# LEBESGUE CONSTANTS ON CANTOR TYPE SETS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

# LEBESGUE CONSTANTS ON CANTOR TYPE SETS

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The properties of compact subsets of the real line which are in the class of Bounded Lebesgue Constants (BLC) are investigated. Knowing that any such set must have 1-dimensional Lebesgue measure zero and nowhere density, and the fact that there are examples of countable sets both inside and outside of the class BLC, families of Cantor-type sets were focused on. Backed up by numerical experiments (up to degree 128) and analytical results, the conjecture “there exists no perfect set in BLC” was put forward.

*Keywords:* Lebesgue Constants, Cantor type Sets, Faber Basis, Lagrange Interpolation.

# ÖZET

# KANTOR TİPİ KÜMELERDE LEBESGUE SABİTLERİ

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Sınırlı Lebesgue Sabitleri (SLS) sınıfının içinde bulunan gerçel eksenin kompakt alt kümelerinin özellikleri incelenmiştir. Bu tip kümelerin tek boyutlu Lebesgue ölçütlerinin sıfır olduğu, hiç bir yerde yoğun olmamaları ve bu sınıfın hem içinde hem dışında sayılabilir küme örnekleri olduğu gerçeği göz önünde bulundurularak, Kantor tipi kümlere odaklanılmıştır. Sayısal deneylerle (128 dereceye kadar) ve analitik sonuçlarla desteklenen, “SLS sınıfı içinde mükemmel küme bulunmamaktadır.” hipotezi ortaya konulmuştur.

*Anahtar sözcükler*: Lebesgue Sabitleri, Kantor tipi Kümeler, Faber Bazı, Lagrange İnterpolasyonu.

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# Chapter 1

# Introduction

Interpolation is the method of approximating what lies beyond the discrete collected data. It is called *Lagrange interpolation* when continuous functions are interpolated by polynomials. It is perhaps the most commonly studied branch of interpolation theory as polynomials are perfect, simple tools for approximation.

Given a compact set $K \subset \mathbb{R}$ and a triangular matrix with distinct entries in each row $X = (x_{k,N})_{k=1,N=1}^{N,\ \infty} \subset K$, the corresponding Lagrange interpolatory polynomials are denoted by $(L_N)_{N\in\mathbb{N}}$ where for each $N \in \mathbb{N}$, $L_N(f, X, .) \in \mathcal{P}_{N-1} =$ $\{$algebraic polynomials of degree less than or equal to $N-1\}$ and

$$L_N(f, X, x_{k,N}) = f(x_{k,N}), \quad \text{for } k = 1, 2, ..., N. \tag{1.1}$$

Lebesgue constants are the operator norms of Lagrange interpolatory polynomials. They are denoted by $(\Lambda_N(X, K))_{N\in\mathbb{N}}$ where

$$\Lambda_N(X, K) = ||L_N(., X, .)|| = \sup_{||f||=1} \sup_{x\in K} |L_N(f, X, x)|.$$

Due to Weierstrass Theorem, we know that in the space of continuous functions on compact sets (denoted as $\mathcal{C}(K)$), any function can be approximated (via convergence with respect to the supremum norm) by polynomials. However, it turns out the additional condition of interpolation (1.1) disrupts this fact.

Due to Lebesgue Lemma, the growth of the Lebesgue constants is crucially related to the accuracy of Lagrange interpolation.

From the famous result of Faber [10], on a compact interval, for any system of nodes, Lebesgue constants follow at least logarithmic growth. In chapter 2, following the classical literature, we take a look at the growth of Lebesgue constants on a compact interval for some significant system of nodes. More specifically, we see that when the nodes are distributed equidistantly, Lebesgue constants $(\Lambda_N(E, [-1, 1]))_{N\in\mathbb{N}}$ grow exponentially, which corresponds to rapidly growing error terms $||L_N(f, E, .) - f(.)||_{[-1,1]}$. Taking the nodes to be the zeros of Chebyshev polynomials, the corresponding Lebesgue constants $(\Lambda_N(T, [-1, 1]))_{N\in\mathbb{N}}$ follow a logarithmic growth and later on we demonstrate that they are infact very close to being optimal.

In the final part of the chapter, we follow the notation introduced on [3] and say $K \in BLC$ (Bounded Lebesgue Constants) if there exists a system of nodes in $K$ of which the Lebesgue constants are bounded. We go over the relevant results and some necessary conditions of the compact set $K$ to be in the class $BLC$. Perhaps the most important one (at least for our research aims) being the condition of $K$ having Lebesgue measure zero and nowhere density on the real line, proven by Szabados, Vertesi [22].

In chapter 3, having our search narrowed down by these conditions, we first take a look at countable sets where by Obermaier [17] the family of sets $S_q = \{q^n : n \in \mathbb{N}\} \cup \{0\}$ for $0 < q < 1$ are in the class of $BLC$. This result was later generalized by him together with Szwarc [18] where sets consisting of monotone sequences with their limit points are shown to be in $BLC$ if they satisfy geometric progression or a faster convergence. On the other hand, Privalov [19] showed that there exist countable sets where Lebesgue constants are unbounded for every system of nodes.

Turning our attention to Cantor-type sets (perfect, nowhere dense), we see that

Korovkin [14] consturcted a perfect set with a bounded subsequence of Lebesgue constants, which imply that the continous functions can be approximated on that set, by that subsequence of Lagrange interpolatory polynomials.

In second part of chapter 3, we adress Mergelyan's book [16] dated 1951, where he attains some important results in the theory of complex approximation. In one of his supplementary theorems, Mergelyan asserts that geometrically symmetric Cantor sets, if they are sufficiently small, are in the class $BLC$. We look at his proof and show that it is not correct. Although his theorem can be utilized to show that, infact for sufficienty small (smaller than what he considered) geometrically symmetric Cantor type sets, we have a bounded subsequence of Lebesgue constants.

Finally, we introduce a range of geometrically symmetric Cantor-type sets and analyze the Lebesgue constants on these sets. With the support of numerical results regarding these families, we conjecturize that perfect sets are outside of the class BLC.

# Chapter 2

# Lagrange Interpolation and Lebesgue Constants

## 2.1 Notations and Definitions

Let $K \subset \mathbb{R}$ be a compact set and $X = (x_{k,N})_{k=1,N=1}^{N,\ \infty} \subset K$ be an *interpolatory matrix* (we use the notation $X \subset K$ to indicate that the entries of the matrix $X$ are elements of $K$), i.e, a triangular matrix such that every row consists of distinct entries (called *interpolation nodes*). Moreover, WLOG assume $x_{k,N} < x_{k+1,N}$ for convenience later on.

For such $K$, $X$ and $N \in \mathbb{N}$ and for a fixed $k \in \{1, 2, ..., N\}$, the corresponding *fundemental polynomial of Lagrange interpolation*, denoted as $l_{k,N}$ is the unique polynomial of degree $N-1$, satisfying $l_{k,N}(x_{j,N}) = \delta_{k,j}$ where $\delta$ is the Kronicker delta. It can be written explicitly as

$$l_{k,N}(X, x) := \frac{\omega_N(X, x)}{(x - x_{k,N})\ \omega'_N(X, x_{k,N})}\ , \tag{2.1}$$

where $\omega_N(X, x) = \prod\limits_{k=1}^{N} (x - x_{k,N})$. The *N'th Lagrange interpolatory polynomial* is defined as

$$L_N(f, X, x) := \sum_{k=1}^{N} f(x_{k,N}) l_{k,N}(X, x) \ , \quad f \in \mathcal{C}(K), \tag{2.2}$$

the *Lebesgue function* as $\lambda_N(X, x) := \sum_{k=1}^{N} |l_{k,N}(X, x)|$ and the *Lebesgue constant* as $\Lambda_N(X, K) := \max_{x \in K} \lambda_N(X, x)$.

Some basic properties of Lebesgue functions are as following (see [15]) : $\lambda_N$ is a piecewise polynomial (where pieces are $[x_{i,N}, x_{i+1,N}]$, $i = 1, 2, ...N-1$) for $N \geq 2$. For all $x \in \mathbb{R}$ we have $\lambda_N(X, x) \geq 1$ with equality only on the interpolation nodes. In each interval $[x_{i,N}, x_{i+1,N}]$, $i = 1, 2, ...N-1$, $\lambda_N$ has a single maximum and in $(-\infty, x_{1,N})$ and $(x_{N,N}, \infty)$ it is convex and monotone.

We see that the $N$'th Lagrange interpolatory polynomial can be thought of as a projection

$$L_N(., X, .) : \mathcal{C}(K) \to \mathcal{P}_{N-1},$$

where $\mathcal{P}_{N-1} = \{$algebraic polynomials of degree less than or equal to $N-1\}$.

Regarding the operator norm of $L_N(., X, .)$, it is a simple exercise to show that

$$||L_N|| = \max_{||f|| \leq 1} \max_{x \in K} |L_N(f, X, x)| = \max_{x \in K} \lambda_N(X, x) = \Lambda_N(X). \tag{2.3}$$

These operator norms of $(L_N)_{N \in \mathbb{N}}$ play a crucial role in the convergence of the interpolatory polynomials to the corresponding functions of $\mathcal{C}(K)$. Namely, by Lebesgue Lemma, we have

$$|L_N(f, X, x) - f(x)| \leq (\Lambda_N(X) + 1) E_{N-1}(f), \quad x \in K \tag{2.4}$$

where $E_{N-1}(f) = \min_{P \in \mathcal{P}_{N-1}} ||f - P||_K$.

We can see from here that if

$$\lim_{N \to \infty} \Lambda_N(X, K) E_{N-1}(f) = 0 \tag{2.5}$$

is satisfied, then $L_N(f, X)$ uniformly converges to $f$ on $K$. As a consequence, boundedness of the Lebesgue constants and Weierstrass Approximation Theorem imply that the Lagrange interpolatory polynomials uniformly converge to each continuous function on the corresponding compact set.

From now on, some arguments of $L_N, \lambda_N, \Lambda_N$, etc. might be omitted where it doesn't create a confusion.

## 2.2 Classical Results for Interpolation on Intervals

Due to Weierstrass Approximation Theorem, the expectation in the mathematical world regarding polynomial interpolation in compact intervals was highly positive at late 19th, early 20th century, which is why Faber's result in 1914 came as a mild shock and perhaps a let down. Faber [10] showed that for any interpolatory matrix $X \subset [-1, 1]$, there exists a function $f \in \mathcal{C}[-1, 1]$ such that $\limsup_{N\to\infty} ||L_N(f, X)|| = \infty$, where $||.||$ denotes the supremum norm in $\mathcal{C}[-1, 1]$. In fact he proved that the corresponding Lebesgue constants followed at least logarithmic growth.

Regardless of this fact, the exigency of uses of interpolation on intervals is so strong that it is used often and in any science or engineering where there are discrete data points and the need to fill in the blanks in between them. Arguably, the most essential problem that attracted mathematicians until this day, has been to obtain the optimal (in the sense of smallest corresponding Lebesgue constants) set of interpolation nodes on an interval. Although many properties about these nodes are now known, their explicit formula still constitutes an open problem.

Let us do a quick review of the classical literature corresponding to Lagrange interpolation and Lebesgue constants on compact intervals. Before moving on further, we need to note that the fundemental polynomials are invariant under affine transformations, so the results obtained for a certain interval, apply to all of them. Throughout this section, $K = [-1, 1]$ and will be ommited as an argument when possible.

### 2.2.1 Equidistant Nodes

The first negative result regarding Lagrange interpolation was due to Runge in 1901. Runge [20] showed that there exists a function such that its uniform distance to the corresponding Lagrange interpolatory polynomials (using equidistant nodes) diverges.

