EXPLORATIONS TO REFINE AIZERMAN MALISHEVSKI'S REPRESENTATION FOR PATH INDEPENDENT CHOICE RULES

A Ph.D. Dissertation

by
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To my family

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The Graduate School of Economics and Social Sciences
of
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by

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İHSAN DOĞRAMACI BİLKENT UNIVERSITY
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September 2020

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ABSTRACT

EXPLORATIONS TO REFINE AIZERMAN MALISHEVSKI'S REPRESENTATION FOR PATH INDEPENDENT CHOICE RULES

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This dissertation consists of four main parts in which we explore Aizerman Malishevski's representation result for path independent choice rules. Each path independent choice rule is known to have a maximizer-collecting (MC) representation: There exists a set of priority orderings such that the choice from each choice set is the union of the priority orderings' maximizers (Aizerman and Malishevski, 1981). In the first part, we introduce the maximal and prime sets to characterize all possible MC representations and show that the size of the largest anti-chain of primes determines its smallest size MC representation. In the second part, we focus on q-acceptant and path independent choice rules. We introduce prime atoms and prove that the number of prime atoms determines the smallest size MC representation. We show that q-responsive choice rules require the maximal number of priority orderings in their smallest size MC representations among all q-acceptant and path independent choice rules. In the third part, we aim to generalize q-responsive choice rules and introduce responsiveness as a choice axiom. In order to provide a new representation for responsive and path independent choice rules we introduce weighted responsive choice rules. Then, we show that all responsive and path independent choice rules are weighted responsive choice rules with an additional regularity condition. In the final part we focus on assignment problem. In this problem Probabilistic Serial assignment is always sd-efficient and sd–envy-free. We provide a sufficient and almost necessary condition for uniqueness of sd-efficient and sd–envy-free assignment via a connectedness condition over preference profile.

Keywords: Choice Rules, Path Independence, Prime Sets, Responsive Rules, Substitutability.

ÖZET

YOLDAN BAĞIMSIZ SEÇME KURALLARININ AIZERMAN VE MALISHEVKSI TEMSİLİNİ İNCELTMEYE YÖNELİK ARAŞTIRMALAR

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Bu çalışma yoldan bağımsız seçim kuralları için Aizerman Malishevski'nin temsilinin araştırıldığı üç ana kısımdan oluşmaktadır. Yoldan bağımsız her seçim kuralının, belirli bir öncelik sıralaması kümesi sayesinde, verilen her seçim kümesi için elimizdeki öncelik sıramalarındaki en iyi elemanların kümesi olarak göstermemizi sağlayacak bir en-iyileri-biriktiren (EB) temsilinin olduğu bilinmektedir (Aizerman and Malishevski, 1981). İlk kısımda bütün yoldan bağımsız seçim kurallarını inceliyoruz. Bu seçim kurallarının mümkün bütün EB temsillerini bulmak için maksimal ve asal küme kavramlarını tanımlıyoruz. Daha sonra asal kümelerin en geniş anti-zincirinin büyüklüğünün mümkün EB temsilleri arasında en kısasının büyüklüğünü verdiğini gösteriyoruz. İkinci kısımda q-(kapasite) dolduran ve yoldan bağımsız seçim kurallarına odaklanıyoruz. En küçük asal kümeleri asal atomlar olarak tanımlayıp, bu kuralların en kısa EB temsilinin tam olarak asal atomların sayısı kadar sıralama içerdiğini gösteriyoruz.

Bu sonucu kullanarak, q-(sıralamaya) duyarlı seçim kurallarının en kısa EB gösteriminin bütün q-(kapasite) dolduran ve yoldan bağımsız seçim kuralları arasında mümkün olabilecek en uzun gösterimi olduğunu gösteriyoruz. Üçüncü kısımda q-(sıralamaya) duyarlı seçim kurallarını genellemek amacıyla (sıralamaya) duyarlılığı bir seçim beliti olarak tanımlıyoruz. Yoldan bağımsız ve (sıralamaya) duyarlı seçim kuralları için ağırlıklı (sıralamaya) duyarlı seçim kuralları olarak isimlendirdiğimiz yeni bir temsil tanımlayıp bu yeni temsilin ağırlıklar üzerindeki ek bir kısıt altında bütün (sıralamaya) duyarlı ve yoldan bağımsız seçim kuralları kümesine denk olduğunu gösteriyoruz. Dördüncü kısımda ise atama problemine odaklandık. Bu problemde Eşit Hızla Yedirme algoritmasının her zaman olasılıksal-verimli ve olasılıksal-kıskançlıksız bir dağıtım verdiği bilinmektedir. Biz de biricik olasılıksal-verimli ve olasılıksal-kıskançlıksız dağılımın olduğu tercih profillerini belirlemek için yeterli ve neredeyse gerekli bir koşul getirdik.

Anahtar Kelimeler: Asal Kümeler, (Sıralamaya) Duyarlı Kurallar, İkame Edilebilirlik, Seçim Kuralları, Yoldan Bağımsızlık.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Recent advances in market design have called for a better understanding of how institutions choose, or how they should choose, when faced with a set of alternatives. For example, in the context of assigning students to schools, it is important to understand the structure of plausible choice rules a school can use as an admissions policy. Although the relevant restrictions on choice rules vary across applications, *path independence*, *substitutability*, *q-acceptance* and *responsiveness* are some of the prominent conditions that are satisfied by choice rules in application. In this study, we will mainly try to provide refined representations for *path independent* choice rules and seek the possible improvements for known representations in order to make them simpler.

We consider a decision maker who encounters a choice problem for possible choice sets. A choice rule, at each choice set chooses some of the alternatives from the choice set. A choice rule is *substitutable* if it chooses an alternative from a smaller set whenever it is chosen from a larger set. In the literature this condition is also referred as *Sen's* α, *Chernoff's condition* or *independence of irrelevant* *alternatives*. *Substitutability* has been a standard requirement in market design literature following the seminal work of Kelso and Crawford (1982). Hatfield and Milgrom (2005) show that *substitutability* of choice rules guarantees the existence of a *stable* matching, which is a central desideratum for applications. Hatfield and Kojima (2008) show that *substitutability* of choice rules is an "almost necessary" condition for the non-emptiness of the core and the existence of *stable* allocations. Similarly, several classical results of matching literature have been generalized with *substitutable* choice rules (Roth and Sotomayor (1990), Alkan and Gale (2003), Hatfield and Milgrom (2005)).

We focus on *path independent* choice rules (Plott, 1973) in which a choice over the union of a collection of sets is same as the choice over the union of chosen elements of these sets. It is known that such choice rules are the choice rules that satisfy *substitutability* and *independence of rejected alternatives* (*IRA*), the latter of which requires that removing any of the rejected alternatives has no effect on the choice. Among others, Plott (1973), Moulin (1985), Johnson (1990), and Johnson (1995) study the structure of path independent choice rules. Johnson and Dean (2001) and Koshevoy (1999) provide a lattice theoretic characterization of path independent choice rules.

We say that a choice rule has a maximizer-collecting (MC) representation if there exists a set of priority orderings[1] such that the choice from each choice set is obtainable by collecting the maximizers of the priority orderings. It follows from

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[1] A priority ordering is a complete, transitive, and anti-symmetric binary relation over all possible alternatives.

Aizerman and Malishevski (1981) that each *path independent* choice rule has an MC representation. However, the size of a smallest size MC representation of a choice rule and how to construct such representations have been unknown.

Our results in this thesis together with the constructions in the proofs are of interest for applications in which *path independent* choice rules are adopted. A prominent example is the school choice problem, in which each school specifies its admission policy in the form of a *path independent* choice rule that reconciles the objective of admitting students with high exam scores and affirmative policies for females, ethnic minorities, or neighborhood students. A curious question is what is the simplest way of communicating the choice rule to the public. Since such a choice rule can be represented as an MC choice rule, it is natural to assume that as the size of this representation decreases, the communication can be easier.

In Chapter 3, we provide a necessary and sufficient condition for an MC representation to represent any given *path independent* choice rule. We say a set is *maximal* if there is no larger set which makes the same choice. We consider a relation over maximals and lattice formed by these.[2] A maximal set is a *prime*, if there exist an alternative not belonging to this set that is chosen when it is added, but fails to be chosen whenever any other alternative is added. In Theorem 1, we prove that a set of priority orderings represent a *path independent* choice rule if and only if every prime appears as a lower contour set[3] of some priority ordering

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[2] It was first noted by Johnson (1990) that each path independent choice rules induces a specific choice lattice over maximal sets. Alkan (2001) and Chambers and Yenmez (2017) use a similar lattice in their proofs. We are grateful to Ahmet Alkan for bringing this to our attention.

[3] A lower contour set of an alternative in a priority ordering is the set of alternatives that are worse than given alternatives. So it corresponds to some bottom block of ordering.

and all lower contour sets of all priority orderings are maximal. Following this result we also provide an answer for the size of the minimal MC-representation by Theorem 2. It turns out this number is equal to the cardinality of the largest anti-chain[4] of primes.

The prime set notion that we have formulate has a central role in answering the questions analyzed in this thesis. The same notion is also used in my joint work Doğan et al. (2020). We believe that this notion will be fruitful for further research on this topic.

In Chapter 4, we impose *q-acceptance* on *path independent* choice rules and refine our results for such rules. A *q-acceptant* choice rule has capacity q. If it is possible, then the choice rule fills the capacity and chooses exactly q alternatives, otherwise it accepts all alternatives from the choice sets which have no more alternatives than the capacity. This is a reasonable restriction in many applications where institutions prefer to fill their positions whenever it is possible.[5] The implication of Theorem 2 sharpens when we additionally assume *q-acceptance*. We call a prime set with cardinality $q-1$ as *prime atom*. In Theorem 3, we show the number of orderings in the minimal size representation is equal to the number of prime atoms.

Well-known examples of *q-acceptant* and *path independent* choice rules include

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[4] A collection of sets is called anti-chain if there is no two sets such that one contains the other.

[5] In the matching literature, *q-acceptance* is also referred to as *capacity-filling* which terminology has been increasingly popular in the recent literature. Alkan (2001) is the first study which uses the “filling" terminology where he uses the term *quota filling*.

q-responsive choice rules which have been studied particularly in the two-sided matching context (Gale and Shapley, 1962). A choice rule is *q-responsive* if there exists a priority ordering such that the choice from each choice set is obtainable by choosing the highest priority alternatives until the capacity is reached or no alternative is left.

In Theorem 4, we show that the upper bound on the number of prime atoms of a *q-acceptant* choice rule is achieved by *q-responsive* choice rules. That is, the size of the smallest size MC representation rendered by *q-responsive* choice rules is largest among all *q-acceptant* and *path independent* choice rules.

Our Theorem 4 highlights the gap between *q-responsive* choice rules and MC-representations. In order to provide a neater representation we introduce a generalized responsive notion which also includes *q-responsive* choice a rules as a special case. Roth (1985) introduced responsive preferences over subsets and Gale and Shapley (1962) studied *q-responsive* choice rules in matching and college admission contexts. We formulate responsiveness as a choice axiom: A choice rule is *responsive* if there exist a priority ordering such that for any choice set chosen alternatives are preferred to rejected alternatives with respect to this priority ordering. In order to represent *path-independent* and *responsive* rules we introduce a new class of choice rules that is called *weighted responsive rules.* Suppose we have a priority ordering over alternatives and each alternative has a positive real valued weight (cost) vector over some index set such that we have a unit quota of each coordinate in the index set. Given a priority ordering and the weight vectors, a *weighted responsive* choice rule chooses the alternatives with the highest priority until none of the quotas are exceeded. In Theorem 5, we show that the *path independent* and *responsive choice* rules are *weighted responsive* rules which satisfy an additional regularity condition.

In Chapter 6, we focused a question on assignment problem. We tried to figure out when Probabilistic Serial assignment is the unique sd-efficient and sd–envy-free assignment. For this purpose we introduced a connectedness condition which is a necessary condition for uniqueness. We also introduced a betweenness condition, when satisfied connectedness became a sufficient condition.

CHAPTER 2

PRELIMINARIES

Let A be a nonempty finite set of n elements and a choice set S be a subset of A. Let $\mathcal{A}$ denote the collection of all choice sets, which is the set of all subsets of A. A *choice rule* $C : \mathcal{A} \to \mathcal{A}$ associates with each nonempty choice set $S \in \mathcal{A}$, a nonempty set of elements $C(S) \subset S$, notice $C(\emptyset) = \emptyset$ for all choice rules. We analyze choice rules that satisfy the following properties that are well-known in the literature.

Substitutability: If an element is chosen from a choice set, then it is also chosen from any subset of the choice set that contains the element. Formally, for each $S, T \in \mathcal{A}$ such that $a \in T \subset S$, if $a \in C(S)$, then $a \in C(T)$.

Notice that for any $a \in C(S)$, by *substitutability* we get $a \in C(C(S))$. Therefore, we have $C(S) \subset C(C(S))$. By the definition of choice rules, we also know $C(C(S)) \subset C(S)$, therefore we get $C(C(S)) = C(S)$.

Independence of Rejected Alternatives *(IRA)*: Removing a set of unchosen elements does not affect the chosen elements. Formally, for each $S, T \in \mathcal{A}$, if $C(S) \subset T \subset S$, then $C(T) = C(S)$.

Path Independence: For any two choice sets, the chosen elements from the union of these sets is same as the chosen elements among the union of the chosen elements of these sets (Plott, 1973). Formally, for each $S, T \in \mathcal{A}$ we have $C(S \cup T) = C(C(S) \cup C(T))$.

One direct implication of *path independence* when we let $T = \emptyset$ is $C(S) = C(S \cup \emptyset) = C(C(S) \cup C(\emptyset)) = C(C(S))$. So *path independence* implies one basic observation of *substitutability*. Actually we know more fundamental relations between these properties from the literature. Following corollary states equivalence between *path independent* choice rules with choice rules that satisfy *substitutability* and *IRA*.

Corollary. *(Aizerman and Malishevski, 1981) A choice rule C is path independent if and only if it satisfies substitutability and IRA.*

Another observation about *path independence* is that, if $C(S) = C(T)$ then

$$C(S \cup T) = C(C(S) \cup C(T)) = C(C(S)) = C(S).$$

Thus if $\overline{S} = \bigcup\limits_{C(S')=C(S)} S'$, we get $C(\overline{S}) = C(S)$.

Suppose $C(T) = C(S)$, then by definition of choice rules we know $C(S) \subset T$ and by definition of $\overline{S}$ we have $T \subset \overline{S}$ thus $C(S) \subset T \subset \overline{S}$. Moreover for any T with $C(S) \subset T \subset \overline{S}$, since $C(\overline{S}) = C(S)$ all elements in $\overline{S} \setminus C(S)$ are rejected, thus by *IRA* we get $C(T) = C(S)$. These results will give us the following useful remark which helps us throughout the thesis.

Remark 1 (Koshevoy (1999)). Let $C : \mathcal{A} \to \mathcal{A}$ be a *path independent* choice rule and for any $S \in \mathcal{A}$ define $\overline{S} = \bigcup\limits_{C(S')=C(S)} S'$. Then $C(T) = C(S)$ if and only if $C(S) \subset T \subset \overline{S}$.

This remark gives us helpful insights about the structure of all *path independent* choice rules. For any S in the image of a *path independent* choice rule C, there exists a unique maximal set $\overline{S}$ such that inverse image of S consists of sets which are superset of S and subset of $\overline{S}$. Since C is a well-defined function, these inverse images must be non-intersecting thus, these collections must be a partition of $\mathcal{A}$. One additional observation about that result is about the cardinalities of these inverse image sets. One can get,

$$\{T \in \mathcal{A} : C(S) \subset T \subset \overline{S}\} = \{C(S) \cup T_0 \in \mathcal{A} : T_0 \subset \overline{S} \setminus C(S)\}$$

since $\mathcal{A}$ is the power set of A

$$|\{T \subset A : C(S) \subset T \subset \overline{S}\}| = 2^{|\overline{S} \setminus C(S)|}.$$

Thus, the cardinality of the inverse image of any S in $\mathcal{A}$ is always a power of 2.

q-Acceptance: For a given capacity q with $1 \leq q \leq |A|$, an element is rejected from a choice set at the capacity q only if the capacity is full. Formally, for each $S \in \mathcal{A}$, $|C(S)| = \min\{|S|, q\}$.

