DEFORMATIONS OF SOME BISET-THEORETIC CATEGORIES

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

DEFORMATIONS OF SOME BISET-THEORETIC CATEGORIES

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We define the subgroup category, a category on the class of finite groups where the morphisms are given by the subgroups of the direct products and the composition is the star product. We also introduce some of its deformations and provide a criteria for their semisimplicity. We show that biset category can be realized as an invariant subcategory of the subgroup category, where the composition is much simpler. With this correspondence, we obtain some of the deformations of the biset category. We further our methods to the fibred biset category by introducing the subcharacter partial category. Similarly, we also realize the fibred biset category and some of its deformations in a category where the composition is more easily described.

Keywords: biset functor, fibred biset functor, subgroup category, partial category, semisimplicity, semisimple deformation .

ÖZET

İKİLİ KÜME KURAMLI BAZI KATEGORİLERİN DEFORMASYONLARI

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Objeleri sonlu gruplar, morfizmaları direk çarpımların alt grupları olan, kompozisyonu ise yıldız çarpımı ile verilen altgrup kategorisini tanımlıyoruz. Bu kategorinin bazı deformasyonlarını oluşturup yarıbasitlikleri için bir kriter veriyoruz. İkili küme kategorisinin, daha basit kompozisyona sahip olan altgrup kategorisinin bir değişmez kategorisi olarak görülebileceğini gösteriyoruz. Bu bağlantı sayesinde ikili küme kategorisinin bazı deformasyonlarını elde ediyoruz. Ayrıca, alt karakter kısmi kategorisini tanımlayarak yöntemlerimizi lifli ikili küme kategorisine genişletiyoruz. Benzer şekilde, lifli ikili küme kategorisini ve bazı deformasyonlarını, kompozisyonu daha basit olan bir kategoride elde ediyoruz.

Anahtar sözcükler: ikili etki izleci, lifli ikili etki izleci, altgrup kategorisi, kısmi kategori, yarı basitlilik, yarı basit deformasyon.

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Contents

Chapter 1

Introduction

The Burnside ring $B(G)$ of a finite group G is defined to be the Grothendieck ring of isomorphism classes of finite G-sets. It has relations with some fundamental properties of G; a result due to Dress appearing in [12] states that G is solvable if and only if 0 and 1 are the only idempotents of $B(G)$.

An approach towards studying the structure of $B(G)$ involves embeddings into a suitable ghost ring; i.e., a ring where the multiplicative relations are easier to obtain. An example of this would be the embedding of $B(G)$ into $\prod_{H\leq G}\mathbb{Z}$ via the mark homomorphism.

The double Burnside group $B(G,H)$ of two finite groups G and H is defined to be the Grothendieck group of finite G-H-bisets; the sets with a left G action and a right H action that commute with each other. Biset functors, brought out by Bouc in [10], is mainly concerned with studying the biset category $\mathcal{B}$, where the double Burnside groups constitute the morphism sets.

Certain subcategories of $\mathcal{B}$ attracted some attention, one example is the bifree biset category $\mathcal{B}^{\Delta}$, which is the subcategory of $\mathcal{B}$ formed by allowing only the bifree bisets as morphisms, that is, by replacing $B(G,H)$ with $B^{\Delta}(G,H)$; the bifree double Burnside group. Although the result has independently been proven by other authors, In [9], Boltje and Danz utilized the ghost ring theme to obtain the semisimplicity of $\mathcal{B}^\Delta$. A striking result by Ragnarsson and Stancu in [16] is the correspondance between the saturated fusion systems on a finite p-group P and the idempotents of $\mathbb{Z}_{(p)}B^\Delta(P,P)$, where p denotes a prime.

Letting A be a multiplicatively written abelian group, a more general version of the Burnside ring, called the monomial Burnside ring $B^A(G)$ is given by Dress in [13]. This was achieved by equipping the G-sets with one-dimensional characters, which are called fibres. A consequence of this structural enrichment is the possibility of realizing some representation theoretic rings, such as the Green ring $R_{\mathbb{F}}(G)$ or the trivial source ring $T(G)$, as subrings of $B^A(G)$ via the linearization map.

By adopting Dress' generalization for the Burnside ring, one can define the fibred biset category $\mathcal{B}^A$, which was studied by Boltje and Coşkun in [7]. It could possibly be the playground for some of the representation theoretic categories, such as the trivial source category $\mathcal{T}$ or the bifree trivial source category $\mathcal{T}^\Delta$.

We are interested in simplifying the multiplicative difficulties that are present in the biset and the fibred biset category. Similar to Boltje and Danz, we overcome this by introducing ghost algebras. These algebras will also allow us to obtain the deformations.

Let $\mathbb{K}$ be a field of characteristic zero. We shall study a family of categories that are indexed by monoid homomorphisms $\ell : \mathbb{N}^+ \mapsto \mathbb{K}^\times$ whose role will be to control the coefficients that appear in the composition. These will consitute the deformations of the **subgroup category** $\mathcal{S}$. It is a category where the objects are the class of finite groups and letting F, G and H be objects of $\mathcal{S}$, its morphisms are given by the subgroups of the direct products. For morphisms $U \leq F \times G$ and $V \leq G \times H$ of $\mathcal{S}$, their composition is given as $U * V \leq F \times H$, which is the subgroup consisting of elements $f \times h$ for which there exists a $g \in G$ such that $f \times g \in U$ and $g \times h \in V$. Letting $\mathcal{K}$ denote any collection of finite groups, we will also focus on the full subcategories $\mathcal{S}_{\mathcal{K}}$ where objects is given by $\mathcal{K}$. We will see that the subcategory $\mathcal{S}_{\mathcal{K}}$ is not semisimple if $\mathcal{K}$ contains a nontrivial group. However, we also will show that this is more or less an isolated instance and one can recover the semisimplicity by considering an ℓ that is algebraically independent over $\mathcal{K}$, that is, the set $\{\ell(q)\}$ is algebraically independent over $\mathbb{Q}$.

In the particular case of ℓ being the inclusion of natural numbers, we will see that $\mathcal{B}$ can be realized as a subcategory, and one side of the Bouc's theorem which says, $\mathcal{B}_{\mathcal{K}}$ is semisimple if and only if every $G \in \mathcal{K}$ is cyclic, can be generalized.

We will extend our approach to the fibred biset category $\mathcal{B}^A$ by equipping the morphisms of $\mathcal{S}$ with subcharacters to define the subcharacter partial category $\mathcal{S}^A$. Similary, ℓ will determine the deformations. We will realize $\mathcal{B}^A$ and its deformations as invariant subcategories. However, the resolution of the semisimplicity results that are analogous to those of the subgroup category remains for future studies.

The thesis is organized as follows;

In Chapter 3, we define the subgroup category and obtain the deformations of the biset category.

In Chapter 4, we adopt the theme initiated in the previous chapter to the fibred biset category.

We finish by a chapter on the semisimplicity of the deformations of the biset category, we prove that the subgroup category is semisimple if ℓ is algebraically independent and we will also show that, in the case where every element in $\mathcal{K}$ is cyclic, the semisimplicity holds for the certain deformations of the biset category that are not algebracally independent.

Let us note that Chapters 3 and 5 are based on [4] whereas Chapter 4 closely follows the paper [5].

Chapter 2

Preliminaries

2.1 Subgroup structure of the direct and star products of groups

Let F, G and H be groups and $U \leq F \times G$ and $V \leq G \times H$ two subgroups. One can obtain the following subgroups of G and H by using the structure of U:

"Left **projection** of U": $p_1(U) = \{f \in F : f \times g \in U \,\text{for some}\, g \in G\}$,

"Right **projection** of U": $p_2(U) = \{g \in G : f \times g \in U \,\text{for some}\, f \in F\}$,

"Left **kernel** of U": $k_1(U) = \{f \in F : f \times 1 \in U\}$,

"Right **kernel** of U": $k_2(U) = \{g \in G : 1 \times g \in U\}$.

It is clear that $k_1(U) \trianglelefteq p_1(U)$, $k_2(U) \trianglelefteq p_2(U)$ and $k_1(U) \times k_2(U) \trianglelefteq U$. Moreover, we have

$$\frac{p_1(U)}{k_1(U)} \cong \frac{U}{k_1(U) \times k_2(U)} \cong \frac{p_2(U)}{k_2(U)}$$

where the isomorphisms are given by the canonical projections onto the first and the second coordinates. With this observation, we can make one more definition about U:

$$\text{"}\mathbf{Thorax}\text{ of }U\text{"}\colon q(U) = \frac{p_1(U)}{k_1(U)} \cong \frac{U}{k_1(U) \times k_2(U)} \cong \frac{p_2(U)}{k_2(U)}.$$

Goursat's Theorem provides a way to classify the subgroups of a direct product by the five subgroups given above. We omit the straightforward proof.

Theorem 2.1.1 (Goursat)**.** *Let F and G be two groups. Then there is a bijective correspondence between the subgroups $U \leq F \times G$ and the quintuples $(P_1, K_1, \kappa, K_2, P_2)$ such that $K_1 \trianglelefteq P_1 \leq F$, $K_2 \trianglelefteq P_2 \leq G$ and $\kappa : P_1/K_1 \leftarrow P_2/K_2$ is an isomorphism. The correspondance is such that $U \leftrightarrow (p_1(U), k_1(U), \kappa_U, k_2(U), p_2(U))$ where $xk_1(U) = \kappa_U(yk_2(U))$ for $x \times y \in U$.*

From now on, whenever we need to indicate more details about a subgroup of a direct product, we will use the notation $U = \Delta(p_1(U), k_1(U), \kappa_U, k_2(U), p_2(U))$ and for a group G we also set $\Delta(G) = \Delta(G, 1, \mathrm{id}, 1, G)$ where id denotes the identity homomorphism on G.

Before we define an operation between subgroups of direct products, let us recall a fundamental lemma which will allow us to understand their structure more thoroughly. Proofs of the next two lemmas are easy.

Lemma 2.1.2 (Zassenhaus' Butterfly Lemma)**.** *Given groups*

$$k_2(U) \trianglelefteq p_2(U) \leq G \geq p_1(V) \trianglerighteq k_1(V)$$

we have

$$\frac{(p_2(U) \cap p_1(V))k_2(U)}{(p_2(U) \cap k_1(V))k_2(U)} \cong \frac{p_2(U) \cap p_1(V)}{(p_2(U) \cap k_1(V))(k_2(U) \cap p_2(V))} \cong \frac{(p_2(U) \cap p_1(V))k_1(V)}{(k_2(U) \cap p_1(V))k_1(V)}.$$

We let the star product of U and V, denoted $U * V$, to be the subgroup $W \leq F \times H$, where $W = \{f \times h : f \in p_1(U), h \in p_2(V) \text{ s.t. } f \times g \in U, g \times h \in V \text{ for some } g \in G\}$. A quick application of the butterfly lemma yields;

Lemma 2.1.3. $\dfrac{p_1(W)}{k_1(W)} \cong \dfrac{p_2(U) \cap p_1(V)}{(p_2(U) \cap k_1(V))(k_2(U) \cap p_2(V))} \cong \dfrac{p_2(W)}{k_2(W)}.$

Given any group X, we let $[X]$ to denote its isomorphism class. We say X is a factor group of a group Y when X is isomorphic to a quotient of a subgroup of Y, in which case we write $[X] \leq [Y]$.

Next proposition follows directly from Lemma 2.1.3.

Proposition 2.1.4. *Letting U, V be as before and $W = U * V$, we have $[q(U)] \geq [q(W)] \leq [q(V)]$*

Proposition 2.1.5. *For every $W \leq F \times H$, there exists $U \leq F \times q(W), V \leq q(W) \times H$ such that $W \leq U * V$*

Proof. Pick $U = \Delta(p_1(W), k_1(W), \phi, 1, q(W))$ and $V = \Delta(q(W), 1, \psi, k_2(W), p_2(W))$ where ϕ and ψ are the isomorphisms such that $\varphi\psi = \kappa_W$. □

2.2 Bisets and fibred permutation sets

2.2.1 Bisets

For a more detailed discussion, see Section 2 of [11].

Letting F, G and H be finite groups, we define an F-G-**biset** X to be a set with left F action and a right G action such that the actions commute.

Letting Y be a G-H-biset, the product $X \times_G Y$ of bisets X and Y is defined to be the set of G-orbits of the cartesian product $X \times Y$ with the G-action given by

$$g \cdot (x \times y) = xg^{-1} \times gy.$$

The product becomes an F-H-biset via the actions

$$f(x \times_G y)h = (fx) \times_G (yh).$$

Let G^{op} denote the group obtained by reversing the group operation of G, that is $f \cdot_{op} g = gh$ for the elements f and g from the underlying set of G. We can see an $F - G$-biset X as an $F \times G^{op}$-set. So the transitive bisets are of the form $(F \times G)/U_x$ for some $U_x \leq F \times G$. One can decompose X into transitive bisets, that is;

$$X = \bigsqcup_{x \in [F \backslash X / G]} (F \times G)/U_x$$

where x runs over the representatives of F-G-orbits on X and U_x denotes the stabilizer of x in $F \times G$.

We let $\left[\frac{F \times G}{U}\right]$ to denote the isomorphism class of the transitive F-G-biset with point stabilizer U.

The product of isomorphism classes of two bisets can be given more explicitly in the transitive case by the following statement, which is Lemma 2.3.24 in [11].

Lemma 2.2.1 (Mackey product formula). *Let F, G and H be finite groups and let $U \leq F \times G$ and $V \leq G \times H$. Then*

$$\left[\frac{F \times G}{U}\right] \cdot \left[\frac{G \times H}{V}\right] = \sum_{p_2(U) g p_1(V) \subseteq G} \left[\frac{F \times H}{U * {}^{(g,1)}V}\right]$$

where g is running over the double coset representatives.

The Burnside group of G, denoted with $B(G)$, is the Grothendieck group of the isomorphism classes of finite G-sets where for G-sets X and Y, we can define addition on their isomorphism classes as their disjoint union, that is,

$$[X] + [Y] = [X \sqcup Y] .$$

Similarly, we can make $B(G)$ into a commutative ring by defining their multiplication as

$$[X] \cdot [Y] = [X \times Y] .$$

We will use $B(F, G)$ to denote the Burnside group $B(H \times G^{op})$, which corresponds to the Grothendieck group of isomorphism classes of finite F-G-bisets and is usually called the double Burnside group.

