END-OF-LIFE INVENTORY MANAGEMENT PROBLEM: NEW RESULTS AND INSIGHTS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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We consider a manufacturer who controls the inventory of spare parts in the end-of-life phase and takes one of three actions at each period: (1) place an order, (2) use existing inventory, or (3) stop holding inventory and use an outside/alternative source. Two examples of this source are discounts for a new generation product and delegating operations. The novelty of our study is allowing multiple orders while using strategies pertinent to the end-of-life phase. Demand is described by a non-homogeneous Poisson process, and the decision to stop holding inventory is described by a stopping time. After formulating this problem as an *optimal stopping problem with additional decisions* and presenting its dynamic programming algorithm, we use martingale theory to facilitate the calculation of the value function. Comparison with benchmark models and sensitivity analysis show the value of our approach and generate several managerial insights. Next, in a more special environment with a single order and a deterministic time to stop holding inventory, we present structural properties and analytical insights. The results include the optimality of (s, S) policy, and the relation between S and the time to stop holding inventory. Finally, we tackle the issue of selecting the intensity function by allowing it to be a stochastic process. The demand process can be constructed by using a Poisson random measure and an intensity process being measurable with respect to the Skorokhod topology. We show the necessary properties of this process including Laplace functional, strong Markov property and its compensated random measure. In case the intensity process is unobservable, we construct a non-linear filter process and reduce the problem to one with complete observation.

Keywords: Spare parts, end-of-life, inventory control, optimal stopping, Poisson processes, stochastic intensity.

ÖZET

SON AŞAMADA ENVANTER YÖNETİMİ: YENİ SONUÇLAR VE ÇIKARIMLAR

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Bir perakendeci, yaşam döngüsünün son aşamasında bulunan ürünler için yedek parça envanteri tutmaktadır. Bu çalışmada perakendecinin yedek parça envanterini kontrol etme problemi ele alınmaktadır. Perakendeci her periyotta üç karardan birini seçer: (1) yedek parça ısmarlamak, (2) var olan envanteri kullanmak, veya (3) envanter tutmayı bırakarak alternatif bir kaynak kullanmak. Bu kaynağa örnek olarak ürünün yeni versiyonu için müşteriye indirim sunmak veya operasyonu üçüncü dereceden bir tedarikçiye havale etmek verilebilir. Bu çalışmanın özgünlüğü birden fazla siparişe izin verilirken son aşama ile ilgili stratejilerin kullanılmasıdır. Talep homojen olmayan Poisson süreci ile gelmekte ve perakendecinin envanter tutmayı bırakma kararı bir durma zamanı ile ifade edilmektedir. Bu problem dinamik programlama ile modellenirken martingal kayışı teorisi yardımıyla amaç fonksiyonu hesaplanmaktadır. Kıyaslama yapılan modeller ve hassasiyet analizi, yaklaşımımızın değerini göstermekte ve yöneticiler için çıkarımlar sunmaktadır. Özel olarak, perakendecinin bir kez sipariş verdiği ve durma zamanının deterministik olduğu durum incelenmiştir. Bu şartlar altında karakteristik özellikler ve analitik çıkarımlar sunulmuştur. (s, S) politikasının en iyi ısmarlama politikası olduğu gözlemlenmiş ve S ile durmadan önce geçen zaman arasındaki ilişki kurulmuştur. Son olarak Poisson sürecinin hız fonksiyonunun seçilmesi problemi ele alınmış ve hızın rassal süreç olduğu varsayılmıştır. Yeni bir talep süreci tanımlanırken bir Poisson rassal ölçü ile birlikte Skorokhod topolojisine göre ölçülebilen rassal bir hız süreci kullanılmıştır. Laplace dönüşümü, güçlü Markov özelliği ve kompanse edilmiş rassal ölçüsü, tanımlanan yeni sürecin çalışılan özellikleri arasındadır.

Anahtar sözcükler: Yedek parça yönetimi, son aşamada envanter kontrolü, Poisson süreçleri, rassal hız süreci.

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Chapter 1

Introduction and Literature Review

While rapid technological developments have been shortening the life-cycle of products sold in the market, competition and customer satisfaction have made the firms increase the warranty periods of those products. To fix a product in case of failures, a firm holds spare parts inventory for long periods and even after the product is no longer produced. This leads to a challenging problem of inventory management of spare parts in the end-of-life phase – a time frame within the product's life-cycle that begins when the product is no longer produced and that ends at the expiration date of all customers' warranties [1].

Original equipment manufacturers strive to properly manage the inventory in the end-of-life phase since the spare parts are held for long periods although the demand for them can be quite low. For instance, in electronics industry, a manufacturer may need to keep the spare parts from 4 years to 30 years after the product is discontinued from production [2]. It might seem tempting to pile up abundant inventory to obey customer warranties, however, this may result in excessive holding and scrapping costs given that the demand is expected to be low. Indeed, HP suffered from huge obsolescence cost due to end-of-life write-offs [3], and in general, after-sales services can be a significant profit source for the firms [4]. As a result, several strategies have been developed to control inventory and mitigate the risk of over- and under-stocking of spare parts in the end-of-life phase.

Early approaches for inventory control in the end-of-life phase attempt to use classical inventory models while aiming to calibrate the parameters pertinent to this phase. For instance, [5, p. 363, Subsection 8.5.1] reviews the studies that develop extensions of the economic order quantity (EOQ) model while assuming a deterministic and decreasing demand rate. Those studies find how many replenishments to make as well as the timing and sizes of replenishments. Simultaneously, several studies are motivated by the intermittent demand structure in this phase, devising inventory models with stochastic demand. Extensions of the newsvendor model, for example, are developed where the parameters (e.g., mean and standard deviation of demand) are estimated from available data. Such studies are reviewed in [5, p. 364, Subsection 8.5.2] as well.

More recent approaches often assume that the original equipment manufacturer can place a single order at the beginning of the end-of-life phase, and they propose complementary business strategies. The motivation behind the single order assumption is that a component manufacturer might decide to stop producing certain spare parts, thereby requiring the original equipment manufacturer to place a final order. This final order is also called last-time buy, final buy, end-of-life buy, or all-time requirements. On the other hand, complementary strategies aim to support the final buy in case of a discrepancy between realized demand and the order quantity.

A wide literature on business strategies complementing a final order includes, but is not limited to, repairing defective spare parts collected from customers [6, 7] (while repairing may not be feasible for some of them [8]), buying back functional or dysfunctional used products to take them apart and obtain the recoverable spare parts [9, 10], considering budget constraints [11] or multiple spare parts in the bill-of-materials of a main product [12], extending customer contracts [13, 14], designing a new product to replace the obsolete one (design refresh) [15, 16], partially scrapping spare parts in case of over-stocking [17], differentiating customers based on demand criticality or service contracts [18], re-manufacturing [4, 19], finding outside/alternative sources [17, 20, 21, 22], and finally, obviating the need to place a final order [23, 2, 24]. We focus on the last two strategies in this study.

The benefit of complementary strategy that finds an outside/alternative source, instead of holding spare parts inventory, can be two-fold. On the one hand, in case the demand for spare parts exceeds the inventory on-hand, the manufacturer can start using the outside source as a back-up source and avoid underage costs. On the other hand, it can be used to get rid of excess inventory in case of an insufficient amount of demand, decreasing overage costs. Some examples of this outside/alternative source can be expedited spare parts supply from a third party supplier, replacing the failed product with a new generation product [18, 20, 21, 22], or substitution of another spare part having the same functionality [16]. Besides, if the cost of such a source decreases over time (for instance, due to price erosion of a new generation product), this strategy can become truly valuable.

Among the studies incorporating an outside/alternative source, [18] considers a manufacturer who places a final order at time zero and can decide to use outside/alternative source at each time period. After providing the dynamic programming formulation, they present benchmark models (e.g., one that places a final order but does not use this source) to show the value of incorporating such source. In [20], they assume that the manufacturer makes a static decision (made at time zero) on the final order quantity and on the time to stop holding inventory (called switching time). Under such a setting, they show that the objective function is convex in the final order quantity for any fixed switching time. [25] extends the model in [20] with more general parameters and describe the decision to stop holding inventory by a stopping time, solving an optimal stopping problem by means of a dynamic programming algorithm. Also, the value of outside/alternative source is shown in different environments, such as products with short life-cycles [26].

We could find a few recent studies which analyze the benefit of providing flexibility in placing orders in the end-of-life phase, although early inventory control approaches consider such flexibility. It is reasonable to accept the existence of a time point when the manufacturer places a final order. Still, such a time point may need to be found after completing an in-depth analysis, since after all, the component manufacturers might be willing to produce the spare parts as long as it is profitable to do so.

Among the studies allowing a flexibility in placing orders in the end-of-life phase, [23] analyze the effects of delaying a final order rather than placing it at time zero, and determine the optimal timing of the final buy from an aggregated supply chain perspective including both the manufacturer and the supplier. They also characterize the delay benefits under different demand scenarios. [27] proposes a dynamic programming model to help manufacturers who can place extra production/procurement orders as well as remanufacture the recoverable spare parts. [28] further explores [27] and devises an advanced heuristic that provides near-optimal solutions and that can quickly solve real-life problem instances. [2] devises a continuous-time solution when demand is described by a Poisson process with constant rate, and finds optimal base-stock policy where order-up-to levels decrease over a finite time horizon. [24] also provides a continuous-time formulation and their model mainly differs from [2] in that partial obsolescence is allowed, that is, intensity rate drops to a lower level at a known future time instance. Also see [29] for a dynamic programming approach when demand is deterministic. To the best of our knowledge, none of the above studies combine flexibility in placing orders and an outside/alternative source.

Chapter 2 of this thesis analyzes the value of providing flexibility in placing orders while making use of strategies related to the end-of-life phase. The novelty of our study is the incorporation of the following main features.

- Timing of the final order can be found. This is a time point that the manufacturer does not choose to place an order afterwards.
- Multiple orders opportunity. Instead of being required to place a single

order at time zero, the manufacturer has the flexibility of placing orders at each time period by paying a large setup cost.

- Opportunity to stop holding inventory and use an alternative/outside source. This source has a relatively higher per-unit cost to satisfy spare part demand, however, it is useful in avoiding excessive penalty and holding costs. The manufacturer's decision to stop holding inventory is described by a stopping time adapted to the filtration of demand process.

- Demand variability. Demand for spare parts is described by a non-homogeneous Poisson process with a non-increasing intensity function. We provide numerical analyses with intensity functions shaped similar to convex, concave and linear functions, and also a constant function as a benchmark.

Therefore, the manufacturer's problem is to make one of the three decisions at each period: (1) place an order for spare parts, (2) do nothing and use existing inventory to satisfy demand, or (3) stop holding inventory permanently and use outside/alternative source. We cast this combined inventory control and optimal stopping problem as an *optimal stopping problem with additional decisions* that can be solved by means of a stochastic dynamic programming (DP) algorithm [30]. After posing the problem and providing its DP formulation, we first use martingale theory to facilitate the calculation of the value function. Next, we benchmark our DP model with special cases that resemble the ideas presented in previous studies.