Let $E \subset [-1, 1]$ be the interpolatory matrix of equidistant nodes, i.e $E = (x_{k,N})_{k=1,N=1}^{N,\ \infty}$ with $x_{k,N} = -1 + \dfrac{2(k-1)}{N-1}$ for $k = 1, 2, ..., N$ and $N \geq 2$.

Moreover let $||.||_{[-1,1]}$ denote the sup-norm in $\mathcal{C}[-1, 1]$.

**Theorem 2.2.1** (Runge). *Let* $f(x) = \dfrac{1}{1+25x^2}$ *for* $x \in [-1, 1]$. *Then*

$$\lim_{N\to\infty} ||L_N(f, E) - f||_{[-1,1]} = \infty.$$

Admitting fundemental importance to Runge's demonstration, 17 years later, Bernstein [1] showed that even a function as basic as $f(x) = |x|$, $x \in [-1, 1]$ when interpolated equidistantly, not only couldn't be uniformly approximated but the process diverged, in the sense of pointwise limits, almost everywhere.

**Theorem 2.2.2** (S.N Bernstein). *For* $f(x) = |x|, \quad x \in [-1, 1]$ *and every* $x_0 \in (-1, 1) \setminus \{0\}$, *the sequence* $\{L_N(f, E, x_0), N = 1, 2, ...\}$ *diverges.*

The exact asymptotic estimations for the Lebesgue constants corresponding to equidistant nodes came many years later. Turetskii [23] and later Schönhage [21] estimated that these Lebesgue constants grow exponentially. Namely

$$\Lambda_N(E) \sim \frac{2^{N+1}}{eN(\log N + \gamma)}, \tag{2.6}$$

where $\gamma$ is the Euler constant, i.e.

$$\gamma = \lim_{n\to\infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \log n \right) \approx 0.577. \tag{2.7}$$

### 2.2.2 Chebyshev and Extended Chebyshev Nodes

From Runge and Bernstein's examples, after some further analysis, it was evident that the equidistant interpolation caused more problems near the end points compared to the mid part of the interval. Due to this observation, it was only natural to increase the density of the nodes around the end points. Thus came the nodes corresponding to the zeros of Chebyshev polynomials.

Let $T = (x_{k,N})_{k=1,N=1}^{N,\ \infty} \subset [-1,1]$ with $x_{k,N} = -\cos\left(\frac{(2k-1)\pi}{2N}\right)$ for $k = 1, 2, ..., N$ and $N \in \mathbb{N}$.

The sequence of Lebesgue constants corresponding to these nodes $\{L_N(T), N \in \mathbb{N}\}$ was subject to many asymptotic estimations and improvements of those estimations, without going into too much historical detail, starting with Bernstein [1] and realising its final upper and lower bounds by Günttner [11]. The corresponding Lebesue constants were found follow a logarithmic growth, more specifically

$$a_o + \frac{2}{\pi}\log N < \Lambda_N(T) < 1 + \frac{2}{\pi}\log N, \tag{2.8}$$

where $a_o = \frac{2}{\pi}\left(\gamma + \log\frac{8}{\pi}\right) = 0.9625...$ .

After the observation $\Lambda_N(T) = \lambda_N(T,1)$ by Lutmann and Rivlin [15], the idea of making the interpolatory matrix *cannonical*, i.e. taking the end points of the interval in question as interpolation nodes on every step, occured. The extended Chebyshev nodes is the linear transformation of the Chebyshev nodes such that $[x_1, x_N]$ is mapped into $[-1,1]$ with $x_1 \mapsto -1$ and $x_N \mapsto 1$. Explicitly

$\hat{T} = (x_{k,N})_{k=1,N=1}^{N,\ \infty}$ with

$$x_{k,N} = -\frac{\cos\left(\dfrac{(2k-1)\pi}{2N}\right)}{\cos\left(\dfrac{\pi}{2N}\right)} \quad k = 1, 2, ..., N \ , \ N \in \mathbb{N} \ .$$

Brutman [4] found an upper bound corresponding to these nodes as

$$\Lambda_N(\hat{T}) < 0.7219 + \frac{2}{\pi} \log N, \tag{2.9}$$

and later on Günttner [12] proved the existence of an asymptotic expansion, for $k = 1, 2, ...$:

$$\Lambda_N(\hat{T}) = \frac{2}{\pi} \log N + b_0 + \frac{b_1}{\log N} + ... + \frac{b_k}{(\log N)^k} + R_N^{(k)} \tag{2.10}$$

where

$$b_0 = 0.5381... \ , \quad b_1 = 0.006371... \ , \dots \ , R_N^{(k)} = O\left(\frac{1}{(\log N)^{k+1}}\right) \ .$$

### 2.2.3 Optimal Nodes

A natural question that arises is: what is the best choice of nodes for Lagrange interpolation on the interval, i.e. $X^* \subset [-1, 1]$ such that $\Lambda_N(X^*) = \inf_{X \subset [-1,1]} \Lambda_N(X)$, for each $N \in \mathbb{N}$? Existence of such nodes is easy to prove (see [22], pp. 94).

It was initially Faber [10] who proved that any sequence of Lebesgue constants on a compact interval would follow at least logarithmic growth. In 1931, Bernstein [2] put forward a conjecture that the Lebesgue function corresponding to the optimal nodes *equioscillates*, i.e. values the function attains at each maximum are the same and he estimated asymptotically

$$\Lambda_N(X^*) \sim \frac{2}{\pi} \log N. \tag{2.11}$$

In 1950, Erdös [8] added as a conjecture that given that the interpolatory matrix is cannonical (we don't lose generality if so), the optimal nodes are unique and that

$$m_N(X) \leq \Lambda_N(X^*) \leq M_N(X), \quad N \in \mathbb{N}, \quad \text{for every } X \subset [-1,1] \ , \tag{2.12}$$

where $m_N(X)$ and $M_N(X)$ are the minimal and maximal local maximums of $\lambda_N(X)$, respectively.

The results in the direction of these conjectures started coming after 1976, which is when T. Kilgore and E.W. Cheney [5] proved the existence of interpolatory nodes which the corresponding Lebesgue function equioscillates. Two years later, T. Kilgore [13] and DeBoor and Pinkus [6] proved rest of the conjectures.

Utilizing (2.12), also known as Erdös inequality, Brutman [4] has shown that for $N \geq 1$:

$$\frac{1}{2} + \frac{2}{\pi} \log N < m_N(\hat{T}) < \Lambda_N(X^*) < M_N(\hat{T}) < \frac{3}{4} + \frac{2}{\pi} \log N. \tag{2.13}$$

Although we have a nice characterization of the Lebesgue function for the optimal nodes, their explicit forms of these nodes are unknown to this date. On the other hand, Rivlin in his monograph "Chebyshev Polynomials" states: "The readily available extended Chebyshev array $\hat{T}$ is, for all practical purposes as useful as the optimal nodes". We can infer from here that the extended Chebyshev nodes are considered to be "almost optimal".

## 2.3 Divergence of Lagrange Interpolation

As we saw in the previous section, interval is not exactly an ideal set for Lagrange interpolation of continuous functions in general. In order to obtain a converging Lagrange interpolatory process, one must return to the inequality (2.4).

There are three obvious frontiers to attack this problem from. First one is restricting the class of functions that are going to be interpolated, (i.e. attain a more rapidly decreasing sequence $E_N(f)$ for every $f$ in that restricted family) with criteria such as smoothness, Lipschitz continuity, etc.

Second one is to loosen restrictions on the degrees of interpolating polynomials. Without going into too much detail, it was found by Erdös [7] that given an interpolatory matrix $X$, for any function $f \in \mathcal{C}[-1,1]$ and $\varepsilon > 0$, there exists a sequence of interpolatory polynomials $p_N(f) \in \mathcal{P}_{N(1+\varepsilon)}$ such that $p_N$ interpolates $f$ at $N$ distinct nodes ($N$'th row of $X$) and

$$\lim_{N\to\infty} ||f - p_N(f)||_{[-1,1]} = 0 \tag{2.14}$$

if and only if $X$ satisfies what is known as the Erdös conditions. For an extended review of results in this direction, we refer the reader to [22], pp. 37 – 69.

Third one is to restrict the domain of the functions, i.e. look for a compact set $K$ and an interpolatory matrix $X \subset K$ such that the corresponding Lebesgue constants are small, preferably bounded. In the next chapter, such pairs of sets and matrices will be investigated. First, let us state some results that will direct us through our search of such pairs.

In 1918, Bernstein [1] proved the following theorem regarding pointwise divergence of Lagrange polynomials for arbitrary system of nodes:

**Theorem 2.3.1** (Bernstein). *For any $X \subset [-1,1]$ there exist a point $x_0 \in [-1,1]$ and $f \in \mathcal{C}[-1,1]$ such that*

$$\overline{\lim}_{N\to\infty} |L_N(f, X, x_0)| = \infty \tag{2.15}$$

The next result by Erdös in 1958 proved that the sequence of Lebesgue functions for any interpolatory matrix can only be bounded in a set of measure zero.

**Theorem 2.3.2** (Erdös)**.** *Let $\varepsilon$ and $A$ be any given positive numbers and $X \subset [-1,1]$ any interpolatory matrix. Then, the measure of the set*

$$\{x \in \mathbb{R} : \lambda_N(X, x) \leq A \text{ for } N \geq N_0(A, \varepsilon)\}$$

*is less than $\varepsilon$.*

In 1980, a theorem that seems like continuation of the previous one was proven by Erdös and Vertesi [9].

**Theorem 2.3.3** (Erdös, Vertesi)**.** *Let $X \subset [-1,1]$, then there exists $f \in \mathcal{C}[-1,1]$ such that*

$$\limsup_{N\to\infty} |L_N(f, X, x_0)| = \infty,$$

*for almost all $x_0 \in [-1,1]$.*

From (2.5) we know that boundedness of Lebesgue constants implies uniform convergence of Lagrange interpolatory polynomials on the corresponding compact set. The next proposition by Bilet, Dovgoshey, Prestin [3] show the inverse implication is also true.

**Proposition 2.3.4** (Bilet, Dovgoshey, Prestin)**.** *Let $K \subset \mathbb{R}$ be infinite and compact, and let $X \subset K$ be an interpolatory matrix. Then the following are equivalent:*

1. *The inequality*

   $$\limsup_{N\to\infty} \Lambda_N(X) < \infty$$

   *holds.*

2. *The limit relation*

   $$\lim_{N\to\infty} ||L_N(f, X) - f||_K = 0$$

   *is true for every $f \in \mathcal{C}(K)$.*

3. *The inequality*

$$\limsup_{N\to\infty} ||L_N(f,X)||_K < \infty$$

*holds for every* $f \in \mathcal{C}(K)$.

*Proof.* By (2.4) we have (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), assume (3) is true, then the sequence $(||L_N(f,X)||_K)_{N\in\mathbb{N}}$ is bounded for all $f \in \mathcal{C}(K)$. Since $\mathcal{C}(K)$ is a Banach space and $L_N(.,X,.) : \mathcal{C}(K) \to \mathcal{C}(K)$ is a continuous, linear operator, the Banach-Steinhaus theorem gives

$$\sup_{N\in\mathbb{N}} ||L_N(.,X,.)||_K < \infty.$$

$\square$

There is also an analog of the pointwise version of the previous proposition.

**Proposition 2.3.5** (Bilet, Dovgoshey, Prestin)**.** *Let* $K \subset \mathbb{R}$ *be infinite and compact, and let* $X \subset K$ *be an interpolatory matrix and* $x \in K$*. Then the following are equivalent:*

1. *The inequality*

$$\limsup_{N\to\infty} \lambda_N(X,x) < \infty$$

*holds.*

2. *The limit relation*

$$\lim_{N\to\infty} L_N(f,X,x) = f(x)$$

*is true for every* $f \in \mathcal{C}(K)$.