Another fact is that *q-acceptance* together with *substitutability* imply *IRA*. This can be proven by using the definitions. Suppose $C(S) \subset T \subset S$. If $|C(S)| < q$ then by *q-acceptance* $|S| < q$ and $C(S) = S$ so $T = S$ and $C(T) = C(S)$. If $|C(S)| \geq q$ then it must be equal to q, and by *substitutability* elements in $C(S)$ should still belong to $C(T)$ therefore $C(S) \subset C(T)$. Since $|C(T)| \leq q$ that means $C(T) = C(S)$ therefore C satisfies *IRA*. Thus a choice rule satisfies *q-acceptance* and *substitutability* if and only if it is *q-acceptant* and *path independent.*

Aizerman and Malishevski (1981) show that a choice rule is *path independent* if and only if there exists a set of priority orderings such that the choice from each choice set is the union of the highest priority elements in the priority orderings.[1] Next, we formally define and add more structure on these choice rules that we call Maximizer-Collecting (MC) choice rules.

A *priority ordering* $\succ$ is a complete, transitive, and anti-symmetric binary relation over A. Suppose π is defined as $\pi = \{\succ_1, \ldots, \succ_m\}$, for some $m \in \mathbb{N}$, is a set of distinct priority orderings. Let Π denote the collection of all priority ordering sets. Given $S \in \mathcal{A}$ and a priority ordering $\succ$, let $\max(S, \succ) = \{a \in S : \forall b \in S \setminus \{a\}, a \succ b\}$.

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[1] In the words of Aizerman and Malishevski (1981), each *path independent* choice rule can be generated by some mechanism of collected extremal choice.

For any given priority ordering set π, *Maximizer-Collecting* rule of π, $MC(\pi)$ is obtained by collecting maximizers of the priority orderings in π, that is

$$MC(\pi)(S) = \bigcup_{\succ \in \pi} \max(S, \succ).$$

If the number of priority orderings in π is m then $MC(\pi)$ is also called m-maximizer collecting.

Example 1. **(i).** Suppose $\pi = \{\succ, \succ'\}$ with $1 \succ 2 \succ 3 \succ \cdots \succ n$ and $n \succ' n{-}1 \succ' n-2 \cdots \succ' 1$ then for any $S \subset \{1, 2, \cdots, n\}$, $MC(\pi)(S) = \{\max(S), \min(S)\}$. In this example $MC(\pi)$ is a 2-maximizer collecting rule.

(ii). As an another example consider $\pi' = \{\succ_a \colon a \in A\}$ with top element in $\succ_a$ is a and rest is ordered arbitrarily. Now clearly for any $a \in S$, $\max(S, \succ_a) = \{a\}$ since a is top element, so $MC(\pi')(S) = S$ for any $S \in \mathcal{A}$.

A choice rule C has a *maximizer-collecting (MC) representation* of size $m \in \mathbb{N}$ (or simply *m-maximizer-collecting*) if there exists $\{\succ_1, \ldots, \succ_m\} \in \Pi$ such that for each $S \in \mathcal{A}$, $C(S)$ is obtained by collecting the maximizers of the priority orderings in S, that is,

$$C(S) = \bigcup_{i \in \{1, \ldots, m\}} \max(S, \succ_i).$$

Following theorem is the main theorem which we aim to explore further.

Theorem. *(Aizerman and Malishevski, 1981) A choice rule C satisfies path independence if and only if it has a* MC *representation, that is there exists $\pi \in \Pi$ such that for each $S \in \mathcal{A}$, $C(S) = MC(\pi)(S)$.*

Let us consider the following example as a demonstration of above theorem.

Example 2. Let $A = \{1, 2, 3, 4, 5\}$ and *path independent* $C : 2^A \rightarrow 2^A$ be defined as

$$C(\{1,2,3,4,5\}) = \{1,2\},\ C(\{1,3,4,5\}) = \{1\},\ C(\{2,3,4,5\}) = \{2,3\},$$

$$C(\{3,4,5\}) = \{3,4\},\ C(\{2,4,5\}) = \{2,5\},\ C(\{2,4\}) = \{2\},\ C(\{3,5\}) = \{3\},$$

$$C(\{4,5\}) = \{4,5\},\ C(\{4\}) = \{4\},\ C(\{5\}) = \{5\}, C(\emptyset) = \emptyset.$$

Clearly $\{1, 2, 3, 4, 5\}$ is a maximal set. By Remark 1, we know $C(T) = \{1, 2\}$ if $\{1, 2\} \subset T \subset \{1, 2, 3, 4, 5\}$. There are 8 such T sets so this covers 8 possible sets in the domain of C. Similarly from the information in the first row we can deduce the values of C extended to 8+8+4 values. The values in the second row can be extended to 2+2+2+2 values. The last ones cannot be extended anymore because there are no rejected elements in these therefore only 4 values have been determined in the last row. In total these make 32 values where 32 is the number of subsets of A. One can verify these sets are also non overlapping so given values above define a unique well-defined path-independent choice rule. Now lets consider $\pi = \{\succ_1, \succ_2\}$ as defined in 2.1

Table 2.1: MC-represenation for Example 2

$\succ_1$	$\succ_2$
1	2
3	1
5	4
2	3
4	5

We claim that for any $S \subset \{1, 2, 3, 4, 5\}$, we get $C(S) = MC(\pi)(S)$. For instance, $\max(\{2, 3, 4, 5\}, \succ_1) = \{3\}$ and $\max(\{2, 3, 4, 5\}, \succ_2) = \{2\}$ so $MC(\pi)(\{2, 3, 4, 5\}) = \{2, 3\}$. Similarly $MC(\pi)(\{1, 3, 4, 5\}) = \{1\}$ and $MC(\pi)(\{3, 4, 5\}) = \{3, 4\}$. One can verify $MC(\pi)$ and C coincide at every value in $\mathcal{A}$.

CHAPTER 3

PATH INDEPENDENT CHOICE RULES

In the previous chapter, with Example 2 we have demonstrated an application of the Aizerman-Malishevksi theorem. Aizerman and Malishevski (1981) state the theorem but lack a proof. Moulin (1985) gives a proof for the theorem but in the literature there does not exist an algorithm or theorem which provide an optimal or minimal representations for *path independent* choice rules in general. In this section, we first provide a full characterization for all representations of a given *path independent* choice rule. Then, we find the size of minimal MC-representation by using our characterization result. In order to do this we need to make some auxiliary definitions.

3.1 Characterization of MC-representations

Definition 1. A choice set $S \in \mathcal{A}$ is a **maximal** if for all $T \supsetneq S$, $C(S) \neq C(T)$. Moreover let $\mathcal{M}(C)$ denote the collection of *maximal* sets of C. i.e. $\mathcal{M}(C) = \{S \subset A : \nexists T \supsetneq S, C(S) = C(T)\}$.

Here by Remark 1 we can deduce for every set in the image of C there exists a unique maximal, thus $|\mathcal{M}(C)|$ is equal to the cardinality of image of C or $|C(\mathcal{A})|$. Naming this notion as maximal is consistent with literature but we can provide an equivalent definition which might be more useful later on.

Lemma *M1*. *S is maximal if and only if for any $a \notin S$, $a \in C(S \cup \{a\})$.*

Proof. We can prove both implications by contrapositive. Suppose S is not maximal, then there exist $T \supsetneq S$ with $C(S) = C(T)$. So $C(S) \cap (T \setminus S) = \emptyset$, let $a \in T \setminus S$, by *IRA* $C(S \cup \{a\}) = C(S)$ and $a \notin C(S \cup \{a\})$ so there exists $a \notin S$ with $a \notin C(S \cup \{a\})$. For the reverse implication if there exists $a \notin S$ such that $a \notin C(S \cup \{a\})$ then by *IRA* $C(S \cup \{a\}) = C(S)$ thus, S is not maximal. □

We will use this lemma instead of maximality definition in some of the further proofs. Next lemma is quite useful and crucial to determine all of maximals via utilizing maximality of universal set A.

Lemma *M2*. *Let S be a maximal. $S \setminus \{a\}$ is maximal if and only if $a \in C(S)$.*

Proof. If $S \in \mathcal{M}(C)$ and $a \notin C(S)$ then by *IRA*, $C(S \setminus \{a\}) = C(S)$ thus $S \setminus \{a\}$ is not maximal by definition. Now suppose $a \in C(S)$ and let $T = S \setminus \{a\}$. Suppose T is not maximal; then by Lemma *M1*, there exists $b \notin T$ such that $C(T) = C(T \cup \{b\})$, apparently $a \neq b$ since $a \in C(S)$. Consider $C(T \cup \{a, b\})$. If b is chosen here then this would be contradicted by $b \notin C(T \cup \{b\})$ since C is *substitutable*, so $b \notin C(T \cup \{a, b\})$ and by *IRA*, $C(T \cup \{a, b\}) = C(T \cup \{a\})$. But now this is contradicted by $T \cup \{a\} = S$ being maximal. Therefore T must be maximal as well. □

We define the following binary relation on $\mathcal{M}(C)$ by utilizing Lemma *M2*. For each $S, S' \in \mathcal{M}(C)$, S is a *parent* of S', denoted by $S \to S'$, if there exists $a \in S$ such that $S' = S \setminus \{a\}$. Lemma *M2* tells us for any $S \in \mathcal{M}(C)$ and for any $a \in C(S)$, we have $S \to S \setminus \{a\}$.

Lemma *M3*. *If $T \subsetneq S$ and S, T are maximal with $|S \setminus T| = k$, then there exists a sequence of maximals $S_0 = T, S_1, S_2, \cdots, S_k = S$ such that for each $i = 1, 2, \cdots, k$, we have $S_{i-1} \subset S_i$ and $|S_i \setminus S_{i-1}| = 1$.*

Proof. We claim $C(S) \cap (S \setminus T) \neq \emptyset$. If it is empty then by *IRA* we get $C(S) = C(T)$ by removing alternatives in $S \setminus T$ and this would contradict maximality of T. Let $a \in C(S) \cap (S \setminus T)$. Now define $S_{k-1} = S \setminus \{a\}$. By Lemma *M2*, S_{k-1} is maximal and by choice of a clearly $T \subset S_{k-1}$. By iterating similarly we can reach T after k steps and construct a sequence described as in lemma. □

By Lemma *M3* we can consider transitive closure of *parent* relation on $\mathcal{M}(C)$. For each $S, S' \in \mathcal{M}(C)$, S is an *ancestor* of S', denoted by $S \searrow^C S'$, if there exists a collection of sets in $S_1, \ldots, S_k \in \mathcal{M}(C)$ such that $S \to S_1 \to \cdots \to S_k \to S'$. Lemma *M3* tells us $S \searrow^C S'$ if and only if $S, S' \in \mathcal{M}(C)$ and $S' \subset S$. Since the binary relation $\searrow^C$ is transitive, $(\mathcal{M}(C), \searrow^C)$ is a partially ordered set. Moreover universal set A is the unique maximal element of this partially ordered set while $\emptyset$ is the unique minimal element.

Since for any $S \in \mathcal{M}(C)$ we have $S \subset A$, by Lemma *M3*, we can construct a sequence of maximals from A to S and Lemma *M2* tells us how this can be done. This means all maximals can be found via subtracting a sequence of elements from A which are defined as in Lemma *M2*.

Maximal sets are quite important in our work but for an exact characterization result we need to consider another notion that we call *prime* which will turn out to be a refinement of maximals.

Definition 2. A choice set $S \in \mathcal{A}$ is a **prime** if there exists an element $a \notin S$ such that $a \in C(S \cup \{a\})$ and there is no $T \supsetneq S \cup \{a\}$ with $a \in C(T)$. Moreover, let $\mathcal{P}(C)$ denote the collection of all **prime** sets of C.

Here, observe that if C is *path independent* then every prime set is also maximal.

Lemma *P1*. *Every prime is maximal.*

Proof. Let S be a prime set then there exists $a \notin S$ as given in the definition. Since $a \in C(S \cup \{a\})$ and for all $T \supsetneq S \cup \{a\}$ we have $a \notin C(T)$, we can deduce $C(T) \neq C(S \cup \{a\})$, so $S \cup \{a\}$ is maximal by definition. Now, since $a \in C(S \cup \{a\})$, by Lemma *M2* we get $S \in \mathcal{M}(C)$ as well. This proves every prime must also be maximal, that is $\mathcal{P}(C) \subset \mathcal{M}(C)$. □

Above lemma shows primes are maximals but primes have one more definitive property among maximals. Following lemma specifies that.

Lemma *P2*. *A set is prime if and only if it is a maximal with unique parent. Formally,* $S \in \mathcal{P}(C)$ *if and only if* $S \in \mathcal{M}(C)$ *and there exists unique* $a \notin S$ *such that* $S \cup \{a\} \in \mathcal{M}(C)$.

Proof. Suppose S is a prime. By Lemma *P1* S is maximal. We also got by definition there exists $a \notin S$ such that for any $T \supsetneq S \cup \{a\}$, $a \notin C(T)$ so by definition $S \cup \{a\}$ is maximal as well. Suppose there exists $c \notin S \cup \{a\}$ such that $S \cup \{c\} \in \mathcal{M}(C)$. So $S \cup \{a\}$ and $S \cup \{c\}$ are maximal. Consider $C(S \cup \{a, c\})$. Since a does not belong to this set, by *IRA* , $C(S \cup \{a, c\}) = C(S \cup \{c\}))$ is contradicted by $S \cup \{c\} \in \mathcal{M}(C)$, therefore, there is no such c.

Now we will prove that if S has a unique *parent* then it should be a prime. Let $S \cup \{a\}$ be that *parent*. Suppose S is not a prime; then there exists $T \supsetneq S \cup \{a\}$ such that $a \in C(T)$. Let $\overline{T}$ be defined as in Remark 1, that is $\overline{T} = \bigcup\limits_{C(T')=C(T)} T'$. We know $\overline{T} \in \mathcal{M}(C)$ and $a \in C(T) = C(\overline{T})$. By Lemma *M2* we get $T \setminus \{a\} \in \mathcal{M}(C)$ and since $S \subset (T \setminus \{a\})$ by Lemma *M3* there exists a sequence of maximal sets such that $S_0 = S, S_1, S_2, \cdots, S_k = T \setminus \{a\}$. Clearly none of these sets includes a so $S_1 \neq S \cup \{a\}$. This implies S_1 and $S \cup \{a\}$ are two different *parents* of S which contradicts our initial assumption. Therefore S must be a prime. $\square$

This lemma gives us equivalent definition for primes which can be easier to verify on partially ordered set $(\mathcal{M}(C), \searrow^C)$. A choice set $S \in \mathcal{M}(C)$ is a *prime* of C if S has a unique *parent*, that is, there exists a unique $S' \in \mathcal{M}(C)$ such that $S' \rightarrow S$.

Now, we need to introduce a couple more definitions related to MC-representations, which will turn out to be closely related to maximal and prime sets in our characterization theorem.

For any $a \in A$ and $\succ$ priority ordering on A, let us define

$$L(a, \succ) = \{b \in A : a \succ b\} \text{ and } L(\succ) = \{L(a, \succ) : a \in A\}.$$

And for any set of priority orderings π, let $L(\pi) = \bigcup\limits_{\succ \in \pi} L(\succ)$. Now Let us reconsider the choice rule given in the Example 2 and evaluate maximals and primes for it.