In the case $F = G$, one can define the double Burnside ring $B(G, G)$ with the product $\times_G$ given above.

We finish this section by giving a descriptive picture for the double Burnside group $B(F, G)$.

Remark 2.2.2. As an abelian group,

$$B(F, G) = \bigoplus_{U \in_{F \times G} F \times G} \mathbb{Z}\left[\frac{F \times G}{U}\right].$$

where U runs over the representatives of the conjugacy classes of subgroups of $F \times G$.

2.2.2 Fibred permutation sets

Monomial Burnside rings were introduced by Dress in [13]. We will provide a brief coverage that is concerned mainly with subcharacters. We refer the reader to [2] for further details.

Let A be a multiplicatively written abelian group and G be a finite group. Let AG denote the set $A \times G = \{(a, g), a \in A, g \in G\}$. An A-free AG-set with finitely many A-orbits is called an A-**fibred** G-set. We call A-orbits **fibres**.

Let AX be an A-fibred G-set where X is a set of representatives of fibres. Letting $A \setminus AX$ denote the set of fibres of AX, we say that AX is transitive as an AG-set if and only if $A \setminus AX$ is transitive as a G-set.

A group homomorphism $\mu : A \leftarrow G$ is called an A-**character** of G. Given $V \leq G$ and an A-character ν of V, we call the pair (V, ν) an A-**subcharacter**.

The group G acts on A-subcharacters via conjugation, that is ${}^g(V, \nu) = ({}^gV, {}^g\nu)$ and ${}^g\nu({}^gv) = \nu(v)$ for all $v \in V$.

Let $A_\nu G/V$ denote a transitive A-fibred G-set such that V is the stabilizer of a fibre Ax and $vx = \nu(v)x$ for all $v \in V$.

Remark 2.2.3. Given A subcharacters (V, ν) and (W, ω) then $A_\nu V/V$ is isomorphic to $A_\omega G/W$ if and only if (V, ν) is G-conjugate to (W, ω). Moreover, every transitive A-fibred G-set is of the form $A_\nu G/V$.

Now suppose $S = AX$ and $T = AY$ are both A-fibred G-sets. The addition on their isomorphism classes is defined by their disjoint union, that is,

$$[S] + [T] = [S \sqcup T]$$

The multiplication is defined as $[S] \cdot [T] = [S \times T]$ where

$$S \times T = \{s \times t : s \in S, t \in T, s \times t = (as, a^{-1}t),\ a \in A\}$$

that is, $S \times T$ is generated by the A-orbits. This construction makes $S \times T$ an A-fibred G-set. The monomial Burnside ring is defined to be the Grothendieck ring $B_A(G)$ generated by the isomorphism classes of A-fibred G-sets with operations; $[AX] + [AY] = [AX \sqcup AY] = [A(X \sqcup Y)]$ and $[AX] \cdot [AY] = [AX \times AY] = [AXY]$.

Remark 2.2.4. As an abelian group,

$$B_A(G) = \bigoplus_{[V,\nu]} \mathbb{Z}\,[A_\nu G/V]$$

where $[V, \nu]$ runs over the representatives of conjugacy classes of A-subcharacters.

A version of the two sided constructions that were introduced for $B(G)$ can be defined similarly, that is we let $B^A(F, G) = B^A(F \times G)$. The isomorphism class of an A-fibred $F \times G$-biset where the stabilizer of an A-orbit is (U, μ) will be denoted with $\left[\frac{F \times G}{U, \mu}\right]$.

2.3 Projective modules and the structure of finite dimensional algebras

In this section, we will briefly cover the structure of finitely generated algebras, focusing mainly on the non-semisimple case. The content is well known, more elaborate discussions can be found in Chapter 3 of [1] or Chapter 7 of [18]. We shall omit all the proofs.

Throughout this section, unless we state otherwise, we will let A be a unital finite-dimensional algebra over an algebraically closed field k of characteristic p.

There is a strong connection between the structure of the regular A-module and the category of A-modules. When A is semisimple, this connection yields the famous Artin-Wedderburn Theorem, which decomposes A into direct sum of matrix algebras over division algebras over k.

In the case where A is not semisimple, the relation, though not as explicit, still survives. Plainly, A still decomposes as a direct sum, but of indecomposable modules, which are, in this case, not necessarily simple. We are going to see that this decomposition can be associated with the simple A-module via projective indecomposable summands.

For an A-module P, let $J(P)$ denote the Jacobson radical of P. The following important property of projective modules gives us some insight on their relation with simple A-modules:

Proposition 2.3.1. *Let P be a projective A-module. Then $J(P)$ is the unique maximal ideal of P, hence $P/J(P)$ is simple and each simple A-module arises in this way.*

It is possible to provide the relation given above in a more explicit way by making use of idempotents that appear in the following decomposition:

Proposition 2.3.2. *Let A be a ring. There is a bijection between the decompositions ${}_AA = A_1 \oplus \cdots \oplus A_n$ of the regular module into submodules and the decomposition $1 = e_1 + \cdots + e_n$ of the identity into orthogonal idempotents given by $A_i = Ae_i$. Moreover, A_i is indecomposable if and only if e_i is primitive.*

The next theorem is an explicit description of the simple A-modules and their projective covers:

Theorem 2.3.3. *Up to isomorphism, there is a bijection between the simple A-modules S and the projective indecomposable modules P_S given by $P_S/J(P_S) \cong S$. Moreover, P_S is the projective cover of S if and only if $P_S = Ae$ and $eS \neq 0$ for a primitive idempotent e that annihilates any simple A-module T that is not isomorphic to S.*

Assembling everything together, we obtain this final description for A:

Theorem 2.3.4. *Each P_S is isomorphic to a summand of ${}_AA$. More explicitly,*

$$ {}_AA \cong \bigoplus_S (P_S)^{n_S} $$

is a decomposition of the regular A-module into projective indecomposable A-modules where n_S is the multiplicity of S in $A/J(A)$.

2.4 Posets and möbius inversion

Posets will play an important role in the semisimplicity results that we will obtain in the upcoming sections. We are going to provide an account on them that is more or less self contained. The reader can use [17] for more details on the subject.

We call a set $\mathcal{P}$ a **partially ordered set**, or poset in short, if there exists a binary relation $\leq_{\mathcal{P}}$ which satisfies,

- For all $x \in \mathcal{P}$, $x \leq_{\mathcal{P}} x$,
- For $x, y, z \in \mathcal{P}$, if $x \leq_{\mathcal{P}} y$ and $y \leq_{\mathcal{P}} x$ then $x = y$,
- For $x, y, z \in \mathcal{P}$, if $x \leq_{\mathcal{P}} y$ and $y \leq_{\mathcal{P}} z$ then $x \leq_{\mathcal{P}} z$.

We will drop the subscript of $\leq_{\mathcal{P}}$ when no confusion can arise.

Given two posets $\mathcal{P}$ and $\mathcal{Q}$, a function $\rho : \mathcal{Q} \leftarrow \mathcal{P}$ is called a **poset map** if ρ is order preserving, that is, given $x, y \in \mathcal{P}$, if $x \leq y$, then $\rho(x) \leq \rho(y)$.

We sometimes will restrict our attention to subsets $\mathcal{P}'$ of $\mathcal{P}$, which naturally will inherit the poset structure, so we call $\mathcal{P}'$ a **subposet** of $\mathcal{P}$.

Some natural subposets to consider are the ones that are defined by considering intervals such as,

- $[x, z] = \{y \in \mathcal{P} : x \leq y \leq z\}$ a closed interval
- $(x, z) = \{y \in \mathcal{P} : x < y < z\}$, an open interval.

A subposet $\mathcal{P}'$ of $\mathcal{P}$ is called **convex** if for every $x, z \in \mathcal{P}'$, the interval $[x, z]$ is contained in $\mathcal{P}'$.

The **lower set** of x in $\mathcal{P}$, denoted with $_{\leq \mathcal{P}}(x)$ is defined as the set $\{y \in \mathcal{P} : y \leq x\}$.

We call a nonempty subset $I \subseteq \mathcal{P}$ an **ideal** of $\mathcal{P}$ if for every $x \in I$, the lower set $_{\leq \mathcal{P}}(x)$ is contained in I and for every pair of elements $x, y \in I$, there exists an element $z \in I$, such that, $x, y \in_{\leq \mathcal{P}} (z)$.

Now let us focus on locally finite posets; the posets where every interval consists of finitely many elements. The main tool we use to understand a locally finite poset will be the Möbius function $\text{möb}_{\mathcal{P}} : \mathbb{Z} \leftarrow \mathcal{P} \times \mathcal{P}$, which is defined by the relations

- $\text{möb}_{\mathcal{P}}(x, x) = 1$ for all $x \in \mathcal{P}$,
- $\sum_{x \leq y \leq z} \text{möb}_{\mathcal{P}}(x, z) = 0$ for all $x < z$ in $\mathcal{P}$.

We call an ideal I principal order ideal if it has the form $I =_{\leq \mathcal{P}} (x)$ for some $x \in \mathcal{P}$.

The following is 3.7.1 in [17].

Proposition 2.4.1. *Let $\mathcal{P}$ be a poset in which every principal order ideal is finite. Let A be a multiplicatively written abelian group and $\sigma, \tau : A \leftarrow \mathcal{P}$ two functions. Then the following are equivalent:*

- $\sigma(x) = \sum_{y \leq x} \tau(y)$ *for all* $x \in \mathcal{P}$
- $\tau(x) = \sum_{y \leq x} \sigma(y) \text{möb}(y, x)$ *for all* $x \in \mathcal{P}$.

We call σ the **sum function** of τ and τ the **totient function** of σ.

A subposet $\mathcal{C} \subseteq \mathcal{P}$, where every pair of elements is comparable, that is, if $x, y \in \mathcal{C}$, then $x \leq y$ or $y \leq x$, is called a **chain**.

Now suppose that $\mathcal{P}$ satisfies the assumption of 2.4.1. The **height** of an element $x \in \mathcal{P}$ is defined to be the height of $\leq_{\mathcal{P}} (x)$, the maximum cardinality of the chains $\mathcal{C}$ contained in the lower set of x.

Given two posets $\mathcal{P}$ and $\mathcal{Q}$, their direct product $\mathcal{P} \times \mathcal{Q}$ is a poset with relation $(p, q) \leq_{\mathcal{P} \times \mathcal{Q}} (p', q')$ if and only if $p \leq_{\mathcal{P}} p'$ and $q \leq_{\mathcal{Q}} q'$.

The next result is 3.8.2 in [17].

Proposition 2.4.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two posets with the property that every interval $[a, b]$ has finitely many elements. If $(p, q) \leq_{\mathcal{P} \times \mathcal{Q}} (p', q')$, then*

$$\text{möb}_{\mathcal{P} \times \mathcal{Q}}((p, q), (p', q')) = \text{möb}_{\mathcal{P}}(p, p') \text{möb}_{\mathcal{Q}}(q, q').$$

A poset map $\rho : \mathcal{P} \leftarrow \mathcal{P}$ is called a **retraction** if for all $u \in \mathcal{P}$ we have $\rho^2(u) = \rho(u) \leq u$. We end this section with a result that provides a way to refine the poset maps via retractions, it is Lemma 4.1 from [9].

Lemma 2.4.3 (Boltje-Danz). *Let $\mathcal{P}$ be a poset in which every principal order ideal is finite and A be an abelian group. Let $\sigma : A \leftarrow \mathcal{P}$ be a map such that, for all $u \in \mathcal{P}$, we have, $\sigma(u) = \sigma(\rho(u))$. Let σ' be a map on $\mathcal{P}'$ where $\rho(\mathcal{P}) = \mathcal{P}'$ such* *that $\sigma' = \sigma$ on $\mathcal{P}'$, τ and τ' the totient functions of σ and σ'. Then τ vanishes for all $v \in \mathcal{P} - \mathcal{P}'$ and $\tau(v) = \tau'(v)$ for all $v \in \mathcal{P}'$.*

Proof. Let $v \in \mathcal{P} - \mathcal{P}'$, let $v' = \rho(v)$, so $v' \leq v$. Then we have

$$0 = \sigma(v) - \sigma(\rho(v)) = \sigma(v) - \sigma(v') = \sum_{u \leq_{\mathcal{P}} v} \tau(v) - \sum_{r \leq_{\mathcal{P}} v'} \tau(r) = \tau(v) + \sum_{w < v, w \not\leq v'} \tau(w).$$

Since ρ is a poset map and $w \leq v$, we have $\rho(w) \leq \rho(v) = v'$, so $\rho(w) \neq w$ which implies $w \notin \mathcal{P}'$, because otherwise for some $t \in \mathcal{P}$, $\rho(t) = w$, $\rho(\rho(t)) = \rho(w)$ so $w = \rho(w)$. Suppose that $\tau(s) = 0$ for every s which has lower height than v. Then, $\sum_{s<v, s \not\leq v'} \tau(s) = 0$, so by induction on the height of v, we obtain $\tau(v) = 0$. Now let $v \in \mathcal{P}'$, refining the index of $\sum_{w<v, w \not\leq v'} \tau(w)$ we get

$$\sigma(v) = \tau(v) + \sum_{w <_{\mathcal{P}'} v} \tau(w) = \sum_{w \leq_{\mathcal{P}'} v} \tau(w) = \sigma'(v) = \tau'(v) + \sum_{w <_{\mathcal{P}'} v} \tau'(w).$$

Similar to above, supposing that $\tau(s) = \tau'(s)$ for every s which has lower height than v, we get $\sum_{s <_{\mathcal{P}'} v} \tau(s) = \sum_{w <_{\mathcal{P}'} v} \tau'(w)$ so by induction on the height of v we obtain $\tau(v) = \tau'(v)$. $\square$

2.5 Local semisimplicity and corner subalgebras

In order to be able to exploit the structural properties we have previously presented for finite-dimensional algebras in the infinite-dimensional case, we are going to introduce the concept of local semisimplicity.