Table 1.1 and Table 1.2 summarizes the benchmark models and the related literature. To keep track of different formulations, we use the notation $a/b/c$ which describes the main features. $a = D$ means that the decision of stopping holding inventory is made dynamically (hence it is a stopping time adapted to the demand filtration); $a = S$ denotes that such decision is static and made at time zero (also called switching time in this study); $a = T$ implies that we do not stop until end-of-horizon T (in case there is no outside/alternative source). Moreover, if $b = M$ for some $M \in \mathbb{Z}_+$, then the manufacturer can place at most

M orders throughout the horizon; $b = \infty$ means that the manufacturer can place an order at each period with no restriction. Finally, $c = Z$ means that first order must be placed at time zero; and $c = F$ means that the manufacturer is free to place the first order at any time.

Time to Stop Holding Inventory
S (Static decision – made at time 0)
D (Decision is made dynamically)
T (Do not stop until the end of horizon T)
Max. Number of Orders $\equiv M$
1
∞ (unrestricted)
Order Time
Z (First order must be placed at time zero)
F (First order can be placed at any time)

Table 1.1: Notation for benchmark models. For instance, our main DP model can be denoted by $D/\infty/F$. This notation is helpful to keep track of different formulations while comparing them.

Model	Explanation	Related Study
$D/\infty/F$	Multiple orders and stopping time	This study
$D/1/F$	Single order at any time and stopping time	
$D/1/Z$	Single order at time zero and stopping time	[17, 25, 26]
$S/\infty/F$	Multiple orders and switching time	
$S/1/F$	Single order at any time and switching time	
$S/1/Z$	Single order at time zero and switching time	[20]
$T/\infty/F$	Multiple orders without outside source	[2, 24, 27, 28]
$T/1/F$	A delayed single order without outside source	[23]
$T/1/Z$	Single order at time zero without outside source	[31]

Table 1.2: Benchmark models which resemble the ideas presented in previous studies. Notation is presented in Table 1.1. We note that the following models are trivial: $D/\infty/Z$, $S/\infty/Z$, $T/\infty/Z$.

To the best of our knowledge, the closest study to Chapter 2 of this thesis is [16] in that they develop a DP algorithm allowing multiple orders until a fixed last-time-buy date as well as consider a design refresh program, which redesigns the product and the spare part. One difference in our study is that we describe the demand by using a non-homogeneous Poisson process with a non-increasing intensity function, thereby calculating the costs in continuous-time. One motivation for continuous-time calculation of costs is that the manufacturer may not review the inventory for long periods, so we may miss correct representation of costs. For instance, in our model, we describe the exact time that the inventory on hand hits zero and lost sales is observed within a period, by using a stopping time denoted by σ_x. Also there is a difference between the usage of an outside/alternative source and design refresh program; they also consider the decisions to manage the inventory of new spare parts after when the design refresh program is initiated. In our study, the manufacturer stops holding inventory and does not put an effort into the usage of an outside/alternative source. This source can be another option in case the manufacturer does not want to redesign products or spare parts, since such redesign may cannibalize design resources that could otherwise be used for designing new products [32].

Chapter 3 of this thesis provides structural properties and analytical insights for the benchmark model $S/1/Z$ in Table 1.2. According to this model, the manufacturer places a single order at time zero and can stop holding inventory, yet the time of stopping is decided at time zero and it is deterministic (rather than a stopping time adapted to the demand filtration). This deterministic time is called switching time in this study. The motivation behind such a model is that the manufacturer may desire to know the time to stop holding inventory so that better strategic plans can be developed in advance.

By utilizing the results and expressions in [20], we (1) provide a rigorous proof that for a fixed switching time, the optimal policy characterizing the ordering decision is an (s, S)-policy, (2) find expressions for the values of the re-order level s and the order-up-to level S, (3) generate an analytical insight that S is a non-decreasing function of the switching time as long as the demand rate and the cost of the outside source are high, (4) find conditional upper and lower bounds on the best switching time when the inventory level is fixed.

Chapter 4 presents numerical analyses. We first verify our code and compare the main DP model $D/\infty/F$ with selected benchmark models, showing the value of incorporating the main features. Next, we provide sensitivity analyses on problem parameters as well as the intensity function, generating several managerial insights. The analyses and insights mainly revolve around the following remarks. Assuming that a final order must be placed at time zero can be a very strong assumption; the dynamic selection of time to stop (via stopping time) can be valuable; and allowing for multiple orders can be valuable. Moreover, depending on the initial inventory and setup cost values, it might be wise to encourage customers come earlier, invest in outside/alternative source, extend the warranty period, and announce a very large penalty for not satisfying the demand to attract customers.

Another common issue in managing end-of-life inventory is forecasting and modeling the demand for spare parts [31, 33]. Since historical demand data is often lacking in the end-of-life phase, standard forecasting techniques cannot be applied [34]. One solution approach in the literature is to devise advanced forecast models by using, for example, the available information on products in the market (installed base), failure rate of spare parts, and the return of used products (phase-out returns). Indeed, there is a wide literature focusing on how to estimate and describe the demand in this phase; [35, 36] motivate the relevant problems and review the existing approaches.

Motivated by this common issue, we also focus on the problem that there might be errors while selecting the intensity function of the non-homogeneous Poisson process. As a solution approach, we aim to extend non-homogeneous Poisson processes by allowing intensity rate to be a stochastic process as well. This extended process is called conditional Poisson process, or doubly stochastic Poisson process, or Cox process. Although there are some studies which construct this process, we could not find a version that has the necessary properties enabling us to model the demand in the end-of-life phase. Among the studies that construct the conditional Poisson process, [38, p. 211, Sec. 3.14] and [39] define the conditional Poisson process by assuming the form of conditional Laplace transform, and [40, p. 169, Sec. 6.2.] assume the form of conditional probability; they also argue the equivalence of these two forms. Moreover, [41, p. 58, Sec. 1.8.2] define the conditional Poisson process by assuming that the intensity process is a compensator of the conditional Poisson process.

Chapter 5 of this thesis provides a new construction of the conditional Poisson process which can be used in an optimal stopping problem related to end-of-life inventory context. One novel feature of our construction is that we do not assume a finite state space for the intensity process. We first prove the necessary properties of our construction, including Laplace functional, strong Markov property, and its compensated random measure. Next, we describe the demand by the conditional Poisson process and use its properties in the solution of continuous-time version of the model $D/1/Z$ (see Table 1.2 for notation); this model assumes that the decision to stop holding inventory is described by a stopping time. Since the manufacturer may not be able to use an alternative/outside source immediately in a continuous-time model, we further assume that there is a delay between when the decision is made and when the source can be used, leading to an optimal stopping problem with delay introduced by [42]. We reduce this optimal stopping problem with delay to a classical optimal stopping problem. Finally, in case the intensity process is unobservable, we construct a non-linear filter process and reduce the problem to one with complete observation. We also discuss the solution of this reduced optimal stopping problem.

In Chapter 6, concluding remarks are provided and future research directions are discussed.

Chapter 2

The End-of-Life Inventory Problem under Fixed Ordering Cost, Multiple Orders and Stopping Time

2.1 Problem Definition

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space and let $T \in \mathbb{R}_+$. We start by assuming that the demand for spare parts is described by a non-homogeneous Poisson process $N : \Omega \times [0, T] \to \mathbb{Z}_+$ with a non-increasing intensity function $\lambda : [0, T] \to \mathbb{R}_+$ and mean value function $\Lambda(t) = \int_0^t \lambda(u)du$. Most of the results in this thesis can be recovered without the assumption that λ is non-increasing. Still, such assumption can be more appropriate to describe the demand for spare parts in the end-of-life phase. The manufacturer periodically reviews the inventory level and for brevity of notation, we assume that length of time periods are identical. Our model can be easily adjusted for non-identical period lengths. At each time period $k \in \mathbb{T} := \{0, 1, 2, \ldots, T\}$, the manufacturer observes the current inventory level $x \in \mathbb{Z}_+$ and decides whether to stop or continue holding inventory. If the manufacturer continues to hold inventory, an order $\mu_k(x) \in \mathbb{Z}_+$ can be placed where a function $\mu_k : \mathbb{Z}_+ \to \mathbb{Z}_+$ specifies the order amount. The order cost function $c : \mathbb{Z}_+ \to \mathbb{R}_+$ is given by

$$c(m) := \begin{cases} K + \bar{c}\,m, & \text{if } m > 0 \\ 0, & \text{if } m = 0 \end{cases} \tag{2.1.1}$$

where $\bar{c} \in \mathbb{R}_+$ is the per unit purchase cost and $K \in \mathbb{R}_+$ is the fixed ordering cost. We assume that the order cost at time k is discounted by $e^{-\delta k}$ and the lead time is zero. After placing an order at time k, the manufacturer continues operations and reaches to the next period $k+1$. During the period $[k, k+1]$, holding cost accrues with rate $c_1 \in \mathbb{R}_+$ so that the expected inventory holding cost is given by

$$c_1 \mathbb{E}\left[\int_k^{k+1} e^{-\delta(u-k)}(x - (N_u - N_k))^+ du\right].$$

If the inventory level hits zero during $[k, k+1]$ and a defective part arrives, then the manufacturer replaces the part by paying a time-dependent per unit cost $c_2 : [0, T] \to \mathbb{R}_+$. Such replacement cost is given by

$$\mathbb{E}\left[\int_{(k+1)\wedge\sigma_x^k}^{k+1} e^{-\delta(u-k)} c_2(u) dN_u\right]$$

where $(k+1)\wedge\sigma_x^k = \min\left\{k+1, \sigma_x^k\right\}$, and $\sigma_x^k = \inf\{u > k : N_u - N_k \geq x\}$ denotes the arrival time of the x^{th} item after time k. We denote $\sigma_x := \sigma_x^0$. Combining the two terms above, the one-period operation cost for time k and inventory level x can be written as

$$\begin{aligned} C(k, x) :=& c_1 \mathbb{E}\left[\int_k^{k+1} e^{-\delta(u-k)}(x - (N_u - N_k))^+ du\right] \\ &+ \mathbb{E}\left[\int_{(k+1)\wedge\sigma_x^k}^{k+1} e^{-\delta(u-k)} c_2(u) dN_u\right] \end{aligned}$$

On the other hand, if the stopping decision is chosen, the manufacturer scraps the inventory with per unit cost $c_4 \in \mathbb{R}$. Future defective parts, if any, are replaced with a time-dependent per unit cost $c_3 : [0, T] \to \mathbb{R}_+$. Some examples of this per unit cost can be expedited supply from a third party supplier, offering a new generation product [20], or substitution of another spare part having the same functionality [16]. Therefore, the cost of stopping holding inventory is given by

$$S(k,x) := c_4 x + \mathbb{E}\left[\int_k^T e^{-\delta(u-k)} c_3(u) dN_u\right] \tag{2.1.2}$$

where $\delta \in [0,1]$ is the discount rate of continuous compounding. We assume that $\bar{c} > -c_4$ holds since otherwise, the manufacturer can place an infinite order and then scrap inventory at the same time. Moreover, it is natural to interpret that $c_2(u) \geq c_3(u)$ holds for every $u \in [0,T]$. The per unit cost c_2 incurs when the manufacturer unexpectedly replaces a defective part without being able to use the inventory. On the other hand, the per unit cost c_3 incurs when the manufacturer uses an outside source that is prepared beforehand.