Example 3. Let $A = \{1, 2, 3, 4, 5\}$ and *path independent* $C : \mathcal{A} \to \mathcal{A}$ be defined as in Example 2. $\{1, 2, 3, 4, 5\}$ is trivially a maximal set. By Lemma *M2* we know that if S is maximal and $a \in C(S)$ then $S \setminus \{a\}$ is also maximal. We can verify that all given sets in Example 2 are maximals. For simplicity let us remove commas in set definitions and denote choice sets by strings of integers. So we have,

$$\mathcal{M}(C) = \{12345, 1345, 2345, 345, 245, 24, 35, 45, 4, 5, \emptyset\}.$$

Considering *parent* and *ancestor* relations, we get the partially ordered set in the figure. Here we have denoted the chosen elements as bold and underlined.

Notice that, Figure 3.1 contains complete information of C in one lattice. Those 11 points of $\mathcal{M}(C)$ have their values given and this is enough for us to determine outcome of any choice set because these are all sets in the image of C. We could deduce the chosen elements even if they are not denoted. By Lemma *P2* we know a maximal set is prime if it has a unique *parent* so finding $\mathcal{P}(C)$ will not be much

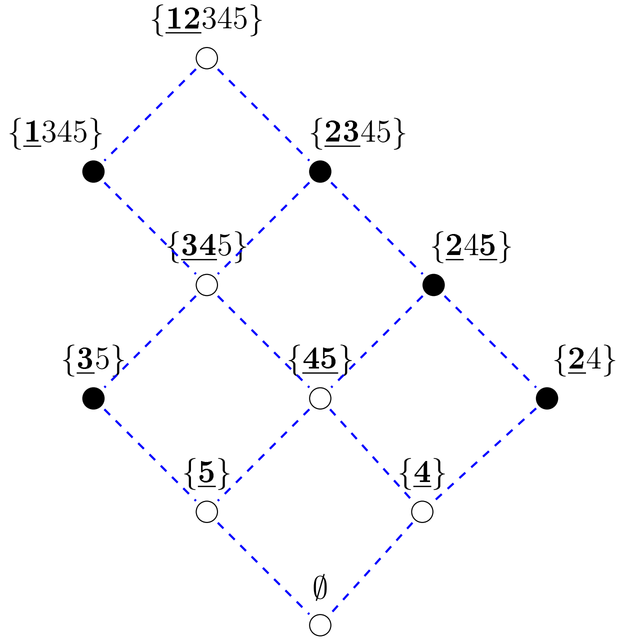


Figure 3.1: Lattice representation of the choice rule in Example 3

of a challenge. We get,

$$\mathcal{P}(C) = \{1345, 2345, 245, 24, 35\}.$$

In Figure 3.1, primes are denoted by black nodes, while white nodes are non-prime maximals.

As an additional demonstration of (Aizerman and Malishevski, 1981) Theorem, we can represent C via different sets of preferences without any redundancies as shown in Table 3.1.

Let $\pi_1 = \{\succ_1, \succ_2\}$, $\pi_2 = \{\succ_3, \succ_4, \succ_5\}$. One can verify that $MC(\pi_1)(S) = MC(\pi_2)(S) = C(S)$ for any $S \in \mathcal{A}$. Moreover, we can also show that for both π_1 and π_2, we cannot remove any of the priorities and still represent C. All of these priorities are necessary for some choice sets so these representations are minimal

Table 3.1: MC-representation for Example 3

π_1		π_2		
$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$
1	2	1	2	1
3	1	3	1	2
5	4	5	3	4
2	3	2	4	3
4	5	4	5	5

in some sense. We can also find $L(\pi_1)$ and $L(\pi_2)$ as a demonstration such that,

$L(\succ_1) = \{2345, 245, 24, 4, \emptyset\}$, $L(\succ_2) = \{1345, 345, 35, 5, \emptyset\}$ and

$L(\succ_3) = \{2345, 245, 24, 4, \emptyset\}$, $L(\succ_4) = \{1345, 345, 45, 5, \emptyset\}$,

$L(\succ_5) = \{2345, 345, 35, 5, \emptyset\}$. It follows that,

$L(\pi_1) = \{2345, 1345, 245, 345, 24, 35, 4, 5, \emptyset\}$ and

$L(\pi_2) = \{2345, 1345, 245, 345, 24, 45, 35, 4, 5, \emptyset\}$.

Theorem 1. *Let C be a path independent choice rule and π be a set of priority orderings. π produces an MC representation of C if and only if $\mathcal{P}(C) \subset L(\pi) \subset \mathcal{M}(C)$.*

Proof. (*Only if part*) Suppose $MC(\pi) = C$. First we prove $\mathcal{P}(C) \subset L(\pi)$. If $S \in \mathcal{P}(C)$, by definition $\exists a \notin S$ such that $\forall T \supsetneq S \cup \{a\}$, $a \notin C(T)$. Since $a \in C(S \cup \{a\})$ there exists some $\succ \in \pi$ such that $a = \max(S \cup \{a\}, \succ)$ and since $\forall T \supsetneq S \cup \{a\}$, $a \neq \max(T, \succ)$, we get $L(a, \succ) = S$. Therefore, $S \in L(\succ) \subset L(\pi)$ which means $\mathcal{P}(C) \subset L(\pi)$.

Now let $S \in L(\pi)$ then, there exists $\succ \in \pi$ with $S \in L(\succ)$. Clearly $\forall a \notin S$, we get $a = \max(S \cup \{a\}, \succ)$; so $a \in c(S \cup \{a\})$. By Lemma *M1* we get $S \in \mathcal{M}(C)$. Therefore we get $L(\pi) \subset \mathcal{M}(C)$. These two results conclude the first part of the proof.

(*If part*) Suppose $\mathcal{P}(C) \subset L(\pi) \subset \mathcal{M}(C)$. First we prove $MC(\pi) \subset C$. For any $S \in \mathcal{A}$, let $a \in MC(\pi)(S)$. So there exists some $\succ \in \pi$ with $a = \max(S, \succ)$. We want to deduce $a \in C(S)$. Let $S^a = L(a, \succ) \cup \{a\}$, clearly $S \subset S^a$. By assumption $L(a, \succ)$ and S^a are maximal. Since $S^a \setminus \{a\} = L(a, \succ)$, by Lemma *M2* we can deduce $a \in C(S^a)$; then by *substitutability* we get $a \in C(S)$. Therefore for any $S \in \mathcal{A}$, $MC(\pi) \subset C$.

Now we prove $C \subset MC(\pi)$. For any $S \in \mathcal{A}$ and $a \in C(S)$, let T be a maximal set which satisfies $S \subset T$, $a \in C(T)$ and for any $T' \supsetneq T$, $a \notin C(T')$. By definition $T \setminus \{a\}$ is prime, since $\mathcal{P}(C) \subset L(\pi)$, for some $\succ \in \pi$ we have $T \setminus \{a\} \in L(\succ)$. Now by Lemma *P2*, T is the unique maximal set that can be found via adding a single element to $T \setminus \{a\}$. $T \setminus \{a\}$ appears as a lower contour set of $\succ$. Since lower contour set just above this should be maximal there is only one possible choice and this means $T \in L(\succ)$ and $L(a, \succ) = T \setminus \{a\}$. Since $a \in S \subset T$, we have $a = \max(S, \succ)$ and $a \in MC(\pi)(S)$. Therefore we get $C \subset MC(\pi)$ for all $S \in \mathcal{A}$. That concludes the proof of the final part. □

3.2 Minimal MC-representations

The above theorem gives us an exact characterization for all MC representations of a given *path independent* C. We also want to determine the size of the minimal such representation. In order to reach such a result, we will utilize Dilworth's theorem (Dilworth, 1950) on minimal chain decompositions. Introduction of chains is quite natural because the structures of $L(\pi)$ and $\mathcal{M}(C)$ are unions of chains and $L(\succ)$ is exactly a chain.

In our context, a *chain* is a collection of sets where for any two sets in our chain there should be one containing other, that is if S, T are in chain ω then either $S \subset T$ or $T \subset S$. Let Ω denote the collection of all chains of $\mathcal{A}$. Note that for any $\succ \in \Pi$, $L(\succ) \subset \mathcal{A}$ is also a chain so $L(\succ) \in \Omega$. We can easily observe if ω is a chain then $\omega \cup \{A\}$ and $\omega \setminus \{A\}$ are also chains. Also we know any subset of a chain is also a chain.

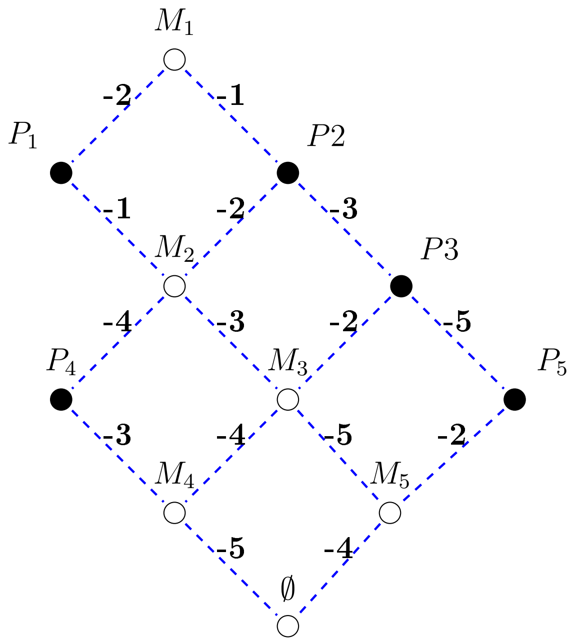
A chain $\omega \in \Omega$ is called *full* if there is no $\omega' \in \Omega$ with $\omega \subsetneq \omega'$, that induces $A, \emptyset \in \omega$ and $|\omega| = |A| + 1$. Notice that ω is a full chain if and only if there exists $\succ \in \Pi$ such that $\omega = L(\succ) \cup \{A\}$.

There are some additional nice properties satisfied by $\mathcal{M}(C)$ related to chains that we have not mentioned yet. We will present these in the following remark.

Remark 2. Let C be a *path independent* choice rule and $\omega_0 = \{S_1, S_2, \cdots S_k\} \subset \mathcal{M}(C)$ be a chain of maximals with $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k$. Then there exists a full chain ω such that $\omega_0 \subset \omega \subset \mathcal{M}(C)$. This also means there exists $\succ \in \Pi$ with $\omega_0 \subset L(\succ) \cup \{A\} \subset \mathcal{M}(C)$. In order to show that we will utilize Lemma *M3*. Since $A, \emptyset \in \mathcal{M}(C)$ and $S_0 = \emptyset \subset S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subset S_{k+1} = A$, by Lemma *M3* if $|S_{i+1} \setminus S_i| > 1$ we can find appropriate maximals to fill between S_i and S_{i+1} and by this way we can expand our initial chain to a full chain. So for any $S \in \mathcal{M}(C)$ there is a full chain in $\mathcal{M}(C)$ that is descending from A to S and then from S to $\emptyset$. That means $\mathcal{M}(C)$ is actually a union of chains.

Let $\Pi_C = \{\succ \in \Pi : L(\succ) \subset \mathcal{M}(C)\}$. So we have $L(\Pi_C) = \mathcal{M}(C)$ and $\mathcal{P}(C) \subset L(\Pi_C) \subset \mathcal{M}(C)$ is satisfied, therefore $MC(\Pi_C) = C$. But Π_C is the list of all possible priority orderings we will be able to use for the representation of C. Anything outside Π_C would violate $L(\pi) \subset \mathcal{M}(C)$ condition and would fail to be utilized to represent C. Therefore that is the most wasteful representation possible. What we need to accomplish is to cover all elements in $\mathcal{P}(C)$ by using priority orderings in Π_C efficiently. Now, let us try to find all possible representations for our Example 2.

Example 4. Let C be the choice rule defined in Example 2. We know the partially ordered set of $\mathcal{M}(C)$ from Example 3.



$M_1 = \{\underline{\mathbf{12}}345\}$, $M_2 = \{\underline{\mathbf{34}}5\}$, $M_3 = \{\underline{\mathbf{45}}\}$, $M_4 = \{\underline{\mathbf{5}}\}$, $M_5 = \{\underline{\mathbf{4}}\}$
$P_1 = \{\underline{\mathbf{1}}345\}$, $P_2 = \{\underline{\mathbf{23}}45\}$, $P_3 = \{\underline{\mathbf{2}}4\underline{\mathbf{5}}\}$, $P_4 = \{\underline{\mathbf{3}}5\}$, $P_5 = \{\underline{\mathbf{2}}4\}$

Figure 3.2: Lattice representation of the choice rule in Example 4

Let's call maximal sets defined as in Figure 3.2. Notice that the collection of primes is $\mathcal{P}(C) = \{P_1, P_2, P_3, P_4, P_5\}$. Also denote the subtracted elements on figure as well. That is if $S \to S \setminus \{a\}$ then we write $-a$ on the edge that connects S to $S \setminus \{a\}$. In this example we can write all possible full chains that is descending to $\emptyset$ from A and list corresponding priority orderings. That is $\Pi_C = \{\succ \in \Pi : L(\succ) \subset \mathcal{M}(C)\}$. Notice that this corresponds to the list of all possible routes that descends to $\emptyset$ from A.

Table 3.2: List of all feasible priority orderings(Π_C) and covered primes by these.

Π_C								
$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_8$	$\succ_9$
2	2	2	1	1	1	1	1	1
1	1	1	2	2	2	3	3	3
4	3	3	4	3	3	2	2	5
3	4	5	3	4	5	4	5	2
5	5	4	5	5	4	5	4	4

$P_1, P_4 \in L(\succ_1)$, $P_1 \in L(\succ_2)$
$P_1 \in L(\succ_3)$, $P_2, P_4 \in L(\succ_4)$
$P_2 \in L(\succ_5)$, $P_2 \in L(\succ_6)$
$P_2, P_3 \in L(\succ_7)$, $P_2, P_3 \in L(\succ_8)$
$P_2, P_3, P_5 \in L(\succ_9)$

As it seems in the Table 3.2, $|\Pi_C| = 9$. Any MC-representation of C must be a subset of Π_C. Now let us figure out these representations. Since for any $\pi \subset \Pi_C$, $L(\pi) \subset \mathcal{M}(C)$ is satisfied. All we need to ensure is $\mathcal{P}(C) \subset L(\pi)$ to hold.

There is only one chain that is passing through P_5, that is $P_5 \in L(\succ_i)$ if and only if $i = 9$, so in any representation $\succ_9$ must be used. But $\succ_9$ also passing through P_2 and P_3, that is $P_2, P_3 \in L(\succ_9)$. So $\succ_9$ covers 3 sets in $\mathcal{P}(C)$ and we are left with only P_1 and P_4. $P_4 \in L(\succ_1)$ and $P_4 \in L(\succ_4)$ implying that we need to use either $\succ_1$ or $\succ_4$. Note that $P_1 \in L(\succ_1)$; so $\{\succ_1, \succ_9\}$ provides a representation. That is the representation we have given in Example 2. If we do not use $\succ_1$ then we should use $\succ_4$ to cover P_4. Now we are only left with P_1 and $\succ_1, \succ_2, \succ_3$ are passing through P_1. But we will not use $\succ_1$ so $\{\succ_2, \succ_4, \succ_9\}$ or $\{\succ_3, \succ_4, \succ_9\}$ are two other possible representations. Former is one of the representations we have provided in Example 3. Finally we have found all representations of C without any redundant priorities. These are

$$\{\succ_1, \succ_9\},\ \{\succ_2, \succ_4, \succ_9\} \text{ and } \{\succ_3, \succ_4, \succ_9\}.$$

That means $MC(\pi) = C$ if and only if $[\{\succ_1, \succ_9\} \subset \pi$ or $\{\succ_2, \succ_4, \succ_9\} \subset \pi$ or $\{\succ_3, \succ_4, \succ_9\} \subset \pi]$ and $\pi \subset \Pi_C$. All three of these representations are minimal in set inclusion sense but we want to find cardinally minimal representation; so in this case our unique minimal representation is $\{\succ_1, \succ_9\}$. Therefore we can say C has a 2-maximizer collecting representation.

In order to find the cardinally minimal representation we will utilize Dilworth's theorem. So we need to know about anti-chains. A collection of sets is called *anti-chain* if no set includes another. For any finite collection, Dilworth's theorem gives the equality of the size of the minimal chain decomposition and size of the largest anti-chain.