Let A be a ring and B be a subring of A. We call B a **corner subring** if it satisfies $BAB \leq B$. We call a ring monomorphism $\nu : A \leftarrow C$ a **corner embedding** if $\nu(C)$ is a corner subring of A. We call A **locally unital** if every finite subset of A is contained in a subring of the form eAe where e is an idempotent of A.

The following proposition follows from the Theorem 6.2g in [14] .

Proposition 2.5.1 (Green)**.** *Let B be a corner subring of a locally unital ring A. Then B is locally unital. Moreover, there is a bijective correspondence between* *the isomorphism classes of the simple A-modules $[S]$ satisfying $BS \neq 0$ and the simple B-modules $[T]$, determined by the condition $T \cong BS$.*

We call A **locally semisimple** if A is locally unital and eAe is semisimple for every idempotent e of A.

Now let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let us briefly provide a way to see a $\mathbb{K}$-linear category as an algebra over $\mathbb{K}$. Let $\mathcal{C}$ be a small category and $\mathbb{K}\mathcal{C}$ denote its $\mathbb{K}$-linearization. Then the algebra $\mathcal{C}_{\mathrm{alg}}$ associated with $\mathbb{K}\mathcal{C}$ is the algebra with elements $\bigoplus_{x,y\in \mathrm{Obj}(\mathcal{C})} \mathcal{C}(x,y)$ that inherits its multiplication from the composition defined on the morphisms of $\mathbb{K}\mathcal{C}$. Furthermore, in the case where we are dealing with two incompatible morphisms, for example when the codomain and the domain do not match, the product is defined to be zero.

The next remark will allow some structural properties of the finite dimensional algebras to be realized by some particular infinite dimensional algebras.

Remark 2.5.2. Let $\mathcal{C}$ be a small $\mathbb{K}$-linear category. Let $\mathcal{O}$ be a set formed by some of the objects of $\mathcal{C}$ and let $\mathcal{C}_{\mathcal{O}}$ denote the full subcategory of $\mathcal{C}$ on $\mathcal{O}$. Then the following are equivalent:

- $\mathcal{C}$ is locally semisimple,
- Every full subcategory of $\mathcal{C}$ is locally semisimple,
- $\mathcal{C}_{\mathcal{O}}$ is semisimple for every finite set $\mathcal{O}$ of objects of $\mathcal{C}$.

Chapter 3

Deformations of the biset category

3.1 The subgroup category

We define the **subgroup category** to be the category $\mathcal{S}$ where the objects are finite groups. For two finite groups R and S, the $\mathcal{S}$-morphisms $R \leftarrow S$ are the subgroups of $R \times S$. If R, S and T are finite groups, $U \leq R \times S$ and $V \leq S \times T$ are subgroups, then their composition is given by their star product $U * V$.

Let $\mathcal{K}$ be any nonempty set of finite groups and let $\mathcal{S}_{\mathcal{K}}$ denote the full subcategory of $\mathcal{S}$ on $\mathcal{K}$.

In order to consider the deformations of the subgroup category, letting $\mathbb{K}$ to be a field of characteristic zero, we will be using a monoid homomorphism ℓ from $\mathbb{N}^+$ to the unit group $\mathbb{K}^\times$, which, in some cases will be taken as algebraically independent with respect to $\mathcal{K}$, that is, the set $\{\ell(q)\}$ where q runs over the prime divisors of the orders of the groups that are contained in $\mathcal{K}$ does not satisfy any nontrivial polynomial equation with coefficients from the minimal subfield $\mathbb{Q}$ of $\mathbb{K}$.

Consider elements $F, G, H, I \in \mathcal{K}$ and subgroups $U \leq F \times G, V \leq G \times H, W \leq H \times I$. For ease of notation, set $\ell(F) = \ell(|F|)$ for any $F \in \mathcal{K}$.

We define an associative algebra whose composition $*_\sigma$ is given as

$$U *_\sigma V = \sigma(U,V) U * V$$

where

$$\sigma(U,V) = \ell(k_2(U) \cap k_1(V)).$$

For associativity we need

$$\begin{aligned}(U *_\sigma V) *_\sigma W &= \sigma(U,V)(U * V) *_\sigma W \\ &= \sigma(U,V)\sigma(U * V, W) U * V * W\end{aligned}$$

to be equal to

$$\begin{aligned}= U *_\sigma (V *_\sigma W) &= U *_\sigma \sigma(V,W)(V * W) \\ &= \sigma(U, V * W)\sigma(V,W) U * V * W.\end{aligned}$$

That equality holds by the following relation, which is Proposition 3.5 from [9].

Proposition 3.1.1 (Boltje-Danz)**.** *$\sigma(U,V)\sigma(U * V, W) = \sigma(U, V * W)\sigma(V,W)$.*

Proof. Let A be the collection of paths one can take to obtain the element 1×1 in the $*$-product $U * V * W$, that is,

$$A = \{g \times h : 1 \times g \in U, g \times h \in V, h \times 1 \in W\}.$$

The projections and kernels of A can be expressed as

$$p_1(A) = k_2(U) \cap k_1(V * W) \quad k_1(A) = k_2(U) \cap k_1(V),$$

$$p_2(A) = k_2(U * W) \cap k_1(W) \quad k_2(A) = k_2(V) \cap k_1(W).$$

Since $q(A) \cong \frac{p_1(A)}{k_1(A)} \cong \frac{p_2(A)}{k_2(A)}$, we get

$$\frac{k_2(U) \cap k_1(V * W)}{k_2(U) \cap k_1(V)} \cong \frac{k_2(U * W) \cap k_1(W)}{k_2(V) \cap k_1(W)}$$

giving us the equality

$$|k_2(U) \cap k_1(V)| \cdot |k_2(U * W) \cap k_1(W)| = |k_2(U) \cap k_1(V * W)| \cdot |k_2(V) \cap k_1(W)|.$$

$\square$

The **twisted category algebra**, $\Lambda_{\mathcal{K}} = \mathbb{K}_\sigma \mathcal{S}_{\mathcal{K}}$, is defined to be the algebra over $\mathbb{K}$ with a basis consisting of formal elements $s_U^{F,G}$ where $U \leq F \times G$, with its multiplication $*_\sigma$ given by

$$s_U^{F,G} *_\sigma s_V^{G,H} = \sigma(U,V) s_{U*V}^{F,H}.$$

The identity element on G, denoted with $\mathrm{id}_G^{\Lambda_{\mathcal{K}}}$, is $s_{\Delta(G)}^{G,G}$.

Let us note that we can extend the above construction to the subgroup category. When we do so, we will drop the subscripts and let $\Lambda = \mathbb{K}_\sigma \mathcal{S}$. Given a $\Lambda_{\mathcal{K}}$-module M, its evaluation at G, expressed as $M(G)$, is given as $\mathrm{id}_G^{\Lambda_{\mathcal{K}}}.M$

We call the $\mathbb{K}$-basis $\{s_U^{F,G} : F, G \in \mathcal{K}, U \in \mathcal{S}_K(F,G)\}$ the **square basis**. Given $G \in \mathcal{K}$, we let $\mathrm{End}_\Lambda(G)$ denote the subalgebra of $\Lambda_{\mathcal{K}}$ generated by the elements $\{s_U^{G,G} : U \leq G \times G\}$.

The main tool that we use when investigating the structure of $\Lambda_{\mathcal{K}}$ will be the concept of seeds. We define an **$\mathcal{S}$-seed** for $\mathcal{K}$ to be the pair (E, W) where E is a factor group of an element of $\mathcal{K}$, and W is a simple $\mathbb{K}\mathrm{Aut}(E)$-module. The equivalence of two seeds (E, W) and (E', W') is determined by the existence of a group homomorphism $\theta : E \leftarrow E'$ such that $W \cong \mathrm{Iso}^\theta(W')$ where $\mathrm{Iso}^\theta(W') = \mathrm{Ind}^\theta(W') = \mathrm{Res}^{\theta^{-1}}(W')$ and Ind^θ and $\mathrm{Res}^{\theta^{-1}}$ denotes the induction and restriction of modules via θ.

The following is a classification of simple $\Lambda_{\mathcal{K}}$-modules by seeds, but with a relatively strong assumption:

Theorem 3.1.2. *Assume $\mathcal{K}$ owns an isomorphic copy of every factor group of every element. Every $\mathcal{S}$-seed for $\mathcal{K}$ can be realized with a group in $\mathcal{K}$, that is, for each $\mathcal{K}$-seed, there exists an equivalent $\mathcal{S}$-seed (E, W) with $E \in \mathcal{K}$. Moreover, there is a bijective correspondence between the isomorphism classes $[S]$ of simple $\Lambda_{\mathcal{K}}$-modules and the equivalence classes $[E, W]$ of $\mathcal{S}$-seeds for $\mathcal{K}$ which can be expressed with the condition that $[E]$ is minimal with respect to the partial ordering on factor groups such that $S(E) \neq 0$ and $W \cong \mathrm{Res}^{\mu_E}(S(E))$ where $\mu_E : \mathrm{End}_\Lambda(E) \leftarrow \mathbb{K}\mathrm{Aut}(E)$ is the algebra monomorphism induced by extending the map $s_{\Delta(\theta)}^{E,E} \leftarrow \theta$ $\mathbb{K}$-linearly for $\theta \in \mathrm{Aut}(E)$.*

Proof. The first statement follows from the fact that given any seed (E', W'), by the closure assumption on $\mathcal{K}$, one can work with the equivalent seed (E, W) where $E \in \mathcal{K}$. The rest of the statement follows from Theorem 2.4 of [6], because by Proposition 2.1.5, every morphism $F \leftarrow G$ factorizes through a group $T \in \mathcal{K}$ such that $[F] \geq [T] \leq [G]$. □

The next lemma tells us precisely when a simple module vanishes:

Lemma 3.1.3. *Let $\mathcal{K}$ satisfy the assumption of the Theorem 3.1.2. Also let $[S]$ and $[E, W]$ be the correspondents of each other in the sense of the same theorem. Then $S(G) \neq 0$ if and only if $[E] \leq [G]$.*

Proof. If $S(G) \neq 0$, then, by the minimality of $[E]$, we must have $[E] \leq [G]$. So suppose that there exists an isomorphism $\phi : E \leftarrow B/Y$ where $Y \trianglelefteq B \leq G$. Also, let us work with the equivalent seed where representative E is included in $\mathcal{K}$. Let x be a nonzero element from $S(E)$. Let $I = \Delta(E, 1, \phi, Y, B)$ and $J = \Delta(B, Y, \phi^{-1}, 1, E)$ Then

$$s_I^{E,G} s_J^{G,E} x = \sigma(I, J) s_{I*J}^{E,E} x = \ell(Y) s_{\Delta(E)}^{E,E} x = \ell(Y) x \neq 0.$$

Hence $S(G) \neq 0$ because $s_J^{G,E} x \in S(G)$. □

Our next aim is to generalize Theorem 3.1.2. Let $\mathcal{K}$ be any collection of finite groups and let $\mathcal{K}' \supseteq \mathcal{K}$ be a collection which satisfies the closure hypothesis of the Theorem 3.1.2. In that case, given $\mathcal{S}$-seed (E, W) for $\mathcal{K}$, there is a corresponding simple $\Lambda_{\mathcal{K}'}$-module S'. By Proposition 2.5.1 and Lemma 3.1.3, $\Lambda_{\mathcal{K}} S'$ is a simple module for Λ_K. For us to be able to work with this module we are going to need the following lemma on the independence of the choice $\mathcal{K}'$.

Lemma 3.1.4. *Let $\mathcal{K}, \mathcal{K}'$ and S' be as above. Then the simple module $\Lambda_{\mathcal{K}} S'$ is independent of the choice of collection $\mathcal{K}'$.*

Proof. Suppose $\mathcal{K}''$ is another such collection, let $\mathcal{K}''' = \mathcal{K}' \cup \mathcal{K}''$. Let S'' and S''' be the simple $\Lambda_{\mathcal{K}''}$ and $\Lambda_{\mathcal{K}'''}$ modules corresponding to the seed (E, W). Then

$S' = \Lambda_{\mathcal{K}'} S'''$ and $S'' = \Lambda_{\mathcal{K}''} S'''$. Repeated use of Proposition 2.5.1 gives

$$\Lambda_{\mathcal{K}} S' \cong \Lambda_{\mathcal{K}} \Lambda_{\mathcal{K}'} S''' \cong \Lambda_{\mathcal{K}} S''' \cong \Lambda_{\mathcal{K}} \Lambda_{\mathcal{K}''} S''' \cong \Lambda_{\mathcal{K}} S''$$

□

Now we are ready to generalize the Theorem 3.1.2:

Theorem 3.1.5. *Let $\mathcal{K}$ be any collection of finite groups. Then $[S] \leftrightarrow [E, W]$ characterizes a bijective correspondence between the isomorphism classes $[S]$ of simple $\Lambda_{\mathcal{K}}$-modules and the equivalence classes $[E, W]$of $\mathcal{S}$-seeds for $\mathcal{K}$.*

Proof. Letting $\mathcal{K}' \supseteq \mathcal{K}$ be as above, the simple $\Lambda_{\mathcal{K}'}$-module S' corresponding to the seed (E, W) yields to the simple $\Lambda_{\mathcal{K}}$-module $\Lambda_{\mathcal{K}} S'$. To finish the proof, observe that every simple $\Lambda_{\mathcal{K}}$-module is realized this way by Proposition 2.5.1 and there is a bijective correspondence with the simple $\Lambda_{\mathcal{K}'}$-modules. □

Next, we will provide a way to describe simple $\Lambda_{\mathcal{K}}$-modules more explicitly. Let $G \in \mathcal{K}$, $Y \trianglelefteq B \leq G$ and $\phi : E \leftarrow B/Y$ be an isomorphism. We let $\mu_\phi : \mathrm{End}_{\Lambda_{\mathcal{K}}}(G) \leftarrow \mathbb{K}\mathrm{Aut}(E)$ to be the algebra monomorphism that satisfies

$$\mu_\phi(\epsilon) = \ell(Y)^{-1} s^{G,G}_{\Delta(B,Y,\phi^{-1}\epsilon\phi,Y,B)}$$

for $\epsilon \in \mathrm{Aut}(E)$.