Let $\mathcal{T}$ denote the set of all stopping times of the filtration generated by the demand process $\{N_t, t \in [0,T]\}$ and taking values in $\mathbb{T}$. We introduce the notation π for a policy that specifies both an order amount $\mu_k(x_k)$ for every $k \in \mathbb{T}$ and $x_k \in \mathbb{Z}_+$ as well as a stopping time $\tau \in \mathcal{T}$. Let Π denote the set of all admissible policies. Then, each $\pi \in \Pi$ is defined by

$$\pi = (\tau, \mu_1, \mu_2, \ldots, \mu_T)$$

for some $\tau \in \mathcal{T}$ and $\mu_k : \mathbb{Z}_+ \to \mathbb{Z}_+$, $k \in \mathbb{T}$. Moreover, under an arbitrary policy $\pi \in \Pi$, the inventory level X_k at time $k \in \mathbb{T}$ can be expressed by using the recursive relation

$$X_{k+1} = \Big(X_k + \mu_k(X_k) - (N_{k+1} - N_k)\Big)^+, \quad X_0 = x \in \mathbb{Z}_+, \tag{2.1.3}$$

where $(x)^+ = \max\{0, x\}$. We use the notation X_{k+1}^π for X_{k+1} only when a policy $\pi \in \Pi$ is not an arbitrary policy, for brevity of notation. The manufacturer's problem is to determine both the optimal order amount and the optimal time to stop the process in order to minimize the total costs. We formulate this problem as

$$V^*(x) = \inf_{\pi\in\Pi} \mathbb{E}\left[\sum_{k=0}^{\tau-1} e^{-\delta k}\Big(c(\mu_k(X_k)) + C(k, X_k + \mu_k(X_k))\Big) + e^{-\delta\tau} S(\tau, X_\tau)\,\middle|\, X_0 = x\right], \quad x \in \mathbb{Z}_+. \tag{2.1.4}$$

This formulation yields an *optimal stopping problem with additional decisions* [30] and can be solved by means of the following dynamic programming (DP) algorithm. Define the backward dynamic programming algorithm for each $k \in \{T-1, T-2, \ldots, 0\}$ and $x_k \in \mathbb{Z}_+$ by

$$\begin{aligned} &V(k, x_k) \\ &= \min\Big\{S(k, x_k), \inf_{m\in\mathbb{Z}_+} \{c(m) + C(k, x_k + m) \\ &\qquad + e^{-\delta}\mathbb{E}\left[V(k+1, \left(x_k + m - (N_{k+1} - N_k)\right)^+)\right]\}\Big\}. \end{aligned} \tag{2.1.5}$$

Also define the terminal condition by

$$V(T, x_T) = S(T, x_T) = c_4 x_T, \quad x_T \in \mathbb{Z}_+.$$

Then, $V(0, x) = V^*(x)$ for every $x \in \mathbb{Z}_+$ [30]. Moreover, an optimal stopping time τ^* is the one that stops the process if $S(k, x_k) \le \inf_{m\in\mathbb{Z}_+} \{\ldots\}$ in (2.1.5). Furthermore, the optimal order amount $\mu_k^*(x_k)$ is equal to m^* where m^* attains the infimum $\inf_{m\in\mathbb{Z}_+} \{\ldots\}$ in (2.1.5) [30]. The next section provides the analytical results to calculate $V(0, x)$.

2.2 Analytical Results for the Value Function

This section converts the value function V, one-step operation cost C and stopping cost S into a new form so that $V(0, x)$ in (2.1.5) can be calculated. We proceed as in [20, 25] with modifications since, here, the inventory process is a controlled process and there is no repairability option. Subsection 2.2.1 reduces the computation of the stopping cost by re-formulating the problem $V^*(x)$ in (2.1.4). Subsection 2.2.2 provides analytical results to calculate the one-step operation cost in this reformulated problem.

2.2.1 Reformulation of the Value Function

This subsection simplifies the computation of $S(k, x_k)$ defined by (2.1.2) by reformulating $V^*(x)$ in (2.1.4). For every $u \in [k, k+1]$, define

$$X_u^{k,y} := (y - (N_u - N_k))^+$$

as the position of inventory process at time $u \in [k, k+1]$, given that the position at time $k \in \mathbb{T}$ is equal to $y \in \mathbb{Z}_+$. Then, we can write

$$X_u := X_u^{0,x} = X_1^{0,x} + X_2^{1, X_1 + \mu_1(X_1)} + \cdots + X_u^{k, X_k + \mu_k(X_k)}, \quad x \in \mathbb{Z}_+$$

for every $k \in \mathbb{T}$, $u \in [k, k+1]$ and $\pi = (\tau, \mu_1, \ldots, \mu_T) \in \Pi$. Therefore, for an arbitrary $\pi \in \Pi$, we can write

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{\tau-1} e^{-\delta k} C(k, X_k + \mu_k(X_k)) + e^{-\delta \tau} S(\tau, X_\tau) \,\middle|\, X_0 = x \right] \\
=&\mathbb{E}\Bigg[c_1 \int_0^\tau e^{-\delta u} X_u \, du + \int_{\tau \wedge \sigma_x}^\tau e^{-\delta u} c_2(u) dN_u \\
&\quad + c_4 e^{-\delta \tau} X_\tau + \int_\tau^T e^{-\delta u} c_3(u) dN_u \,\Bigg|\, X_0 = x \Bigg] \\
=&c_1 \mathbb{E}\Bigg[\int_0^\tau e^{-\delta u} X_u du \\
&\quad + \int_0^\tau e^{-\delta u} c_2(u) dN_u - \int_0^{\tau \wedge \sigma_x} e^{-\delta u} c_2(u) dN_u \\
&\quad + \int_0^T e^{-\delta u} c_3(u) dN_u - \int_0^\tau e^{-\delta u} c_3(u) dN_u + c_4 e^{-\delta \tau} X_\tau \,\Bigg|\, X_0 = x \Bigg]
\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\left[c_1 \int_0^{\tau} e^{-\delta u} X_u du + \int_0^{\tau} e^{-\delta u}[c_2(u) - c_3(u)] dN_u\right. \\
&\left.- \int_0^{\tau\wedge\sigma_x} e^{-\delta u} c_2(u) dN_u + c_4 e^{-\delta\tau} X_\tau \,\middle|\, X_0 = x\right] + A
\end{aligned}
$$

where

$$
A = \mathbb{E}\left[\int_0^T e^{-\delta u} c_3(u) dN_u\right] \in \mathbb{R}_+. \tag{2.2.1}
$$

Now, to provide the alternative formulation, define for each $k \in \mathbb{T}$ and $x \in \mathbb{Z}_+$ the cost of stopping by

$$
\widetilde{S}(k, x) := c_4 x
$$

and define the one-period operation cost by

$$
\begin{aligned}
\widetilde{C}(k, x) :=& c_1 \mathbb{E}\left[\int_k^{k+1} e^{-\delta(u-k)} \big(x - (N_u - N_k)\big)^+ du\right] \\
&+ \mathbb{E}\left[\int_k^{k+1} e^{-\delta(u-k)} [c_2(u) - c_3(u)]\, dN_u\right] \\
&- \mathbb{E}\left[\int_k^{(k+1)\wedge\sigma_x^k} e^{-\delta(u-k)} c_2(u)\, dN_u\right].
\end{aligned} \tag{2.2.2}
$$

Then, by using the strong Markov property of Poisson processes [43, p. 296, VI.5.18], it is possible to see that for every $\pi \in \Pi$,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{\tau-1} e^{-\delta k}\Big(c(\mu_k(X_k)) + C(k, X_k + \mu_k(X_k))\Big) + e^{-\delta\tau} S(\tau, X_\tau) \,\middle|\, X_0 = x\right] \\
=& \mathbb{E}\left[\sum_{k=0}^{\tau-1} e^{-\delta k}\Big(c(\mu_k(X_k)) + \widetilde{C}(k, X_k + \mu_k(X_k))\Big) + e^{-\delta\tau} \widetilde{S}(\tau, X_\tau) \,\middle|\, X_0 = x\right] + A.
\end{aligned}
$$

Hence, it is possible to see that $V^*(x)$ is equivalent to the problem

$$
\begin{aligned}
\widetilde{V}^*(x) = \inf_{\pi\in\Pi} \mathbb{E}&\left[\sum_{k=0}^{\tau-1} e^{-\delta k}\Big(c(\mu_k(X_k)) + \widetilde{C}(k, X_k + \mu_k(X_k))\Big)\right. \\
&\left. + e^{-\delta\tau} \widetilde{S}(\tau, X_\tau) \,\middle|\, X_0 = x\right], \quad x \in \mathbb{Z}_+.
\end{aligned} \tag{2.2.3}
$$

in the sense that $V^*(x) = \widetilde{V}^*(x) + A$ for every $x \in \mathbb{Z}_+$. Therefore, we aim to solve the problem $\widetilde{V}^*(x)$. Again, as stated by [30], the following dynamic programming algorithm can solve the problem $\widetilde{V}^*(x)$. Define the backward dynamic programming algorithm for each $k \in \{T-1, T-2, \dots, 0\}$ and $x_k \in \mathbb{Z}_+$ by

$$\begin{aligned}
&\widetilde{V}(k, x_k) \\
&= \min\Bigg\{\widetilde{S}(k, x_k), \inf_{m \in \mathbb{Z}_+} \Big\{c(m) + \widetilde{C}(k, x_k + m) \\
&\qquad\qquad + e^{-\delta}\mathbb{E}\left[\widetilde{V}\left(k+1, \left(x_k + m - (N_{k+1} - N_k)\right)^+\right)\right]\Big\}\Bigg\} \qquad (2.2.4)
\end{aligned}$$

with the terminal condition

$$\widetilde{V}(T, x_T) = \widetilde{S}(T, x_T) = c_4 x_T, \quad x_T \in \mathbb{Z}_+.$$

Then, $\widetilde{V}(0, x) = \widetilde{V}^*(x)$ for every $x \in \mathbb{Z}_+$ [30]. Moreover, an optimal stopping time τ^* is the one that stops the process if $\widetilde{S}(k, x_k) \leq \inf_{m \in \mathbb{Z}_+} \{\dots\}$ in (2.2.4). Furthermore, the optimal order amount $\mu_k^*(x_k)$ is equal to m^* where m^* attains the infimum $\inf_{m \in \mathbb{Z}_+} \{\dots\}$ in (2.2.4) [30]. The next subsection provides the analytical results to calculate $\widetilde{C}$.

2.2.2 Analytical Results for the One-Step Operation Cost

This subsection provides the analytical results to calculate $\widetilde{C}(k, x_k)$ in (2.2.2) for each $k \in \{0, 1, \dots, T\}$ and $x_k \in \mathbb{Z}_+$. Lemma 2.2.1 below introduces the martingale property for the non-homogeneous Poisson process and it helps us convert Poisson integrals into Lebesgue integrals.

Lemma 2.2.1. *Let $H : \Omega \times [0, T] \to \mathbb{R}_+$ be a positive predictable process such that $\mathbb{E}\left[\int_0^t H_u \lambda(u) du\right] < \infty$ for every $t \in [0, T]$. Then, the process $L : \Omega \times [0, T] \to \mathbb{R}$ defined by*

$$L_t = \int_0^t H_u \, dN_u - \int_0^t H_u \lambda(u) \, du$$

is a martingale with respect to the filtration generated by $\{N_t, t \in [0,T]\}$. *Moreover, for each stopping time* $\tau \in \mathcal{T}$,

$$\mathbb{E}\left[\int_0^\tau H_u \, dN_u\right] = \mathbb{E}\left[\int_0^\tau H_u \lambda(u) \, du\right].$$

Proof. See [43, p. 299, VI.6.4]. $\square$

The following Lemma 2.2.2 is helpful while converting the one-period operation cost into a new form that can be calculated.