Theorem. *(Dilworth, 1950) Let $\mathcal{S}$ be a collection sets. The size of the maximal anti-chain in $\mathcal{S}$ equals the size of a minimal chain decomposition of $\mathcal{S}$.*

Following result depends heavily on this theorem as well as some results we have mentioned above.

Theorem 2. *The minimum number of preferences to represent a path independent choice rule as a collected maximizer is the size of the largest anti-chain of prime sets.*

Proof. Recall that $L(\succ)$ is always a chain. So $L(\succ) \cap \mathcal{P}(C)$ is also a chain. Notice $\{L(\succ) \cap \mathcal{P}(C) : \succ \in \pi\}$ is collection of chains, and their union will give us $\mathcal{P}(C)$ since $\mathcal{P}(C) \subset L(\pi)$. So, each representation will correspond to a chain decomposition of $\mathcal{P}(C)$ with size $|\pi|$.

Consider any chain $\omega \subset \mathcal{P}(C)$, let $\omega = \{S_1, S_2, \cdots, S_k\}$ with $S_1 \subsetneq \cdots \subsetneq S_k$. Since $\mathcal{P}(C) \subset \mathcal{M}(C)$ all primes are also maximal, by Remark 2, there exists $\succ \in \Pi_C$ such that $\omega \subset L(\succ) \cup \{A\} \subset \mathcal{M}(C)$. Thus for any chain decomposition of $\mathcal{P}(C)$, we can find a priority ordering for each chain in this decomposition. Let π be the set of priority orderings we have found here.

Clearly $\mathcal{P}(C) \subset L(\pi)$ since each prime set belongs to at least one chain in chain decomposition and each chain is a subset of some $L(\succ)$ with $\succ \in \pi$. Moreover $L(\pi) \subset \mathcal{M}(C)$ since for every $\succ \in \pi$ we have $L(\pi) \subset \mathcal{M}(C)$. Therefore by Theorem 1 we deduce $MC(\pi) = C$.

Now that we have established the relationship between a chain decomposition of $\mathcal{P}(C)$ and MC-representation of C, we can utilize Dilworth's theorem.

By Dilworth's theorem the minimal size of this chain decomposition is the size of the maximal anti-chain therefore this is the minimal number of priority orderings to represent a *path independent* choice rule.

□

CHAPTER 4

ACCEPTANT CHOICE RULES

Although it follows from Aizerman and Malishevski (1981) that a choice rule C is *path independent* if and only if it is MC, they remain silent about the minimal size of the MC representation and construction of the priority profile. A study which I have coauthored, currently revised and resubmitted, Doğan et al. (2020), answers this question for q-*acceptant* and *path independent* choice rules by providing a construction for canonical representation.

The choice rule defined in the following example is called q-**responsive** choice rule. It is the most well known q-*acceptant* and *path independent* choice rule and widely used in school choice models.

Example 5. Let $A = \{1, 2, \cdots, n\}$ be the universal set, $1 \leq q \leq n$ be the capacity and $\succ$ be defined as $1 \succ 2 \succ \cdots \succ n$. For any $S \subset A$ we choose best available alternatives with respect to $\succ$ until we fill our capacity q, formally, $C(S) = \{a \in \ S : |\{b \in S : b \succ a\}| < q\}$.

Let $S \in \mathcal{M}(C)$ with $S = \{a_1, a_2, \cdots, a_k\}$ and $a_1 \succ a_2 \succ \cdots \succ a_k$. If $k < q$ then S is trivially maximal by *q-acceptance* and Lemma *M1*. If $k \geq q$ then we need all elements below a_q in S that is $L(a_q, \succ) \subset S$. So S is maximal if and only if $a \in S$ and $|\{b \in S : b \succ a\}| = q - 1$ implies $L(a, \succ) \subset S$.

4.1 Minimal Size Representation For q-acceptant Rules

In order to reach the minimal size of the MC representation of a given *acceptant* choice rule we introduce the concept of a prime atom of a choice rule, which will be the key in finding the minimal number of priorities needed for an MC representation. Given a *q-acceptant* and *substitutable* choice rule C, a choice set is a prime atom of C if the number of elements in the choice set is equal to the one less than capacity and it is a prime. The formal definition is as follows.

Definition 3. A choice set $S \in \mathcal{A}$ is a **prime atom** if $|S| = q - 1$ and there exists an element $a \notin S$ such that $a \notin C(S \cup \{a, b\})$ for each $b \notin S \cup \{a\}$. Also let $\mathcal{P}_A(C)$ denote the collection of all prime atoms.

Notice that $q - 1$ is the minimum possible cardinality for a prime. In order to see that suppose $|S| \leq q - 2$ then by *q-acceptance* for any $a, b \notin S$ we get $a, b \in C(S \cup \{a\})$. That means we cannot find an appropriate a as in the definition of primes. Therefore prime atoms are just primes with minimal possible size which is $q - 1$.

Also recall that since C is *q-acceptant* we have $C(S \cup \{a\}) = S \cup \{a\}$ and so $a \in C(S \cup \{a\})$ and there is exactly one unchosen element in $S \cup \{a, b\}$. Following result shows that for any *q-acceptant* and *substitutable* choice rule, the number of its prime atoms determines the smallest size MC representation of the choice rule.

Theorem 3. *(Doğan et al. (2020)) For each q-acceptant and substitutable choice rule C,*

i. C has an MC representation of a size equal to the number of its prime atoms.

ii. C does not have an MC representation of any size smaller than the number of its prime atoms.

In order to prove this theorem we will use Theorem 2 but we also need the following lemmas as well.

Lemma *P3*. *Let $S \subsetneq A$ be a maximal set, then S can have at most one prime child.*

Proof. Suppose S has two different prime children, that is, there exists $a, a' \in S$ such that $S \setminus \{a\}$ and $S \setminus \{a'\}$ are both prime. Here we know $a, a' \in C(S)$. Since $S \neq A$, there exists $x \notin S$. Consider $C(S \cup \{x\})$. Since $S \setminus \{a\}$ is prime, by definition $a \notin C(S \cup \{x\})$. Similarly $a' \notin C(S \cup \{x\})$. So $a, a' \in (C(S) \setminus C(S \cup \{x\}))$. Since there are at least two alternatives rejected in $S \cup \{x\}$ we get $|S| \geq q+1$. This combined with *q-acceptance* we get $|C(S \cup \{x\}) \setminus C(S)| \geq 2$; so there exists

$y \neq x$ with $y \in C(S \cup \{x\}) \setminus C(S)$ but this contradicts *substitutability* therefore S can have at most one prime child. $\square$

Notice $S \neq A$ is necessary for this lemma because all children of A are trivially prime.

Lemma *P4*. *Let* $|S| \geq q$ *and* S *be a prime. Then there exists a unique* $a \in S$ *such that* $S \setminus \{a\}$ *is also prime.*

Proof. Since S is a prime, by definition there exists $c \notin S$ such that $c \in C(S \cup \{c\})$ and for any $b \notin S \cup \{c\}$, $c \notin C(S \cup \{c, b\})$. By *substitutability* and *q-acceptance* we can say $|C(S \cup \{c\}) \cap C(S)| = q - 1$. That is to say there are $q - 1$ alternatives that are chosen from both $S \cup \{c\}$ and S. We know $c \in C(S \cup \{c\})$ and $|S| \geq q$; so there should exist exactly one element that is chosen from S while available but not chosen from $S \cup \{c\}$. Let this be a, that means $C(S) \setminus C(S \cup \{c\}) = \{a\}$. We claim $S \setminus \{a\}$ is prime and it is the only prime child of S.

Suppose $S \setminus \{a\}$ is not a prime. Then there should exist $b \notin S$ such that $a \in C(S \cup \{b\})$. Since $a \notin C(S \cup \{c\})$ we get $b \neq c$. Now consider $C(S \cup \{b, c\})$. We know $c \notin C(S \cup \{b, c\})$, so by *IRA* we get $C(S \cup \{b, c\}) = C(S \cup \{b\})$ and this yields $a \in C(S \cup \{b, c\})$. By *substitutability* we get $a \in C(S \cup \{c\})$ which is a contradiction since $a \in C(S) \setminus C(S \cup \{c\})$. That means $S \setminus \{a\}$ should be a prime. Since S is prime and A is not prime we get $S \neq A$. By Lemma *P3* we know S can have at most one prime child; therefore S has a unique prime child. $\square$

This lemma requires *q-acceptance* to hold. For instance in Example 3 we can see

$\{1,3,4,5\}$ is prime but it has no prime child. Lemma *P4* guarantees that from each prime we can descend to a unique prime atom through a path of primes by removing one element at each step. Let us consider the following example and observe Lemma *P4* on the partially ordered set of $\mathcal{M}(C)$ which has been depicted in Figure 4.1.

Example 6. Let $A = \{1,2,3,4,5,6\}$ and consider the priority ordering set $\{\succ_\alpha, \succ_\beta, \succ_\gamma, \succ_\delta\}$. Let C be the 2-*acceptant* choice rule that is MC of this priority profile.

Table 4.1: MC-representation for Example 6

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
1	1	2	2
3	2	3	3
4	4	4	1
5	5	5	6
6	6	6	5
2	3	1	4

The choice lattice $(\mathcal{M}(C), \searrow)$ associated with C is depicted in Figure 4.1.[1] Since prime atoms are basically primes with cardinality $q-1$, we can say C has 4 prime atoms, namely $\{1\},\{4\},\{3\},\{2\}$, which gives the lower bound on the number of preferences for a representation of C .

Proof of Theorem 3. Let S be a prime atom. By Lemma *P2*, S has a unique maximal parent. Let us call that S_1. If S_1 is prime then similarly we get S_2. We go on similarly up until we reach a non-prime set. Since A is not prime we should hit a non-prime maximal set at some point. Let S_k be the last prime we

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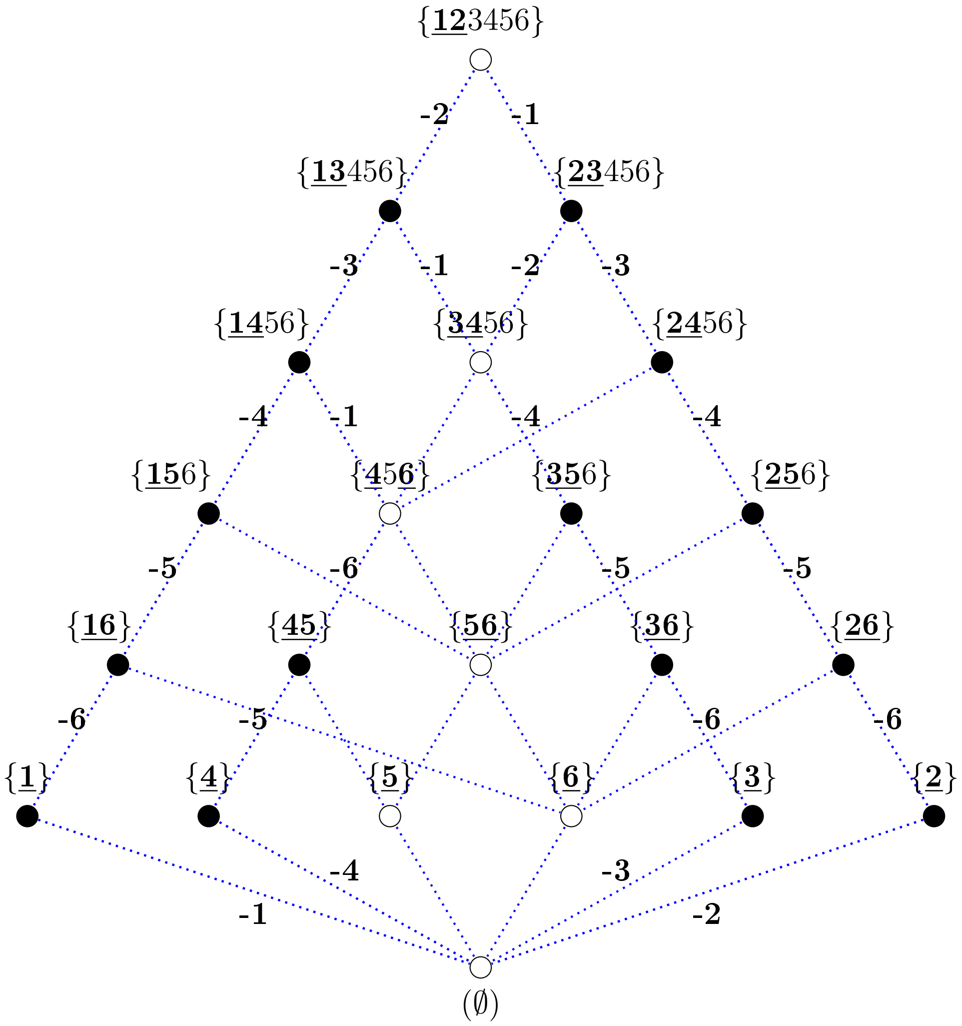
[1] Prime choice sets are colored as black.



$$\mathcal{M}(C) = \{123456, 13456, 23456, 1456, 3456, 2456,$$
$$156, 456, 356, 256, 16, 45, 56, 36, 26, 1, 4, 5, 6, 2, 3, \emptyset\}$$
$$\mathcal{P}(C) = \{13456, 23456, 1456, 2456, 156, 356, 256, 16, 45, 36, 26, 1, 4, 3, 2\}$$

Figure 4.1: Lattice representation of the choice rule in Example 6

can reach while ascending from S. Now let us define $\omega_S = \{S, S_1, \cdots, S_k\}$ chain of primes.

We claim $\{\omega_S : S \in \mathcal{P}_A(C)\}$ is a chain decomposition of $\mathcal{P}(C)$. Let $T \in \mathcal{P}(C)$, If $|T| \geq q$, by Lemma *P4* we know there exists $a \in T$ such that $T \setminus \{a\} \in \mathcal{P}(C)$. So we can go on similarly and reach some $S' \in \mathcal{P}(C)$ with $|S'| = q - 1$. So S' is a prime atom and $T \in \omega_{S'}$. Therefore $\{\omega_S : S \in \mathcal{P}(C), |S| = q - 1\}$ is a chain decomposition of primes where each chain has exactly one prime atom. In the proof of Theorem 2 we have shown that every chain in a chain decomposition of primes produces a priority ordering and the set of these priority orderings provides an MC representation of C; so this proves first part of the theorem.

Now notice that the collection of prime atoms is an anti-chain because between two different sets with the same cardinality one cannot include the other. Actually this is the largest anti-chain of primes but we do not need to prove it here. Dilworth's theorem ensures that minimal chain decomposition must have at least as much chains as the number of prime atoms. So this completes the proof of the second part of the theorem. □

4.2 An Upper Bound on the Size of Minimal Representations

As a follow up to Theorem 3, we can find an upper bound for the number of prime atoms among all *q-acceptance* choice rules. In order to calculate that we need the following lemma on the partially ordered set $\mathcal{M}(C)$ of a *q-acceptance* and *path independent* choice rule.

Lemma 1. *Given a capacity q and a universal set of n alternatives, let C be a q-acceptant and substitutable choice rule. For each $k \in \{q-1, q, \ldots, n\}$, the number of maximal choice sets with cardinality k is $\binom{n-k+q-1}{q-1}$.*

Proof. Let $\mathcal{M}_k$ denote the maximals sets with cardinality k, i.e. $\mathcal{M}_k = \{S \in \mathcal{M}(C) : |S| = k\}$. First, we argue that if for each $k \in \{q, \ldots, n\}$ the following identity holds, then we obtain the desired conclusion.

$$\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \sum_{i=k}^{n} \binom{i-q}{k-q} \binom{n-i+q-1}{q-1} \tag{4.1}$$

To see this, note that $|\mathcal{M}_n| = 1$ and for $k = n-1$, it follows from (4.1) that

$$\binom{n-1-q}{k-q} |\mathcal{M}_{n-1}| + \binom{n-q}{k-q} |\mathcal{M}_n| = \binom{n-1-q}{k-q} \binom{q}{q-1} + \binom{n-q}{k-q} \binom{q-1}{q-1}.$$

Since $|\mathcal{M}_n| = 1 = \binom{q-1}{q-1}$, we have $|\mathcal{M}_{n-1}| = \binom{q}{q-1}$. Similarly for $k = n-2$, we have $|\mathcal{M}_{n-2}| = \binom{q+1}{q-1}$. Proceeding inductively we obtain that $|\mathcal{M}_k| = \binom{n-k+q-1}{q-1}$. In what follows we prove that (4.1) holds in two steps by showing that both sides of the equality are equal to $\binom{n}{k}$.