Theorem 3.1.6. *Let $\mathcal{K}$ be any collection of finite groups and let S be the simple $\Lambda_{\mathcal{K}}$-module with seed (E, W). Then the following determine S:*

- *For any $G \in \mathcal{K}, S(G) \neq 0$ if and only if $[E] \leq [G]$.*
- *Given an idempotent k of $\mathbb{K}\mathrm{Aut}(E)$, then $\mu_\phi(k) S(G) \neq 0$ if and only if $kW \neq 0$.*
- *$[E]$ is minimal such that given an $\mathcal{S}$-seed (E', W') satisfying the above conditions, then $[E] \leq [E']$.*

Proof. Lemma 3.1.3 allows us to assume that $E \in \mathcal{K}$, whence first and the last conditions follow. So we prove the second condition. Recall that $W \cong \mathrm{Res}^{\mu_E}(S(E))$. Therefore $kW \neq 0$ if and only if $\mu_E(k)S(E) \neq 0$. Now suppose that $kW \neq 0$. Let $I = \Delta(E, 1, \phi, Y, B)$, $J = \Delta(B, Y, \phi^{-1}, 1, E)$ and $\epsilon \in \mathrm{Aut}(E)$. Then

$$\mu_E(\epsilon) = s^{E,E}_{\Delta(E)} = s^{E,G}_I s^{G,G}_{\Delta(B,Y,\phi^{-1}\epsilon\phi,Y,B)} s^{G,E}_J \frac{1}{\ell(Y)^2} = s^{E,G}_I \mu_\phi(\epsilon) s^{G,E}_J \frac{1}{\ell(Y)}.$$

Since k is a $\mathbb{K}$-linear combination of such ϵ's, the equality holds for k as well. Thus $s^{E,G}_I \mu_\phi(\epsilon) s^{G,E}_J S(E) \neq 0$, which implies $\mu_\phi(\epsilon) s^{G,E}_J S(E) \neq 0$, so we get $\mu_\phi(\epsilon) S(G) \neq 0$.

For the other direction suppose that $\mu_\phi(k)S(G) \neq 0$, hence $s^{G,E}_J \mu_E(k) s^{E,G}_I S(G) \neq 0$, which implies $\mu_E(k)S(E) \neq 0$. Therefore $kW \neq 0$. $\square$

3.2 $\mathcal{S}$-endomorphisms of cyclic groups of prime order

When $G \cong C_q$ and $\lambda = \ell(q)$ where q is a prime, the complexity of $\mathrm{End}_\Lambda(G)$ is just enough so that it is relatively easy to investigate its structure, at the same time, it is nontrivial enough for us to obtain important consequences.

Our analysis in this section will enable us to decide the semisimplicity of $\Lambda_{\mathcal{K}}$ when $\mathcal{K}$ contains a cyclic group of prime order.

Let us present the statement which encapsulates our calculations regarding $\mathrm{End}_\Lambda(G)$:

Proposition 3.2.1. $\mathrm{End}_\Lambda(G)$ *is not semisimple when* $\lambda = 1$ *and* $G \cong C_q$.

Proof. The basis elements of $\mathrm{End}_\Lambda(G)$ can be given as

$$s_0 = s^{G,G}_{1\times 1}, \quad s_{01} = s^{G,G}_{1\times G}, \quad s_{10} = s^{G,G}_{G\times 1}, \quad s_{11} = s^{G,G}_{G\times G}, \quad s_d = s^{G,G}_{\Delta(d)}$$

where the diagonal subgroup $\Delta(d) = \{(g^d, g) : g \in G)\}$ is determined by the automorphisms of G where the image of g is given by g^d for $d \in (\mathbb{Z}/q)^{\times}$. For ease of tracking, let us introduce a multiplication table for the basis elements:

$*_\sigma$	s_0	s_{01}	s_{10}	s_{11}	s_d
s_0	s_0	s_{01}	s_0	s_{01}	s_0
s_{01}	s_0	s_{01}	λs_0	λs_{01}	s_{10}
s_{10}	s_{10}	s_{11}	s_{10}	s_{11}	s_0
s_{11}	s_{10}	s_{11}	λs_{10}	λs_{11}	s_{10}
s_d	s_0	s_{01}	s_0	s_{01}	

Now let $r = \lambda s_0 - s_{01} - s_{10} + s_{11}$, then

$$
\begin{aligned}
s_0 r &= \lambda s_0 - s_{01} - s_0 + s_{01} \\
s_{01} r &= \lambda s_0 - s_{01} - \lambda s_0 + \lambda s_{01} \\
s_{10} r &= \lambda s_0 - s_{11} - s_{10} + s_{11} \\
s_{11} r &= \lambda s_{10} - s_{11} - \lambda s_{10} + \lambda s_{11} \\
s_d r &= \lambda s_0 - s_{01} - s_0 + s_{01} \\
r s_0 &= \lambda s_0 - s_0 - s_{10} + s_{10} \\
r s_{01} &= \lambda s_{01} - s_{01} - s_{11} + s_{11} \\
r s_{10} &= \lambda s_0 - \lambda s_0 - s_{10} + \lambda s_{10} \\
r s_{11} &= \lambda s_{01} - \lambda s_{01} - s_{11} + \lambda s_{11} \\
r s_d &= \lambda s_0 - s_{10} - s_0 + s_{10}.
\end{aligned}
$$

Letting $\lambda = 1$, we see that all of the equations above vanish, which makes $\mathbb{K}r$ an ideal and $r^2 = 0$. So $\mathbb{K}r$ is nil, Hence $\mathbb{K}r \subseteq J(\mathrm{End}_\Lambda(G))$. The result follows. $\square$

3.3 Deformations of the biset category

In this section, we will construct the deformations of the biset category.

The **biset** category $\mathcal{B}$, introduced by Serge Bouc, is defined to be the category where

- the objects are finite groups,

- the morphism set for finite groups F and G is $B(F,G)$ and the composition operation for $u \in B(F,G)$ and $v \in B(G,H)$ is the element $u \circ v \in B(F,G)$ where $u \circ v = u \times_H v$.

One can extend the coefficients and work with the $\mathbb{K}$-**linear biset** category $\mathbb{K}\mathcal{B}$ objects of which are again the finite groups, the morphisms are however the $\mathbb{K}$-linear extension $\mathbb{K} \otimes_{\mathbb{Z}} B(F,G)$ and the composition is the $\mathbb{K}$-linear extension of composition of $\mathcal{B}$. $\mathbb{K}\mathcal{B}$ is actually a $\mathbb{K}$-linear category since the morphism sets are $\mathbb{K}$-modules and the composition is $\mathbb{K}$-bilinear.

Now we will introduce the $\mathbb{K}$-linear category $\mathbb{K}_\sigma\mathcal{B}$, which will constitute the deformations of $\mathbb{K}\mathcal{B}$. The objects of $\mathbb{K}_\sigma\mathcal{B}$ are finite groups and the morphism sets $\mathbb{K}_\sigma\mathcal{B}(F,G)$ are the $\mathbb{K}$-modules generated by the set $\{d_U^{F,G} : U \in_{F\times G} \mathcal{S}(F,G)\}$ and the composition is given by the formula

$$d_U^{F,G} d_V^{G,H} = \sum_{p_2(U)gp_1(V)\subseteq G} \frac{\ell(k_2(U) \cap {}^g(k_1(V))}{|k_2(U) \cap {}^g(k_1(V))|} d^{F,H}_{U*^{g,1}V}. \tag{3.1}$$

The following two lemmas will give the well definedness of the right hand side of the equation above.

Lemma 3.3.1. *Right hand side of the Equation 3.1 is independent of the coset representatives.*

Proof. Suppose $\widehat{g} = t_1 g t_2^{-1}$ with $(u,t_1) \in U, (t_2, v) \in V$. Then

$$\begin{aligned} {}^{(u,v)}(U * {}^{(g,1)}V) &= {}^{(u,1)}U * {}^{(g,v)}V = {}^{(u,t_1)}U * {}^{(t_1 g t_2^{-1},1)}\big({}^{(t_2,v)}V\big) \\ &= U * {}^{(t_1 g t_2^{-1},1)}V. \end{aligned}$$

This means the $*$-product still correspond to the same basis element upon the changing of the coset representatives. To finish the proof, we compare the coefficient $|k_2(U) \cap {}^{t_1 g t_2^{-1}} k_1(V)|$ with $|k_2(U) \cap {}^{g} k_1(V)|$. We have

$$\begin{aligned} &|k_2(U) \cap {}^{g}(k_1(V))| = |{}^{t_1}k_2(U) \cap {}^{t_1 g}(k_1(V))| \\ &= |{}^{t_1}k_2(U) \cap {}^{t_1 g t_2^{-1}}({}^{t_2}k_1(V))| = |k_2(U) \cap {}^{t_1 g t_2^{-1}} k_1(V)| \end{aligned}$$

because ${}^{t_1}k_2(U) = k_2(U), {}^{t_2}k_1(V) = k_1(V)$. □

Lemma 3.3.2. *The right-hand side of the Equation 3.1 remain unchanged if one replaces U and V with their conjugates.*

Proof. To show the independence on the conjugates, letting $x, y \in G$, it is enough to check whether the $*$-products of $U * {}^{(g,1)}V$ and ${}^{(1,x)}U * {}^{(xgy^{-1},1)(y,1)}V$ and the coefficients of the basis elements corresponding to them are equal. Previous lemma lets us work with the coset represented by $\widehat{g} = xgy^{-1}$. If we take $(m,n) \in U * {}^{(g,1)}V$ and let t be such that $(m,t) \in U, (t,n) \in {}^{(g,1)}V$ then $(m, {}^{x}t) \in {}^{(1,x)}U$ and since ${}^{(xgy^{-1},1)}({}^{(y,1)}V) = {}^{(xg,1)}V$, we also get $({}^{x}t, n) \in {}^{(xgy^{-1},1)}({}^{(y,1)}V)$, so $U * {}^{(g,1)}V = {}^{(1,x)}U * {}^{(xgy^{-1},1)(y,1)}V$. Independence when conjugating on the other coordinates are clear since $\left[\frac{F\times G}{U}\right]$ and $\left[\frac{F\times G}{{}^{f}U^{g}}\right]$ represent the same isomorphism class. The independence of the coefficient $|k_2(U) \cap {}^{g}(k_1(V))|$ follows from the fact that the two sets $\{k_2(U) \cap {}^{g_1}(k_1(V)\}$ and $\{{}^{(1,x)}k_2(U) \cap {}^{xg}k_1(V)\}$ are equal, which can be seen by rewriting the proof above for the situation $(m,n) = (1,1)$. □

If associativity holds, then $\mathbb{K}_\sigma\mathcal{B}$ becomes a $\mathbb{K}$-linear category with identity morphism $d^{G,G}_{\Delta(G)}$, and in the case $\ell(n) = n$ for every $n \in \mathbb{N}^+$, $\mathbb{K}_\sigma\mathcal{B}$ corresponds to $\mathbb{K}\mathcal{B}$ via identifying $d^{F,G}_{U}$ with $\left[\frac{F\times G}{U}\right]$.

As the next step, we will realize $\mathbb{K}_\sigma\mathcal{B}$ as an *invariant category* of $\mathbb{K}_\sigma\mathcal{S}(F,G)$, similar to the theme of Theorem 4.7 in [8].

There is a $\mathbb{K}(F \times G)$-module structure on $\mathbb{K}_\sigma\mathcal{S}(F,G)$ given by the action

$${}^{f\times g}s^{F,G}_{U} = s^{F,G}_{{}^{f\times g}U}.$$

Let $\sigma_G(g) = s^{G,G}_{\Delta(G,g,G)}$ where $\Delta(G,g,G) = \{{}^g b \times b : b \in G\}$. Then the module action can also be given by

$$ {}^{f\times g}x = \sigma_F(f)x\sigma_G(g^{-1}). $$

Since $\mathbb{K}$ is of characteristic zero, $\mathbb{K}G$ is semisimple, the principal block of $\mathbb{K}G$ is $e_G = \sum_{g\in G} \frac{g}{|G|}$. Let

$$ \overline{s}^{F,G}_U = \sigma_F(e_F)s^{F,G}_U\sigma_G(e_G) = \frac{1}{|F|\cdot|G|}\sum_{f\in F, g\in G} s^{F,G}_{{}^{f\times g}U}. $$

Let $\overline{\mathbb{K}_\sigma\mathcal{S}}$ be the category whose objects are finite groups and the morphisms $F \leftarrow G$ are given by the $F\times G$-fixed submodule

$$ \overline{\mathbb{K}_\sigma\mathcal{S}}(F,G) = (\mathbb{K}_\sigma\mathcal{S}(F,G))^{F\times G} = \bigoplus_{U\in_{F\times G}\mathcal{S}(F,G)} \mathbb{K}\overline{s}^{F,G}_U. $$

The identity morphism on G is the element $\sigma_G(e_G) = \overline{s}^{G,G}_{\Delta(G)}$. In order to confirm the identification $\mathbb{K}_\sigma\mathcal{B} \leftrightarrow \overline{\mathbb{K}_\sigma\mathcal{S}}$, we will need the following statement which is Lemma 4.2 in [9].