3. *The inequality*

$$\limsup_{N\to\infty} |L_N(f,X,x)| < \infty$$

*holds for every* $f \in \mathcal{C}(K)$.

In the light of this result, Theorem 2.3.3 has the following corollary:

**Corollary 2.3.6.** *Let $K \subset \mathbb{R}$ be infinite and compact and $X \subset K$ be an interpolatory matrix. If*

$$\limsup_{N\to\infty} \Lambda_N(X, K) < \infty,$$

*then $K$ has one dimensional Lebesgue measure equal to zero.*

Now let us introduce a definition that was first put forward by Bilet, Dovgoshey, Prestin [3]. Let us say that the infinite, compact set $K \subset \mathbb{R}$ is in the family of *Bounded Lebesgue Constants* ($K \in BLC$), if there exists an interpolatory matrix $X \subset K$ such that

$$\limsup_{N\to\infty} \Lambda_N(X, K) < \infty. \tag{2.16}$$

**Corollary 2.3.7.** *Let $K \in BLC$. Then $K$ is nowhere dense in $\mathbb{R}$ and its one dimensional Lebesgue measure is zero.*

*Proof.* $K$ has measure zero by Theorem 2.3.3 and Proposition 2.3.5.

Since it is compact we have $\overline{K} = K$. Due to its zero measure we get $Int(\overline{K}) = Int(K) = \emptyset$. □

Now, we are confined in our search to compact sets $K$ with zero measure and nowhere density. Let us take a slight detour and see another significance of this problem, which is constructing an interpolating polynomial basis of the space $\mathcal{C}(K)$, with strict degrees.

# Chapter 3

# Faber and Lagrange Bases

## 3.1 Definitions and Relevance

In what follows, we will see the relevance of Faber bases to Lagrange interpolatory processes. In this section, to a large extent, we follow the survey by Bilet, Dovgoshey, Prestin. Thus for most of the proofs (will be stated otherwise) we refer the reader to [3].

Let us recall first the definition of a *Schauder basis*. Let $V$ be a Banach space over field $F$. Then, a countable set $\{b_n : n \in \mathbb{N}\} \subset V$ is called a Schauder basis if for every $v \in V$, there exist a unique sequence $(\alpha_n)_{n\in\mathbb{N}} \subset F$ such that

$$v = \sum_{n=1}^{\infty} \alpha_n b_n,$$

where the convergence is with respect to the norm topology in V.

**Definition 3.1.1.** Let $K \subset \mathbb{R}$ be infinite and compact. A polynomial Schauder basis $(P_N)_{N\in\mathbb{N}}$ of $\mathcal{C}(K)$ is called a *Faber basis* if $\deg P_N = N - 1$ for all $N \in \mathbb{N}$.

It is a well known result by Faber [10] that there exists no Faber basis for $\mathcal{C}[a,b]$. Let us assume that for a given $\mathcal{C}(K)$ we have a Faber basis $(P_N)_{N\in\mathbb{N}}$. For

$N \in \mathbb{N}$, define the operator $S_N : \mathcal{C}(K) \to \mathcal{P}_N$ as the partial sum

$$S_N(f) = \sum_{k=1}^{N} a_k P_k \tag{3.1}$$

where $f = \sum_{k=1}^{\infty} a_k P_k$.

Notice that, similar to the Lagrange interpolatory operators $(L_N(.,X,.))_{N\in\mathbb{N}}$, each $S_N$ is a linear, continuous projection onto the set of algebraic polynomials of degree at most $N-1$.

We say that a Faber basis $(P_N)_{N\in\mathbb{N}}$ of $\mathcal{C}(K)$ is *interpolating* with respect to the sequence of distinct points $(x_k)_{k\in\mathbb{N}} \subset K$ if

$$S_k f(x_k) = f(x_k) \tag{3.2}$$

holds for all $f \in \mathcal{C}(K)$ and $k \in \mathbb{N}$.

**Lemma 3.1.2** (Bilet, Dovgoshey, Prestin)**.** *A Faber basis $(P_N)_{N\in\mathbb{N}}$ of $\mathcal{C}(K)$ is interpolating with respect to the sequence $(x_k)_{k\in\mathbb{N}}$ if and only if*

$$P_k(x_k) \neq 0 \quad \textit{and} \quad P_k(x_j) = 0 \tag{3.3}$$

*for every $k \in \mathbb{N}$ and $j < k$.*

*Proof.* ($\Leftarrow$) Assume (3.3) holds. Then we have for any $f \in \mathcal{C}(K)$ and $k \in \mathbb{N}$

$$f(x_k) = \sum_{j=1}^{\infty} a_j P_j(x_k) = \sum_{j=1}^{k} a_j P_j(x_k) = S_k f(x_k).$$

Thus $(P_N)_{N\in\mathbb{N}}$ is interpolating with respect to $(x_k)_{k\in\mathbb{N}}$.

($\Rightarrow$) Now assume $(P_N)_{N\in\mathbb{N}}$ is interpolating with respect to $(x_k)_{k\in\mathbb{N}}$. Firstly, it is clear that $P_1 \not\equiv 0$. For $k > 1$, from uniqueness of the representation $P_k = \sum_{j=1}^{\infty} a_j P_j$, we have $a_j = \delta_{j,k}$. Thus $S_j P_k \equiv 0$ for every $j < k$. Since $(P_N)_{N\in\mathbb{N}}$ is interpolating with respect to $(x_k)_{k\in\mathbb{N}}$, $P_k(x_j) = S_j P_k(x_j) = 0$ for every $j < k$.

Now $P_k \in \mathcal{P}_{k-1} \setminus \mathcal{P}_{k-2}$, so $P_k$ attains zero at $x_k$ only if $P_k \equiv 0$, which gives a contradiction. So (3.3) follows. $\square$

**Corollary 3.1.3.** *Let $(P_N)_{N\in\mathbb{N}}$ be an interpolating Faber basis of $\mathcal{C}(K)$ with nodes $(x_k)_{k\in\mathbb{N}}$ and let $(\mu_N)_{N\in\mathbb{N}}$ be any sequence of nonzero, real numbers. Then*

$$(\mu_N P_N)_{N\in\mathbb{N}}$$

*is also an interpolating Faber basis with same nodes.*

*Conversely, if there exist two interpolating Faber bases $(P_N)_{N\in\mathbb{N}}$ and $(Q_N)_{N\in\mathbb{N}}$ with the same nodes, then*

$$P_N = \mu_N Q_N$$

*for some sequence of nonzero real numbers $(\mu_N)_{N\in\mathbb{N}}$.*

Now we can uniquely determine an interpolating Faber basis $(P_N)_{N\in\mathbb{N}}$ with a certain set of nodes $(x_k)_{k\in\mathbb{N}}$, by normalization

$$P_N(x_N) = 1 \qquad N \in \mathbb{N}. \tag{3.4}$$

**Definition 3.1.4.** An interpolating Faber basis $(P_N)_{N\in\mathbb{N}}$ with nodes $(x_k)_{k\in\mathbb{N}}$ is called a *Lagrange basis* if (3.4) hold for all $N \in \mathbb{N}$.

Note that if we have such a Lagrange basis, for $f = \sum_{k=1}^{\infty} a_k P_k$ we have

$$a_k = f(x_k) - \sum_{j=1}^{k-1} a_j P_j(x_k), \qquad k = 1, 2, ... \tag{3.5}$$

The next result demonstrates the equivalence of admitting an interpolating Faber basis and having a convergent interpolatory process.

**Theorem 3.1.5** (Bilet, Dovgoshey, Prestin)**.** *Let $K \subset \mathbb{R}$ be infinite and compact and let $X = (x_{k,N})_{k=1,N=1}^{N,\ \infty} \subset K$ be an interpolatory matrix. The following are equivalent:*

1. *The space $\mathcal{C}(K)$ admits a Faber basis such that the equality*

$$S_N = L_N(., X, .) \tag{3.6}$$

*holds for every $N \in \mathbb{N}$.*

2. *The sequence $(\Lambda_N(X, K))_{N\in\mathbb{N}}$ is bounded and there is a sequence $(x_k)_{k\in\mathbb{N}}$ of distinct nodes such that for any $N \geq 2$ the tuple $(x_{1,N}, ..., x_{N,N})$ is a permutation of the set $\{x_1, ..., x_N\}$.*

Observe that in the interpolating matrix $X$, each interpolation node is carried over to higher degrees, i.e. $x_{k,N} = x_{k_1,N+1} = x_{k_2,N+2} = ...$ , for every $x_{k,N} \in X$. This restrics our system of nodes if we are looking for an interpolating Faber basis rather than a convergent Lagrange interpolatory process.

## 3.2 Results for Countable Sets

Due to Corollary 2.3.7, in order to attain a convergent interpolatory process, we must turn our attention to small sets in terms of measure and density. The first class of sets that come to mind is the class of countable sets. There are two major results showing us that countability is not strongly linked to "being in the class of BLC".

First one is due Obermaier [17] in 2003, where he proves that convergent, geometrically progressing sequences together with their limit points are of BLC.

**Theorem 3.2.1** (Obermaier)**.** *Let $S_q = \{q^n : n \in \mathbb{N}\} \cup \{0\}$ for $0 < q < 1$. Then there exist a Lagrange basis of $\mathcal{C}(S_q)$ with respect to the sequence $(q^n)_{n\in\mathbb{N}_0}$.*

After this theorem, Obermaier states that the same is not true for any set of similar structure. Namely for $S^r = \{(k+1)^{-r} \ : \ k \in \mathbb{N}_0\} \cup \{0\}, \ \ 0 < r < \infty$, there is no Lagrange basis of $\mathcal{C}(S^r)$ with respect to the sequence $((n+1)^{-r})_{n\in\mathbb{N}_0}$. Clearly, this does not mean that there exist no Lagrange basis on $\mathcal{C}(S^r)$, there could exist another sequence on the set such that the corresponding Lebesgue constants are bounded.

In any ways, we see here that $S_q \in BLC$. Two years later, Obermaier and Szwarc [18] improve this result in the following way.

**Theorem 3.2.2** (Obermaier, Szwarc)**.** *Assume $(s_k)_{k\in\mathbb{N}_0}$ is a strictly increasing or strictly decreasing sequence and $S = \{s_0, s_1, ...\} \cup \{\sigma\}$ where $\sigma = \lim_{k\to\infty} s_k$. Then there exist a Lagrange basis of $\mathcal{C}(S)$ with respect to the sequence $(s_k)_{k\in\mathbb{N}_0}$ if and only if there exist $0 < q < 1$ with*

$$|\sigma - s_{k-1}| \leq q|\sigma - s_k| \quad \textit{for all } k \in \mathbb{N}_0. \tag{3.7}$$

Again, the "only if" in the previous theorem does not imply the non-existence of a Lagrange basis without (3.7), but non-existence with respect to these nodes.

The second result is due Privalov [19], where he finds an example of a countable set out of the class $BLC$.

**Theorem 3.2.3** (Privalov)**.** *Let $F = \{n^{-1/2} : n \in \mathbb{N}\} \cup \{0\}$. Then for every $X \subset [-1, 1]$ there exists a positive constant $c(X)$ such that for every $N \in \mathbb{N}$, we have*

$$\Lambda_N(X, F) > c(X) \log N. \tag{3.8}$$

Thus, as long as countability is concerned, there are sets in and out of the class $BLC$ that are countable.

## 3.3 Results for Perfect Sets

In the previous section, we established that being countable gives no information about being in $BLC$, second class of sets that we consider is perfect sets. Korovkin [14] constructed a perfect set $K \subset [-1, 1]$ and interpolatory matrix $X \subset K$ such that

$$\Lambda_{N^2}(X, K) < \infty. \tag{3.9}$$

While this result does not imply that $K \in BLC$, it means that every $f \in \mathcal{C}(K)$ can be approximated by Lagrange interpolating polynomials $(L_{N^2}(f, X, .))_{N\in\mathbb{N}}$.

