathcal{A}$ *is Schur stable if* $H(|E_i|)$, i = 1, ..., D *are Schur stable.*

Proof. The result follows by (11), Theorem 1. ii, and (2).

6. CONCLUSIONS

We have examined robust stability of matrix polytopes and demonstrated tht elementary porperties of the field of values directly yield many existing results and some others such as Facts 1, 3-7. In view of the encouraging reports as in Palazoglu and Khanmbanonda (1989) concerning the graphical computation of the field of values, (15) can be used to graphically check the satbility of a matrix polytope.

We have not consumed all applications. A refinement of the inequality (9) as in Horn and Jhonson (1991) and diagonal scaling yield circular disks in which the eigenvalues are inscribed. In general, these regions neither contain nor are contained in the union of the Gershgorin circles and hence they can be used to give alternative sufficient conditions for the stability of the polytope in terms of the radius of the disks obtained for vertex matrices. We have not pursued such alternative approaches to e.g. Sezer and Siljak (1994).l

A limitation of the field of values approach is clear. Like the Gershgorin circles, the field of values yield regions in the complex plane where the eigenvalues lie in. The field of values like Gershgorin circles can not capture a full information on the spectrum. Unlike the Gershgorin's theorem or its extensions, however, there are stronger links between the type of a matrix and ist field of values as witnessed by the properties listen in Section 2.

Finally, as pointed out in Barmish (1994), construction of parametric Lyapunov functions for matrix polytopes is a research direction not yet fully exploited. A field of values approach to parametric Lyapunov seems also possible in view of some results in Horn and Johnson (1991).

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