Lemma 3.3.3. *With the notation above,*

$$ |U|\cdot|V| = |p_2(U)p_1(V)|\cdot|k_2(U)\cap k_1(V)|\cdot|U*V|. $$

Proof. Let $\Gamma(U,V) = \{f\times g\times h : f\times g\in U, g\times h\in V\}$. Given an element $f\times g\times h\in\Gamma(U,V)$, $f'\in F, h'\in H$ then $f'\times g\times h'\in\Gamma(U,V)$ if and only if $f'f^{-1}\times 1\times h'h^{-1}\in\Gamma(U,V)$, that is $f'f^{-1}\in k_1(U)$ and $h'h^{-1}\in k_2(V)$. Therefore

$$ |\Gamma(U,V)| = |p_2(U)\cap p_1(V)|\cdot|k_1(U)|\cdot|k_2(V)| $$

because

$$ p_2(U)\cap p_1(V) = \{g : \exists f,h \,\text{s.t.}\, f\times g\times h\in\Gamma(U,V)\}. $$

Similarly, given $g'\in G$ then $f\times g'\times h\in\Gamma(U,V)$ if and only if $1\times g'g^{-1}\times 1\in\Gamma(U,V)$, that is, $g'g^{-1}\in k_2(U)\cap k_1(V)$ and we get

$$ |\Gamma(U,V)| = |k_2(U)\cap k_1(V)|\cdot|U*V| $$

since

$$U * V = \{f \times h : \exists g \in G \text{ s.t. } f \times g \times h \in \Gamma(U, V)\}.$$

By a similar argument to above, we also have $|U| = |p_2(U)| \cdot |k_1(U)|$ and $|V| = |p_1(V)| \cdot |k_2(V)|$. Bringing everything together, we obtain

$$|\Gamma(U, V)| = \frac{|U| \cdot |V|}{|p_2(U)| \cdot |p_1(V)|} \cdot |p_2(U) \cap p_1(V)| = |k_2(U) \cap k_1(V)| \cdot |U * V|.$$

Hence

$$\begin{aligned}|U| \cdot |V| &= \frac{|p_2(U)| \cdot |p_1(V)|}{|p_2(U) \cap p_1(V)|} \cdot |k_2(U) \cap k_1(V)| \cdot |U * V| \\ &= |p_2(U)p_1(V)| \cdot |k_2(U) \cap k_1(V)| \cdot |U * V|.\end{aligned}$$

□

Let the $\mathbb{K}$-linear isomorphism $\nu^{F,G} : \overline{K_\sigma \mathcal{S}}(F, G) \leftarrow K_\sigma \mathcal{B}(F, G)$ be given by

$$\nu^{F,G}(d_U^{F,G}) = \frac{|G| \bar{s}_U^{F,G}}{|U|}.$$

Now we are ready to prove the associativity of the composition of $K_\sigma \mathcal{B}$ by making use of the fact that we can see $\overline{K_\sigma \mathcal{S}}$ as an invariant category of $K_\sigma \mathcal{S}$.

Theorem 3.3.4. *Composition is associative on $K_\sigma \mathcal{B}$, $\nu^{F,G}$ is an isomorphism of $\mathbb{K}$-linear categories which acts as identity on objects.*

Proof. We are to show that

$$\nu^{F,G}(d_U^{F,G})\nu^{G,H}(d_V^{G,H}) = \nu^{F,H}(d_U^{F,G} d_V^{G,H})$$

so that ν is an algebra map. We have

$$\begin{aligned}s_U^{F,G}\sigma_G(e_G)s_V^{G,H} &= s_U^{F,G} \frac{1}{|G|} \sum_{g \in G} s_{\Delta(G,g,G)}^{G,G} s_V^{G,H} \\ &= s_U^{F,G} \sum_{g \in G} s_{g \times 1 V}^{G,H} \frac{1}{|G|} \\ &= \frac{1}{|G|} \sum_{g \in G} \ell(k_2(U) \cap k_1({}^{g \times 1}V)) s_{U * {}^{g \times 1}V}^{F,H}.\end{aligned}$$

Then

$$
\begin{aligned}
\overline{s}_U^{F,G}\overline{s}_V^{G,H} &= \sigma_F(e_F)s_U^{F,G}\sigma_G(e_G)\sigma_G(e_G)s_V^{G,H}\sigma_H(e_H) \\
&= \sigma_F(e_F)s_U^{F,G}\sigma_G(e_G)s_V^{G,H}\sigma_H(e_H) \\
&= \frac{1}{|G|}\sum_{g\in G}\sigma_F(e_F)\ell(k_2(U)\cap k_1({}^{g\times 1}V))s^{F,H}_{U*^{g\times 1}V}\sigma_H(e_H) = \frac{1}{|G|}\sum_{g\in G}\overline{s}^{F,H}_{U*^{g\times 1}V} \\
&= \frac{1}{|G|}\sum_{p_2(U)gp_1(V)\subseteq G}|p_2(U)gp_1(V)|\ell(k_2(U)\cap k_1({}^{g\times 1}V))\overline{s}^{F,H}_{U*^{g\times 1}V} \\
&= \frac{1}{|G|}\sum_{p_2(U)gp_1(V)\subseteq G}\frac{|U|\cdot|{}^{g}(V)|}{|k_2(U)\cap k_1({}^{g}V)|\cdot|U*{}^{g\times 1}V|}\ell(k_2(U)\cap k_1({}^{g\times 1}V))\overline{s}^{F,H}_{U*^{g\times 1}V}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\overline{s}_U^{F,G}\overline{s}_V^{G,H}|G|\cdot|H|}{|U|\cdot|V|} &= \sum_{p_2(U)gp_1(V)\subseteq G}\frac{|H|}{|U*{}^{g\times 1}V|}\cdot\frac{\ell(k_2(U)\cap k_1({}^{g\times 1}V))}{|k_2(U)\cap k_1({}^{g}V)|}\overline{s}^{F,H}_{U*^{g\times 1}V} \\
&= \nu^{F,H}(d_U^{F,G}d_V^{G,H}).
\end{aligned}
$$

The isomorphism ν gives the associativity of the composition of $\mathbb{K}_\sigma\mathcal{B}$ since the composition of $\mathbb{K}_\sigma\mathcal{S}$ is already associative. $\square$

Chapter 4

Deformations of the fibred biset category

4.1 The partial category of subcharacters

In this section, we are going to generalize the theme we have initiated on the subgroup category and that will allow us to realize the deformations of the fibred biset category.

In our generalization, we will equip the subgroup category with some additional structure by incorporating subcharacters. In addition to the requirement of the agreement of codomain and domain, we will impose some more conditions on compatibility of morphisms.

Let X be a set. A set $\mathcal{P}$, equipped with a relation $\sim$ and a multiplication, that is, a map $\mathcal{P} \ni \phi\psi \leftarrow (\phi, \psi)$ on the elements $(\phi, \psi) \in \mathcal{P}$ which satisfy $\phi \sim \psi$, is called a **small partial category** on X if the following conditions are satisfied:

- Given $\theta, \phi, \psi \in \mathcal{P}$, such that $\theta \sim \phi$ and $\phi \sim \psi$, then $\theta \sim \phi\psi$ if and only if $\theta\phi \sim \psi$, in which case $\theta(\phi\psi) = (\theta\phi)\psi$. When this associativity condition holds, we will write $\theta\phi\psi$ as ambiguities can no longer arise.

- $\mathcal{P}$ contains a family of idempotents $\{\mathrm{id}_x^{\mathcal{P}} : x \in X\}$ which satisfy $\mathrm{id}_x^{\mathcal{P}} \sim \mathrm{id}_x^{\mathcal{P}}$, $\mathrm{id}_x^{\mathcal{P}}\mathrm{id}_x^{\mathcal{P}} = \mathrm{id}_x^{\mathcal{P}}$ and for all $\phi \in \mathcal{P}$, a filtration condition, existence of unique elements $x, y \in X$ such that $\mathrm{id}_x^{\mathcal{P}} \sim \phi \sim \mathrm{id}_y^{\mathcal{P}}$ and $\mathrm{id}_x^{\mathcal{P}} \cdot \phi = \phi \cdot \mathrm{id}_y^{\mathcal{P}}$. Note that in this case we identify x and y as the codomain and domain of ϕ and write $\mathrm{cod}(\phi) = x$ and $\mathrm{dom}(\phi) = y$.

We can identify the $\mathcal{P}$-morphisms $x \leftarrow y$ with the set

$$\mathcal{P}(x, y) = \{\phi \in \mathcal{P} : \mathrm{cod}(\phi) = x, \mathrm{dom}(\phi) = y\}.$$

In the case when $\phi \sim \psi$ for all $\phi, \psi \in \mathcal{P}$ satisfying $\mathrm{dom}(\phi) = \mathrm{dom}(\psi)$ (e.g. in the subgroup category) , $\mathcal{P}$ becomes a small category.

Now let us define the partial category $\mathcal{S}^A$, a generalization of the subgroup category. The objects of $\mathcal{S}^A$ are finite groups. The morphism set $\mathcal{S}^A(F, G)$ consists of the subcharacters of $F \times G$. Given A-subcharacters $(U, \mu), (V, \mu), (W, \omega)$ from $\mathcal{S}^A(F, G), \mathcal{S}^A(G, H)$ and $\mathcal{S}^A(H, I)$, we let $(U, \mu) \sim (V, \nu)$ if $\mu(1 \times g)\nu(g \times 1) = 1$ for all $g \in k_2(U) \cap k_1(V)$, whence we define $\mu * \nu$ to be the A-character of $U * V$ given by $(\mu * \nu)(f \times h) = \mu(f, g)\nu(g, h)$. To finish our definition, we give the following proposition on the associativity of the composition.

Proposition 4.1.1. *Defining the composition by $(U, \mu) * (V, \nu) = (U * V, \mu * \nu)$ when $(U, \mu) \sim (V, \nu)$ makes $\mathcal{S}^A$ become a partial category.*

Proof. We are to show that the composition is associative and identity morphisms exist. For associativity, we need

$$(U * V, \mu * \nu) * (W, \omega) = (U, \mu) * (V * W, \nu * \omega)$$

One thing to check is that the compatibility relations of the subcharacters that appear on both sides should agree, that is, if $(U, \mu) \sim (V, \nu)$and $(U * V, \mu * \nu) \sim (W, \omega)$, then, we must have, $(V, \nu) \sim (W, \omega)$ and $(U, \mu) \sim (V * W, \nu * \omega)$ and the implication should work in the opposite direction as well.

So suppose the "if" direction, then $\mu(1\times g)\nu(g\times 1) = 1$ for all $g \in k_2(U)\cap k_1(V)$ and

$$1 = (\mu * \nu)(1 \times h)\omega(h \times 1) = \mu(1 \times g)\nu(g \times h)\omega(h \times 1)$$

for all $h \in k_2(V) \cap k_1(W)$ and $g \in G$. Letting $g = 1$, we obtain

$$1 = \mu(1 \times 1)\nu(1 \times h)\omega(h \times 1) = \nu(1 \times h)\omega(h \times 1)$$

which gives $(V, \nu) \sim (W, \omega)$. Then

$$\mu(1 \times g)(\nu * \omega)(g \times 1) = \mu(1 \times g)\nu(g \times h)\omega(h \times 1)$$

and letting $h = 1$ and $g \in k_2(U) \cap k_1(V)$, we get

$$\mu(1 \times g)\nu(g \times 1)\omega(1 \times 1) = \mu(1 \times g)\nu(g \times 1) = 1.$$

Therefore $(U, \mu) \sim (V * W, \nu * \omega)$. The other direction follows from similar arguments.

Hence if we have incompatibility of subcharacters on one side of the equation, we will have it on the other side as well and both sides will vanish. It remains to prove associativity in the case when all sucharacters appearing are compatible, assuming which gives

$$\begin{aligned}(\mu * \nu) * \omega(f \times i) &= (\mu * \nu)(f \times h)\omega(h \times i) = \mu(f \times g)\nu(g \times h)\omega(h \times i)\\ &= \mu(f \times g)(\nu * \omega)(g \times i) = (\mu * (\nu * \omega))(f \times i).\end{aligned}$$

To finish the proof, we observe that, given a finite group G, we have $\mathrm{id}_G^{\mathcal{S}^A} = s^{G,G}_{\Delta(G),1}$ where $\Delta(G) = \{y\times y : y \in G\}$ and 1 denotes the trivial A-character. □

Let us note that we can work with the situation where the set of objects is a set $\mathcal{G}$ and in that case we shall be working with the small partial category $\mathcal{S}^A_{\mathcal{G}}$.

Let $s^{F,G}_{U,\mu}$ denote the subcharacter (U, μ) as an element of $\mathcal{S}^A(F, G)$.

As a set, we define $R_\ell\mathcal{S}^A$ to be such that $R_\ell\mathcal{S}^A(F, G) = \bigoplus_{(U,\mu)\in\mathcal{S}^A(F,G)} Rs^{F,G}_{U,\mu}$.

In order for us to be able to introduce the deformations of $\mathcal{S}^A$, we need to provide its linearization so that a coefficient before $s^{F,G}_{U,\mu}$ makes sense. One can see $R_\ell\mathcal{S}^A$ as an R-linear category by considering the $R(F\times G)$-module structure of $\mathcal{S}^A(F,G)$ given by the evident action of $f\times g\in F\times G$ on the subcharacter (U,μ) to define the action on $s^{F\times G}_{U,\mu}$.

Taking the multiplication to be given by

$$s^{F,G}_{U,\mu} *_\sigma s^{G,H}_{V,\nu} = \sigma(U,V)s^{F,H}_{U*V,\mu*\nu}$$

we shall obtain deformations of the partial category.

4.2 Deformations of the fibred biset category

We let the R-linear A-fibred biset category $R\mathcal{B}^A$ to be the category whose objects are finite groups and whose morphisms are given as

$$RB^A(F,G) = \bigoplus_{[U,\mu]\in_{F\times G}\mathcal{S}^A(F,G)} \left[\frac{F\times G}{U,\mu}\right].$$

The composition is defined by

$$\left[\frac{F\times G}{U,\mu}\right]\cdot\left[\frac{G\times H}{V,\nu}\right] = \sum_g \left[\frac{F\times G}{U*{}^gV,\mu*{}^g\nu}\right]$$

where g runs over those double coset representatives satisfying $p_2(U)gp_1(V)\subseteq G$ such that $(U,\mu)\sim {}^g(V,\nu)$. The identity $R\mathcal{B}^A$-morphism on G is $\left[\frac{G\times G}{\Delta(G),1}\right]$ where 1 denotes the trivial character.

Now let

$$R_\ell B^A(F,G) = \bigoplus_{[U,\mu]\in_{F\times G}\mathcal{S}^A(F,G)} Rd^{F,G}_{U,\mu}$$

where $d^{F,G}_{U,\mu}$ is a formal symbol determined uniquely by F,G and $[U,\mu]$.

We shall realize the fibred biset category as an invariant subalgebra of $R_\ell\mathcal{S}^A$.