Step 1. We show that $\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \binom{n}{k}$. To see this, first, consider $\mathcal{K} = \{S \in \mathcal{A} : |S| = k\}$. Then, consider the partition of $\mathcal{K}$ such that for each $S, S' \in \mathcal{K}$, S and S' belong to the same part if and only if $C(S) = C(S')$. First, we show that

$$\mathcal{K} = \bigcup_{i=k}^{n} \bigcup_{S' \in \mathcal{M}_i} \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}. \tag{4.2}$$

Since for each $S \in \mathcal{A}$, there exists a unique $S' \in \mathcal{M}$ such that $C(S) = C(S')$ and $S \subset S'$, we get

$$\mathcal{K} = \bigcup_{S' \in \mathcal{M}} \{S \in \mathcal{A} : |S| = k, C(S) = C(S')\}. \tag{4.3}$$

Since $\{\mathcal{M}_i\}_{i=k}^n$ partitions $\{S' \in \mathcal{M} : |S'| \geq k\}$, we can rewrite (4.3) as

$$\mathcal{K} = \bigcup_{i=k}^{n} \bigcup_{S' \in \mathcal{M}_i} \{S \in \mathcal{A} : |S| = k, C(S) = C(S')\}. \tag{4.4}$$

Finally, note that for each $S' \in \mathcal{M}$ and $S \in \mathcal{K}$, if $C(S') \subset S \subset S'$, then *substitutability* implies that $C(S') = C(S)$. Therefore, $\{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} = \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}$. This observation together with (4.4) implies that (4.2) holds.

Now, if we count both sides of (4.2), then we obtain

$$\binom{n}{k} = \sum_{i=k}^{n} \sum_{S' \in \mathcal{M}_i} |\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}|. \tag{4.5}$$

Next, we argue that for each $i \in \{k, \ldots, n\}$, and $S' \in \mathcal{M}_i$,

$$|\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}| = \binom{i-q}{k-q}. \tag{4.6}$$

To see this, for each $i \in \{k, \ldots, n\}$, and $S' \in \mathcal{M}_i$, consider the set $\{T \subset S' \setminus C(S') : |T| = k - q\}$. Since $S' \in \mathcal{M}_i$, $|S' \setminus C(S')| = i - q$. It directly follows that $|\{T \subset S' \setminus C(S') : |T| = k - q\}| = \binom{i-q}{k-q}$. To show that (4.6) holds, we argue that $F = \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}$ is isomorphic[2] to $F' = \{T \subset S' \setminus C(S') : |T| = k - q\}$. To see this we define the mapping g such that for each $S \in F$, $g(S) = S \setminus C(S')$. Since for each $S \in F$, $C(S') \subset S \subset S'$ and $|S| = k$, we have $|g(S)| = k - q$ and $g(S) \in F'$. Thus $g : F \to F'$. Since for each distinct $S_1, S_2 \in F$, $S_1 \setminus C(S') \neq S_2 \setminus C(S')$, g is one-to-one. Since for each

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[2] That is, there is a bijection between the two sets.

$T \in F'$, $g(T \cup C(S')) = T$, g is onto. Therefore g is a bijection between F and F'. Thus, we obtain that (4.6) holds. Finally, if we combine (4.5) and (4.6), then we directly obtain that $\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \binom{n}{k}$.

Step 2. We show that $\sum_{i=k}^{n} \binom{i-q}{k-q} \binom{n-i+q-1}{q-1} = \binom{n}{k}$. To see this consider the set $\{1, 2, \ldots, n\}$ and let $\mathcal{K}$ be the collection of its subsets that contain k alternatives. Since there are $\binom{n}{k}$ such subsets, $|\mathcal{K}| = \binom{n}{k}$. Since $k > q$, for each $S \in \mathcal{K}$, there exists $q(S) \in S$ that is the q^{th} highest number in S. Now, consider the partition of $\mathcal{K}$ such that for each $S, S' \in \mathcal{K}$, S and S' belong to the same part if and only if $q(S) = q(S')$. We denote this partition of $\mathcal{K}$ by $\mathcal{L}$. Now, note that for each $S \in \mathcal{K}$, $q(S) \in \{k+1-q, \ldots, n+1-q\}$. Next, for each $j \in \{k+1-q, \ldots, n+1-q\}$, we count the number of $S \in \mathcal{K}$ such that $q(S) = j$. If $q(S) = j$, then there are $k - q$ numbers in S that are less than j, and $q - 1$ numbers that are greater than j. It follows that S can be chosen in $\binom{j-1}{k-q}\binom{n-j}{q-1}$ different ways. This observation together with $\mathcal{L}$ partitions $\mathcal{K}$ implies that

$$\binom{n}{k} = \sum_{j=k+1-q}^{n+1-q} \binom{j-1}{k-q} \binom{n-j}{q-1}. \tag{4.7}$$

A standard change of variables with $j = i + 1 - q$ here yields that the right-hand side of (4.7) equals $\sum_{i=k}^{n} \binom{i-q}{k-q} \binom{n-i+q-1}{q-1}$. Thus, we obtain the desired equality. □

Now we can find the upper bound for the number of priorities to represent a *q-acceptant* and *path independent* choice rule.

Theorem 4. *The number of preferences required in a minimal MC-representation of a path independent and q-acceptant choice rule cannot exceed* $\binom{n-1}{q-1}$. *Moreover* *this upper bound is attained by q-responsive choice rules.*

Proof. By combining Theorem 3 and Lemma 1 we can say that the number of required priorities must be less than or equal to $|\mathcal{M}_{q-1}| = \binom{n}{q-1}$. Actually we can find a better and exact bound by using Lemma *P3*. Every maximal in $\mathcal{M}_q$ can have at most one prime child and prime children of maximals in $\mathcal{M}_q$ are prime atoms; so there can exist at most $|\mathcal{M}_q| = \binom{n-1}{q-1}$ prime atoms.

In order to show that this is the exact upper bound we provide an example where this bound holds. Actually we have provided this with Example 5. If C is a *q-responsive* choice rule with respect to $\succ$ then S is a prime atom if and only if $|S| = q-1$ and $n \notin S$. Note that $n \in C(S \cup \{n\})$ and for any $x \notin S \cup \{n\}$ we get $n \notin C(S \cup \{n, x\})$; so, S is a prime atom and there are $\binom{n-1}{q-1}$ such sets. Therefore the upper bound we have found is satisfied with equality for *q-responsive* choice rules and this concludes our proof. $\square$

Even though *q-responsive* choice rule is one of the easiest *path independent* choice rules to explain, Theorem 4 shows that it has one of the most complicated MC-representations. This looks like a weakness of MC-representation.

CHAPTER 5

RESPONSIVE CHOICE RULES

Our results in the previous chapter highlights that it is difficult to represent *q-responsive* choice rules in MC form. Building blocks of MC-representations are actually 1-*responsive* choice rules. So in order to overcome this difficulty we aim to generalize *responsive* rules and then if possible to have an alternative representation which is the union of these *responsive* choice rules.

If a choice rule chooses a set of top block among given alternatives with respect to a priority ordering, we say this choice rule is *responsive* to the priority ordering. If a choice rule is *responsive* to some priority ordering, then we call this choice rule a *responsive* choice rule. Formal definition is as follows.

Definition 4. Let C denote a choice function and $\succ$ denote a priority ordering. C is **responsive to** $\succ$ if $\forall a, b, S$, $a \in C(S)$ and $b \in S \backslash C(S)$ implies $a \succ b$. C is **responsive** if there exists $\succ$ such that C is *responsive* to $\succ$.

The *responsive* rules have been used in the literature, but the responsiveness axiom we formulate here have not been studied previously. In order to find a characterization for *path independent* and *responsive* choice rules we propose a family of choice rules which is called *weighted responsive* choice rules.

First let us introduce **simple weighted responsive choice rules**. Let $\succ$ be any linear order on A and a vector $w \in [0,1]^A$. For any given choice set $S \subset A$ we define *simple weighted responsive* choice $\succ_w (S)$ by choosing top elements in S with respect to $\succ$ until we fill our quota 1 by considering weights of each alternative. Formally we define $\succ_w$: $\mathcal{A} \to \mathcal{A}$ as

$$\succ_w (S) = \{a \in S : \sum_{b \in S, b \succeq a} w_b \leq 1\}.$$

Here notice that if $\forall a \in A$, we choose $w_a = 1/q$ then resulting *simple weighted responsive* choice rule will be *q-responsive.* Let us give another example of a simple weighted choice rule.

Example 7. Let $A = \{1,2,3,4,5,6\}$ and $w = (1/3, 1/3, 1/3, 1/2, 1/2, 1/2)$ and $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$. So we get $\succ_w (\{1,2,3,4,5,6\}) = \{1,2,3\}$, $\succ_w (\{2,3,4,5,6\}) = \{2,3\}$ or $\succ_w (\{3,4,5,6\}) = \{3,4\}$. We can see $| \succ_w (S)| = 3$ only if $1,2,3 \in S$ and $| \succ_w (S)| = \min\{2, |S|\}$ otherwise. Since sum of three weights does not exceed 1 only if these weights are w_1, w_2, w_3. In all other cases we can choose at most 2 elements and if S has more than one element we can always choose two alternatives since greatest weight is $1/2$.

Now define *weighted responsive* choice rules. In order to define such a rule we need a weight vector for each alternative instead of a scalar as we did for simple weighted choice rules.

Definition 5. Let $\succ$ be any linear order on A and a matrix $w \in [0,1]^{A \times K}$ with some index set K. A **weighted responsive choice rule** $\succ_w$: $2^A \to 2^A$ such that

$$\succ_w (S) = \{a \in S : \forall k \in K, \sum_{b \in S, b \succeq a} w_{bk} \leq 1\}.$$

In order to understand this rule we can adhere to the following interpretation. Suppose we have k different budget constraints and every alternative a has w_{ak} cost for constraint $k \in K$. An element is chosen from S if that element and all above it in S chosen will not violate any of the constraints. If at least one of the constraints is violated then this choice is not feasible.

Weighted responsive choice rules are *responsive* by construction. One can see they are also *substitutability.* To see that let $a \in S$ with $a \in \succ_w (S)$ for some $\succ$ and w. If $a \in T \subset S$ then clearly $\sum_{b \in T, b \succeq a} w_{bk} \leq \sum_{b \in S, b \succeq a} w_{bk}$, so we can deduce $a \in \succ_w (S)$.

In order to have a *path independent* and *responsive* choice rule we also need *IRA* to be satisfied but *weighted responsive* rules fail to satisfy this property. Following example demonstrates such a situation.

Example 8. Let $A = \{1,2,3,4,5\}$, $w = \{1/2, 1, 1/2, 1/2, 1/2\}$ and $1 \succ 2 \succ 3 \succ$

$4 \succ 5$. Now $\succ_w (\{1,2,3,4,5\}) = \{1\}$ but $\succ_w (\{1,3,4,5\}) = \{1,3\}$. Clearly 2 is rejected from A but removing 2 have changed the choice; so $\succ_w$ does not satisfy *IRA*.

Here it seems *weighted responsive* choice rules are not *path independent* in general. But we can impose additional conditions on w to make it *path independent.*

Condition-ρ : We say w and $\succ$ satisfies **condition-ρ** if any infeasible set cannot be made feasible by worsening its worst element. Formally, for a given weight matrix $w \in [0,1]^{A\times K}$ and linear order $\succ$ on A, we say $(w, \succ)$ satisfies **condition-ρ** if for any $S \subset A$ with $b \in S$ denoting its the worst element with respect to $\succ$, if there exists $k \in K$ such that $\sum_{a\in S} w_{ak} > 1$, then for any c with $b \succ c$, there exists $k' \in K$ such that $\sum_{a\in S\setminus\{b\}} w_{ak'} + w_{ck'} > 1$). If $(w, \succ)$ satisfies condition-ρ then we say $\succ_w$ is a ρ-compatible *weighted responsive* rule

Theorem 5. *A choice rule C is path independent and responsive if and only if it is a ρ-compatible weighted responsive rule.*

Proof. $(\Leftarrow)$ By construction, any *weighted responsive* rule is *responsive* and *substitutable.* Now let us prove if C is ρ-compatible *responsive* choice rule then it satisfies *IRA* as well.

Let $S \subset A$ and $b = \max(S \setminus C(S), \succ)$ so there exist $k \in K$ such that $1 < \sum_{a\in S, a\succeq b} w_{ak}$. Now pick any $c \in S \setminus C(S)$ and let us call $S \setminus c = S'$.

If $c \neq b$, for any $d \in S' \setminus C(S)$ we can see $1 < \sum_{a \in S, a \succeq b} w_{ak} \leq \sum_{a \in S', a \succeq d} w_{ak}$ so d should be rejected, that means all rejected elements still will be rejected in S'. If $c = b$ condition-ρ ensures the rejection of the object just below b in S. Therefore any condition-ρ *responsive* choice rule satisfies *IRA*.

($\Rightarrow$) Let $\mathcal{M}$ denote the collection of maximal choice sets for C; then let us define $w \in [0,1]^{A \times \mathcal{M}}$ as

$$w_{aT} = \begin{cases} 0 & \text{if} \quad a \notin T \\ 1/|C(T)| & \text{if} \quad a \in C(T) \\ \epsilon & \text{if} \quad a \in T \backslash C(T) \end{cases}$$

for any $a \in A$ and $T \in \mathcal{M}$. We can easily verify that $(w, \succ)$ satisfies condition-ρ. Now let us prove $\succ_w (S) = C(S)$ for any $S \subset A$.

Let $a \in S \setminus C(S)$, there exists some $T \in \mathcal{M}$ such that $C(S) = C(T)$ with $S \subset T$. Clearly $w_{aT} = \epsilon$ and $\forall b \in C(S)$ we have $w_{bT} = 1/|C(T)|$, so $\sum_{b \in C(S)} w_{bT} = 1$. Also by responsiveness, for any $b \in C(S)$ we have $b \succ a$; thus $\sum_{b \succ a, b \in S} w_{bT} \geq w_{aT} + \sum_{b \in C(S)} w_{bT} = 1 + \epsilon > 1$; therefore a is not chosen. That means $\succ_w (S) \subset C(S)$.

Now we claim for any $T \in \mathcal{M}$, $\sum_{b \in C(S)} w_{bT} \leq 1$. Suppose contrary, that means there exists some $T \in \mathcal{M}$ with $\sum_{b \in C(S)} w_{bT} > 1$. This is possible only if $C(T) \subsetneq C(S)$ and $(T \setminus C(T)) \cap C(S) \neq \emptyset$. This means every alternative in $T \setminus C(S)$ is a rejected element from T. Thus by *IRA*, $C(T) = C(T \cap C(S))$. Moreover by *substitutability*, $C(C(S)) = C(S)$ and for any $S' \subset C(S)$, $C(S') = S'$. So we have $C(T \cap C(S)) = T \cap C(S)$, which yields $C(T) = T \cap C(S)$. Therefore $T \setminus C(T) = T \setminus (T \cap C(S)) = T \setminus C(S)$ and this means $(T \setminus C(T)) \cap C(S) = \emptyset$. which is a contradiction; thus our claim is true which means $C(S)$ is a subset of $\succ_w (S)$. These two conclude the proof. $\square$

CHAPTER 6

ON THE UNIQUESNESS OF PROBABILISTIC SERIAL ASSIGNMENT

The results in this chapter have been published in an article (Doğan et al., 2018) that I have co-authored. In this chapter we study the assignment problem in which n objects are to be allocated among n agents such that each agent receives an object and monetary compensations are not possible. Applications include assigning houses to agents or students to schools. Motivated by fairness concerns, probabilistic assignments (lotteries over sure assignments) have been extensively studied in the literature.