The other result that is often referred to in this topic is Mergelyan [16]. Mergelyan claimed that a wide range of Cantor-type sets belong to the class $BLC$. Here, we will show a mistake in his proof and then prove that his result is incorrect by a counter example and finally put forward the following conjecture:

**Conjecture 3.3.1.** There exists no perfect set in $BLC$.

Before starting, let us explain some of the notations regarding the processes of geometrically symmetric Cantor-type sets. Let $(\ell_s)_{s=0}^{\infty}$ be a sequence of positive numbers such that $\ell_0 = 1$ and $0 < 3\ell_{s+1} \leq \ell_s$ for $s \in \mathbb{N}_0$. Let $K$ be the Cantor set associated with the sequence $(\ell_s)_{s=0}^{\infty}$, that is

$$K = \bigcap_{s=0}^{\infty} E_s$$

where $E_0 = [0, 1] = I_{1,0}$, $E_s$ is the union of $2^s$ closed basic intervals $I_{j,s}$ of length $\ell_s$ and $E_{s+1}$ is obtained by replacing each $I_{j,s}$, $j = 1, 2, ..., 2^s$, by two adjacent subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Let $h_s = \ell_s - 2\ell_{s+1}$ be the distance between them. Note that we have $h_s \geq \ell_{s+1}$.

Also, for each $x \in \mathbb{R}$ and $Z \subset \mathbb{R}$ finite, by $d_{k,Z}(x)$ we denote the distances $|x - z_{j_k}|$ from $x$ to points of Z arranged in the nondecreasing order.

Now, let us consider a specific form of these Cantor-type sets. Let $K_\beta$ for $0 < \beta \leq 1/3$ be the Cantor set associated to sequence $(\ell_s)_{s=0}^{\infty}$ with $\ell_{s+1} = \beta\ell_s$ for $s \in \mathbb{N}_0$. Observe that $K_{1/3}$ is the classical Cantor ternary set.

Now, let us exhibit Mergelyan's result in [16], pg. 64 – 65. For convenience, we translate his result into our notations. Let $w_f$ be the modulus of continuity of a continuous function $f$.

**Theorem 3.3.2** (Mergelyan)**.** *Let $K$ be any geometrically symmetric Cantor-type set with $h_s > \ell_{s+1}$ for every $s \in \mathbb{N}_0$. Let $Y \subset K$ be an interpolatory matrix whose $2^{s+1}$'th row consists of the endpoints of each interval $I_{j,s},\ j = 1, 2, ..., 2^s$. Then, there exists a positive function of integer argument $\varphi_K(n)$ such that for every $f \in \mathcal{C}(K)$ we have the inequality*

$$\max_{x \in K} |f(x) - L_{2^{s+2}}(f, Y, x)| < Cw_f(\varphi_K(2^{s+2})) \tag{3.10}$$

*where $C$ does not depend on $s$.*

First, let us emphasize that (3.10) implies boundedness of the subsequence $(\Lambda_{2^{s+2}}(Y, K))_{s \in \mathbb{N}}$, since the right side of (3.10) is bounded by $2C$ for all continuous functions $f$ with $||f|| \leq 1$. Now, the mistake Mergelyan made, identified by my advisor Alexander Goncharov, is on page 65, 4'th line from below, where Mergelyan estimates from above $M_s = \dfrac{2^{2s+3}\ell_{s+2}}{(2^{s+1}!)\ell_{s+1}^{2^{s+1}}}$ by a bounded value $Cw_f(M_s)$. However, for example taking $K = K_\beta$ for $0 < \beta \leq 1/3$, we have $\ell_s = \beta^s$ (which is $\Delta_{s-1}$ in the author's notation) and by Stirling's formula, the leading term of $\log M_s$ is $2^{s+1}(s+1)[\log \dfrac{1}{\beta} - \log 2]$ which tends to infinity as $s \to \infty$, since $\beta < 1/3$.

Now, let us show that for any $0 < \beta \leq 1/3$, taking $K_\beta$ and as interpolatory matrix $Y \subset K_\beta$ as above, we have

$$\lim_{s \to \infty} \Lambda_{2^s}(Y, K_\beta) = \infty. \tag{3.11}$$

Observe that for any $s \in \mathbb{N}$, $2^{s+1}$'th row of $Y$ (denote this by $Y_{(s)}$) consists of $x_1 = 0, x_2 = \ell_s, ..., x_{2^{s+1}} = 1$. For the next lemma, let $k = 2^s - 1$ so that $x_k = \ell_1 - \ell_s$, also let $\tilde{x} = \ell_{s+1}$.

**Lemma 3.3.3.** *Given $s \geq 2$, we have*

$$|l_{k,2^{s+1}}(\tilde{x})| \geq \frac{\ell_{s+1}(\ell_s - \ell_{s+1})}{\ell_s \ell_1} \left( \frac{1 - \ell_{s+1}}{1 - \ell_1 - \ell_s} \right)^{2^s}. \tag{3.12}$$

*Proof.* We have

$$|l_{k,2^{s+1}}(\tilde{x})| = \pi_1\pi_2 := \prod_{\substack{j=1\\ j\neq k}}^{2^s} \left|\frac{\tilde{x}-x_j}{x_k-x_j}\right| \prod_{j=2^s+1}^{2^{s+1}} \left|\frac{\tilde{x}-x_j}{x_k-x_j}\right|.$$

Let's obtain the lower bounds of these two separately. Observe that $\pi_1$ corresponds to product of ratios of distances of $\tilde{x}$ and $x_k$ to the nodes $Y_{(s)} \cap I_{1,1} \setminus \{x_k\}$ . Thus, $d_{1,Y_{(s)}\setminus\{x_k\}}(\tilde{x}) = d_1(\tilde{x}) = \ell_{s+1},\ d_2(\tilde{x}) = \ell_s - \ell_{s+1}$. For $2 \leq j \leq 2^s - 3$ we have $d_{j+1}(\tilde{x}) = d_j(x_k) + \varepsilon$ with $\varepsilon = \ell_s - \ell_{s+1}$. Indeed for such $j$ we have $d_j(x_k) = d_j(x_2)$ and $x_2 - \tilde{x} = \varepsilon$. Also $d_1(x_k) = \ell_s,\ d_{2^s-2}(x_k) = x_k - x_2 = \ell_1 - 2\ell_s,\ d_{2^s-1}(x_k) = \ell_1 - \ell_s$, and $d_{2^s-1}(\tilde{x}) = \ell_1 - \ell_{s+1}$. Therefore,

$$\pi_1 = \frac{d_1(\tilde{x})d_2(\tilde{x})}{d_1(x_k)} \cdot \prod_{j=2}^{2^s-3} \frac{d_{j+1}(\tilde{x})}{d_j(x_k)} \cdot \frac{d_{2^s-1}(\tilde{x})}{d_{2^s-2}(x_k)d_{2^s-1}(x_k)}.$$

We neglect the product in the middle as all terms are greater than one. Hence,

$$\pi_1 > \frac{\ell_{s+1}(\ell_s - \ell_{s+1})}{\ell_s} \cdot \frac{\ell_1 - \ell_{s+1}}{(\ell_1 - 2\ell_s)(\ell_1 - \ell_s)} > \frac{\ell_{s+1}(\ell_s - \ell_{s+1})}{\ell_s \ell_1}.$$

As for $\pi_2$ i.e. product of ratios of distances to the nodes $Y_{(s)} \cap I_{2,1}$, we have

$$d_j(\tilde{x}) = d_j(x_k) + x_k - \tilde{x} \qquad \text{for} \quad j = 2^s + 1, ..., 2^{s+1}.$$

Here, $x_k - \tilde{x} = \ell_1 - \ell_s - \ell_{s+1}$ and $d_j(x_k) \leq 1 - x_j = 1 - \ell_1 + \ell_s$. Hence,

$$\pi_2 = \prod_{j=2^s+1}^{2^{s+1}} \left(1 + \frac{x_k - \tilde{x}}{d_j(x_k)}\right) \geq \prod_{j=2^s+1}^{2^{s+1}} \left(1 + \frac{\ell_1 - \ell_s - \ell_{s+1}}{1 - \ell_1 + \ell_s}\right) = \left(\frac{1 - \ell_{s+1}}{1 - \ell_1 + \ell_s}\right)^{2^s}.$$

The desired result follows from here. $\square$

**Theorem 3.3.4.** *For any* $0 < \beta \leq 1/3$, *we have* $\Lambda_{2^{s+1}}(Y, K_\beta) \to \infty$ *as* $s \to \infty$.

*Proof.* Applying Lemma 3.3.3 for $\ell_s = \beta^s$ yields $\pi_1 \geq \beta^s(1-\beta)$. It is easy to check that $(1-\beta^{s+1})(1-\beta^2) > 1 - \beta - \beta^s$ for $s \geq 2$. Hence $\pi_2 > (1-\beta^2)^{-2^s}$ and $|l_{k,2^{s+1}}(\tilde{x})| \geq \dfrac{(1-\beta)\beta^s}{(1-\beta^2)^{2^s}}$. Finally, we have

$$\Lambda_{2^s+1}(Y, K_\beta) > \lambda_{2^s+1}(Y, \tilde{x}) > |l_{k,2^{s+1}}(\tilde{x})| > \frac{(1-\beta)\beta^s}{(1-\beta^2)^{2^s}},$$

and the RHS goes to infinity as $s \to \infty$. So, we are done. $\square$

Note that Theorem 3.3.4 does not imply that $K_\beta \notin BLC$. In fact, we believe that just like equidistant nodes for interval, end points for the sets $K_\beta$ is far from the optimal choice, although it gets closer and closer to optimal for smaller and smaller sets.

Now, let us take the classical Cantor ternary set, i.e. $K_\beta$ with $\beta = 1/3$ and observe the behaviour of the Lebesgue functions of degrees $2^s$ for $s = 2, ..., 7$. We denote by $Y_\beta \subset K_\beta$ an interpolatory matrix whose $2^s$'th row consists of the endpoints of the process at step $s-1$.

In the figures below, for each $s \in \{2, 3, ..., 7\}$, the Lebesgue function $\lambda_{2^s}(Y_{1/3}, x)$ was evaluated at every node of $2^{s+1}$'th row of $Y_{1/3}$. So, there are $2^s$ of them where $\lambda_{2^s}(Y_{1/3}, x)$ is exactly one and the same number of them where the function is strictly greater than one. In between the nodes linear extension was used.