We can realize $R_\ell\mathcal{B}^A$ as an R-linear category on $\mathcal{G}$ by defining

$$d^{F,G}_{U,\mu}d^{G,H}_{V,\nu} = \sum_{g} \frac{\ell(k_2(U)\cap k_1(V))}{|k_2(U)\cap k_1(V)|} d^{F,H}_{U*^gV,\mu*^g\nu}$$

where g runs same as before.

Let us present the $\mathbb{K}(F\times G)$-module structure on $\mathbb{K}_\sigma\mathcal{S}(F,G)$ given by the action

$$^{f\times g}s^{F,G}_{U,\mu} = s^{F,G}_{^{f\times g}U,{}^{f\times g}\mu}.$$

and write $^f(s^{F,G}_{U,\mu})^g = {}^{f\times g^{-1}}(s^{F,G}_{U,\mu})$. To complete the realization of the fibred biset category as an invariant category, we introduce $\overline{R_\ell\mathcal{S}^A}$, the category where the objects are finite groups and the morphism sets are given as

$$\overline{R_\ell\mathcal{S}^A}(F,G) = \bigoplus_{[U,\mu]\in_{F\times G}\mathcal{S}^A(F,G)} R\overline{s}^{F,G}_{U,\mu}$$

where $\overline{s}^{F,G}_{U,\mu} = \frac{1}{|F|\cdot|G|}\sum_{f\in F,g\in G} {}^f(s^{F,G}_{U,\mu})^g$.

Let us briefly present the map $\nu : \overline{R_\ell\mathcal{S}^A} \leftarrow R_\ell\mathcal{B}^A$ which is given by the collection of maps $\nu_{F,G} : \overline{R_\ell\mathcal{S}^A}(F,G) \leftarrow R_\ell B^A(F,G)$ where $\nu_{F,G}(d_{U,\mu}) = \frac{|G|\overline{s}^{F,G}_{U,\mu}}{|U|}$. The next theorem describes the composition of these basis elements and will imply that ν is an algebra map:

Theorem 4.2.1. *Let $F,G,H\in\mathcal{G}$. Let $[U,\mu]\in\mathcal{S}^A[F,G]$ and $[V,\nu]\in\mathcal{S}^A[G,H]$. Then*

$$\frac{\overline{s}^{F,G}_{U,\mu}}{|U|}\cdot\frac{\overline{s}^{G,H}_{V,\nu}}{|V|} = \frac{1}{|G|}\sum_{g}\frac{\ell(k_2(U)\cap k_1(^gV))}{|k_2(U)\cap k_1(^gV)|}\cdot\frac{\overline{s}^{F,H}_{U*^gV,\mu*^g\nu}}{|U*{}^gV|}$$

where g runs over representatives of the double cosets $p_2(U)gp_1(V)\subseteq G$ such that $(U,\mu)\sim{}^g(V,\nu)$.

Proof. Observing that $(U,\mu)*(V,\nu) = (U,\mu)^{g^{-1}}*^g(V,\nu)$ for all $g\in G$ and applying the definition for the basis elements yields

$$\overline{s}^{F,G}_{U,\mu}\overline{s}^{G,H}_{V,\nu} = \frac{1}{|F|\cdot|G|}\cdot\sum_{f\in F,g\in G}{}^{f}(s^{F,G}_{U,\mu})^{g}\frac{1}{|G|\cdot|H|}\sum_{g\in G,h\in H}{}^{g}(s^{G,H}_{V,\nu})^{h}$$
$$= \frac{1}{|F|\cdot|G|\cdot|H|}\sum_{f\in F,g\in G,h\in H}{}^{f}(s^{F,G}_{U,\mu})^{g}(s^{G,H}_{V,\nu})^{h}$$
$$= \frac{1}{|G|}\sum_{y}\ell(k_2(U)\cap k_1({}^{y}V))\overline{s}^{F,H}_{U*{}^{y}V,\mu*{}^{y}\nu}$$

where y runs over those elements of G such that $(U,\mu)\sim{}^{y}(V,\nu)$. We have $(U,\mu)\sim{}^{y'}(V,\nu)$ and $|k_2(U)\cap k_1({}^{y}V)| = |k_2(U)\cap k_1({}^{y'}V)|$ for all $y'\in p_2(U)yp_1(V)$. So

$$\overline{s}^{F,G}_{U,\mu}\overline{s}^{G,H}_{V,\nu} = \frac{1}{|G|}\sum_{g}|p_2(U)gp_1(V)|\ell(k_2(U)\cap k_1({}^{g}V))\overline{s}^{F,H}_{U*{}^{g}V,\mu*{}^{g}\nu}\ .$$

Since $|p_2(U)gp_1(V)| = |p_2(U)p_1({}^{g}V)|$ and $|{}^{g}V| = |V|$, the result follows from Lemma 3.3.3. □

Now we are ready to complete the definition of $R_\ell\mathcal{S}^A(F,G)$ by obtaining that its composition is associative.

Theorem 4.2.2. *The composition for $R_\ell\mathcal{B}^A$ is associative and $R_\ell\mathcal{B}^A$ is an R-linear category on $\mathcal{G}$. The maps $\nu_{F,G}$, for $F,G\in\mathcal{G}$, determine an object-identical isomorphism of R-linear categories $\nu : \overline{R_\ell\mathcal{S}^A} \leftarrow R_\ell\mathcal{B}^A$.*

Proof. Theorem 4.2.1 implies that $\nu_{F,G}(d^{F,G}_{U,\mu})\cdot\nu_{G,H}(d^{G,H}_{V,\nu}) = \nu_{F,H}(d^{F,G}_{U,\mu}\cdot d^{G,H}_{V,\nu})$. By R-linearity, the composition is associative. The identity $R_\ell\mathcal{B}^A$-morphism on G is $d^{G,G}_{\Delta(G),1}$. □

The following is the immediate consequence, let us mention that it appears as Corollary 5.2 in [5], but that version contains a typo. Below is the corrected version:

Corollary 4.2.3. *$R\mathcal{B}^A$ can be realized as an invariant category of $R\mathcal{S}^A$ by letting $\ell(n) = n$ for all $n\in\mathbb{Z}^+$, that is, $\overline{R_\ell\mathcal{S}^A}\cong R\mathcal{B}^A$ via identifying $|G|\overline{s}^{F,G}_{U,\mu}\leftrightarrow |U|\left[\frac{F\times G}{(U,\mu)}\right]$.*

Chapter 5

Semisimplicity of the deformations of the biset category

5.1 Semisimplicity and algebraic independence

In this section we are going to define another basis for the subgroup category and use it to obtain a relation between the local semisimplicity of Λ and the algebraic independence of ℓ.

We define $\{t_I^{F,G} : I \leq F \times G\}$ to be the $\mathbb{K}$-basis for $\mathbb{K}_\sigma \mathcal{S}(F,G)$ given by the following relations:

$$s_U^{F,G} = \sum_{I \leq U} t_I^{F,G}, \quad t_I^{F,G} = \sum_{U \leq I} \text{möb}(U,I) s_U^{F,G}$$

for any $U \leq F \times G$ in the first equation and $I \leq F \times G$ in the second equation. Let $\mathcal{S}(U)$ denote the set of subgroups of a finite group U. If we present the functions

$$\sigma : \mathbb{K}_\sigma \mathcal{S}(F,G) \leftarrow \mathcal{S}(F \times G), \quad \sigma(U) = s_U^{F,G},$$

$$\tau : \mathbb{K}_\sigma \mathcal{S}(F,G) \leftarrow \mathcal{S}(F \times G), \quad \tau(I) = t_I^{F,G}.$$

then clearly $\sigma(U) = \sum_{I \leq U} \tau(I)$ and now it follows that the defining relations given above for the basis elements are equivalent by Proposition 2.4.1.

We call the set $\{t_I^{F,G} : I \leq F \times G\}$ the **round basis** of Λ. The point of introducing the round basis is that, in certain cases, we will see that consideration of their products provides significant simplifications. Before we investigate the round basis more in detail, we need to define the machinery we require. Let $\mathcal{P}_K^{I,J} = \{(U,V) \in \mathcal{S}(F \times G) \times \mathcal{S}(G \times H) : K \leq U * V, (U,V) \leq (I,J)\}$. In words, it is the collection of pairs (U,V) in which K appears in their $*$-product. The next definition will describe the coefficient of $t_K^{F,H}$ in the product $t_I^{F,G} t_J^{G,H}$. We let

$$\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{P}_K^{I,J}} \text{möb}(U,I)\text{möb}(V,J)\sigma(U,V).$$

We call the pair (I,J) **compatible**, if their projections match, that is, $p_2(I) = p_1(J)$. For such a pair, notice that we have $p_1(I*J) = p_1(I)$ and $p_2(I*J) = p_2(J)$.

Given an element $W \in \mathcal{S}(F \times H)$, and a subgroup $K \leq W$, we call K an **adequate** subgroup of W if it is projectively W, that is, $p_1(K) = p_1(W), p_2(K) = p_2(W)$. Let $\text{ad}(W)$ denote the set of all adequate subgroups of W. Notice that if (I,J) is compatible and $K \in \text{ad}(I * J)$, then $p_1(K) = p_1(I), p_2(K) = p_2(J)$. The next result can be seen as combination of the statements 3.9, 4.2, 4.3 of [9].

Theorem 5.1.1 (Boltje-Danz)**.** *Let F, G, H be finite groups, $I \in \mathcal{S}(F \times G)$ and $J \in \mathcal{S}(G \times H)$. Then*

• $t_I^{F,G} t_J^{G,H} = \sum_{K \in \mathcal{S}(F \times H)} \tau_K^{I,J} t_K^{F,H}$.

•*For any $K \in \mathcal{S}(F \times H)$, if $t_K^{I,J}$ appears in the above multiplication, then (I,J) is compatible and $K \in \text{ad}(I * J)$.*

•*If $t_I^{F,G} t_J^{G,H} \neq 0$, then (I,J) is compatible, in which case, for all $K \in \text{ad}(I * J)$, one can refine the index of the coefficient $\tau_K^{I,J}$ to the compatible elements of $\mathcal{P}_K^{I,J}$, that is, $\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{R}_K^{I,J}} \text{möb}_{\mathcal{R}_K^{I,J}}((U,V),(I,J))\sigma(U,V)$ where $\mathcal{R}_K^{I,J} = \{(U,V) \in \mathcal{P}_K^{I,J} : p_2(U) = p_1(V)\}$.*

Proof. We directly calculate

$$\begin{aligned} t_I^{F,G} t_J^{G,H} &= \sum_{U \in \mathcal{S}(I)} \text{möb}(U, I) s_U^{F,G} \cdot \sum_{V \in \mathcal{S}(J)} \text{möb}(V, J) s_V^{G,H} \\ &= \sum_{U \in \mathcal{S}(I), V \in \mathcal{S}(J)} \text{möb}(U, I)\text{möb}(V, J) s_{U*V}^{F,H} \sigma(U, V) \\ &= \sum_{(U,V) \leq (I,J)} \text{möb}(U, I)\text{möb}(V, J)\sigma(U, V) \cdot \sum_{K \leq U*V} t_K^{F,H}. \end{aligned}$$

Collecting the coefficient of $t_K^{F,H}$ yields

$$= \sum_{K \in \mathcal{S}(F \times H)} \sum_{(U,V) \in \mathcal{P}_K^{I,J}} \text{möb}(U, I)\text{möb}(V, J)\sigma(U, V) t_K^{F,H} = \sum_{K \in \mathcal{S}(F \times H)} \tau_K^{I,J} t_K^{F,H}.$$

Now let $\Gamma_K(U, V) = \{f \times g \times h \in F \times G \times H : f \times g \in U, f \times h \in K, g \times h \in V\}$ and consider the retraction given by

$$\rho : (S_{U,V}, T_{U,V}) \leftarrow (U, V) \in \mathcal{P}_K^{I,J}$$

where $S_{U,V} = \{f \times g : f \times g \times h \in \Gamma_K(U, V)\}$ and $T_{U,V} = \{g \times h : f \times g \times h \in \Gamma_K(U, V)\}$. To make an appeal to Lemma 2.4.3 we need to check a few things. Firstly, the codomain of ρ lies in $\mathcal{P}_K^{I,J}$ since by definition $S_{U,V} * T_{U,V} \geq K$ and $(S_{U,V}, T_{U,V}) \leq (U, V) \leq (I, J)$. Secondly, $\sigma(U, V) = \sigma(S_{U,V}, T_{U,V})$ since letting $1 \times g \in U, g \times 1 \in V$, then $1 \times g \times 1 \in \Gamma_k(U, V)$ so $1 \times g \in S_{U,V}, g \times 1 \in T_{U,V}$. Lastly, we have $\rho^2(U, V) = \rho(U, V) = (S_{U,V}, T_{U,V}) \leq (U, V)$. Moreover, from Proposition 2.4.2, we obtain

$$\text{möb}(U, I)\text{möb}(V, J) = \text{möb}_{\mathcal{P}_K^{I,J}}((U, V), (I, J)).$$

Hence $\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{P}_K^{I,J}} \text{möb}_{\mathcal{P}_K^{I,J}}((U, V), (I, J))\sigma(U, V)$ so $\tau_K^{I,J}$ is the totient function of $\sigma(U, V)$, therefore by Lemma 2.4.3, $\tau_K^{I,J} = 0$ for all (I, J) that is not of the form $(S_{U,V}, T_{U,V})$. So $\tau_K^{I,J} \neq 0$ implies (I, J) to be compatible and $K \in \text{ad}(I * J)$.