Lemma 2.2.2. *For any* $x \in \mathbb{Z}_+$ *and* $k \in \mathbb{T}$,

$$\mathbb{E}[(x - N_k)^+] = \sum_{n=0}^{x-1} \mathbb{P}\{N_k \leq n\}.$$

Proof. Note that

$$\begin{aligned}
&\mathbb{E}[(x - N_k)^+] - \mathbb{E}[(x - 1 - N_k)^+] \\
&= \sum_{n=0}^{x-1} (x - n)\mathbb{P}\{N_k = n\} - \sum_{n=0}^{x-2} (x - 1 - n)\mathbb{P}\{N_k = n\} \\
&= \sum_{n=0}^{x-1} \mathbb{P}\{N_k = n\} \\
&= \mathbb{P}\{N_k \leq x - 1\}.
\end{aligned}$$

Iterating this equality yields that

$$\begin{aligned}
&\mathbb{E}[(x - N_k)^+] \\
&= \mathbb{E}[(x - 1 - N_k)^+] + \mathbb{P}\{N_k \leq x - 1\} \\
&= \mathbb{E}[(x - 2 - N_k)^+] + \mathbb{P}\{N_k \leq x - 2\} + \mathbb{P}\{N_k \leq x - 1\} \\
&= \dots \\
&= \sum_{n=0}^{x-1} \mathbb{P}\{N_k \leq n\}.
\end{aligned}$$

$\square$

Finally, Proposition 2.2.3 below converts one-period operation $\widetilde{C}(k,x)$ in (2.2.2) into a new form that can be calculated.

Proposition 2.2.3. *For every $k \in \mathbb{T}$, the one-period operation cost $\widetilde{C}(k,x)$ in* (2.2.2) *can be written as*

$$\widetilde{C}(k,0) = \int_k^{k+1} e^{-\delta(u-k)}[c_2(u) - c_3(u)]\lambda(u)\,du,$$

and for $x \geq 1$,

$$\begin{aligned}\widetilde{C}(k,x) =& c_1 \sum_{n=0}^{x-1}\sum_{i=0}^{n}\int_k^{k+1} e^{-\delta(u-k)} e^{-(\Lambda(u)-\Lambda(k))}\frac{(\Lambda(u)-\Lambda(k))^i}{i!}\,du \\ &+ \int_k^{k+1} e^{-\delta(u-k)}[c_2(u) - c_3(u)]\lambda(u)\,du \\ &- \sum_{i=0}^{x}\int_k^{(k+1)} e^{-\delta(u-k)} c_2(u)\lambda(u) e^{-\Lambda(u)-\Lambda(k)}\frac{(\Lambda(u)-\Lambda(k))^i}{i!}\,du. \qquad (2.2.5)\end{aligned}$$

Proof. Let us convert the terms of $\widetilde{C}(k,x)$ in (2.2.2) one by one. First,

$$\begin{aligned}&\mathbb{E}\left[\int_k^{k+1} e^{-\delta(u-k)}\big(x-(N_u-N_k)\big)^+\,du\right] \\ =& \int_k^{k+1} e^{-\delta(u-k)}\mathbb{E}\left[\big(x-(N_u-N_k)\big)^+\right]du \quad \text{(Fubini's Theorem)} \\ =& \int_k^{k+1} e^{-\delta(u-k)}\sum_{n=0}^{x-1}\mathbb{P}\{N_u-N_k \leq n\}\,du \quad \text{(Lemma 2.2.2)} \\ =& \sum_{n=0}^{x-1}\int_k^{k+1} e^{-\delta(u-k)}\mathbb{P}\{N_u-N_k \leq n\}\,du \\ =& \sum_{n=0}^{x-1}\sum_{i=0}^{n}\int_k^{k+1} e^{-\delta(u-k)}\mathbb{P}\{N_u-N_k = i\}\,du \\ =& \sum_{n=0}^{x-1}\sum_{i=0}^{n}\int_k^{k+1} e^{-\delta(u-k)} e^{-(\Lambda(u)-\Lambda(k))}\frac{(\Lambda(u)-\Lambda(k))^i}{i!}\,du,\end{aligned}$$

and $\Lambda(u) - \Lambda(k) = \int_k^u \lambda(s)ds$. Now, let us focus on the next term of $\widetilde{C}$ in (2.2.2).

It follows immediately from Lemma 2.2.1 that

$$\mathbb{E}\left[\int_k^{k+1} e^{-\delta(u-k)}[c_2(u) - c_3(u)]\, dN_u\right] = \int_k^{k+1} e^{-\delta(u-k)}[c_2(u) - c_3(u)]\lambda(u)\, du$$

Finally, the last term of $\widetilde{C}$ in (2.2.2) is

$$\begin{aligned}
&\mathbb{E}\left[\int_k^{(k+1)\wedge\sigma_x^k} e^{-\delta(u-k)} c_2(u)\, dN_u\right]\\
=&\mathbb{E}\left[\int_k^{(k+1)\wedge\sigma_x^k} e^{-\delta(u-k)} c_2(u)\lambda(u)\, du\right] \quad \text{(Lemma 2.2.1)}\\
=&\int_k^{k+1} \mathbb{E}\left[1_{\left\{u<\sigma_x^k\right\}}\right] e^{-\delta(u-k)} c_2(u)\lambda(u)\, du \quad \text{(Fubini's Theorem)}\\
=&\int_k^{k+1} \mathbb{P}\left\{N_u - N_k \leq x\right\} e^{-\delta(u-k)} c_2(u)\lambda(u)\, du\\
&\quad \text{(Definition of } \sigma_x^k\text{)}\\
=&\sum_{i=0}^{x} \int_k^{k+1} e^{-\delta(u-k)} c_2(u)\lambda(u) e^{-(\Lambda(u)-\Lambda(k))} \frac{(\Lambda(u) - \Lambda(k))^i}{i!}\, du.
\end{aligned}$$

□

2.3 Benchmark Models

In this section, we develop benchmark models for our main model presented in Section 2.1. To keep track of different formulations, we use the notation $V^{a/b/c}$ to describe the main features: $a = D$ means that the decision of stopping holding inventory is made dynamically; $a = S$ denotes that such decision is static and made at time zero; $a = T$ implies that we do not stop until end-of-horizon T. Moreover, if $b = M$ for some $M \in \mathbb{Z}_+$, then the manufacturer can place M orders throughout the horizon; $b = \infty$ means that the manufacturer can place an order at each period with no restriction. Finally, $c = Z$ means that the first order must be placed at time zero; $c = F$ means that the manufacturer is free to place the first order at any time. Table 1.1 in Chapter 1 summarizes the notation. In numerical analyses (Chapter 4), we will compare these models as well as our main model to show the value of our approach.

2.3.1 D/1/F - Single Order Opportunity at Any Time and Stopping Time

This benchmark dynamic programming formulation analyzes the effects of delaying a single order and [23] presents this idea in a different setting. Let $z \in \{0,1\}$ be the number of remaining orders that the manufacturer can place. The following dynamic programming algorithm describes this formulation. Define the terminal cost for each $x_T \in \mathbb{Z}_+$ by

$$V^{D/1/F}(T, x_T, z) = \widetilde{S}(T, x_T), \quad z \in \{0,1\}.$$

If $z = 0$, then define the backward dynamic programming algorithm for each $k \in \{T-1, T-2, \ldots, 0\}$ and $x_k \in \mathbb{Z}_+$ by

$$\begin{aligned} &V^{D/1/F}(k, x_k, 0) \\ &= \min\left\{\widetilde{S}(k, x_k), \widetilde{C}(k, x_k) + e^{-\delta}\mathbb{E}\left[V^{D/1/F}\left(k+1, \left(x_k - (N_{k+1} - N_k)\right)^+, 0\right)\right]\right\}. \end{aligned} \tag{2.3.1}$$

If $z = 1$, then define the backward dynamic programming algorithm for each $k \in \{T-1, T-2, \ldots, 0\}$ and $x_k \in \mathbb{Z}_+$ by

$$\begin{aligned} &V^{D/1/F}(k, x_k, 1) \\ &= \min\Big\{\widetilde{S}(k, x_k),\ \widetilde{C}(k, x_k) + e^{-\delta}\mathbb{E}\left[V^{D/1/F}(k+1, (x_k - (N_{k+1} - N_k))^+, 1)\right], \\ &\qquad \inf_{m \in \mathbb{Z}_+} \{c(m) + \widetilde{C}(k, x_k + m) \\ &\qquad\qquad + e^{-\delta}\mathbb{E}\left[V^{D/1/F}(k+1, (x_k + m - (N_{k+1} - N_k))^+, 0)\right]\}\Big\}. \end{aligned} \tag{2.3.2}$$

2.3.2 D/1/Z - Single Order Opportunity at Time Zero and Stopping Time

A prevalent assumption in the literature is that a final order has to be placed at time zero. Therefore, we develop a dynamic programming algorithm to reflect the manufacturer's decision when only one order can be placed at time zero and the manufacturer can stop holding inventory at any time. This model resembles the one presented by [17]. It follows from Section 2.2 that the dynamic programming algorithm to solve this problem is the following. Define the backward dynamic programming algorithm for each $k \in \{T-1, T-2, \ldots, 0\}$ and $x_k \in \mathbb{Z}_+$ by

$$\begin{aligned}&\dot{V}^{D/1/Z}(k, x_k) \\ &= \min\left\{\widetilde{S}(k, x_k), \widetilde{C}(k, x_k) + e^{-\delta}\mathbb{E}\left[\dot{V}^{D/1/Z}(k+1, \left(x_k - (N_{k+1} - N_k)\right)^+)\right]\right\}\end{aligned}$$

Also define the terminal condition by

$$\dot{V}^{D/1/Z}(T, x_T) = \widetilde{S}(T, x_T)$$

The optimal order quantity at time zero and the value of this dynamic program is found by calculating

$$V^{D/1/Z}(x) = \inf_{m \in \mathbb{Z}_+}\left\{c(m) + \dot{V}^{D/1/Z}(0, x+m)\right\}, \quad x \in \mathbb{Z}_+, \tag{2.3.3}$$

where $c(m)$ is defined by equation (2.1.1). It is possible to see the following relation between $D/1/F$ and $D/1/Z$. While solving the model $D/1/F$, if we decide to place an order at time k, then we solve the model $D/1/Z$ with a different time horizon that is equal to $T - k$.

2.3.3 S/∞/F or S/1/F or S/1/Z

$S/\infty/F$ can be formulated a special case of $D/\infty/F$ whose value function is $\widetilde{V}$, see (2.2.4). For each switching time $k \in \mathbb{T}$, we implement the dynamic programming algorithm and select the best switching time k^*.

Moreover, $S/1/F$ can be formulated as a special case of $D/1/F$ presented in Subsection 2.3.1. We modify the value function $V^{D/1/F}$ by eliminating the stopping option with cost $\widetilde{S}(k, x_k)$ and solve the DP algorithm.

Finally, $S/1/Z$ can be formulated as a special case of $D/1/Z$ presented in Subsection 2.3.2. For every $t \in \mathbb{T}$, we implement $D/1/Z$ without being able to stop.

2.3.4 T/∞/F or T/1/F or T/1/Z

These benchmark models are further special cases of $S/\infty/F$, $S/1/F$ and $T/1/F$. They resemble the classical inventory models, which can be solved by means of standard DP algorithms. For instance, see [44, Chapter 4].