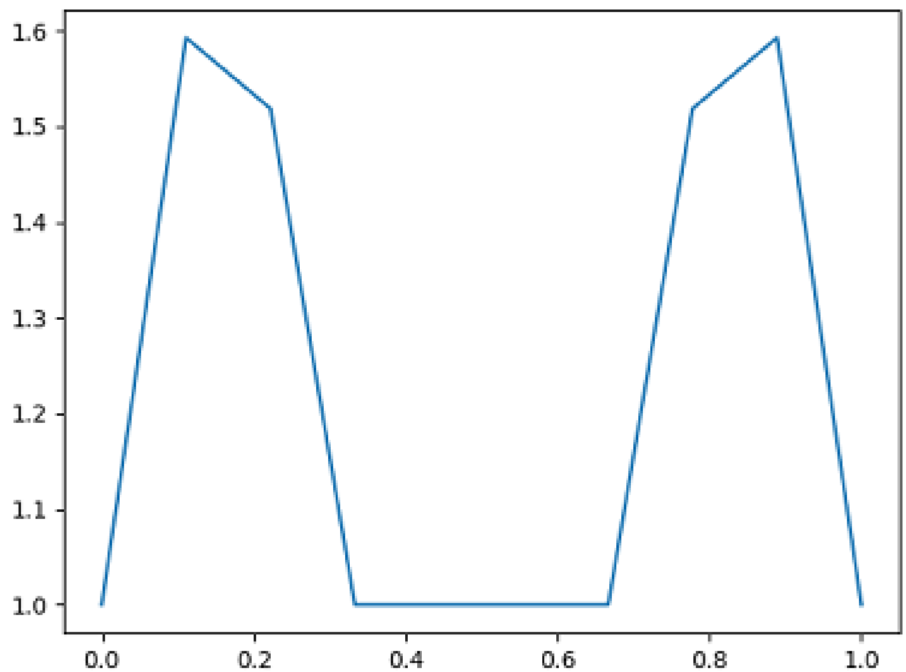


Figure 3.1: $\lambda_{2^2}(Y_{1/3}, .)$

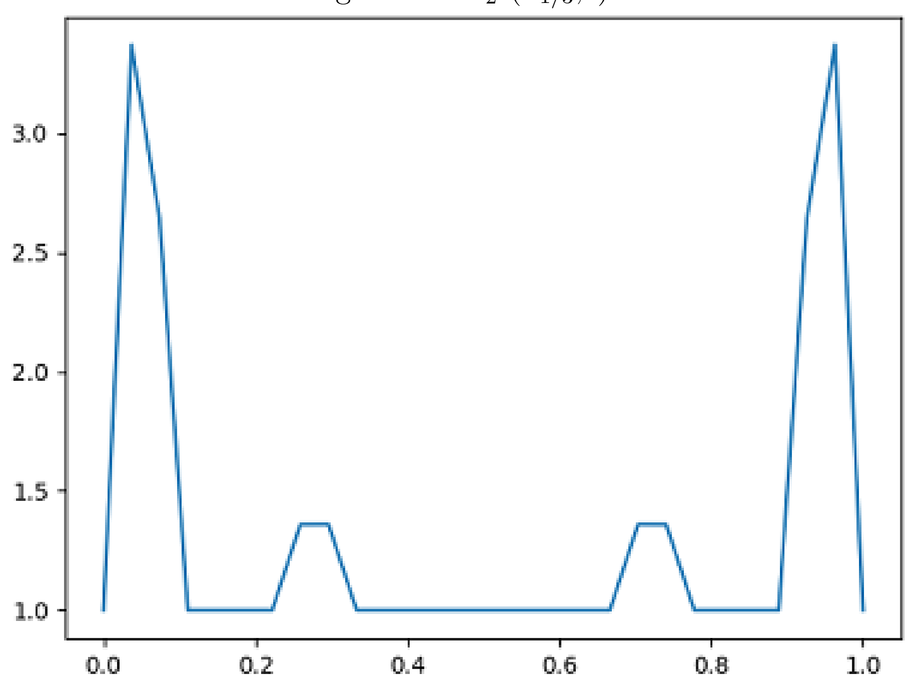


Figure 3.2: $\lambda_{2^3}(Y_{1/3}, .)$

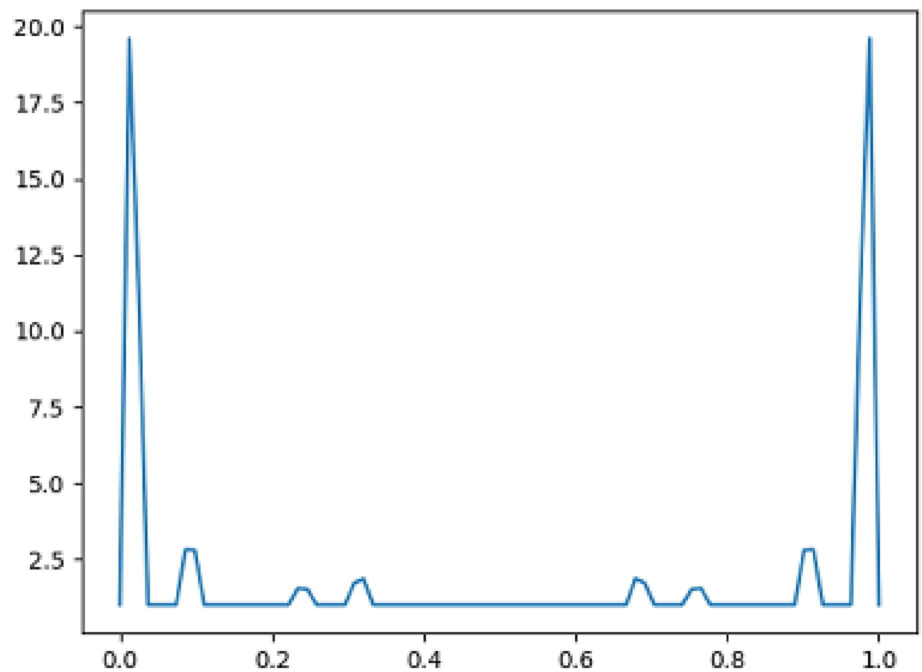


Figure 3.3: $\lambda_{2^4}(Y_{1/3}, .)$

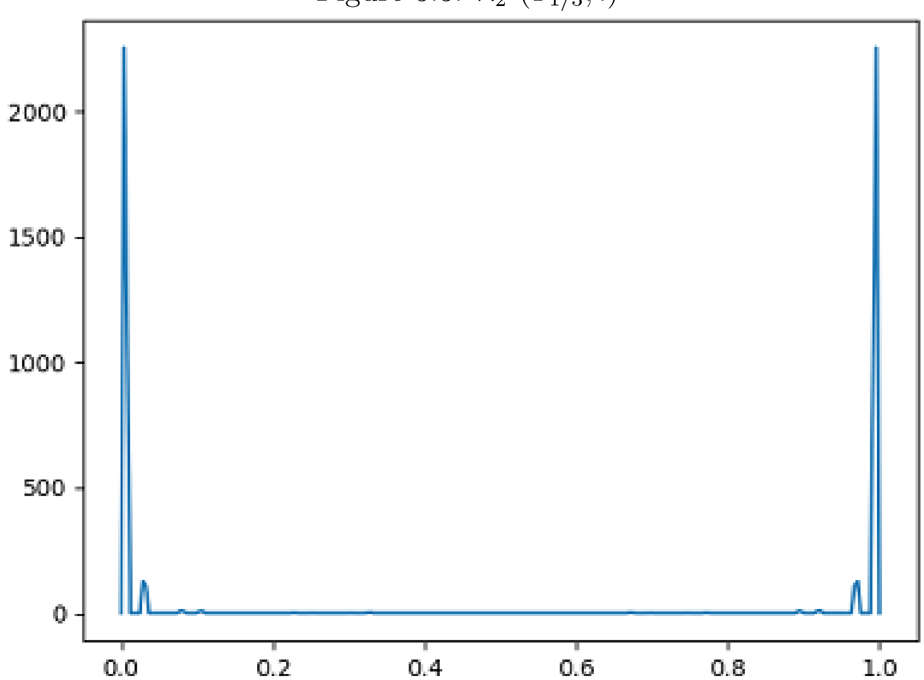


Figure 3.4: $\lambda_{2^5}(Y_{1/3}, .)$

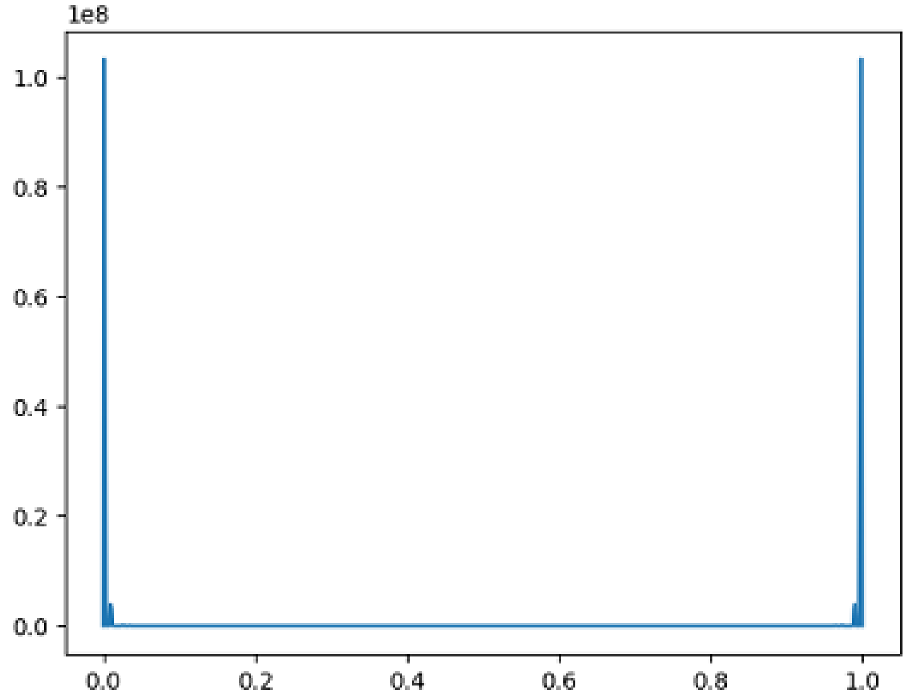


Figure 3.5: $\lambda_{2^6}(Y_{1/3}, .)$

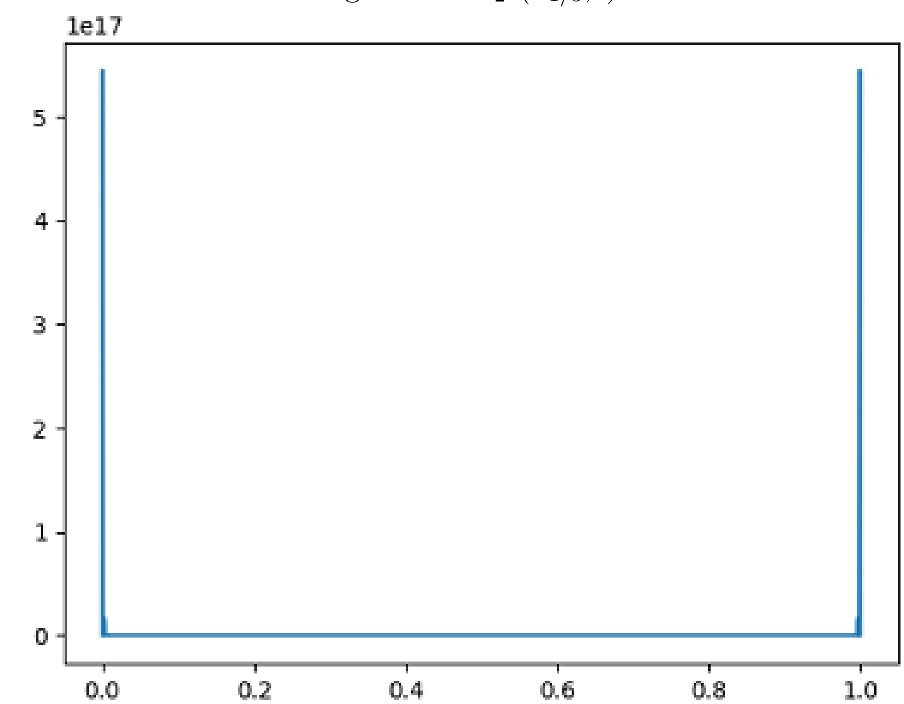


Figure 3.6: $\lambda_{2^7}(Y_{1/3}, .)$

As we can see from the figures above, the Lebesgue functions are maximized in the first and last intervals $I_{1,s}$ and $I_{2^s,s}$ for $s \in \mathbb{N}$. Moreover, just like equidistant nodes, the maxima in these two intervals follow exponential growth and are incomparably large with respect to middle ones.