Starting with the seminal study by Hylland and Zeckhauser (1979), the vast majority of the literature assumes that each agent derives a utility for being assigned an object, and his ex-ante evaluation of a probabilistic assignment is his expected utility for that probabilistic assignment. In other words, agents are endowed with von-Neumann–Morgenstern (vNM) preferences over probabilistic assignments. In this setup, a natural efficiency requirement for a probabilistic assignment is *ex-ante efficiency*: the probabilistic assignment maximizes the sum of the expected utilities. Obviously, evaluating the *ex-ante efficiency* of a probabilistic assignment requires knowledge of the vNM preferences. However, ordinal allocation mechanisms that elicit only preferences over sure objects have been particularly studied in the literature.[1] When an ordinal mechanism is used, agents are asked to report their preference orderings over objects.[2] Therefore, the efficiency of an assignment has to be evaluated based only on the ordinal preference information. The common method in the literature to ordinally evaluate the efficiency of a probabilistic assignment is based on first order stochastic dominance. This efficiency notion, introduced by Bogomolnaia and Moulin (2001), is called *sd-efficiency*: a probabilistic assignment is sd-efficient if it is not stochastically dominated by any other assignment.[3] McLennan (2002) shows that an assignment is sd-efficient if and only if there is a utility profile at which it is ex-ante efficient. Pathak (2008) compares the performance of probabilistic serial assignment to the random serial dictatorship by using the data of student placement in public schools in New York City. For 44% of the students the probabilistic assignments generated by the two mechanisms are not comparable with respect to first order stochastic dominance.

In this study, we revisit an extensively studied probabilistic assignment mechanism, namely the Probabilistic Serial (PS) mechanism. Bogomolnaia and Moulin (2001) introduce the PS mechanism and show that it always chooses a fair, sd-efficient and envy-free assignment.[4] Given this observation, an important ques-

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[1] See Bogomolnaia and Moulin (2001) for several justifications for observing ordinal mechanisms in practice.

[2] Part of the literature focuses on strict preferences such that each agent reports a complete, transitive, and anti-symmetric ordering over objects. Unless otherwise noted, we allow for weak preferences that are not necessarily anti-symmetric.

[3] Bogomolnaia and Moulin (2001) refers to sd-efficiency as "ordinal efficiency." Here, we use the terminology of Thomson (2010).

[4] See Section 6.2 for the formal definition of fairness.

tion is "Is it possible to have an sd-efficient and envy-free assignment other than the serial assignment?". To answer this question, we consider a directed graph, the configuration of which depends on the given ordinal preference profile. We show that a special connectedness property of this graph plays a critical role in understanding at which preference profiles the serial assignment yields the unique sd-efficient and envy-free assignment.

6.1 The framework

Let N be a set of n agents and A be a set of n objects. For each $i \in N$, the preference relation of i, which we denote by R_i, is a **weak order** on A, i.e. it is transitive and complete. Given a pair of objects $a, b \in A$, we write $a\ P_i\ b$ when $a\ R_i\ b$ but not $b\ R_i\ a$; we write $a\ I_i\ b$ when $a\ R_i\ b$ and $b\ R_i\ a$, and call I_i the associated indifference relation. Let $\mathcal{R}_i$ denote the set of all possible preference relations for i, and $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$ denote the set of all possible preference profiles, which we also call the **weak preference domain**. Let $\mathcal{R}_i^S \subset \mathcal{R}_i$ denote the set of all possible strict preference relations for i, i.e. the set of all anti-symmetric preference relations in $\mathcal{R}_i$, and $\mathcal{R}^S \equiv \times_{i \in N} \mathcal{R}_i^S$ denote the set of all possible strict preference profiles, which we also call the **strict preference domain**.[5] Note that for each $R_i \in \mathcal{R}_i^S$ and each pair of objects $a, b \in A$, $a\ I_i\ b$ implies that $a = b$.

A **deterministic assignment** is a one-to-one function from N to A. A deterministic assignment can be represented by an $n \times n$ matrix with rows indexed

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[5] Since we allow our agents to be indifferent among objects, *multi-unit assignment problems*, in which each object may have multiple copies, turn out to be a special case of our framework. In that we can consider preference profiles such that each agent is indifferent between different copies of the same object.

by agents and columns indexed by objects, and having entries in $\{0, 1\}$ such that each row and each column has exactly one 1. Such a matrix is called a **permutation matrix**. For each $(i, a) \in N \times A$, having 1 in the (i, a) entry indicates that i is assigned a. A probabilistic assignment (an **assignment** hereafter) is a probability distribution over deterministic assignments. An assignment can be represented by an $n \times n$ matrix having entries in $[0, 1]$ such that the sum of the entries in each row and each column is 1. Such a matrix is called a **doubly stochastic matrix**. For each assignment π, and each pair $(i, a) \in N \times A$, the entry π_{ia}, which we also write as $\pi_i(a)$ or $\pi(i, a)$, indicates the probability that i is assigned to a at π. Since each doubly stochastic matrix can be represented as a convex combination of permutation matrices (Birkhoff (1946) and Von Neumann (1953)), the set of all doubly stochastic matrices is the set of all assignments. Let Π be the set of all doubly stochastic matrices.

We denote the collection of all lotteries over A by $\mathcal{L}(A)$. For each $i \in N$, a von-Neumann–Morgenstern (vNM) utility function u_i is a real valued mapping on A, i.e. $u_i : A \to \mathbb{R}$. For each $i \in N$ with preferences $R_i \in \mathcal{R}_i$, a vNM utility function u_i is **consistent** with R_i if for each pair $(a, b) \in A$, we have $u_i(a) \geq u_i(b)$ if and only if $a \ R_i \ b$. We obtain the corresponding preferences of i over $\mathcal{L}(A)$ by comparing the expected utilities, where the expected utility from $\pi_i \in \mathcal{L}(A)$ is $\sum_{a \in A} \pi_i(a) u_i(a)$.

Next, we define the sd-efficiency of an assignment. The formulation of sd-efficiency is independent of any vNM utility specification consistent with the ordinal preferences. Let $\pi, \pi' \in \Pi$, $i \in N$, and $R \in \mathcal{R}$. We say that π_i stochastically dominates π'_i at R_i, or simply π_i **sd-dominates** π'_i at R_i, if for each $a \in A$,

$$\sum_{b:bR_ia} \pi_i(b) \geq \sum_{b:bR_ia} \pi'_i(b).$$

We say that π stochastically dominates π' at R, or simply π **sd-dominates** π' at R, if $\pi \neq \pi'$ and for each $i \in N$, π_i sd-dominates π'_i at R_i. An assignment $\pi \in \Pi$ is **sd-efficient** at R if no assignment sd-dominates π at R. Let $P^{sd}(R)$ denote the set of sd-efficient assignments at R.

6.2 Uniqueness of the Probabilistic Serial Mechanism

An assignment mechanism is a function $\varphi : \mathcal{R} \to \Pi$, associating an assignment with each preference profile. On the strict preference domain, a widely studied probabilistic assignment mechanism is the probabilistic serial (PS) mechanism. At each $R \in \mathcal{R}^S$, the PS assignment is computed by the following algorithm. Consider each object as an infinitely divisible good with a one unit supply that will be eaten by agents in the time interval $[0, 1]$ through the following steps:

Step 1: Each agent eats from his most preferred object. Agents eat at the same speed. When an object is completely eaten, proceed to the next step.

Steps $s \geq 2$: Each agent eats from his most preferred object from among the ones that have not yet been completely eaten. Agents eat at the same speed. When an object is completely eaten, proceed to the next step.

The algorithm terminates when all the objects are exhausted (or equivalently when each agent has eaten in total exactly one unit of objects), and the probability that an agent receives an object in the *PS* assignment is defined as the amount of the object the agent has eaten. We denote the *PS* assignment at R by $\pi^{ps}(R)$.

Given $R \in \mathcal{R}^S$, $a \in A$, and $t \in [0, 1]$, we say that a is **exhausted** at time t in the *PS* algorithm at R if at the end of the step that ends when a is completely eaten, each agent has eaten in total t units of the objects. Note that for each pair $a, b \in A$, if a and b are exhausted at different times in the PS algorithm at R, then for each $i, j \in N$ with $\pi^{ps}(i, a) > 0$ and $\pi^{ps}(j, b) > 0$, we have $\pi^{ps}(i, \{c \in A : c\ R_i\ a\}) \neq \pi^{ps}(j, \{c \in A : c\ R_j\ b\})$.

Bogomolnaia and Moulin (2001) show that the *PS* mechanism chooses an *sd-efficient* assignment at each strict preference profile. Another well-known probabilistic assignment mechanism is the *random serial dictatorship (RD) mechanism*, which draws at random an ordering of the agents from the uniform distribution, then lets them choose successively their best remaining object (the first agent in the ordering is assigned to his best object, the second agent to his best among the remaining objects, and so on). Bogomolnaia and Moulin (2001) show that *RD* mechanism does not always choose an sd-efficient assignment. Manea (2009) shows that the inefficiency of the *RD* mechanism prevails even in the large markets, since the probability that the resulting assignment is sd-efficient converges to zero as the number of object types becomes large. However, as Bogomolnaia and Moulin (2001) shows, there are preference profiles at which RD and PS mechanisms choose different assignments such that neither sd-dominates the other. Empirical observations, by Pathak (2008), in the context of school choice problem, indicate that this indeterminacy arises so often that comparing RD and PS mechanisms based on sd-dominance becomes difficult.

Besides *sd-efficiency*, the PS mechanism also satisfies *sd–envy-freeness* (Bogomolnaia and Moulin Bogomolnaia and Moulin (2001)), which has been a central fairness requirement in the probabilistic assignment literature: an assignment π is **sd–envy-free** at R if for each pair of agents $i, j \in N$, π_i sd-dominates π_j at R_i. Our next example shows that there is a strict preference profile for which there is an sd–envy-free assignment other than PS assignment.

Example 9. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Consider the following preference profile.

Table 6.1: The assignment π, which is sd-envy-free at R

R_1	R_2	R_3
a	a	b
b	c	c
c	b	a

$\pi^{ps}(R)$	a	b	c
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	0	$\frac{3}{4}$	$\frac{1}{4}$

π	a	b	c
1	$\frac{1}{2}$	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	0	$\frac{1}{2}$	$\frac{1}{2}$

Consider the PS assignment $\pi^{ps}(R)$ and an object assignment, namely π, both of which are depicted above. It is easy to check that π is sd–envy-free.

6.2.1 Necessity

As the previous example shows, there are strict preference profiles for which there is an sd–envy-free and sd–efficient assignment other than PS assignment. Now, we ask when is it unique? To answer this question, given $R \in \mathcal{R}^S$, we define a directed graph $G(R)$ as follows:

Definition 6. For each $R \in \mathcal{R}^S$, $G(R)$ is a directed graph where each agent-object pair is a vertex and for each vertex pair $(i, a), (j, b)$, there is an edge from (i, a) to (j, b), denoted by $(i, a) \rightarrow (j, b)$, if for each pair of objects $x, y \in A$ such that $x \; R_i \; a$ with $\pi^{ps}(i, x) > 0$ and $b \; P_j \; y$ with $\pi^{ps}(j, y) > 0$, we have $x \; P_j \; y$.[6]

To paraphrase the definition for $(i, a) \rightarrow (j, b)$, let $U(R_i, a)$ denote the upper contour set of R_i at a, that is, $U(R_i, a) = \{b \in A : b \; R_i \; a\}$. Let $U^+(R_i, a)$ denote the set of objects in $U(R_i, a)$ that are assigned to i with positive probability at the PS assignment. Then, $(i, a) \rightarrow (j, b)$ if $U^+(R_i, a) \subset U(R_j, y)$ for each object y such that $b \; P_j \; y$ with $\pi^{ps}(j, y) > 0$. To put it more compactly, let y^{jb} be the best object at R_j such that $b \; P_j \; y$ with $\pi^{ps}(j, y) > 0$. Then, we have $(i, a) \rightarrow (j, b)$ if $U^+(R_i, a) \subset U(R_j, y^{jb})$. The following figure illustrates the $G(R)$ for Example 9. Note that if $(i, a) \rightarrow (j, b)$, then for each $y \in A$ with $b \; P_j \; y$, $(i, a) \rightarrow (j, y)$. The bold edges are the *critical edges*, in the sense that if $(i, a) \rightarrow (j, b)$, then there is no $z \; P_j \; b$ with $(i, a) \rightarrow (j, z)$. The dotted edges are the ones that are not critical.

Roughly speaking $G(R)$ provides an ordinal account of "whose assignment is about to fail to be envy-free for whom and for which object at the PS assignment." This interpretation of $G(R)$ follows from Lemma 2 in which we show that for each (i, a) and (j, a) such that a is assigned to i and j with positive probability,

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[6] Note that if there exists no $x \in A$ such that $x \; R_i \; a$ and $\pi^{ps}(i, x) > 0$, or if there exists no $y \in A$ such that $b \; P_j \; y$ and $\pi^{ps}(j, y) > 0$, then, trivially, there is an edge from (i, a) to (j, b).

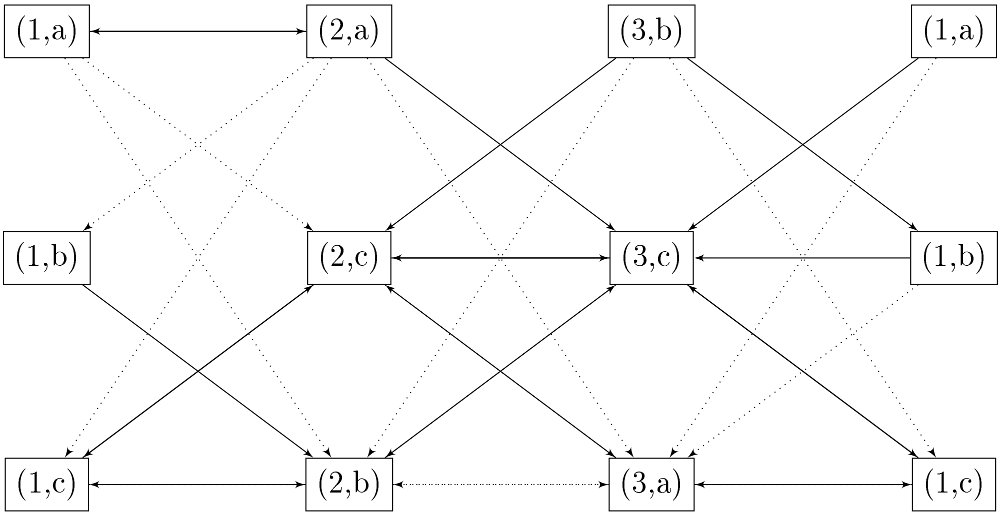

Figure 6.1: G(R) for the Example 9

$(i, a) \to (j, a)$ implies that if we increase the probability that a is assigned to i in the PS assignment, then the assignment of j no longer stochastically dominates that of i. Moreover, $G(R)$ provides an ordinal account of these relationships, since configuration of $G(R)$ depends on whether an object is assigned to an agent or not in the PS assignment, but it is independent of the particular assignment probabilities.

We observe that a special connectedness property of $G(R)$ plays a critical role in understanding when the PS assignment is unique sd–envy-free and sd-efficient assignment. In graph theoretic language, a vertex (i, a) is said to be **connected** to another vertex (j, b) in $G(R)$ if there is a **path**, a sequence of vertices $v_1, v_2, \dots, v_k$ such that $(i, a) \to v_1 \to v_2 \to \cdots v_k \to (j, b)$. Next, we introduce the connectedness property that will be key for our results.