2.3.5 S/M/F or D/M/F for M>1

These models can be formulated by using a similar idea to $D/1/F$ in Subsection 2.3.1 by representing the number of setups as a state variable.

2.3.6 Newsvendor Formulation

Newsvendor formulations have been used for a long time in end-of-life inventory context [5, p. 364, Subsection 8.5.2] as well as other environments. Hence, to provide a newsvendor formulation of our problem, we assume that a single order of size $m \in \mathbb{Z}_+$ can be placed at time zero and a deterministic switching time $k \in \mathbb{T} = \{0, 1, \ldots, T\}$ is found at time zero. Let the holding cost in this newsvendor formulation be the average holding cost for one unit of inventory, which is given by

$$H(k) := c_1 \int_0^k e^{-\delta u} du = \frac{c_1}{\delta}\left[1 - e^{-\delta k}\right].$$

Moreover, any positive inventory is scrapped at time k with per unit cost $c_4 \in \mathbb{R}_+$. Furthermore, we assume that any unsatisfied demand at time k has the penalty $c_2(k)$. The optimal order quantity $m \in \mathbb{Z}_+$ is found for each switching time $k \in \{0, 1, \dots, T\}$ and initial inventory level $x \in \mathbb{Z}_+$ by solving

$$\begin{aligned} &TC^{NV}(k,x) \\ &= \inf_{m \in \mathbb{Z}_+} \left\{ c(m) + c_o(k)\mathbb{E}\left[(x + m - N_k)^+\right] + c_u(k)\mathbb{E}\left[(N_k - x - m)^+\right] \right\}, \end{aligned} \tag{2.3.4}$$

where $c_o(k) = H(k) + e^{-\delta k} c_4$ and $c_u(k) = e^{-\delta k} c_2(k)$ and $c(m)$ is defined by (2.1.1). It is well-known that (s, S) policy is optimal to find the order amount. The value of the newsvendor formulation is found by solving

$$V^{NV}(x) = \inf_{k \in \mathbb{T}} \left\{ TC^{NV}(k,x) + \mathbb{E}\left[\int_k^T e^{-\delta u} c_3(u) dN_u\right] \right\}, \quad x \in \mathbb{Z}_+. \tag{2.3.5}$$

Note that c_o underestimates the total holding cost as we only consider holding cost for the items on-hand at the switching time. Similarly, c_u underestimates the total penalty to be paid as underage cost is only charged at the switching time, whereas it could have been charged earlier. It is also possible to see that $TC^{NV}(k,x)$ is not necessarily monotone in k.

Chapter 3

The End-of-Life Inventory Problem under Fixed Ordering Cost, One Order and Switching Time - Analytical Results

3.1 Problem Definition

This chapter provides analytical results regarding the model $S/1/Z$ presented in the previous Chapter 2: single order opportunity at time zero and deterministic switching time. For the sake of completeness, we describe the problem here as well. For ease of notation, we denote the switching time by τ and present an equivalent expression of the objective function. The model $S/1/Z$ deals with two trade-offs while selecting the best order amount x and switching time τ. On the one hand, for a fixed τ, there is a trade-off between holding cost and underage cost. This trade-off is relatively more classical in the inventory theory. On the other hand, for a fixed x, there is a trade-off between underage plus holding cost and the cost of outside/alternative source. Indeed, if τ is too large, the risk of over-stocking and under-stocking increases. If τ is too small, however, excessive usage of outside source increases the expected total costs since it is presumably an expensive option to replace spare parts.

We start by assuming that the demand for spare parts is described by a non-homogeneous Poisson process $N : \Omega \times [0, T] \to \mathbb{Z}_+$ with a non-increasing and right-continuous intensity function $\lambda : [0, T] \to \mathbb{R}_+$ and mean value function $\Lambda(t) = \int_0^t \lambda(u)du$. At time zero, the manufacturer owns $x_0 \in \mathbb{Z}_+$ items in the inventory and makes an order decision. The order cost function $c : \mathbb{Z}_+ \to \mathbb{R}_+$ is given by

$$c(m) := \begin{cases} K + \bar{c}\, m, & \text{if } m > 0 \\ 0, & \text{if } m = 0 \end{cases}$$

where $\bar{c} \in \mathbb{R}_+$ is per unit purchase cost and $K \in \mathbb{R}_+$ is the fixed ordering cost. Moreover, the following costs incur in $[0, \tau]$ before the switching time. For each unit, holding cost accrue with rate $c_1 \in \mathbb{R}_+$ so that expected inventory holding costs are given by

$$c_1 \mathbb{E} \int_0^\tau e^{-\delta u}(x - N_u)^+ du$$

where $\delta \in [0, 1]$ is the discount rate of continuous compounding. If the inventory level hits zero during $[0, \tau]$ and a defective part arrives, the manufacturer replaces the part by paying a time-dependent per unit cost $c_2 : [0, T] \to \mathbb{R}_+$. Such replacement cost is given by

$$\mathbb{E} \int_{\tau \wedge \sigma_x}^\tau e^{-\delta u} c_2(u) dN_u$$

where $\tau \wedge \sigma_x = \min\{\tau, \sigma_x\}$ and $\sigma_x = \inf\{u > 0 : N_u \geq x\}$ denotes the arrival of x^{th} defective spare part. For each inventory on hand at switching time τ, if any, the manufacturer scraps the inventory with per unit cost $c_4 \in \mathbb{R}_+$, hence the scrapping cost is given by

$$c_4 e^{-\delta\tau} \mathbb{E}[(x - N_\tau)^+]$$

and we assume that $c_1 \int_\tau^T e^{-\delta u}\, du \geq c_4 e^{-\delta\tau}$ since otherwise holding one spare part in $[\tau, T]$ yields a lower cost than scrapping at τ. Finally, in $[\tau, T]$, defective parts are replaced with a time-dependent per unit cost $c_3 : [0, T] \to \mathbb{R}_+$ and the cost is given by

$$\mathbb{E}\int_{\tau}^{T} e^{-\delta u}c_3(u)dN_u$$

In this chapter, we assume that $c_2(u) = \tilde{c}_2(u) + c_3(u)$ for $\tilde{c}_2 : [0,T] \to \mathbb{R}_+$. We interpret $\tilde{c}_2$ as a penalty to be paid while using the outside source with cost c_3. Moreover, we also assume that $\tilde{c}_2$ and c_3 are piece-wise continuous non-increasing functions, as the manufacturer becomes more prepared to use outside source over time. Finally, λ, $\tilde{c}_2$ and c_3 are differentiable except at finitely many points, at which they can have downward discontinuities as well.

Combining the terms above, we write the total expected operation cost as

$$\begin{aligned} C(x,\tau) :=& c_1\mathbb{E}\left[\int_0^{\tau} e^{-\delta u}(x-N_u)^+ du\right] + \mathbb{E}\left[\int_{\tau\wedge\sigma_x}^{\tau} e^{-\delta u}c_2(u)dN_u\right] \\ &+ \mathbb{E}\left[\int_{\tau}^{T} e^{-\delta u}c_3(u)dN_u\right] + c_4 e^{-\delta\tau}\mathbb{E}[(x-N_\tau)^+]. \end{aligned} \tag{3.1.1}$$

The problem of finding optimal order quantity m and switching time τ is

$$V^{S/1/Z}(x_0) = V(P) := \inf_{\substack{m\in\mathbb{Z}_+ \\ 0\le\tau\le T}} \Big\{c(m) + C(m+x_0,\tau)\Big\}, \quad x_0 \in \mathbb{Z}_+. \tag{3.1.2}$$

It is possible to see that the formulation in equation (3.1.2) is same with the formulation in mentioned in Subsection 2.3.3. We use the properties of $C(x,\tau)$ in the next sections.

3.2 Optimality of *(s,S)* Policy

This section shows that for a fixed switching time τ, the optimal policy characterizing the order decision at time zero is (s,S) policy. That is, there exist s and S in $\mathbb{Z}_+$ such that if initial inventory x_0 is below a threshold s, then $S - x_0$ items are ordered to elevate inventory up; otherwise, if x_0 is as large as s, then an order is not placed. The main problem in relation (3.1.2) can be re-written as

$$V(P) = \inf_{\substack{m\in\mathbb{Z}_+ \\ 0\le\tau\le T}} \left\{ c(m) + C(m + x_0, \tau) \right\} = \inf_{\tau\le T} V^\tau(P),$$

where $V^\tau(P)$ denotes the problem of finding optimal order quantity m for a fixed $\tau \in [0, T]$, defined by

$$\begin{aligned} V^\tau(P) &:= \inf_{m\in\mathbb{Z}_+} \{c(m) + C(m + x_0, \tau)\} \qquad (3.2.1) \\ &= \min\{C(x_0, \tau), \inf_{m\in\mathbb{N}} \{K + \bar{c}m + C(m + x_0, \tau)\}\} \\ &= \min\{C(x_0, \tau), V^\tau(\tilde{P})\}, \end{aligned}$$

and $V^\tau(\tilde{P})$ is defined as

$$\begin{aligned} V^\tau(\tilde{P}) &:= \inf_{m\in\mathbb{N}} \{K + \bar{c}m + C(m + x_0, \tau)\} \\ &= K + \inf_{m\in\mathbb{N}} \{\bar{c}m + C(m + x_0, \tau)\} \qquad (3.2.2) \end{aligned}$$

The problem $V^\tau(\tilde{P})$ is solved by Frenk et al. [20] when initial inventory $x_0 = 0$ and $K = 0$. They prove and utilize the discrete convexity of $m \to \bar{c}m + C(m + x_0, \tau)$. Optimal order quantity $m^*(\tau)$ for the problem $V^\tau(\tilde{P})$ is

$$m^*(\tau) := \min\{m \in \mathbb{N} : \bar{c} + \Delta_x C(m + x_0, \tau) \ge 0\} \qquad (3.2.3)$$

where Δ_x is the first order difference operator defined by

$$\Delta_x C(m + x_0, \tau) := C(m + 1 + x_0, \tau) - C(m + x_0, \tau)$$

By using total enumeration over all possible initial inventory x_0 values, optimal order quantity $m^*(\tau)$ for the problem $V^\tau(P)$ might be found by comparing the values of $C(x_0, \tau)$ and $V^\tau(\tilde{P})$. On the other hand, the optimality of (s, S) policy reduces this computational burden by characterizing the ordering policy for any initial inventory level.

By proceeding as in [45], we first choose an order-up-to level S^* and a re-order level s^* in Lemma 3.2.1 and Lemma 3.2.3, respectively. Next, we argue in Proposition 3.2.4 that (s^*, S^*) policy yields an optimal policy for the ordering decision at time zero. Lemma 3.2.1 chooses S^* based on the observation that if an order has to be placed, then there exists a best order-up-to level for all initial inventory levels.