Now, let us return to Mergelyan's work. Notice that the sequence $M_s = \dfrac{2^{2s+3}\ell_{s+2}}{(2^{s+1}!)\ell_{s+1}^{2^{s+1}}}$ for $s \in \mathbb{N}$ depends solely on the lenghts of basic intervals of the Cantor process. For the set $K_\beta$ they are unbounded. However, if we were to look at Cantor processes where the ratios $\dfrac{\ell_{s+1}}{\ell_s}$ are sufficiently small, then Mergelyan's conclusion would be true. In other words, we would attain a Cantor-type set with a subsequence of bounded Lebesgue constants. Let us construct another family of Cantor-type sets, to examine this.

Let $\alpha = (\alpha_s)_{s\in\mathbb{N}}$ be a real sequence with $\alpha_s > 1$ for all $s \in \mathbb{N}$ and assume we have the Cantor-type set $K^\alpha$ defined by $\ell_1 \leq 1/3$ and $\ell_{s+1} = \ell_s^{\alpha_s}$ for $s \in \mathbb{N}$. Observe that the class $\{K_\beta : 0 < \beta \leq 1/3\}$ is a subclass of $\{K^\alpha : \alpha_s > 1,\ \ell_1 \leq 1/3\}$. In fact $K^\alpha = K_\beta$ if and only if

$$\alpha_1 = 2\,\frac{\log\beta}{\log\ell_1} \quad \text{and} \quad \alpha_s = \frac{s+1}{s} \quad \text{for } s = 2, 3, \ldots .$$

Now, let's look at the set $K^\alpha$ with $\alpha_s \geq 2^s$ for every $s \in \mathbb{N}$. We have

$$M_s = \frac{2^{2s+3}\ell_{s+2}}{(2^{s+1}!)\ell_{s+1}^{2^{s+1}}} \leq \frac{2^{2s+3}}{(2^{s+1}!)}, \quad s \in \mathbb{N}.$$

Thus, Mergelyan's theorem is applicable to these very small Cantor-type sets. On the other hand, the value $M_s$ was attained a little generously, without consideration of the Cantor structure of the set, which caused a loss of precision. To see this, let's take again $K^\alpha$, this time with $\alpha_s \equiv c$ for some constant $c > 1$. For this set, just like $K_\beta$, the sequence $(M_s)_{s\in\mathbb{N}}$ is unbounded.

However, let us inspect the results on this set of the same numerical experiment that was previously applied on $K_\beta$. Namely, let $Y^\alpha \subset K^\alpha$ be an interpolatory matrix whose $2^s$'th row consists of the endpoints of the Cantor process (corresponding to $K^\alpha$) at step $s-1$. For $\alpha \equiv 2$ and $\ell_1 = 1/3$ we have:

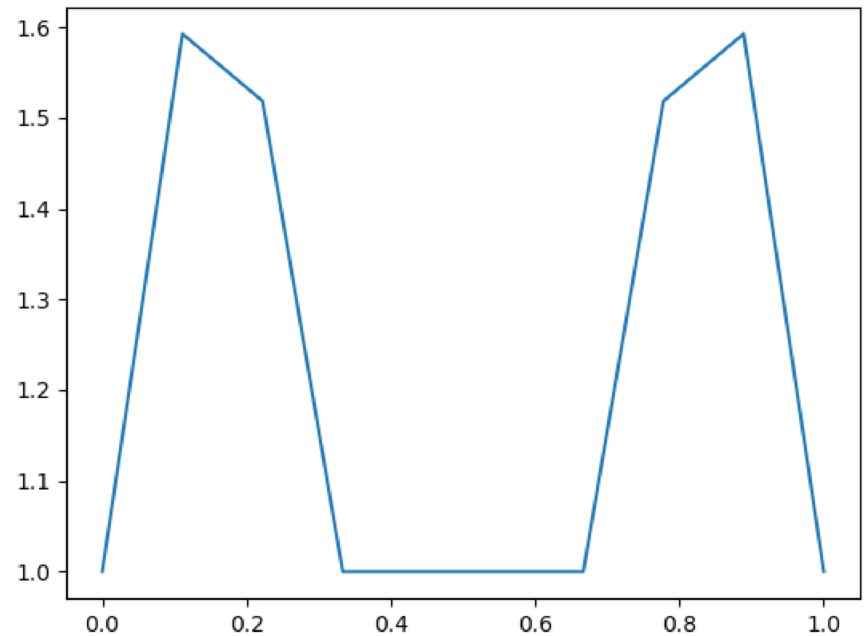


Figure 3.7: $\lambda_{2^2}(Y^2, .)$

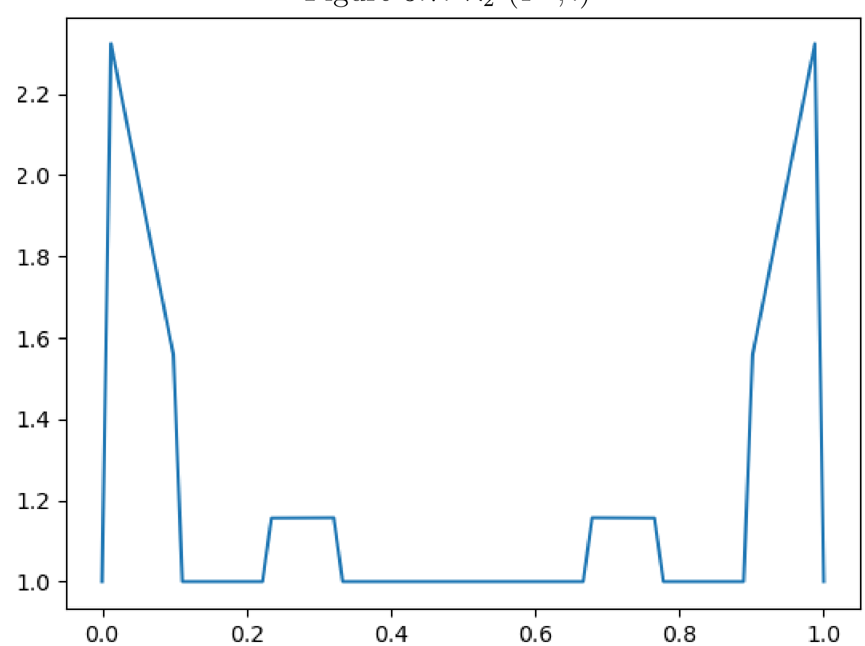


Figure 3.8: $\lambda_{2^3}(Y^2, .)$

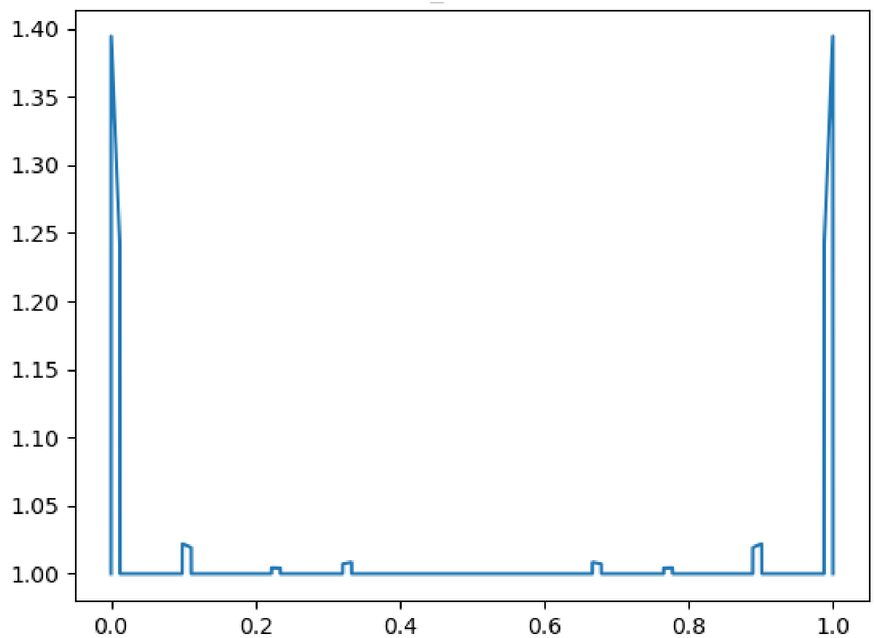


Figure 3.9: $\lambda_{2^4}(Y^2,.)$

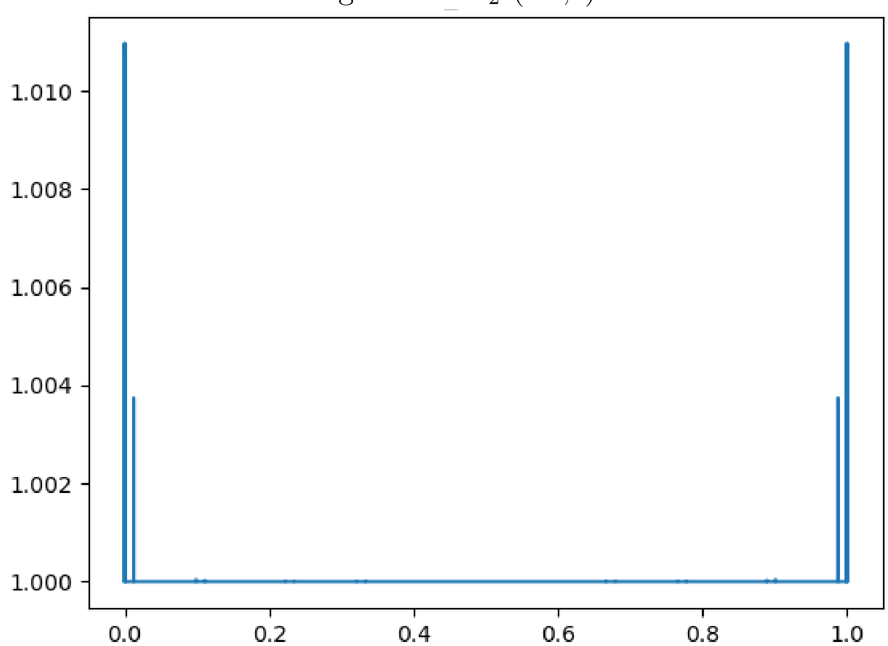


Figure 3.10: $\lambda_{2^5}(Y^2,.)$

As we can see, for $s \geq 2$, this subsequence of Lebesgue constants $(\Lambda_{2^s}(K^2, Y^2))_{s\in\mathbb{N}}$ seems to be decreasing down to its minimum possible value, which is one. Note that for constant $\alpha$, $K^\alpha$ is polar if and only if $\alpha \geq 2$. So, before making inferences, let us look at the graphs of $\lambda_{2^s}(Y^\alpha, .)$ with $\alpha < 2$, $s = 2, 3, 4$.