To finish the proof, observe that $p_2(S_{U,V}) = p_1(T_{U,V})$ and if $p_2(U) = p_1(V)$, then $S_{U,V} = U$ and $T_{U,V} = V$ so $\rho(\mathcal{P}_K^{I,J}) = \mathcal{R}_K^{I,J}$. □

Now we are ready to discuss a relationship between the local semisimplicity of Λ and the semisimplicity of its objects. Next lemma describes the multiplication of the round basis elements for some particular cases;

Lemma 5.1.2. *Let F, G, H be finite groups, $I \in \mathcal{S}(F \times G)$ and $J \in \mathcal{S}(G \times H)$.*

- *Suppose $I = \Delta(A, 1, \phi, 1, B)$ and $J = \Delta(B, Y, \psi, Z, C)$. Then $t_I^{F,G} t_J^{G,H} = t_K^{F,H}$ and $K = I * J = \Delta(A, \phi(Y), \underline{\phi}\psi, Z, C)$ where $\underline{\phi} : A/\phi(Y) \leftarrow B/Y$ is the isomorphism induced by ϕ.*

- *The result above is symmetric, that is, if $I = \Delta(A, X, \phi, Y, B)$ and $J = \Delta(B, 1, \psi, 1, C)$, then $t_I^{F,G} t_J^{G,H} = t_K^{F,H}$ where $K = I * J = \Delta(A, X, \phi\underline{\psi}, \psi^{-1}(Y), C)$ where $\underline{\psi} : B/Y \leftarrow C/\psi^{-1}(Y)$ is the isomorphism induced by ϕ.*

Proof. We will show that there is only one adequate subgroup of $I * J$, that can be obtained as a result of $*$-products of the subpairs $(I', J') \leq (I, J)$, itself, then the previous theorem will imply that there is only one nonzero term in $t_I^{F,G} t_J^{G,H}$. Take $L \in \mathrm{ad}(I * J)$. Then $\mathcal{P}_L^{I,J} = \{(I, V) : V \leq J, L \leq I * V\}$ because there is no proper subgroup of I with left projection equal to $p_1(I)$. Now $\tau_L^{I,J} = \sum_{V \leq J} \text{möb}(V, J)$ which is nonzero only if $V = J$ and since $\text{möb}(J, J) = 1$, we get $L = I * J = K$. The form of K can be seen from the Lemma 2.1.3. The second statement can be proven with similar arguments. □

Lemma 5.1.3. *Given $B \leq G \in \mathcal{K}$, then $s_{\Delta(B)}^{G,G} = \sum_{Y \in \mathcal{S}(B)} t_{\Delta(Y)}^{G,G}$ as a sum of mutually orthogonal idempotents of $\mathrm{End}_{\Lambda_{\mathcal{K}}}(G)$.*

Proof. We have $t_{\Delta(Y)}^{G,G} t_{\Delta(Y)}^{G,G} = t_{\Delta(Y) * \Delta(Y)}^{G,G} = t_{\Delta(Y)}^{G,G}$ and $(\Delta(Y), \Delta(X))$ is compatible if and only if $\Delta(Y) = \Delta(X)$ so $t_{\Delta(Y)}^{G,G} t_{\Delta(X)}^{G,G} = 0$ by Theorem 5.1.1. □

Lemma 5.1.4. *Given $A \leq F \in \mathcal{K} \ni G \geq B$, then $\{s_U^{F,G} : U \in \mathcal{S}(A \times B)\}$ and $\{t_I^{F,G} : I \in \mathcal{S}(A \times B)\}$ are $\mathbb{K}$-bases for $s_{\Delta(A)}^{F,F} \Lambda(F, G) s_{\Delta(B)}^{G,G}$.*

Proof. We have $s_{\Delta(A)}^{F,F} t_{I'}^{F,G} s_{\Delta(B)}^{G,G} = 0$ for all $I' \in \mathcal{S}(F \times G) - \mathcal{S}(A \times B)$ because such I''s are incompatible with $\Delta(A)$ and $\Delta(B)$. □

Proposition 5.1.5. *Let $\mathcal{L}, \mathcal{M} \subseteq \mathcal{K}, k : \mathcal{L} \leftarrow \mathcal{M}$ be a function and, for each $G \in \mathcal{M}$, let $k_G : k(G) \leftarrow G$ be a group monomorphism. Then there is a corner embedding $\Lambda_{\mathcal{L}} \leftarrow \Lambda_{\mathcal{M}}$ given by $s_{(k_F \times k_G)(U)}^{k(F),k(G)} \leftarrow s_U^{F,G}$ for $F, G \in \mathcal{M}, U \in \mathcal{S}(F \times G)$.*

Proof. For any $F \in \mathcal{M}$, let $\kappa_F(F)$ denote the isomorphic copy of F in $k(F)$. Now consider the corner subalgebra $s^{k(F),k(F)}_{\Delta(\kappa(F))}\Lambda_{\mathcal{L}}(k(F),k(G))s^{k(G),k(G)}_{\Delta(\kappa(G))}$ of $\Lambda_{\mathcal{L}}$ with basis given by $\{s^{k(F),k(F)}_U : U \leq \kappa_F(F) \times \kappa_G(G)\}$. The result follows from the previous lemma. $\square$

Corollary 5.1.6. *Given $G \in \mathcal{K}$, then the algebra* $\mathrm{End}_{\mathbb{K}\mathcal{S}}(G) = \mathbb{K}\mathcal{S}(G,G)$ *is semisimple if and only if G is trivial. In particular, $\mathbb{K}\mathcal{S}_{\mathcal{K}}$ is locally semisimple if and only if every group in $\mathcal{K}$ is trivial.*

Proof. If G is trivial then $\mathrm{End}_{\mathbb{K}G} \cong \mathbb{K}$ so we have the semisimplicity in that case. Supposing G to be nontrivial, $A \leq G$ to be a subgroup of prime order, we see that $\mathrm{End}_{\mathbb{K}\mathcal{S}}(A)$ is not semisimple by 3.2.1. Since $\mathrm{End}_{\mathbb{K}\mathcal{S}}(A)$ is a corner subalgebra of $\mathrm{End}_{\mathbb{K}\mathcal{S}}(G)$, the result follows. $\square$

Corollary 5.1.7. *Suppose $\mathcal{K}$ has an element H, such that, every element of $\mathcal{K}$ is isomorphic to a subgroup of H. Then Λ is locally semisimple if and only if* $\mathrm{End}_{\Lambda}(H)$ *is semisimple.*

Proof. $\mathrm{End}_{\Lambda}(H)$ being semisimple implies every endomorphism algebra $\mathrm{End}_{\Lambda}(A)$ where $A \leq H$ to be semisimple, therefore every group in $\mathcal{K}$ is trivial. $\square$

Corollary 5.1.8. *Suppose for every pair F, G from $\mathcal{K}$, there exists $H \in \mathcal{K}$ such that F and G are isomorphic to some subgroups of H. Then Λ is locally semisimple if and only if every object of Λ has a semisimple endomorphism algebra.*

Proof. This hypothesis implies the one of the previous corollary in the case where $\mathbb{K}$ is finite. Now the result follows from Remark 2.5.2 $\square$

Our next objective is to describe the relationship between the algebraic independence of ℓ and the semisimplicity of $\Lambda_{\mathcal{K}}$. Given a simple $\Lambda_{\mathcal{K}}$-module S, and its projective cover Λi, semisimplicity of $\Lambda_{\mathcal{K}}$ is equivalent with S being projective or not, that is, $S = \Lambda i$. So let us briefly investigate the projective $\Lambda_{\mathcal{K}}$-modules .

We define the dual of a $\Lambda_{\mathcal{K}}$-module M to be the $\Lambda_{\mathcal{K}}$-module M^* such that, $M^*(G)$ is the dual of $M(G)$ and given $s \in \Lambda(F,G)$ we get $s^\circ : M^*(G) \leftarrow M^*(F)$ as adjoint of the action of $s : M(F) \leftarrow M(G)$.

Lemma 5.1.9. *Given an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, then $S^*_{E,W} \cong S_{E,W^*}$*

Proof. The simple module $S^*_{E,W}$ would have minimal group E because $S_{E,W}(E) \neq 0$ if and only if $S^*_{E,W}(E) \neq 0$. So it remains checking the structure of $\mathrm{Res}^{\mu_E}(S^*(E))$.

We finish the proof by taking $\varphi \in W^*$ and identifying $\varphi \leftrightarrow \psi$ where $\psi \in S^*(E)$ is such that $\psi(s) = \varphi(\mathrm{Res}^{\mu_E}(s))$ for $s \in S(E)$. $\square$

Lemma 5.1.10. *Given a finite group E, then the simple $\Lambda_{\mathcal{K}}$-modules having minimal group E are all projective if and only if they are all injective.*

Proof. Suppose all simple Λ-modules with minimal group E are projective. If we check the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, dotted, "h"] \\ S_{E,W} & \end{tikzcd}$$

where f is a monomorphism, we are to show existence of the map h which makes the diagram commute. After considering the following addition to the diagram

$$\begin{tikzcd} & \mathbb{K} \\ X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[u, "\psi"'] \arrow[dl, dotted, "h"] \\ S_{E,W} \arrow[d, "\varphi"'] & \\ \mathbb{K} & \end{tikzcd}$$

it becomes clear that the map g yields to a map $g^* : X^* \leftarrow S^*_{E,W}$ of duals given by the composition $g \circ \varphi$, similar argument works for a map $f^* : X^* \leftarrow Y^*$ of duals. So we obtain the following diagram:

$$\begin{tikzcd}
X^* & Y^* \arrow[l, "f^*"'] \\
S^*_{E,W} \arrow[u, "g^*"] \arrow[ur, dotted, "h^*"'] &
\end{tikzcd}$$

Since f is a monomorphism, f^* is an epimorphism and since all simple Λ-modules are projective, $S^*_{E,W}$ is also projective, giving us the existence of the map h^*, which gives rise to the map h, hence $S_{E,W}$ is injective.

$\square$

Now let $\mathrm{epi}(E, L)$ denote the set of group epimorphisms $E \leftarrow L$ and let

$$\triangleleft(\phi) = \Delta(E, 1, \underline{\phi}, \ker(\phi), L) \in \mathcal{S}(E, L), \triangleright(\phi) = \Delta(L, \ker(\phi), \underline{\phi}^{-1}, 1, E) \in \mathcal{S}(L, E)$$

where $\underline{\phi}$ is the isomorphism induced by ϕ as before.

For those L that are isomorphic to a subgroup of an element of $\mathcal{K}$ and all factor groups E that are realized as a quotient group of L, we define a square matrix T^L_E where rows and columns are indexed by $\mathrm{epi}(E, L)$ such that $T^L_E(\phi, \psi) = \tau^{\triangleleft(\phi), \triangleright(\psi)}_{\Delta(E)}$, coefficient of $t^{F,H}_{\Delta(E)}$ in $t^{F,G}_{\triangleleft(\phi)} t^{G,H}_{\triangleright(\psi)}$

Next theorem gives a criteria for $\Lambda_{\mathcal{K}}$ to be locally semisimple.

Theorem 5.1.11. *Suppose that the matrix T^L_E is invertible for every finite group L that is isomorphic to a subgroup of an element of $\mathcal{K}$ and every subquotient E of L. Then $\Lambda_{\mathcal{K}}$ is locally semisimple.*

Proof. By the Remark 2.5.2 and Lemma 3.1.4, we can assume that $\mathcal{K}$ is closed under subquotients up to isomorphism. $\Lambda_{\mathcal{K}}$ can be realized as a full subcategory of a collection $\mathcal{K}'$ that satisfy the desired closure condition and the remark gives the equivalence of local semisimplicities of $\Lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}'}$.

Let (E, W) be an $\mathcal{S}$-seed for $\mathcal{K}$. By Theorem 2.3.4 every full subcategory of $\Lambda_{\mathcal{K}}$ obtained by considering a finite set of objects is a direct sum of projective covers, it suffices to prove that the simple $\Lambda_{\mathcal{K}}$-module $S_{E,W}$ is projective because of the relation between local semisimplicities of $\Lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}'}$ in constructed the Remark 2.5.2. Since $\mathcal{K}$ is assumed to contain an isomorphic copy of E, we can progress by assuming $E \in \mathcal{K}$ by the arguments in the beginning.

Suppose that the claim holds for every simple module $S_{E',W'}$ with $[E'] < [E]$, that is, $S_{E',W'}$ is projective, so all simple modules with minimal group $[E']$ are projective, hence by the previous Lemma, $S_{E',W'}$ is injective. As we have done before, let $\mathcal{E} = \mathrm{End}_{\Lambda_{\mathcal{K}}}(E)$, through μ_E, we can regard W as a simple $\mathcal{E}$-module which is annihilated by $\mathcal{E}_<$.

Let i be a primitive idempotent of $\mathcal{E}$ such that $iW \neq 0$. Then i can also be seen as a primitive idempotent of $\Lambda_{\mathcal{K}}$ and $iS_{E,W} \neq 0$, because $iS_{E,W}(E) = iW \neq 0$, by Theorem 2.3.3 we see that this is the condition for the indecomposable projective $\Lambda_{\mathcal{K}}$-module P which satisfies $P = \Lambda_{\mathcal{K}} i$ to be the projective cover of $S_{E,W}$. Now if P is simple, there is nothing to prove, so suppose that the unique maximal submodule Q of P is nonzero, recall that $J(P) = Q$ for projective indecomposable modules by Proposition 2.3.1.

We have $P(E) = \mathrm{id}_E^{\Lambda_{\mathcal{K}}} \Lambda_{\mathcal{K}} i = \mathcal{E} i$, which is the projective cover of W. The unique submodule of $P(E)$ is $Q(E)$ and it satisfies $Q(E) \cong \mathcal{E}_< i$. Moreover, $\mathcal{E}_<$ annihilates every simple $\mathbb{K}\mathrm{Aut}(E)$-module and since $\mathbb{K}\mathrm{Aut}(E)$ is semisimple, we can realize the $\mathbb{K}\mathrm{Aut}(E)$-module W as a quotient, that is $(\mathcal{E}/\mathcal{E}_<)i \cong W$.

Suppose that $Q(E) \neq 0$. Then Q must contain a simple module with the minimal group less than or equal to $[E]$. The "equal" case is not possible by the given form for Q above, because $\mathcal{E}_<$ would annihilate those simple modules. If $S_{E',W'} \subseteq Q$ for $[E'] < [E]$, then $S_{E',W'}$ is injective which is a contradiction. Therefore $Q(E) = 0$ and $\mathcal{E}_<$ annihilates $P(E)$.