Definition 7. For each $R \in \mathcal{R}^S$ and $a \in A$, the graph $G(R)$ is ***a*-connected** if for each $i, j \in N$ such that $\pi^{ps}(R)(i, a) > 0$, (i, a) is connected to (j, a) in $G(R)$.

The graph $G(R)$ is **connected** if it is a-connected for each $a \in A$.

Recall that we interpret the configuration of $G(R)$ as an account of “whose assignment is about to fail to be envy-free for whom and for which object at the PS assignment". Now, suppose $G(R)$ fails to be a-connected for some object a. This means that we can alter the PS assignment at R without violating sd–envy-freeness. In this vein, our next result shows that if PS assignment is sd-efficient and sd–envy-free then $G(R)$ is connected.

Before proceeding with the result, to get some familiarity with the connectedness notion, consider the following two extreme preference profiles. Suppose that in the first profile each agent has the same preference relation, whereas in the second profile each agent top-ranks a distinct object. The PS assignment allocates each object equally between the agents at the first preference profile, and assigns each agent his top choice with probability one at the second preference profile. In both preference profiles, $G(R)$ is connected since for each $a \in A$ and $i, j \in N$ with $\pi^{ps}(i, a) > 0$, we have $(i, a) \to (j, a)$. More specifically, for the first preference profile, where the preferences are exactly the same, observe that for each $a \in A$, if we restrict $G(R)$ to the vertex set $N \times \{a\}$ we obtain the complete graph. Similarly, for the second preference profile, for each $a \in A$, since there is a single agent $i \in N$ with $\pi(i, a) > 0$, if we restrict $G(R)$ to the vertex set $N \times \{a\}$, then we obtain a star-shaped directed graph. Moreover, clearly at both preference profiles, the PS assignment is the unique sd-efficient and sd–envy-free assignment. In fact, we next show that connectedness of $G(R)$ is necessary for the *PS* assignment to be unique sd-efficient and sd–envy-free assignment at R.

Proposition 1. *For each $R \in \mathcal{R}^S$, if PS assignment is the unique sd–envy-free assignment then $G(R)$ is connected.*

Proof. See Section 6.4.1. $\square$

6.2.2 Sufficiency

In this section, we first observe that for a given preference profile R, connectedness of $G(R)$ is not sufficient for the *PS* assignment to be unique sd–envy-free assignment. Next, we introduce the *betweenness* of a preference profile and show that under betweenness assumption, connectedness of $G(R)$ is a sufficient condition.

Example 10. Let $N = \{1, 2, 3, 4\}$ and $A = \{a, b, c, d\}$. Consider the following preference profile.

R_1	R_2	R_3	R_4
a	a	b	c
b	c	c	a
c	d	d	d
d	b	a	b

$\pi^{ps}(R)$	a	b	c	d
1	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$
2	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
3	0	$\frac{3}{4}$	0	$\frac{1}{4}$
4	0	0	$\frac{3}{4}$	$\frac{1}{4}$

First note that objects b and c are exhausted simultaneously at time $3/4$. Since agent 1 ranks c right below b, R violates betweenness. Next, we argue that there is no path that connects the pair $(1, b)$ to $(3, b)$. To see this, first note that only $(4, c)$ is linked to $(3, b)$ and only $(2, c)$ and $(2, a)$ are linked to $(4, c)$. Similarly, note that only $(1, a)$ is linked to $(2, a)$ and $(2, c)$. Since $(1, b)$ is not linked to

$(1, a)$, $(2, c)$, $(4, c)$ or $(3, b)$, there is no path that connects $(1, b)$ to $(3, b)$. Finally we argue that $\pi^{ps}(R)$ is the unique assignment that is sd–envy free and sd-efficient at R. To see this, first note that at any sd–envy-free and sd-efficient assignment at R, a should be shared evenly between agents 1 and 2. Given this, to be sd-efficient 2 and 4 should eat from c. Now, for agent 4 not to envy agent 2, 4 should eat 3/4 of c. Since 1 and 3 rank b over d, 2 and 4 should complete their assignments by equally eating from d. Thus assignments of agents 2 and 4 should be as in $\pi^{ps}(R)$. Next consider the assignment of agent 3. Since a and c are exhausted, 3 can eat from b and d, let p be the amount of b that 3 eats. Now, note that the only value of p that makes 1 and 3 not to envy each other is 3/4. It follows that $\pi^{ps}(R)$ is the unique sd-efficient among sd–envy-free assignments. Thus we show that in the absence of betweenness, although PS assignment is unique sd–envy-free assignment, $G(R)$ may not be connected.

Next, we introduce a property which turns out to be key in understanding when connectedness is necessary for the PS assignment to have unique sd–envy-free assignment.

Definition 8. A preference profile $R \in \mathcal{R}^S$ satisfies **betweenness** if for each pair $a, b \in A$ that are simultaneously exhausted in the PS algorithm at R and for each $i \in N$ with $\pi^{ps}(i, a) > 0$, there exists $c \in A$ such that $\pi^{ps}(i, c) > 0$ and $a \; P_i \; c \; P_i \; b$.

To get some intuition for betweenness, first note that if a pair $a, b \in A$ are simultaneously exhausted in the PS algorithm at R, and for some agent i, we have $\pi^{ps}(i,a) > 0$, then this means i prefers to eat a instead of b. It follows that we necessarily have $a \ P_i \ b$. Betweenness additionally requires the existence of another object that is matched with i, and lies between a and b at P_i.[7]

Remark 3. A specific class of preference profiles that satisfy betweenness is the following. A preference profile R satisfies **distinct exhaustion condition** if for each distinct pair of objects $a, b \in A$, a and b are exhausted at different times in the PS algorithm. For example consider the preference profile at which each agent has the same preference relation over the objects. Then, since each agent-object pair is matched with positive probability in the PS algorithm, the distinct exhaustion condition is directly satisfied. Since for such preference profiles, there is no object pair that are simultaneously exhausted, betweenness is directly satisfied. To see this, note that if a pair of objects a and b are simultaneously exhausted, then agents that exhaust a can not get matched with b.[8]

Our next result shows that if a preference profile R satisfies betweenness, then connectedness of $G(R)$ is sufficient for the PS assignment to be unique sd-efficient and sd–envy-free assignment.

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[7] Note that the betweenness of a preference profile can be directly verified by only using the support of the PS assignment. In section 6.4, we show that betweenness implies the upper semi-continuity of $Sp(\pi^{ps}(R))$, a technical requirement which is critical for proving the result, but rather difficult to verify directly.

[8] To see an example that satisfies betweenness but does not satisfy the distinct exhaustion condition, consider a society with an even number of (at least four agents) such that half of the society adopts the preference relation R_1 and the other half adopts the preference relation R_2. Suppose the only difference between R_1 and R_2 is that $a(b)$ is top-ranked at $R_1(R_2)$, whereas $b(a)$ is bottom-ranked at $R_1(R_2)$. Consider the PS assignment for this society, since half of the society top ranks a and the other half top ranks b, a and b are simultaneously exhausted. Therefore, the distinct exhaustion condition is not satisfied, To see that betweenness is satisfied, first note that only objects a and b are simultaneously exhausted. Since for those agents that eat $a(b)$, $b(a)$ is bottom ranked, there exists another object that is matched with positive probability, and lies between a and b.

Proposition 2. *For each $R \in \mathcal{R}^S$ that satisfy betweenness, if the PS assignment is unique sd–envy-free assignment, then $G(R)$ is connected.*

Proof. See Section 6.4.2.

$\square$

Once we identify when it is possible to find sd–envy-free assignments other than *PS*, the next question is how to obtain such an assignment. The construction in the proof of Proposition 2 implicitly answers this question. Now, we revisit Example 9 to give a rough overview of how can we use this construction to obtain an sd–envy-free assignment other than the *PS* assignment. First, consider the preference profile R and $\pi^{ps}(R)$. One can easily check that each object is exhausted at different times in $\pi^{ps}(R)$. Next, consider the graph $G(R)$. Note that if for each $x \in A$, we restrict the $G(R)$ to the vertex set $N \times \{x\}$, we obtain the three graphs below. It directly follows from their configuration that $G(R)$ is a-connected and c-connected. However, $G(R)$ is not b-connected, since $(1, b)$ is not connected to $(3, b)$. To see this, first note that neither $(1, b)$ nor $(2, b)$ is linked to $(3, b)$. Moreover, since only agent 3 top-ranks b and is assigned to c with positive probability, there is no $(i, x) \in N \times \{a, c\}$ with $(i, x) \to (3, b)$.

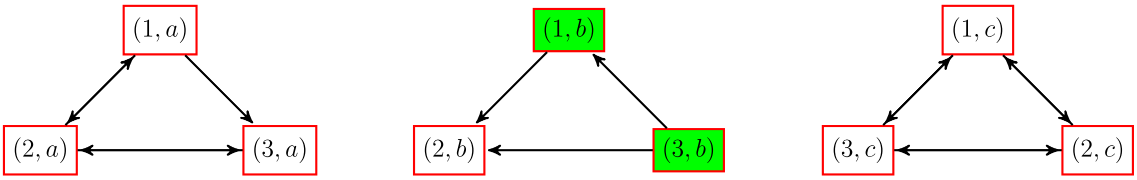

Figure 6.2: A partial reflection of G(R)

Now, since $(1, b)$ is not connected to $(3, b)$, we can transfer some amount of b

from 3 to 1 without violating sd–envy-freeness. Let us transfer the assignment of b from 3 to 1 until any additional transfer makes 3 to envy 1. This way we can transfer one-quarter the probability of b from 3 to 1. Hence agent 1's assignment is finalized, and we can add the c share of agent 1 in $\pi^{ps}(R)$ to the c assignment of agent 3. Thus, we obtain the assignment π in Example 9, which is sd–envy-free and not PS assignment.

6.3 Conclusion

In this study, we focus on the probabilistic serial assignment. We question at which preference profiles the probabilistic serial assignment is unique sd–envy-free and sd-efficient assignments. In Propositions 1 , we show that connectedness of a directed graph induced by the preference profile provides a necessary condition, which turns out to be sufficient if the given preference profile satisfies our betweenness condition by Proposition 2.

6.4 Proofs of Propositions

6.4.1 Proof of Proposition 1

First, we introduce some notation. For each $R \in \mathcal{R}$, $i \in N$, and $a \in A$, let $U(R_i, a)$ and $L(R_i, a)$ denote the upper and the lower contour sets of R_i at a, that is, $U(R_i, a) = \{b \in A : b \; R_i \; a\}$ and $L(R_i, a) = \{b \in A : a \; R_i \; b\}$. Let P_i stand for the strict part of the preference relation R_i. Let $U(P_i, a)$ and $L(P_i, a)$ denote the strict upper and the strict lower contour sets of R_i at a, that is, $U(P_i, a) = \{b \in A : b \; P_i \; a\}$ and $L(P_i, a) = \{b \in A : a \; P_i \; b\}$. Let $V = N \times A$

denote the vertex set of $G(R)$ for each $R \in \mathcal{R}$.

Let π be an sd-efficient probabilistic assignment at R. For a contradiction, suppose that π is an sd–envy-free and sd-efficient assignment other than π^{ps} at R, i.e. $Sp(\pi) \neq Sp(\pi^{ps})$. We will show that, if $G(R)$ is connected, then $\pi = \pi^{ps}$, which will yield a contradiction. In showing that, the following result, which is Theorem 1 of Hashimoto et al. (2014), will be useful: $\pi = \pi^{ps}$ if and only if for each $a \in A$ and $i, j \in N$ such that $\pi(i, a) > 0$, we have $\pi(j, U(R_j, a)) \geq \pi(i, U(R_i, a))$. Next, we show that if $G(R)$ is connected, then for each $a \in A$ and $i, j \in N$ such that $\pi(i, a) > 0$, we have $\pi(j, U(R_j, a)) \geq \pi(i, U(R_i, a))$.

First, we show that for each $(i, a), (j, b) \in V$, if $(i, a) \to (j, b)$, then $\pi(j, U(P_j, b)) \geq \pi(i, U(P_i, a))$. Since $Sp(\pi) \subset Sp(\pi^{ps})$, for each $x \in U(P_i, a)$ and each $y \in L(P_j, b)$ such that $\pi(i, x) > 0$ and $\pi(j, y) > 0$, we have $x \; P_j \; y$. Let z be R_j-best object in $L(P_j, b)$ with $\pi(j, z) > 0$. Since $(i, a) \to (j, b)$, for each $x \in U(R_i, a)$ with $(i, x) \in Sp(\pi)$, $x \in U(P_j, z)$. Since π is envy free, $\pi(j, U(P_j, z)) \geq \pi(i, U(P_j, z))$. Hence we obtain $\pi(j, U(R_j, b)) \geq \pi(i, U(R_i, a))$.

Now, since $G(R)$ is a-connected, there is a path that connects (i, a) to (j, a) in $G(R)$. From the above finding, it follows that $\pi(j, U(R_j, a)) \geq \pi(i, U(R_i, a))$.

6.4.2 Proof of Proposition 2

To prove this result, first we show that for each $a \in A$ and $(i, a), (j, a) \in V$ such that (i, a) is not linked to (j, a) in $G(R)$, in the PS assignment we can increase the probability that i receives a without causing j to envy i.

Lemma 2. *For each $a \in A$ and $(i, a), (j, a) \in V$, if $(i, a) \not\to (j, a)$, then there exists $\epsilon_{ij} > 0$ such that $\pi^{ps}(j, U(R_j, a)) > \pi^{ps}(i, U(R_j, a)) + \epsilon_{ij}$.*

Proof. For the proof we need the following two observations.

Observation 1: Since $(i,a) \not\to (j,a)$, there exits $x \in U(R_i,a)$ such that $\pi^{ps}(i,x) > 0$ and at R_j, j ranks x below some $b \in L(R_j,a)$ such that $\pi^{ps}(j,b) > 0$.

Observation 2: For each $y \in U(P_j,a) \setminus U(R_i,a)$ such that $\pi^{ps}(j,y) > 0$, we have $\pi^{ps}(i,y) = 0$. Suppose not. Since $y \in L(P_i,a)$ and $\pi^{ps}(i,y) > 0$, we have $\pi^{ps}(i,U(R_i,y)) > \pi^{ps}(i,U(R_i,a))$. By definition of π^{ps}, we have $\pi^{ps}(i,U(R_i,a)) = \pi^{ps}(j,U(R_j,a))$. Then, $\pi^{ps}(i,U(R_i,y)) > \pi^{ps}(j,U(R_j,y))$, which contradicts the definition of π^{ps}.

It directly follows from Observation 1 and Observation 2 that $\pi^{ps}(i,U(R_i,a)) \geq \pi^{ps}(i,U(R_j,a)) + \pi_i^{ps}(x)$. Now, choose $\epsilon_{ij} = \frac{\pi_i^{ps}(x)}{2}$. Since by definition of π^{ps} we have $\pi^{ps}(j,U(R_j,a)) = \pi^{ps}(i,U(R_i,a))$, we obtain

$$\pi^{ps}(j,U(R_j,a)) > \pi^{ps}(i,U(R_j,a)) + \epsilon_{ij}$$

. □

Next we introduce an auxiliary allocation mechanism, which is a generalization of the *PS* mechanism to a setup where the available capacity of an object is not necessarily 1 and can be an arbitrary amount. Let $q \in \mathbb{R}_+^A$ be a quota vector, which specifies, for each object, the available amount of the object. For a given $R \in \mathcal{R}^S$, agents eat starting from their most preferred objects at equal speeds as usual. The algorithm terminates when each object is exhausted (note that an agent may end up eating more than or less than 1 unit of objects). We denote the *PS* assignment at (R,q) by $\pi^{ps}(R,q)$. For each $q \in \mathbb{R}_+^A$, let $S(q) = \{a \in A : q_a > 0\}$. As before, $Sp(\pi(R,q))$ denotes the set of agent-object pairs that are assigned with positive probability at $\pi(R, q)$.