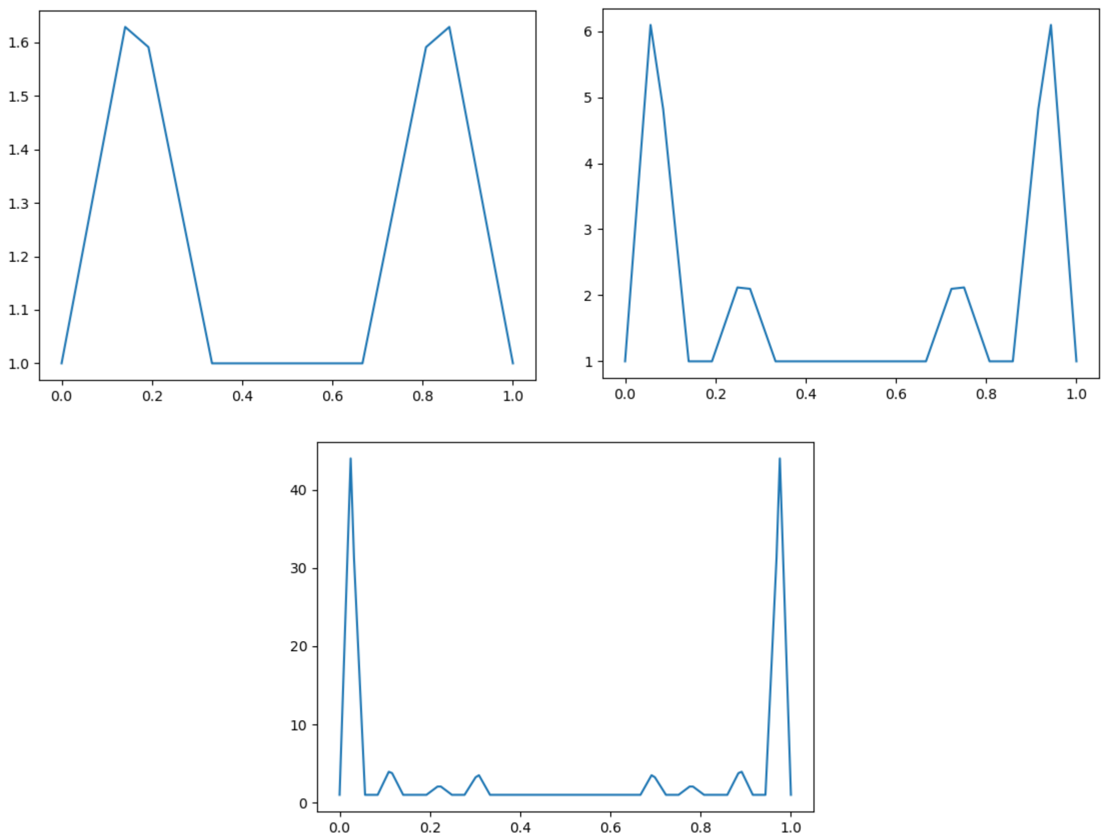

Figure 3.11: $\lambda_{2^2}(Y^{1.5}, .)$ (top left) , $\lambda_{2^3}(Y^{1.5}, .)$ (top right), $\lambda_{2^4}(Y^{1.5}, .)$ (bottom)

Here, we again observe a behavior of the subsequence of Lebesgue constants as we did for $K_{1/3}$. Hence, perhaps polarity plays a role in convergence of this particular subsequence of Lebesgue constants for these types of sets.

Finally, we want to show that for any constant $\alpha > 1$, on the set $K^\alpha$, the sequence of Lebesgue constants with corresponding set of nodes $Y^\alpha$ (defined as before) where $Y^\alpha$ preserves previous nodes, is unbounded.

In order to prove this, it is sufficient to show that $(\Lambda_{2^s-1}(Y^\alpha, K^\alpha))_{s\in\mathbb{N}}$ diverges. So the idea is, we will exclude one node $x_m$ of level $s$, take an interval of $q$'th level that contains $x_m$ ($I_{\cdot,q} \ni x_m$), where we select an admissible $q \leq s$, then estimate $|l_{k,s}(x_m)|$ from below for every $k$ such that $x_k \in I_{\cdot,q}$ and $x_k < x_m$.

First, let us exhibit the graphs that support this divergence of subsequence of Lebesgue constants when for each $s = 2, 3, 4, 5$, $x_m = x_{m(s)} = \sum_{j=0}^{s}(-1)^j \ell_j$ is excluded in between nodes of step $s$ of $K^2$. For $s \in \mathbb{N}$, $k \leq 2^{s+1}$, let $Y_{k,s}^{\alpha} \subset K^{\alpha}$ be an interpolatory matrix such that the $2^{s+1} - 1$'th row of it consists of the endpoints of intervals of level $s$, except $x_k$. Observe that the maximal peaks follow a growth slower than logarithmic (with respect to degree), at precisely $x_m$.

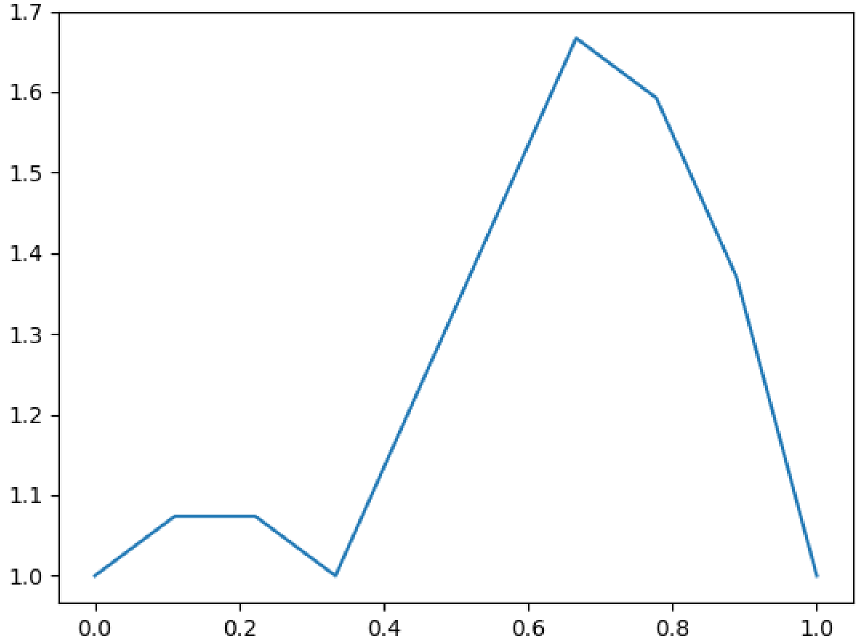


Figure 3.12: $\lambda_{2^2-1}(Y^2_{m,2},.)$

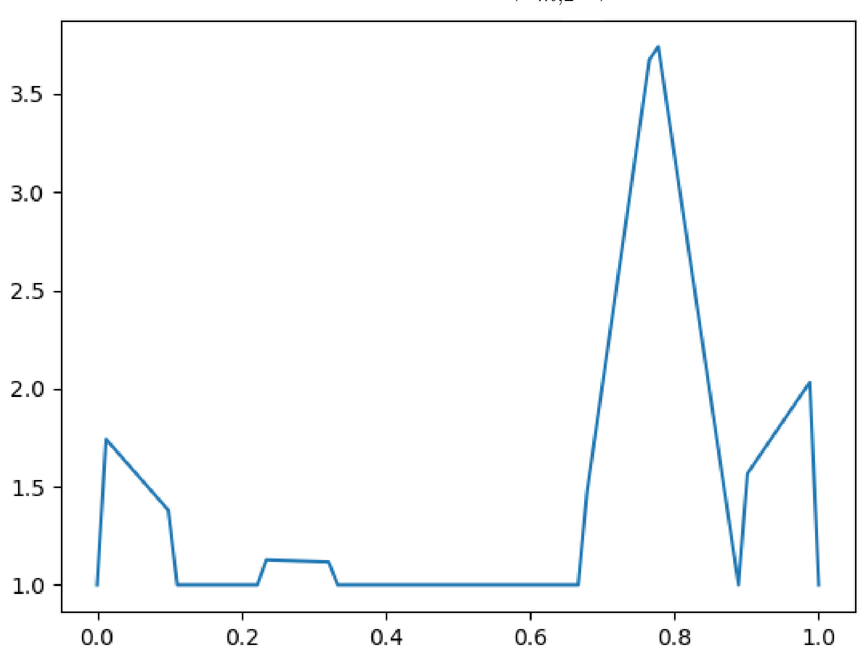


Figure 3.13: $\lambda_{2^3-1}(Y^2_{m,3},.)$

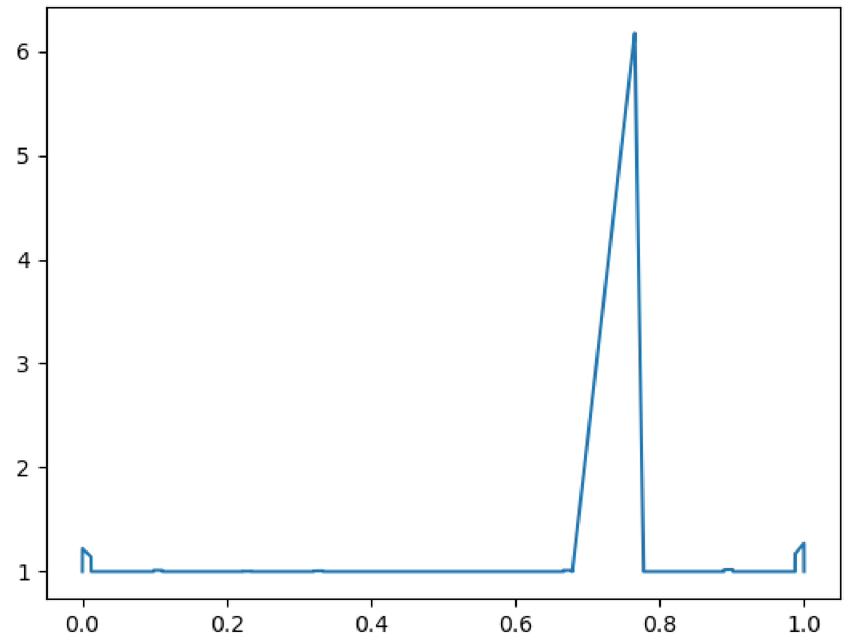


Figure 3.14: $\lambda_{2^4-1}(Y^2_{m,4}, .)$

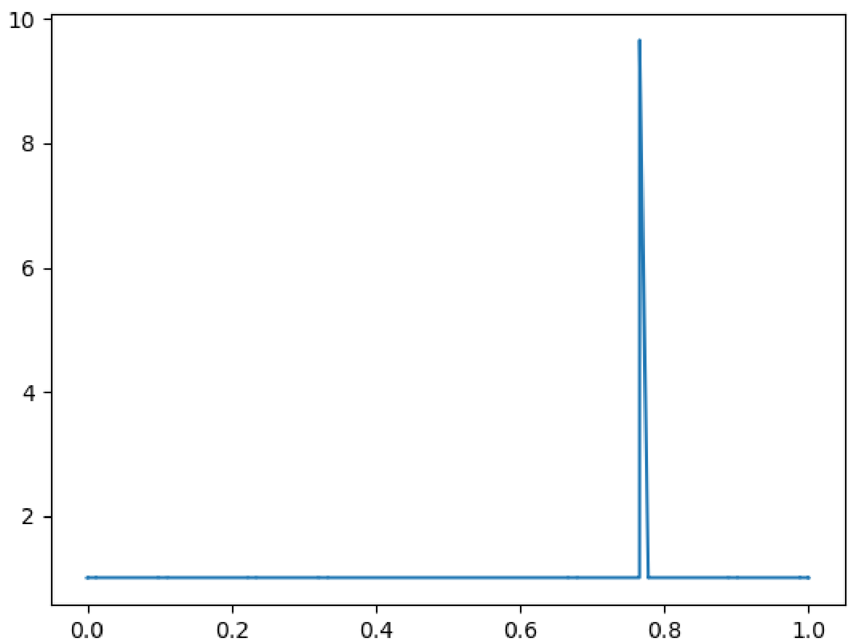


Figure 3.15: $\lambda_{2^5-1}(Y^2_{m,5}, .)$

For the next result, for each $s \in \mathbb{N}$ let $Z = (x_k)_{k=1}^{2^{s+1}}$ be the endpoints of the intervals $(I_{i,s})_{i=1}^{2^s}$ ordered increasingly. Take $m = m(s)$ such that $x_m = \sum_{j=0}^{s}(-1)^j \ell_j$. Assume $x_m \in I_{j_0,s} \subset I_{j_1,s-1} \subset I_{j_2,s-2} \subset ... \subset I_{j_s,0} = I_{1,0} = [0,1]$. Also for a finite set $A$, let $|A|$ denote its cardinality.