So we naturally assume that $Q(G) \neq 0$ where G is of minimal order. Since $Q(G) \subseteq P(G) = \Lambda_{\mathcal{K}}(G, E)i$, letting $v \in \Lambda_{\mathcal{K}}(G, E)$ be such that $vi \neq 0$ and

$vi \in Q(G)$ implies v being a $\mathbb{K}$-linear combination of elements of the form $s_V^{G,E}$ for $V \leq G \times E$ and we have $s_V^{G,E} = s_V^{G,E} s^{E,E}_{\Delta(p_2(V),k_2(V),1,p_2(V),k_2(V))}$.

If $q(V) \ncong E$ then $s^{E,E}_{\Delta(p_2(V),k_2(V),1,k_2(V),p_2(V))} \in \mathcal{E}_<$, so $s_V^{G,E} i = 0$. So those $s_V^{G,E}$'s does not appear in the $\mathbb{K}$-linear combination mentioned above, which forces $k_2(V) = 1$, so the basis elements that appear should satisfy $s_V^{G,E} = \sum_{J \leq V} t_J^{G,E}$ where $k_2(J) = 1$. If $p_2(J) \neq E$, then we can write $t_J^{G,E}$ as $\mathbb{K}$-linear combination of elements $s_{V'}^{G,E}$ where $q(V') < E$, as we have just seen, should vanish after multiplying with i. So we have $p_2(J) = E$ and $k_2(J) = 1$.

Now given $B \leq G$, then by the minimality of G, if $B < G$ then $Q(B) = 0$, therefore $t^{B,G}_{\Delta(B)} vi = 0$, which forces $p_1(J) = G$, since otherwise we would get a term in $Q(B)$ satisfying $t^{B,G}_{\Delta(B)} t_J^{G,E} i \neq 0$ So the left projection of J should match the right projection of $\Delta(G)$, that is,

$$v \in \bigoplus_{\psi \in \mathrm{epi}(E,G)} \mathbb{K} t^{G,E}_{\triangleright(\psi)}.$$

Given $w \in \mathcal{E}$, one can decompose as $w = w_< + w_=$ where $w_< \in \mathcal{E}_<$ and $w_= = \sum_{\epsilon \in \mathrm{Aut}(E)} \partial_{\Delta(\epsilon)}(w) t^{E,E}_{\Delta(\epsilon)}$ for some coefficients $\partial_{\Delta(\epsilon)}$ from $\mathbb{K}$.

The simple factors contained in $\Lambda_{\mathcal{K}} \mathcal{E}_<$ are all isomorphic to $S_{E',W'}$ for some (E', W') with $[E'] < [E]$ because $\mathcal{E}_<$ annihilates those simple modules with minimal group isomorphic to E. Since Q cant realize such $S_{E',W'}$'s, we get $Q \cap \Lambda_{\mathcal{K}} \mathcal{E}_< = \varnothing$. It means Q does not contain $vi_<$ unless $vi_<$ is zero, in which case $vi_=$ must be nonzero, otherwise $vi_< = vi \in Q - \{0\}$. So let

$$vi_= = \sum_{\psi \in \mathrm{epi}(E,G)} v(\psi) t^{G,E}_{\triangleright(\psi)}$$

where $v(\psi) \in \mathbb{K}$.

Let $u = \sum_{\phi \in \mathrm{epi}(E,G)} u(\phi) t^{E,G}_{\triangleleft(\phi)}$ be arbitrary where $u(\phi) \in \mathbb{K}$. Since $vi_= \neq 0$, we get $v(\psi) \neq 0$ for some ψ. Furthermore, $\Lambda_{\mathcal{K}}(E,G)Q(G) = Q(E) = 0$ hence $uvi = 0$ so $uv(i_=) = -uv(i_<) \in \mathcal{E}_<$.

Picking out the coefficients of $t^{E,E}_{\Delta(E)}$ gives

$$0 = \partial_{\Delta(E)}(uvi_=) = \sum_{\phi,\psi\in\mathrm{epi}(E,G)} u(\phi)T^L_E(\phi,\psi)v(\psi).$$

Since $(u(\phi):\phi)$ was arbitrary, and $(v(\psi):\psi)$ is nonzero, it gives a contradiction with T^G_E being invertible since otherwise its columns would not be linearly independent. $\square$

To finish our preparation for proving the local semisimplicity in the case when ℓ is algebraically independent, we need one last statement:

Lemma 5.1.12. *Let $I = \lhd(\phi), J = \rhd(\psi)$ where $\phi \in \mathrm{epi}(A,B)$ and $\psi \in \mathrm{epi}(C,B)$ for subgroups A, B, C of F, G and H. Let $K = \Delta(\theta)$ where $\theta : A \leftarrow C$ is an isomorphism. Suppose that $K \leq I * J$, then $I * J = (\phi\times\psi)(B)$ and $K \in \mathrm{ad}(I * J)$.*

Letting $S \leq B$ run over those subgroups which satisfy $K \leq (\phi \times \psi)(S)$, we get

$$\tau^{I,J}_K = \sum_S \text{möb}(S,B)\ell(\ker(\phi)\cap S\cap\ker(\psi)).$$

Moreover, if $\ker(\phi) = \ker(\psi)$ then $K = I * J$ and $t^{F,G}_I t^{G,H}_J = \tau^{I,J}_K t^{F,H}_K$.

Proof. Firstly, let us note that from the forms that are given for I, J, K clearly K is an adequate subgroup of $I * J$ and $I * J = (\phi\times\psi)(B)$ and in the case $\ker(\phi) = \ker(\psi)$ clearly $K = I * J$ and so by Theorem 5.1.1 we get $t^{F,G}_I t^{G,H}_J = \tau^{I,J}_K t^{F,H}_K$.

So let us prove the formula for $\tau^{I,J}_K$. Let $M = \ker(\phi)$ and $N = \ker(\psi)$. Also let $(R,T) \leq (B,B)$ be such that $K \leq (\phi\times\psi)(R\cap T)$. Now if $K \leq U_R * V_T$ for some $(U_R, V_T) \leq (I,J)$, then we must have

$$U_R = \lhd(\underline{\phi_R}) = \Delta(A, 1, \underline{\phi_R}, M\cap R, R), V_T = \rhd(\underline{\psi_T}) = \Delta(T, T\cap N, \underline{\psi_T^{-1}}, 1, C)$$

where $\underline{\phi_R}$ and $\underline{\psi_T^{-1}}$ are the isomorphisms obtained from restricting ϕ to $R/M\cap R$ and ψ to $T/T\cap N$. Moreover, if we let $S = R\cap T$ and consider the retraction $\rho^{I,J}_K$ on $\mathcal{P}^{I,J}_K$ given by

$$\rho^{I,J}_K(U_R, V_T) = (\Delta(A, 1, \underline{\phi_S}, M\cap S, S), \Delta(S, S\cap N, \underline{\psi_S^{-1}}, 1, C))$$

we will see that it preserves ℓ, because

$$\begin{aligned} k_2(U_R) \cap k_1(V_T) &= M \cap R \cap T \cap N = M \cap S \cap N \\ &= k_2(\Delta(A, 1, \underline{\phi_S}, M \cap S, S)) \cap k_1(\Delta(S, S \cap N, \underline{\psi_S^{-1}}, 1, C)). \end{aligned}$$

Also notice that the image of $\rho_K^{I,J}$ is precisely $\mathcal{R}_K^{I,J}$. Recall that

$$\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{P}_K^{I,J}} \text{möb}(U, I)\text{möb}(V, J)\ell(k_2(U) \cap k_1(V))$$

so applying the Lemma 2.4.3 yields

$$\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{R}_K^{I,J}} \text{möb}(U, I)\text{möb}(V, J)\ell(M \cap p_2(U) \cap N).$$

Since M and N are fixed, there is only one variable $p_2(U)$ (which is equal to $p_1(V)$), and it is bounded above by B, letting $S = p_2(U) = p_1(V)$, 2.4.2 implies

$$\tau_K^{I,J} = \sum_S \text{möb}(S, B)\ell(\ker(\phi) \cap S \cap \ker(\psi)).$$

$\square$

Theorem 5.1.13. *If ℓ is algebraically independent, then $\Lambda_{\mathcal{K}}$ is locally semisimple.*

Proof. We are to show that T_E^L is invertible. Let $\prod_{\mathcal{K}}$ be the set of prime divisors of the orders of the elements of $\mathcal{K}$. Let $\lambda_q = \ell(q)$ for each $q \in \prod_{\mathcal{K}}$. Let $\mathcal{O}$ be the integral domain generated over $\mathbb{Q}$ by λ_q.

Any $o \in \mathcal{O}$ can be expressed uniquely as a polynomial expression in $(\lambda_q : q \in \prod_{\mathcal{K}})$ with coefficients in $\mathbb{Q}$ by algebraic independence of ℓ.

Let $\text{len}(n)$ denote the number of prime factors of n up to multiplicity.

We may assume that $E, L \in \mathcal{K}$. So let $\phi \in \text{epi}(E, L)$. Then by the previous lemma

$$T_E^L(\phi, \phi) = \tau_{\Delta(E)}^{\lhd(\phi), \rhd(\phi)} = \sum_{S \in \mathcal{S}(L) : \Delta(E) \leq \phi(S)} \text{möb}(S, L)\ell(\ker(\phi) \cap S).$$

The term which has the highest degree in this element satisfies $S = L$ because $L/E \cong \ker(\phi)$ and its coefficient, given by $\text{möb}(L, L)$, is 1. So the highest degree d satisfies $d = \text{len}(\frac{|L|}{|E|}) = \text{len}(|\ker(\phi)|)$.

On the off-diagonal entries $\tau_{\Delta(E)}^{\triangleleft(\phi),\triangleright(\psi)}$, that is, when we have $\psi \in \text{epi}(E, L) - \{\phi\}$, the degree is less than d because the element with highest degree is $\text{möb}(L, L)\ell(\ker(\phi) \cap \ker(\psi) \cap L)$ and Lemma 5.1.12 implies $\ker(\phi) \neq \ker(\psi)$ so clearly $\ell(\ker(\phi) \cap \ker(\psi)) < |\ker(\phi)|$. By the structure we obtained for the matrix T_E^L, we see that its determinant can't vanish, so it is invertible.

□

5.2 Semisimplicity of the deformations of the biset category

Now we are ready to prove two of our main results. Let $\Omega_{\mathcal{K}} = \mathbb{K}_\sigma \mathcal{B}_{\mathcal{K}}$. In Theorem 3.3.4, we have seen that $\Omega_{\mathcal{K}}$ can be seen as the corner subalgebra of $\Lambda_{\mathcal{K}}$, so the next statement follows;

Corollary 5.2.1. *If $\Lambda_{\mathcal{K}}$ is locally semisimple, then $\Omega_{\mathcal{K}}$ is locally semisimple.*

We finish this section with a generalization of one side of Serge Bouc's theorem which states that $\mathbb{K}\mathcal{B}_{\mathcal{K}}$ is semisimple if and only if every group in $\mathcal{K}$ is cyclic, which can be found as Theorem 1.1 in [3].

As preparation, let us recall a proposition that is due to Philip Hall [15]:

Proposition 5.2.2 (Philip Hall)**.** *Let $\ell(n) = n^d$ for $d \in \mathbb{Z}^+$ and $\ell(G) = \sum_{U \leq G} \varphi(U)$, equivalently, $\varphi(G) = \sum_{U \leq G} \text{möb}(U, G)\ell(U)$. Then, $\varphi(G)$ is the number of generating sets for G of size d.*

Proof. Given any list of d elements $(g_1, \ldots, g_d)$ of G (not necessarily distinct), let us associate it with the subgroup $U \leq G$ such that $U = \langle g_1, \ldots, g_d \rangle$. Then clearly there are $\varphi(U)$ many d-tuples we associate with U. To finish, observe that the total number of all d-tuples is n^d. $\square$

Now let $\Omega_{\mathcal{K}}^d = \mathbb{K}_\sigma \mathcal{B}_{\mathcal{K}}^d$, that is, the deformation of $\mathbb{K}_\sigma \mathcal{B}_{\mathcal{K}}$ given by homomorphisms where $\ell(n) = n^d$ for a positive integer d, likewise let $\Lambda_{\mathcal{K}}^d$ to denote the deformation of the subgroup category given by the monoid homomorphisms that are obtained in the same way.

Theorem 5.2.3. *If every group in $\mathcal{K}$ is cyclic, then $\Lambda_{\mathcal{K}}^d \cong \Omega_{\mathcal{K}}^d$ and $\Omega_{\mathcal{K}}^d$ is semisimple.*

Proof. If every group in $\mathcal{K}$ is abelian, then clearly $\Omega_{\mathcal{K}} \cong \Lambda_{\mathcal{K}}$. Let E and L be cyclic groups such that $[E] \leq [L]$ and $\mathcal{K}$ contain an isomorphic copy of L . We will to show that the matrix T_E^L is invertible.

Let $\phi, \psi \in \mathrm{epi}(E, L)$ and M be the unique subgroup of L with order $|\frac{L}{E}|$, in which case $\ker(\phi) = M = \ker(\psi)$. Then by Lemma 5.1.12

$$t_{\triangleleft(\phi)}^{E,L} t_{\triangleright(\psi)}^{L,E} = \tau_{\Delta(\theta)}^{E,E} t_{\Delta(\theta)}^{E,E}$$

where $\theta = \phi \circ \psi^{-1}$. If $\phi = \psi$, then, $\Delta(\theta) = \Delta(E)$. Taking S as it appears in the Lemma 5.1.12 we obtain

$$T_E^L(\phi, \psi) = \tau_{\Delta(E)}^{\triangleleft(\phi), \triangleright(\psi)} = \sum_S \text{möb}(S, L)\ell(S \cap M).$$

Considering the retraction $\rho(S) = S \cap M$ and using Proposition 5.2.2 gives

$$\sum_{S \leq M} \text{möb}(S, M)\ell(S) = \varphi(M)$$

which clearly is nonzero since M is a cyclic group so its minimal generating set is of size one. If $\phi \neq \psi$, then $\Delta(\theta) \neq \Delta(E)$ so $t_{\Delta(E)}^{E,E}$ does not appear in the product $t_{\triangleleft(\phi)}^{E,L} t_{\triangleright(\psi)}^{L,E}$ therefore $T_E^L(\phi, \psi) = 0$. So the matrix T_E^L is nonzero multiple of the identity matrix, which clearly is invertible. $\square$

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