Lemma 3.2.1. *Let S^* be a global minimizer of the discrete convex function $x \to \bar{c}x + C(x, \tau)$. Assume that the initial inventory level x_0 is less than S^*. Then, ordering up to S^* yields the lowest unit procurement and operation cost for all initial inventory levels, i.e., for each $x \in \mathbb{Z}_+$ and for each $x_0 \in \{0, 1, \dots, S^*\}$,*

$$\bar{c}(S^* - x_0) + C(S^*, \tau) \leq \bar{c}x + C(x + x_0, \tau). \tag{3.2.4}$$

Proof. Let $x \in \mathbb{Z}_+, x_0 \in \{0, 1, \dots, S^*\}$ be given. Define $S_2 := x + x_0$. Then,

$$\begin{aligned}
&\bar{c}(S^* - x_0) + C(S^*, \tau) \leq \bar{c}x + C(x + x_0, \tau) \\
\iff &\bar{c}(S^* - x_0) + C(S^*, \tau) \leq \bar{c}(S_2 - x_0) + C(S_2, \tau) \\
\iff &\bar{c}S^* - \bar{c}x_0 + C(S^*, \tau) \leq \bar{c}S_2 - \bar{c}x_0 + C(S_2, \tau) \\
\iff &\bar{c}S^* + C(S^*, \tau) \leq \bar{c}S_2 + C(S_2, \tau).
\end{aligned}$$

Finally, since S^* is defined as a global minimizer, it satisfies (3.2.4). $\square$

Remark 3.2.2. The assumption that x_0 is less than S^* is used to logically define S^* as order-up-to level. We will consider the case $x_0 > S^*$ while proving that (s^*, S^*) policy is optimal.

The next lemma is used to choose the re-order level s^*. It is chosen such that an order is placed if only a substantial amount $(S^* - s^*)$ is needed so that paying for setup cost is justified.

Lemma 3.2.3. *Let S^* be given as in Lemma 3.2.1. Let $s^* := \min\{x \in \mathbb{Z}_+ : C(x, \tau) < K + \bar{c}(S^* - x) + C(S^*, \tau)\}$. Then,*

$$C(x_0, \tau) < K + \bar{c}(S^* - x_0) + C(S^*, \tau), \qquad x_0 \in \{s^*, \dots, S^*\}. \tag{3.2.5}$$

Proof. We first re-write the (3.2.5) as

$$\bar{c}x_0 + C(x_0, \tau) < K + \bar{c}S^* + C(S^*, \tau), \qquad x_0 \in \{s^*, \ldots, S^*\}. \tag{3.2.6}$$

Note that $x_0 = s^*$ satisfies (3.2.5) by the definition of s^*. Therefore it satisfies (3.2.6) as well. Then, it suffices to show that

$$\bar{c}x_0 + C(x_0, \tau) \leq \bar{c}s^* + C(s^*, \tau), \qquad x_0 \in \{s^* + 1, \ldots, S^*\}.$$

Let $x_0 \in \{s^* + 1, \ldots, S^*\}$ be given. We may choose $\lambda \in [0, 1]$ such that $x_0 = \lambda s^* + (1 - \lambda)S^*$. Then,

$$\begin{aligned}
&\bar{c}x_0 + C(x_0, \tau) \\
=&\bar{c}(\lambda s^* + (1 - \lambda)S^*) + C(\lambda s^* + (1 - \lambda)S^*, \tau) \\
\leq&\lambda(\bar{c}s^* + C(s^*, \tau)) + (1 - \lambda)(\bar{c}S^* + C(S^*, \tau)) \\
\leq&\bar{c}s^* + C(s^*, \tau)
\end{aligned}$$

where the first inequality is due to the discrete convexity of $x \to \bar{c}x + C(x, \tau)$ and the second inequality is because

$$\bar{c}S^* + C(S^*, \tau) \leq \bar{c}s^* + C(s^*, \tau)$$

from S^* being a minimizer of $x \to \bar{c}x + C(x, \tau)$. $\square$

The next proposition shows the optimality of (s^*, S^*) policy, where s^* and S^* are defined by Lemma 3.2.3 and Lemma 3.2.1, respectively. To that end, let $\mathcal{A}$ denote the set of admissible policies. By definition of our problem, each $\delta \in \mathcal{A}$ should be a function $\delta : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\delta(x_0) = \mu(x_0) + x_0$ where $\mu : \mathbb{Z}_+ \to \mathbb{Z}_+$ denotes the order amount for a given initial inventory x_0. Moreover, let $v_1(\delta, x_0)$ denote the value function evaluated at a given $x_0 \in \mathbb{Z}_+$ and $\delta \in \mathcal{A}$, that is,

$$v_1(\delta, x_0) = c(\delta(x_0) - x_0) + C(\delta(x_0), x_0).$$

Proposition 3.2.4. *Choose s^* and S^* values by using Lemma 3.2.3 and Lemma*

3.2.1, respectively. Then, the policy $\delta^{(s^*,S^*)} : \mathbb{Z}_+ \to \mathbb{Z}_+$ *defined by*

$$\delta^{(s^*,S^*)}(x_0) = \begin{cases} S^*, & \textit{if } x_0 < s^* \\ x_0, & \textit{otherwise} \end{cases} \tag{3.2.7}$$

is optimal for $V^\tau(P)$.

Proof. Let arbitrary $x_0 \in \mathbb{Z}_+$ and $\delta \in \mathcal{A}$ be given.

Case 1: $x_0 < s^*$

Informally, ordering up to S^* yields a lower objective function value than not ordering by Lemma 3.2.3. Given that an order is placed, the best order up to level is S^* by Lemma 3.2.1. Formally, since $x_0 < s^*$, it follows from lemma 3.2.3 that

$$C(x_0, \tau) \geq K + \bar{c}(S^* - x_0) + C(S^*, \tau).$$

Moreover, using Lemma 3.2.1, we obtain

$$K + \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0), \tau) \geq K + \bar{c}(S^* - x_0) + C(S^*, \tau).$$

Therefore,

$$\begin{aligned}
&v_1(\delta^{(s^*,S^*)}, x_0) \\
=&K + \bar{c}(S^* - x_0) + C(S^*, \tau) \\
\leq&C(x_0, \tau)1_{\{\delta(x_0)=x_0\}}(x_0) + \Big(K + \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0), \tau)\Big)1_{\{\delta(x_0)>x_0\}}(x_0) \\
=&v_1(\delta, x_0)
\end{aligned}$$

Case 2: $s^* \leq x_0 \leq S$

Informally, ordering any amount will yield a higher objective function value by Lemma 3.2.1 and Lemma 3.2.3. Formally, whenever $\delta(x_0) > x_0 \geq s^*$, we have

$$\begin{aligned}
C(x_0, \tau) &< K + \bar{c}(S^* - x_0) + C(S^*, \tau) \quad \text{(Lemma 3.2.3)} \\
&\leq K + \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0), \tau). \quad \text{(Lemma 3.2.1)}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& v_1(\delta^{(s^*,S^*)}, x_0) \\
=& C(x_0, \tau) \\
\leq& C(x_0,\tau)1_{\{\delta(x_0)=x_0\}}(x_0) + (K + \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0),\tau))1_{\{\delta(x_0)>x_0\}}(x_0) \\
=& v_1(\delta, x_0).
\end{aligned}
$$

<u>Case 3:</u> $x_0 > S^*$

The function $x \to \bar{c}x + C(x,\tau)$ is discrete convex and S^* is a minimizer. Therefore, for all $x_0 > S^*$, $\Delta_x(\bar{c}x + C(x,\tau))|_{x=x_0} \geq 0$. Thus, for $\delta(x_0) > x_0 > S^*$, we have

$$
\begin{aligned}
& \bar{c}\delta(x_0) + C(\delta(x_0),\tau) \geq \bar{c}x_0 + C(x_0,\tau) \\
\Longleftrightarrow\ & \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0),\tau) \geq C(x_0,\tau) \\
\Longrightarrow\ & K + \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0),\tau) \geq C(x_0,\tau). \qquad \text{(since } K > 0\text{)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& v_1(\delta^{(s^*,S^*)}, x_0) \\
=& C(x_0, \tau) \\
\leq& C(x_0,\tau)1_{\{\delta(x_0)=x_0\}}(x_0) + \big(K + \bar{c}(\delta(x_0) - x_0) + C(\delta(x_0),\tau)\big)1_{\{\delta(x_0)>x_0\}}(x_0) \\
=& v_1(\delta, x_0),
\end{aligned}
$$

which concludes the proof. □

3.3 S is Increasing in Switching Time

This section shows that S increases as we delay the switching time τ as long as demand rate and cost of the outside source are high. This structural insight also reduces the computations by enabling us not to consider smaller order-up-to levels if we delay the switching time and search for a new order-up-to level. Of course, such result is expected since delaying switching time means that more demand should be satisfied before using outside/alternative source. However, in case the demand rate and cost of outside source are both low, we may not purchase an additional spare part and instead, we may use outside source with penalty. Proposition 3.3.3 shows that if $\Lambda(\tau)$ exceeds S, yet the expected outside source cost rate does not decline enough to decrease the underage cost, then the value of S should increase.

To emphasize the relation between S and τ, we denote by $S(\tau)$ the order-up-to-level for a fixed τ in this section. Define the first order difference operator of C by

$$\Delta_x C(x,\tau) := C(x+1,\tau) - C(x,\tau),$$

and second order difference operator of C by

$$\Delta_x^2 C(x,\tau) := \Delta_x C(x+1,\tau) - \Delta_x C(x,\tau).$$

In the sequel, we use the following forms of $C(x,\tau), \Delta_x C(x,\tau)$ and $\Delta_x^2 C(x,\tau)$ shown by [20] in relations (4)-(6) and (14):

$$\begin{aligned} C(x,\tau) =& c_4 x + \mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[-c_4 - c_2(u)]\mathbb{P}\left\{N_u \le x-1\right\}du\right) \\ &+ \int_0^\tau e^{-\delta u}\lambda(u)\tilde{c}_2(u)du \\ &+ (c_1 - \delta c_4)\int_0^\tau e^{-\delta u}\mathbb{E}[(x-N_u)^+]du \\ &+ \int_0^T e^{-\delta u}c_3(u)\lambda(u)du, \end{aligned} \tag{3.3.1}$$

$$C(0,\tau) = \int_0^\tau e^{-\delta u}\lambda(u)\tilde{c}_2(u)du + \int_0^T e^{-\delta u}c_3(u)\lambda(u)du, \tag{3.3.2}$$

$$\begin{aligned} \Delta_x C(x,\tau) =& c_4 + \int_0^\tau e^{-\delta u}\lambda(u)[-c_4 - c_2(u)]\mathbb{P}\left\{N_u = x\right\}du \\ &+ (c_1 - \delta c_4)\int_0^\tau e^{-\delta u}\mathbb{P}\left\{N_u \le x\right\}du, \end{aligned} \tag{3.3.3}$$

$$\begin{aligned}\Delta_x^2 C(x-1,\tau) =& e^{-\delta\tau}\big(c_2(\tau)+c_4\big)\mathbb{P}\left\{N_\tau = x\right\} \\ &+\int_0^\tau e^{-\delta u}\big[c_1 - c_2'(u) + \delta c_2(u)\big]\mathbb{P}\left\{N_u = x\right\} du \\ &-\sum_{i\le m,\, l_i\le\tau} e^{-\delta l_i}\Delta c_2(l_i)\mathbb{P}\left\{N_{l_i} = x\right\}. \end{aligned} \tag{3.3.4}$$

Lemma 3.3.1, Lemma 3.3.2 and Proposition 3.3.3 essentially show that $\Delta_x C(x,\tau)$ is a decreasing function of τ under some conditions on x. Therefore, if τ increases, the so does $S(\tau)$ being the first x value satisfying the first order condition

$$S(\tau) = \min\left\{x \in \mathbb{Z}_+ : \bar{c} + \Delta_x C(x,\tau) \ge 0\right\}. \tag{3.3.5}$$

Lemma 3.3.1. *For every $\epsilon \in [0,T]$ and every $\tau \in [0,T]$ such that*

$$c_1 \le \lambda(\tau+\epsilon)c_3(\tau+\epsilon),$$

we have $\Delta_x C(0,\tau+\epsilon) < \Delta_x C(0,\tau)$.