**Theorem 3.3.5.** *For constant $\alpha > 1$ and $Y^\alpha$ where for each $s \in \mathbb{N}$ the $2^s - 1$'th row of $Y^\alpha$ is missing $x_m$, we have*

$$\lim_{s\to\infty} \Lambda_{2^s-1}(Y^\alpha, K^\alpha) = \infty.$$

*Proof.* Fix $q \in \{1, 2, ..., s-1\}$ and for simplicity, denote $I_q = I_{j_{s-q},q}$ and $I_{q-i} = I_{j_{s-q+i},q-i} \setminus I_{j_{s-q+i-1},q-i+1}$ for $i = 1, ..., q$. Thus $\{I_{q-i},\ i = 0, ..., q\}$ is family of disjoint intervals that cover $E_s$. Clearly we have $|Z \cap I_q| = 2^{s-q+1}$ and $|Z \cap I_{q-i}| = 2^{s-q+i}$ for $i = 1, 2, ..., q$. Let us take some $x_\kappa \in Z \cap I_q$ with $x_\kappa < x_m$ and assume $q$ is even, i.e. $I_{j_{s-q+1},q-1} \supset (I_{j_{s-q},q} \cup I_{j_{s-q}+1,q})$ i.e. $I_{q-1}$ is on the right side of $I_q$.

Then for any $i$ even and $x_j \in I_{q-i}$ , since $|x_m - x_j| > |x_\kappa - x_j|$ , we have

$$\left|\frac{x_m - x_j}{x_\kappa - x_j}\right| > 1$$

and for any $i$ odd and $x_j \in I_{q-i}$ we have

$$\left|\frac{x_m - x_j}{x_\kappa - x_j}\right| = \frac{x_m - x_j + x_\kappa - x_\kappa}{x_\kappa - x_j} = 1 + \frac{x_m - x_\kappa}{x_\kappa - x_j} = 1 - \left|\frac{x_m - x_\kappa}{x_\kappa - x_j}\right| \geq 1 - \frac{\ell_q}{h_{q-i}}.$$

Thus for odd $i$ we have

$$\prod_{x_j \in I_{q-i}} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right| \geq \left(1 - \frac{\ell_q}{h_{q-i}}\right)^{2^{s-q+i}}.$$

Now, we want to estimate $\displaystyle\prod_{x_j\in I_q\setminus\{x_m,x_\kappa\}} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right|$ from below.

In the set $Z\cap I_{j_0,s}$ there exists one point other than $x_m$ of distance to $x_m$ greater or equal to $\ell_s$, in the next set $Z\cap(I_{j_1,s-1}\setminus I_{j_0,s})$ there exist two points with distances greater or equal to $h_{s-1}$, in $Z\cap(I_{j_n,s-n-1}\setminus I_{j_{n-1},s-n+1})$ there exist $2^n$ points with distances greater or equal to $h_{s-n}$, excluding $x_\kappa$ from these points, we get

$$\prod_{x_j\in I_q\setminus\{x_m,x_\kappa\}} |x_m - x_j| \geq \frac{\ell_s\ h_{s-1}^2\ h_{s-2}^{2^2}...\ h_q^{2^{s-q}}}{|x_m - x_\kappa|}$$

Similar argument works for upper bounds of distances to $x_\kappa$ where we have $\ell_n$ instead of $h_n$, excluding $x_m$ from these points gives us

$$\prod_{x_j\in I_q\setminus\{x_m,x_\kappa\}} |x_\kappa - x_j| \leq \frac{\ell_s\ \ell_{s-1}^2\ \ell_{s-2}^{2^2}...\ \ell_q^{2^{s-q}}}{|x_m - x_\kappa|}.$$

Using these, we obtain

$$\prod_{x_j\in I_q\setminus\{x_m,x_\kappa\}} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right| \geq \left(\frac{h_{s-1}}{\ell_{s-1}}\right)^2 \left(\frac{h_{s-2}}{\ell_{s-2}}\right)^{2^2} ... \left(\frac{h_q}{\ell_q}\right)^{2^{s-q}} = \prod_{j=1}^{s-q}\left(\frac{h_{s-j}}{\ell_{s-j}}\right)^{2^j}.$$

Finally, combining all of these inequalities, we have

$$\begin{aligned}
|l_{\kappa,2^s-1}(x_m)| &= \prod_{\substack{j=1\\ j\neq m,\kappa}}^{2^{s+1}} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right| = \left(\prod_{\substack{i=1\\ x_j\in I_{q-i}}}^{q} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right|\right)\left(\prod_{\substack{x_j\in I_q\\ j\neq m,\kappa}} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right|\right)\\
&\geq \left(\prod_{\substack{i=1,odd\\ x_j\in I_{q-i}}}^{q-1} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right|\right)\left(\prod_{\substack{x_j\in I_q\\ j\neq m,\kappa}} \left|\frac{x_m - x_j}{x_\kappa - x_j}\right|\right)\\
&\geq \left(\prod_{i=1,odd}^{q-1}\left(1-\frac{\ell_q}{h_{q-i}}\right)^{2^{s-q+i}}\right)\left(\prod_{j=1}^{s-q}\left(\frac{h_{s-j}}{\ell_{s-j}}\right)^{2^j}\right)\\
&=: A_{s,q}\ B_{s,q}.
\end{aligned}$$

Now, we estimate $A_{s,q}$ and $B_{s,q}$ from below for $K^{\alpha}$ with constant $\alpha$. Let $\alpha > 1$ and $\ell_1 \le 1/3$. Define the set $K^{\alpha}$, as usual, by the condition $\ell_n = \ell_{n-1}^{\alpha} = \ell_{n-2}^{\alpha^2} = \ldots = \ell_1^{\alpha^{n-1}}$ for $n \in \mathbb{N}$. This implies $h_n = \ell_n - 2\ell_{n+1} = \ell_n(1 - 2\ell_n^{\alpha-1})$.

First let us estimate $A_{s,q} = \prod_{i=1,odd}^{q-1} \left(1 - \dfrac{\ell_q}{h_{q-i}}\right)^{2^{s-q+i}}$ from below:

Set $\varepsilon_\alpha := \min\{1 - 2\ell_1, 1 - 2\ell_1^{\alpha-1}\}$. Then for each $k \in \mathbb{N}$ we have

$$h_k \ge \varepsilon_\alpha \ell_k. \tag{3.13}$$

By (3.13), we have

$$\frac{\ell_q}{h_{q-i}} \le \varepsilon_\alpha^{-1} \frac{\ell_q}{\ell_{q-1}} = \varepsilon_\alpha^{-1}\ \ell_1^{\alpha^{q-2}(\alpha-1)}. \tag{3.14}$$

In calculation we consider large enough $q = q(s)$. In particular, we can suppose that $\dfrac{\ell_q}{h_{q-i}} \le 1/2$, since RHS of (3.14) goes to 0 as $q$ goes to infinity. Thus, we can use the bound $\log(1-x) > -2x$ which is valid for $0 < x < 1/2$. Including into the product terms corresponding to even $i$, we get

$$\log(A_{s,q}) > -2 \sum_{i=1}^{q-1} 2^{s-q+i} \varepsilon_\alpha^{-1} \frac{\ell_q}{l_{q-1}} > -\varepsilon_\alpha^{-1}\, 2^{s+1} \ell_1^{\alpha^{q-2}(\alpha-1)}.$$

Let us choose $q \in \left\{ \left[\frac{\log s}{\log \alpha} + 3\right], \left[\frac{\log s}{\log \alpha} + 4\right] \right\}$ (since we want $q$ even), where $[a]$ denotes the greatest integer in $a$. Then $\alpha^{q-2} > s$ and $\ell_1^{\alpha^{q-2}(\alpha-1)} < \ell_1^{s(\alpha-1)} < 2^{-s}$. Therefore,

$$A_{s,q} > \varepsilon_0 := \exp(-2\varepsilon_\alpha^{-1}).$$

On the other hand, for $B_{s,q}$ we have

$$B_{s,q} = \prod_{j=1}^{s-q} \left( \frac{h_{s-j}}{\ell_{s-j}} \right)^{2^j} = \prod_{j=1}^{s-q} \left(1 - 2\ell_{s-j}^{\alpha-1}\right)^{2^j} = (1 - 2\ell_{s-1}^{\alpha-1})^2 \; (1 - 2\ell_{s-2}^{\alpha-1})^{2^2} ... \; (1 - 2\ell_q^{\alpha-1})^{2^{s-q}}$$

$$\geq (1 - 2\ell_q^{\alpha-1})^{2^{s-q+1}}.$$

Thus, if we use the bound $\log(1-x) > -2x$ again we get

$$\log B_{s,q} \geq 2^{s-q+1} \log(1 - 2\ell_q^{\alpha-1}) > -2^{s-q+3}\ell_q^{\alpha-1}.$$

Taking again for large enough $s \in \mathbb{N}$, $q \in \left\{ \left[\frac{\log s}{\log \alpha} + 3\right], \left[\frac{\log s}{\log \alpha} + 4\right] \right\}$ we get $\ell_q^{\alpha-1} < 2^{-s}$, hence

$$\log B_{s,q} > -2^{-q+3}$$

and since $q > 3$

$$B_{s,q} > e^{-1}.$$

Now, remember $|Z \cap I_q| = 2^{s-q+1}$. More than half of the points in $|Z \cap I_q|$ are smaller than $x_m$, so for every $s \in \mathbb{N}$, there are at least $2^{s-q}$ points $x_\kappa \in I_q$ with $x_\kappa < x_m$, which implies $|l_{\kappa,2^s-1}(x_m)| \geq A_{s,q} \; B_{s,q}$. So, for the corresponding Lebesque constant $\Lambda_{2^s-1} = \Lambda_{2^s-1}(Y^\alpha, K^\alpha)$, we have $\Lambda_{2^s-1} \geq 2^{s-q}A_{s,q} \; B_{s,q}$ for every $s \in \mathbb{N}$ and for every even $q \leq s-1$, in particular for $q \approx \left[\frac{\log s}{\log \alpha}\right]$. Finally, using the lower bounds corresponding to $A_{s,q}B_{s,q}$ that we obtained, we get

$$\Lambda_{2^s-1} > 2^{s-q} \exp(-2\varepsilon_\alpha^{-1} - 1).$$

Taking limits as s tends to infinity, we attain

$$\lim_{s\to\infty} \Lambda_{2^s-1} = \infty. \tag{3.15}$$

$\square$

By this result, for the particular exclusion of the node $x_m = \sum_{j=0}^{s}(-1)^j \ell_j$, we have that the corresponding subsequence of Lebesgue constants is unbounded. The reason we have chosen the node $x_m$ in particular is because we think that the following conjecture is true.

**Conjecture 3.3.6.** Let $\Omega_k = \prod_{\substack{i=1 \\ i \neq k}}^{2^{s+1}} |x_k - x_i|$. Then for every $s \in \mathbb{N}$ and $k = 1, 2, ..., 2^{s+1}$, we have

$$\Omega_m \leq \Omega_k.$$

The proof of this conjecture is currently under progress. Assuming Conjecture 3.3.6 is true, we have the following inequality:

$$\lambda_{2^{s+1}-1}(Y^{\alpha}_{m,s}, x_m) = \sum_{\substack{i=1 \\ i \neq m}}^{2^{s+1}} \frac{\Omega_m}{\Omega_i} \leq \sum_{\substack{i=1 \\ i \neq k}}^{2^{s+1}} \frac{\Omega_k}{\Omega_i} = \lambda_{2^{s+1}-1}(Y^{\alpha}_{k,s}, x_k) \tag{3.16}$$

Thus, (3.15) and (3.16) imply that all such subsequences (every choice of exclusion) of Lebesgue constants diverge.

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