Definition 9. Let $R \in \mathcal{R}^S$ and $q \in \mathbb{R}^A_+$. We say that $Sp(\pi^{ps}(R, \cdot))$ is **upper semi-continuous** at $q \in \mathbb{R}^A_+$ if there exists an $\epsilon > 0$ such that for each $q' \in \mathbb{R}^A_+$ with $||q' - q|| < \epsilon$ and $S(q') \subset S(q)$, we have $Sp(\pi^{ps}(R, q')) \subset Sp(\pi^{ps}(R, q))$.

Lemma 3. *Let $\vec{1}$ stand for the unit quota vector in which each object has a quota of 1 unit. For a given preference profile $R \in \mathcal{R}^S$, if R satisfies **betweenness**, then $Sp(\pi^{ps}(R, \cdot))$ is **upper semi-continuous** at $\vec{1} \in \mathbb{R}^A_+$.*

Proof. Consider $\pi^{ps}(R)$ (or equivalently $\pi^{ps}(R, \vec{1})$) and for each object $a \in A$, let $t(\pi^{ps}(R), a)$ be the set of objects that are exhausted before a. It follows from the definition of the PS mechanism that for each agent i and object a, we have $\pi^{ps}(R)(i, a) > 0$ if and only if $U(P_i, a) \subset t(\pi^{ps}(R), a)$. Note that given the weak order of exhaustion times of the objects while running PS at R for different quota vectors q, one can identify the support of $\pi^{ps}(R, q)$. Therefore, given two quota vectors q and q', if the order of exhaustion times of the objects are the same in $\pi^{ps}(R, q)$ and $\pi^{ps}(R, q')$, then $Sp(\pi^{ps}(R, q)) = Sp(\pi^{ps}(R, q'))$.

Now, let ϵ be such that for each $a, b \in A$ that are exhausted at different times in $\pi^{ps}(R)$, $0 < \epsilon < |t_a - t_b|/n$. Note that for each $q' \in \mathbb{R}^A_+$ with $||q' - \vec{1}|| < \epsilon$, none of the exhaustion orders will be reversed while obtaining $\pi^{ps}(R, q')$. That is, if an object a is exhausted before another object b at $\pi^{ps}(R)$, then a is exhausted before b at $\pi^{ps}(R, q')$ too. However, two objects that are exhausted simultaneously at $\pi^{ps}(R)$, may be exhausted at different times at $\pi^{ps}(R, q')$. For the rest, let π stand for $\pi^{ps}(R)$ and π' for $\pi^{ps}(R, q')$.

Now, by contradiction suppose that there exists a pair $(i, a) \in N \times A$ such that $\pi(i, a) = 0$ but $\pi'(i, a) > 0$. It follows that $U(P_i, a) \not\subset t(\pi, a)$ but $U(P_i, a) \subset$

$t(\pi', a)$. We obtain a contradiction by showing that there exists an object $c \in U(P_i, a) \setminus t(\pi', a)$. To see this, first recall that only the objects that are exhausted simultaneously at π may be exhausted at different times at π'. It follows that for each $x \in t(\pi', a) \setminus t(\pi, a)$, x is exhausted at the same time with a at π. From among these let b be the object that i was eating when a is exhausted at π. Since a and b are exhausted simultaneously at π, and $\pi(i, b) > 0$, it follows from betweenness that there exists an object c such that $\pi(i, c) > 0$ and $b \; P_i \; c \; P_i \; a$. Now, since a and b are exhausted simultaneously at π and $b \; P_i \; c$, c is exhausted after a at π. Therefore, by the choice of q', c must be exhausted after a at π'. It follows that although $c \in U(P_i, a)$, $c \notin t(\pi', a)$. □

Lemma 4. *For a given preference profile $R \in \mathcal{R}^S$, let π^t be the partial assignment that is obtained by running PS until time $t \in [0, 1]$ and for each $x \in A$, let $q_x^t = 1 - \pi^t(N, x)$. If $Sp(\pi^{ps}(R, \cdot))$ is upper semi-continuous at $\vec{1}$, then $Sp(\pi^{ps}(R, \cdot))$ is upper semi-continuous at q^t.*

Proof. Let $Sp(\pi^{ps}(R, \cdot))$ be upper semi-continuous at $\vec{1}$. Then, there exists $\epsilon' > 0$ such that for each $q' \in \mathbb{R}_+^A$ with $||q' - q|| < \epsilon'$ and $S(q') \subset S(q)$, we have $Sp(\pi^{ps}(R, q')) \subset Sp(\pi^{ps}(R, q))$. Let ϵ be such that $0 < \epsilon < \epsilon'$. Note that for each q' with $||q^t - q'|| < \epsilon$ and $S(q') \subset S(q^t)$, we have $||(\vec{1} - (\vec{1} - q^t + q')|| < \epsilon$. Since $Sp(\pi^{ps}(R))$ is upper semi-continuous at $\vec{1}$, we obtain $Sp(\pi^{ps}(R, \vec{1} - q^t + q')) \subset Sp(\pi^{ps}(R))$. Note that $Sp(\pi^{ps}(R, \vec{1} - q^t + q')) = Sp(\pi^{ps}(R, q')) \cup Sp(\pi^t)$ and $Sp(\pi^{ps}(R)) = Sp(\pi^{ps}(R, q^t)) \cup Sp(\pi^t)$. Next, we argue that $Sp(\pi^{ps}(R, q')) \subset Sp(\pi^{ps}(R, q^t))$. For this conclusion, it is sufficient to show that for each $(i, a) \in Sp(\pi^{ps}(R, q')) \cap Sp(\pi^t)$, we have $(i, a) \in Sp(\pi^{ps}(R, q^t))$. For each $(i, a) \in Sp(\pi^t)$, if $(i, a) \in Sp(\pi^{ps}(R, q'))$, then $q'_a > 0$. Since $S(q') \subset S(q^t)$, we have $q_a^t > 0$. Thus we have $(i, a) \in Sp(\pi^t)$ and $q_a^t > 0$, note that this is possible only if a is the object that is eaten by agent i at time t. It follows that i first eats a at $\pi^{ps}(R, q^t)$, therefore $(i, a) \in Sp(\pi^{ps}(R, q^t))$. $\square$

Now, we are ready to complete the proof Proposition 2. By contradiction suppose there exists $a \in A$ such that $G(R)$ is not a-connected. First we observe that a cannot be an object that is exhausted last in $\pi^{ps}(R)$. By contradiction, suppose there exists $k \in N$ such that $\pi^{ps}(k, U(R_k, a)) = 1$. Now, we argue that $G(R)$ must be a-connected. To see this note that (1) for each $i \in N$ with $\pi^{ps}(i, a) > 0$, a is the last object that is assigned i in $\pi^{ps}(R)$. (2) for each $j \in N$ with $\pi^{ps}(j, a) = 0$, let b be the last object that j is assigned in $\pi^{ps}(R)$. Since both a and b are exhausted at last, we have $b \ R_j \ a$. It directly follows from (1) and (2) that for each $(i, a), (j, a) \in V$ such that $\pi(i, a) > 0$, $(i, a) \to (j, a)$. In what follows we will construct an assignment $\pi \neq \pi^{ps}$ that is sd–envy-free, sd–efficient and $Sp(\pi) \subset Sp(\pi^{ps})$.

First let us define a partial assignment $\pi' : N \times A \to [0, 1]$ such that for each $i \in N$ and $a \in A$, $\pi'(i, A) \leq 1$ and $\pi'(N, a) \leq 1$. Note that we can still consider sd–envy-freeness of π', and if π' is sd–envy-free then for each $i, j \in N$ we must have $\pi'(i, A) = \pi'(j, A)$. Now let π' be the partial assignment, which is obtained by running the PS algorithm until a is exhausted. That is, if a is exhausted at time $t \leq 1$, then for each $i \in N$ we have (1) for each $b \in A$ such that $\pi^{ps}(i, U(R_i, b)) \leq t$, $\pi'(i, b) = \pi^{ps}(i, b)$, (2) there is at most one object $c \in A$ such that $\pi'(i, c) > 0$ and $\pi^{ps}(i, c) \neq \pi'(i, c)$, (3) $\pi'(i, A) = t$. Note that $\pi'(N, a) = 1$ and for each $b \in A$ that is exhausted after a, $\pi'(N, b) < 1$. Since π' is obtained through running the PS algorithm, all the arguments for the envy-freeness of π^{ps} holds for π'. Thus, we conclude that π' is sd–envy-free. Next, by using the fact that $G(R)$ is not a-connected, we will construct an sd-envy-free partial assignment π'' via making some small perturbations to π' on the assignment probabilities of the objects that are assigned with positive probability at π' just before time t.

Since $G(R)$ is not a-connected, there exist $i^*, j^* \in N$ such that there is no path that connects (i^*, a) to (j^*, a). Let I be the set of all $i \in N$ such that there is a path that connects (i^*, a) to (i, a) and let J be the set of all $j \in N$ such that there is a path that connects (j, a) to (j^*, a). Note that since $(i^*, a) \to (i^*, a)$ and $(j^*, a) \to (j^*, a)$, we have $i^* \in I$ and $j^* \in J$. For each $i \in I$ and $j \in J$, since there is no path that connects (i^*, a) to (j^*, a), there cannot be any path that connects (i, a) to (j, a), so $(i, a) \not\to (j, a)$. It follows from Lemma 2 that for each $i, j \in N$ such that $\pi(i, a) > 0$ and $(i, a) \not\to (j, a)$, there exists $\epsilon_{ij} > 0$ such that $\pi^{ps}(j, U(R_j, a)) > \pi^{ps}(i, U(R_j, a)) + \epsilon_{ij}$. Now, for any $\epsilon \leq min_{\{i,j \in N : \pi(i,a)>0 \text{ and } (i,a) \not\to (j,a)\}} \epsilon_{ij}$, let $2|I|\epsilon_I = 2|J|\epsilon_J = \epsilon$, so we have $\epsilon_I + \epsilon_J \leq \epsilon$. Next, we define π'' as follows:

i. For each $i \in I$, $\pi''(i, a) = \pi'(i, a) + \epsilon_I$ and for any $b \neq a$, $\pi''(i, b) = \pi'(i, b)$.

ii. For each $j \in J$, $\pi''(j, a) = \pi'(j, a) - \epsilon_J$, let b be the next consumable object for j after a, then $\pi''(j, b) = \epsilon_I + \epsilon_J$ and finally for any c except a and b, $\pi''(j, c) = \pi'(j, c)$.

iii. For each $k \notin I \cup J$ with $\pi(k, a) > 0$, let b be the next consumable object for k after a. Now, let $\pi''(k, b) = \epsilon_I$ and for each $c \neq b$, $\pi''(k, c) = \pi'(k, c)$.

iv. Finally for each $k \in N$ with $\pi(k, a) = 0$, let b be the lowest ranked object that is consumed with positive probability in π'.

a. If b is exhausted after a, then let $\pi''(k,b) = \pi'(k,b) + \epsilon_I$ and for each $c \neq b$, $\pi''(k,c) = \pi'(k,c)$.

b. If a and b are exhausted at the same time, then let c be the next object that is consumed by k at π^{ps}. Let $\pi''(k,c) = \pi'(k,c) + \epsilon_I$ and for each $d \neq c$, $\pi''(k,d) = \pi'(k,d)$.

Now, we argue that π'' is envy-free. First, it is easy to see that by our choice of ϵ and Lemma 2, no agent envies another because of a. Second, no agent envies another because of a previously exhausted object, since we kept the probabilities of all such objects as in $\pi^{PS}(R)$, which is envy-free. Finally no agent envies another because of his lowest-ranked object that he is assigned with positive probability, since for each agent the total probability that he is assigned to an object that is at least as good as that object is equal to $t + \epsilon_I$. To see this note that by construction of π'' for each $i \in N$, $\pi''(i,A) = \pi'(i,A) + \epsilon_I = t + \epsilon_I$. Thus, we conclude that π'' is sd–envy-free. Next, note that while $\pi''(N,a) = 1$, for some $b \neq a$ we might have $\pi''(N,b) > 1$. Now, we argue that we can choose ϵ such that for each $b \in A$, $\pi''(N,b) \leq 1$. To see this, first observe that by construction of π'' for each b that is exhausted after a we have $\pi''(N,b) \leq \pi'(N,b) + n \cdot \epsilon$. So, we can choose ϵ so small that $\pi''(N,b) \leq 1$. Hence, we obtain an sd–envy-free partial assignment π'' such that the assignment of a is different from that of $\pi^{ps}(R)$.

Next, we extend the partial assignment π'' to an assignment π^*. Let $q, q'' \in \mathbb{R}^A_+$ be the quota vectors of the objects such that for each $x \in A$, $q_x = 1 - \pi'(N,x)$ and $q''_x = 1 - \pi''(N,x)$ respectively. Define the assignment $\pi^* = \pi'' + \pi^{ps}(R, q'')$. First, we argue that $Sp(\pi^*) \subset Sp(\pi^{ps}(R))$. To see this first note that, by the construction of π'', we have $Sp(\pi'') \subset Sp(\pi^{ps}(R))$. Since R satisfies betweenness, it follows from Lemma 3 and 4 that $Sp(\pi^{ps}(R,\cdot))$ is upper semi-continuous at q. Therefore we can choose ϵ so small that $Sp(\pi^{ps}(R,q'')) \subset Sp(\pi^{ps}(R,q))$. Moreover, since at least $\pi^*(i^*,a) \neq \pi^{ps}(i^*,a)$, π^* is different from $\pi^{ps}(R)$. Since we can easily express π^* as an eating mechanism, it is sd–efficient. Finally, we argue that π^* is sd–envy-free. To see this, note that for each $i,j \in N$ and $x \in A$, we have $\pi^*(i,U(R_i,x)) = \pi''(i,U(R_i,x)) + \pi^{ps}(R,q)(i,U(R_i,x))$ and $\pi^*(j,U(R_i,x)) = \pi''(j,U(R_i,x)) + \pi^{ps}(R,q)(j,U(R_i,x))$. Since π'' and $\pi^{ps}(R,q)$ are sd–envy-free, $\pi^*(i,U(R_i,x)) \geq \pi^*(j,U(R_i,x))$. It follows that π^* is sd–envy-free.

CHAPTER 7

CONCLUSION

In this thesis, we have focused on MC-representations of *path independent* choice rules. Existence of such MC-representations are known from the literature, but there is no study which clarifies how to construct such representations efficiently. This problem is of interest for several applications in which a *path independent* choice rules needs to be communicated to the society. We explored this problem and obtained several results.

In Chapter 3, we find the size of the minimal MC-representation. This result relies heavily upon the prime set notion that we introduced. Most of the other notions that we used in this study have been used in the literature but the prime set notion have been the most helpful one to crack this problem. This notion might deserve more attention and could be helpful to clarify more unanswered questions in related literature.

In Chapter 4 we restricted our focus to *q-acceptant* and *path independent* choice rules. We believe that *q-acceptance* is a natural restriction in many applications where institutions prefer to fill their positions whenever possible. Under this additional restriction it turns out that the number of prime atoms provides the size of the minimal MC-representation. Since finding the minimal size representation is more tractable with *q-acceptance* restriction, we also figured out the upper bound for all *q-acceptant* choice rules. Our final result from this chapter reveals this upper bound is satisfied by one of the well-known choice rules, namely q-responsive choice rules.

In Chapter 5 we have introduced a generalization of q-responsive choice rules for the purpose of enhancing MC-representations and reducing the number of orderings in representations significantly. As we have discussed in the previous chapter, q-responsive choice rules can be expressed by just one priority ordering but MC-representation requires as much orderings as possible. If we can succeed to represent *path independent* choice rules as the union of a set of *responsive* choice rules then that would simplify the representation. Even though this pursuit have not been concluded in this thesis, this study opened avenues which made this seem possible. Along these lines, we have introduced weighted responsive choice rules which encompass all *path independent* and *responsive* choice rules.

In the last chapter we have focused on probabilistic serial mechanism for assignment problem. We have tried to characterize the situation where probabilistic serial is the unique sd-efficient and sd–envy-free assignment. We have provided a connectedness definition over preference profile, which is a necessary condition for our problem. We have also showed that under a betweenness assumption, connectedness is also a sufficient condition.

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