Proof. Using the expression for $\Delta_x C(x,\tau)$ in (3.3.3), we obtain

$$\begin{aligned}&\Delta_x C(0,\tau+\epsilon) - \Delta_x C(0,\tau) \\ =& \int_\tau^{\tau+\epsilon} e^{-\delta u}\lambda(u)\left[-c_4 - c_2(u)\right]\mathbb{P}\left\{N_u = 0\right\} du \\ &+ (c_1 - \delta c_4)\int_\tau^{\tau+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = 0\right\} du \qquad \text{(By (3.3.3))} \\ =& \int_\tau^{\tau+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = 0\right\}\Big(c_1 - \delta c_4 + \lambda(u)[-c_4 - c_3(u) - \tilde{c}_2(u)]\Big) du \\ =& \int_\tau^{\tau+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = 0\right\}\Big(-\delta c_4 + \lambda(u)[\underbrace{-c_4 - \tilde{c}_2(u)}_{<0}]\Big) du \\ +& \int_\tau^{\tau+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = 0\right\}\Big(\underbrace{c_1 - \lambda(u)c_3(u)}_{\le 0}\Big) du \\ <&0\end{aligned}$$

where the inequality $-c_4 - \tilde{c}_2(u) < 0$ holds for every $u \in [\tau, \tau + \epsilon]$ since c_4 and $\tilde{c}_2(u)$ are positive. Moreover, the inequality $c_1 - \lambda(u)c_3(u) \leq 0$ holds for every $u \in [\tau, \tau + \epsilon]$ since

$$\begin{aligned} c_1 \leq & \lambda(\tau+\epsilon)c_3(\tau+\epsilon) \quad \text{(Condition of the lemma)} \\ \leq & \lambda(u)c_3(u), \quad (\lambda \text{ and } c_3 \text{ are non-increasing}) \end{aligned}$$

$\square$

The condition $c_1 \leq \lambda(\tau)c_3(\tau)$ is translated as holding cost rate being less than the cost rate of using outside source, since $c_1 \leq \lambda(\tau)c_3(\tau)$ holds if and only if

$$\lim_{\epsilon \downarrow 0} \int_\tau^{\tau+\epsilon} c_1 \, du \leq \lim_{\epsilon \downarrow 0} \int_\tau^{\tau+\epsilon} c_3(u)\lambda(u) \, du = \lim_{\epsilon \downarrow 0} \mathbb{E} \int_\tau^{\tau+\epsilon} c_3(u) \, dN_u$$

The next lemma is helpful while stating in Proposition 3.3.3 that if (i) expected total demand exceeds the order amount and (ii) c_3 does not decline sufficiently, then the order amount should increase.

Lemma 3.3.2. *For every $x \in \mathbb{Z}_+$, every $\epsilon \in [0,T]$ and every $\tau_2 \in [0,T]$ such that*

$$(i)\, x < \Lambda(\tau_2), \qquad (ii)\, c_1 \leq \left[\frac{\Lambda(u) - x}{\Lambda(u)}\right] \lambda(u)c_3(u) \textit{ for all } u \in [\tau_2, \tau_2 + \epsilon],$$

we have

$$\Delta_x^2 C(x-1, \tau_2 + \epsilon) < \Delta_x^2 C(x-1, \tau_2). \tag{3.3.6}$$

Moreover, letting $\epsilon \downarrow 0$ yields

$$\left.\frac{\partial \Delta_x^2 C(x-1, \tau)}{\partial \tau}\right|_{\tau=\tau_2+} < 0. \tag{3.3.7}$$

Proof. For a non-homogeneous Poisson process N with a right-continuous intensity function λ, the directional derivative of the function $\psi(u) = \mathbb{P}\{N_u = x\}$

exists and it is given by

$$\begin{aligned}
\psi'(u+) :=& \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left[\psi(u+\epsilon) - \psi(u) \right] \\
=& - \lambda(u) e^{-\Lambda(u)} \frac{\Lambda(u)^x}{x!} + e^{-\Lambda(u)} \frac{\Lambda(u)^{x-1}}{(x-1)!} \lambda(u) \\
=& - \lambda(u) \mathbb{P}\left\{ N_u = x \right\} \left[1 - \frac{x}{\Lambda(u)} \right].
\end{aligned}$$

Moreover, we observe that the function Λ is strictly increasing and

$$\psi'(u+) < 0 \text{ for every } u \in [0,T] \text{ such that } \Lambda(u) > x. \tag{3.3.8}$$

After applying chain rule to the function

$$\tau \to \underbrace{e^{-\delta \tau}}_{1} \underbrace{(c_2(\tau) + c_4)}_{2} \underbrace{\mathbb{P}\left\{ N_\tau = x \right\}}_{3}$$

in the expression for $\Delta_x^2 C(x-1,\tau)$ in (3.3.4), we obtain

$$\begin{aligned}
& \Delta_x^2 C(x-1, \tau_2 + \epsilon) - \Delta_x^2 C(x-1, \tau_2) \\
=& - \int_{\tau_2}^{\tau_2+\epsilon} \underbrace{\delta e^{-\delta u}}_{1} \big(c_2(u) + c_4 \big) \mathbb{P}\left\{ N_u = x \right\} du \\
& + \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u} \underbrace{c_2'(u)}_{2} \mathbb{P}\left\{ N_u = x \right\} du + \sum_{i \le m,\, l_i \le \tau} e^{-\delta l_i} \Delta c_2(l_i) \mathbb{P}\{N_{l_i} = x\} \\
& - \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u} \big(c_2(u) + c_4 \big) \underbrace{\lambda(u) \mathbb{P}\left\{ N_u = x \right\} \left[1 - \frac{x}{\Lambda(u)} \right]}_{3} du \\
& + \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u} \big[c_1 - c_2'(u) + \delta c_2(u) \big] \mathbb{P}\left\{ N_u = x \right\} du \\
& - \sum_{i \le m,\, \tau_2 \le l_i \le \tau_2 + \epsilon} e^{-\delta l_i} \Delta c_2(l_i) \mathbb{P}\{N_{l_i} = x\} \qquad \text{(Chain Rule)}
\end{aligned}$$

$$
\begin{aligned}
=\int_{\tau_2}^{\tau_2+\epsilon} & e^{-\delta u}\mathbb{P}\left\{N_u = x\right\} \\
& \times \Bigg[-\delta\big(\underbrace{c_2(u)} + c_4\big) + \underbrace{c_2'(u)} \\
& \quad - \lambda(u)\left[1 - \frac{x}{\Lambda(\tau)}\right](c_2(u)+c_4) \\
& \quad + c_1 - \underbrace{c_2'(u)} + \underbrace{\delta c_2(u)}\Bigg)\Bigg] du \quad \text{(underbraced terms cancel each other)}
\end{aligned}
$$

$$
= \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}\left[(c_1 - \delta c_4) - \lambda(u)\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right](c_2(u)+c_4)\right] du
$$

$$
\begin{aligned}
= & \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}(-\delta c_4) du \\
& + \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}\left[-\underbrace{\lambda(u)}_{\geq \lambda(\tau_2+\epsilon)}\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right]c_4\right] du \quad (\lambda \text{ is non-increasing}) \\
& + \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}\left[-\underbrace{\lambda(u)}_{\geq \lambda(\tau_2+\epsilon)}\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right](\underbrace{\tilde{c}_2(u)}_{\geq \tilde{c}_2(T)})\right] du \quad (\tilde{c}_2 \text{ is non-increasing}) \\
& + \int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\underbrace{\mathbb{P}\left\{N_u = x\right\}}_{=\psi(u)\geq\psi(\tau_2+\epsilon)}\left[c_1 - \lambda(u)\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right]c_3(u)\right] du
\end{aligned}
$$

(Relation (3.3.8) and condition (i))

$$
\begin{aligned}
\leq & -\int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}(\delta c_4) du \\
& - \lambda(\tau_2+\epsilon)c_4\int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right] du \\
& - \lambda(\tau_2+\epsilon)(\tilde{c}_2(T))\int_{\tau_2}^{\tau_2+\epsilon} e^{-\delta u}\mathbb{P}\left\{N_u = x\right\}\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right] du \\
& + e^{-\delta(\tau_2+\epsilon)}\mathbb{P}\left\{N_{\tau_2+\epsilon} = x\right\}\int_{\tau_2}^{\tau_2+\epsilon}\left[c_1 - \lambda(u)\left[\frac{\Lambda(u)-x}{\Lambda(u)}\right]c_3(u)\right] du \qquad (3.3.9) \\
< & 0.
\end{aligned}
$$

In the last expression (3.3.9), the first and second terms are negative due to the assumption $c_4 \in \mathbb{R}_+$ and condition (i) of the lemma. Third term is negative since $\tilde{c}_2(T) \geq 0$ and condition (i) of the lemma. Last term is negative due to condition (ii) of the lemma. This shows the first claim in (3.3.6). Dividing by ϵ and letting $\epsilon \downarrow 0$ gives the second claim in (3.3.7). □

Next, Proposition 3.3.3 shows the main result of this section that S is a non-decreasing function of switching time τ. It essentially shows that if we decide to switch later, if expected total demand exceeds order-up-to level and if the cost of outside source remains high, then order-up-to level cannot decrease.

Proposition 3.3.3. *For every τ_1, τ_2 and ϵ in $[0, T]$ such that*

$(i)\ \tau_1 < \tau_2$ *(Switching time is delayed)*

$(ii)\ S(\tau_1) \geq S(\tau_2)$ *(Order-up-to level has not increased yet, however*

$(iii)\ S(\tau_1) \leq \Lambda(\tau_2)$ *expected total demand exceeds order-up-to level)*

$(iv)\ c_1 \leq \left[\dfrac{\Lambda(u) - S(\tau_1)}{\Lambda(u)}\right] \lambda(u) c_3(u),$ *for all $u \in [\tau_2, \tau_2 + \epsilon]$*

(The cost of outside source is still high in the infinitesimal future)

we have $S(\tau_2) \leq S(\tau_2 + \epsilon)$.

Proof. The function $x \to C(x, \tau)$ being discrete convex implies that the function $x \to \Delta_x C(x, \tau)$ is non-decreasing. Moreover, the first order condition in equation (3.3.5) has to be satisfied by $S(\tau_2)$ and τ_2 as well as $S(\tau_2 + \epsilon)$ and $\tau_2 + \epsilon$, meaning that

$$\bar{c} + \Delta_x C(S(\tau_2), \tau_2) \geq 0 \text{ and } \bar{c} + \Delta_x C(S(\tau_2 + \epsilon), \tau_2 + \epsilon) \geq 0.$$

If we can show that

$$\Delta_x C(S(\tau_2), \tau_2 + \epsilon) < \Delta_x C(S(\tau_2), \tau_2) \tag{3.3.10}$$

then it is possible to see that $S(\tau_2) \leq S(\tau_2 + \epsilon)$ must hold. To show the relation

(3.3.10), we proceed with three steps. First, condition (iv) implies that

$$c_1 \underbrace{\leq}_{(iv)} \left[\frac{\Lambda(\tau_2+\epsilon) - S(\tau_1)}{\Lambda(\tau_2+\epsilon)}\lambda(\tau_2+\epsilon)c_3(\tau_2+\epsilon)\right] \leq \lambda(\tau_2+\epsilon)c_3(\tau_2+\epsilon).$$

By using Lemma 3.3.1, we obtain

$$\Delta_x C(0, \tau_2+\epsilon) < \Delta_x C(0, \tau_2).$$

Next, observe from conditions (iv) and (ii) that for every $u \in [\tau_2, \tau_2+\epsilon]$,

$$c_1 \underbrace{\leq}_{(iv)} \left[\frac{\Lambda(u) - S(\tau_1)}{\Lambda(u)}\right] \lambda(u)c_3(u) \underbrace{\leq}_{(ii)} \left[\frac{\Lambda(u) - S(\tau_2)}{\Lambda(u)}\right] \lambda(u)c_3(u).$$

By using Lemma 3.3.2, we obtain

$$\Delta_x^2 C(S(\tau_2)-1, \tau_2+\epsilon) < \Delta_x^2 C(S(\tau_2)-1, \tau_2)$$

and similarly, for all $x = 1, \ldots, S(\tau_2)-1$, we have

$$\Delta_x^2 C(x-1, \tau_2+\epsilon) < \Delta_x^2 C(x-1, \tau_2).$$

Finally, we obtain

$$\begin{aligned}
\Delta_x C(S(\tau_2), \tau_2+\epsilon) =& \Delta_x C(0, \tau_2+\epsilon) + \sum_{x=0}^{S(\tau_2)-1} \Delta_x^2 C(x, \tau_2+\epsilon) \\
<& \Delta_x C(0, \tau_2) + \sum_{x=0}^{S(\tau_2)-1} \Delta_x^2 C(x, \tau_2) \\
=& \Delta_x C(S(\tau_2), \tau_2).
\end{aligned}$$

□

Remark 3.3.4. If condition (iv) of Proposition 3.3.3 holds, then condition (iii) holds as well since $c_1 \in \mathbb{R}_+$. Still, we write condition (iii) explicitly to understand the necessary structure behind this insight. Moreover, if condition (ii) does not hold, it means that S increases as we delay the switching time, which is the main message of this structural insight.

This insight enables us to reduce computations, as we can use previous order-up-to levels whenever we delay the switching time τ. Moreover, this result can be expected, since delaying the switching time means that inventory will be disposed of at a later time. Therefore, more inventory might be needed to ensure customer warranties until the switching time.

3.4 Conditional Upper and Lower Bounds on the Best Switching Time

In this section, we fix the inventory level x and find conditional upper and lower bounds on the best switching time τ^*. Such bounds are particularly useful when we know that the initial inventory is sufficient and an order is not needed, so that we can reduce the computations regarding the selection of best switching time τ^*. To ensure the existence of derivative of integral terms, we assume in this section that λ, $\tilde{c}_2$ and c_3 are continuous functions. We characterize an upper bound τ^{ub} and a lower bound τ^{lb} by stating, respectively, that $\frac{\partial C(x,\tau)}{\partial \tau} \geq 0$ for every $\tau \geq \tau^{\mathrm{ub}}$ and $\frac{\partial C(x,\tau)}{\partial \tau} \leq 0$ for every $\tau \leq \tau^{\mathrm{lb}}$.

To that end, we first show in the following Lemma 3.4.1 that an upper bound on τ^* exists since the intensity rate λ is non-increasing. We tighten this upper bound in Corollary 3.4.3.

Lemma 3.4.1. *Assume that $\lambda(t) \downarrow 0$ as $t \to \infty$. Then, there exists an upper bound on the best switching time τ^*.*

Proof. By using equation (5.1.2), we write

$$\begin{aligned}
\frac{\partial C(x,\tau)}{\partial \tau} =& e^{-\delta\tau}\lambda(\tau)[-c_4 - c_2(\tau)]\mathbb{P}(N_\tau \le x-1) \\
&+ e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau) \\
&+ e^{-\delta\tau}(c_1 - \delta c_4)\mathbb{E}[(x - N_\tau)^+] \\
\\
=& e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau)\mathbb{P}\{N_\tau \ge x\} \\
&- e^{-\delta\tau}\lambda(\tau)[c_3(\tau) + c_4]\mathbb{P}(N_\tau \le x-1) \\
&e^{-\delta\tau}(c_1 - \delta c_4)\sum_{k=0}^{x-1}\mathbb{P}(N_\tau \le k) \\
\\
=& e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau)\mathbb{P}\{N_\tau \ge x\} \\
&+ e^{-\delta\tau}\mathbb{P}(N_\tau \le x-1)\Big[(c_1 - \delta c_4) - \lambda(\tau)[c_3(\tau) + c_4]\Big] \\
&+ e^{-\delta\tau}(c_1 - \delta c_4)\sum_{k=0}^{x-2}\mathbb{P}(N_\tau \le k)
\end{aligned}$$

Therefore,

$$c_1 - \delta c_4 \ge \lambda(\tau)[c_3(\tau) + c_4] \implies \frac{\partial C(x,\tau)}{\partial \tau} \ge 0, \quad \forall x \in \mathbb{Z}_+$$

Since c_3 is a non-increasing function and $\lambda(t) \downarrow 0$ as $t \to \infty$, there exists $\tau^{\mathrm{ub}} \in \mathbb{R}_+$ such that

$$c_1 - \delta c_4 \ge \lambda(\tau^{ub})[c_3(\tau^{ub}) + c_4] \quad \text{(recall that } c_1 \ge \delta c_4)$$

and for all $t \ge \tau^{ub}$,

$$c_1 - \delta c_4 \ge \lambda(t)[c_3(t) + c_4]$$

Therefore, such τ^{ub} is an upper bound on τ^* □

Lemma 3.4.2 shows a condition which makes the first derivative of $C(x,\tau)$ positive. Intuitively, for fixed x, increasing τ raises the objective function value if penalty cost outweighs the cost of switching at time τ. Moreover, the costs are weighted by under- and over-stock probabilities respectively.

Lemma 3.4.2. *For any $x \in \mathbb{Z}_+$ and $\tau \in [0, T]$, if*

$$\mathbb{P}(N_\tau \geq x)\tilde{c}_2(\tau) \geq \mathbb{P}(N_\tau \leq x-1)[c_3(\tau) + c_4]$$

then $\partial C(x,\tau)/\partial\tau \geq 0$

Proof. By using equation (5.1.2), we see that

$$\begin{aligned}
\frac{\partial C(x,\tau)}{\partial \tau} =& e^{-\delta\tau}\lambda(\tau)[-c_4 - c_2(\tau)]\mathbb{P}\{N_\tau \leq x-1\} \\
&+ e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau) \\
&+ e^{-\delta\tau}(c_1 - \delta c_4)\mathbb{E}[(x - N_\tau)^+] \\
=& e^{-\delta\tau}\lambda(\tau)[-c_4 - c_3(\tau) - \tilde{c}_2(\tau)]\mathbb{P}\{N_\tau \leq x-1\} \\
&+ e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau) \\
&+ e^{-\delta\tau}(c_1 - \delta c_4)\sum_{k=0}^{x-1}\mathbb{P}\{N_\tau \leq k\} \\
=& e^{-\delta\tau}\lambda(\tau)[-c_4 - c_3(\tau)]\mathbb{P}\{N_\tau \leq x-1\} \\
&+ e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau)\mathbb{P}\{N_\tau \geq x\} \\
&+ e^{-\delta\tau}(c_1 - \delta c_4)\sum_{k=0}^{x-1}\mathbb{P}\{N_\tau \leq k\} \\
=& e^{-\delta\tau}\lambda(\tau)\Big(\mathbb{P}\{N_\tau \geq x\}\tilde{c}_2(\tau) - \mathbb{P}\{N_\tau \leq x-1\}[c_3(\tau) + c_4]\Big) \\
&+ e^{-\delta\tau}(c_1 - \delta c_4)\sum_{k=0}^{x-1}\mathbb{P}\{N_\tau \leq k\}
\end{aligned}$$

where second equality uses Lemma 2.2.2. ☐

The following corollary characterizes an upper bound.

Corollary 3.4.3. *For a fixed $x \in \mathbb{Z}_+$, let τ^{ub} be the smallest τ value satisfying*

$$\tilde{c}_2(T) \geq \mathbb{P}\{N_\tau \leq x-1\}[c_2(\tau)+c_4] \tag{3.4.1}$$

If inequality does not hold for any $\tau \in [0,T]$, let $\tau^{ub} = T$. Then, τ^{ub} is an upper bound on the best switching time τ^*.

Proof. We first note that

$$\begin{aligned} &\mathbb{P}\{N_\tau \geq x\}\tilde{c}_2(\tau) \geq \mathbb{P}\{N_\tau \leq x-1\}[c_3(\tau)+c_4] \\ \Longleftrightarrow\ &\tilde{c}_2(\tau) \geq \mathbb{P}\{N_\tau \leq x-1\}[c_3(\tau)+\tilde{c}_2(\tau)+c_4] \\ \Longleftarrow\ &\tilde{c}_2(T) \geq \mathbb{P}\{N_\tau \leq x-1\}[c_3(\tau)+\tilde{c}_2(\tau)+c_4] \end{aligned}$$

where last implication is due to $\tilde{c}_2(.)$ being non-increasing. Since τ^{ub} satisfies inequality (3.4.1), it follows from Lemma 3.4.2 that

$$\left.\frac{\partial C(x,\tau)}{\partial \tau}\right|_{\tau=\tau^{\mathrm{ub}}} \geq 0$$

Therefore, it suffices to show that for any $\tau \geq \tau^{ub}$, $\partial C(x,\tau)/\partial\tau \geq 0$. To see this, we show that the function

$$f(\tau) := \mathbb{P}\{N_\tau \leq x-1\}[c_2(\tau)+c_4]$$

is non-increasing. Taking derivative of $f(\tau)$ yields that

$$\begin{aligned} \frac{\partial f(\tau)}{\partial \tau} &= -\lambda(\tau)\mathbb{P}\{N_\tau = x-1\}[c_2(\tau)+c_4] \\ &\quad + \mathbb{P}\{N_\tau \leq x-1\}c_2'(\tau) \end{aligned}$$

The first term is negative since it is assumed that $c_2(\tau)+c_4 \geq 0$ and that $\lambda(\tau) \geq 0$. The second term is negative since both $c_3(\tau)$ and $\tilde{c}_2(\tau)$ is non-increasing. Therefore, $f(\tau)$ is non-increasing. □

The following Lemma 3.4.4 and Corollary 3.4.5 show a lower bound on τ^*.

Intuitively, we do not stop holding inventory if there is sufficient demand and the cost of switching is higher than the cost of holding and penalty.

Lemma 3.4.4. *If* $\lambda(\tau) \geq 1$ *and*

$$\mathbb{P}\left\{N_\tau \leq x-1\right\}\left[c_2(\tau)+c_4\right] \geq x(c_1-\delta c_4)+\tilde{c}_2(0),$$

then $\partial C(x,\tau)/\partial\tau \leq 0$.

Proof. Using same steps in Lemma 3.4.2 yields

$$\begin{aligned}\frac{\partial C(x,\tau)}{\partial \tau} =& e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau) \\ &+ e^{-\delta\tau}(c_1-\delta c_4)\sum_{k=0}^{x-1}\mathbb{P}\left\{N_\tau \leq k\right\} \\ &- e^{-\delta\tau}\lambda(\tau)[c_2(\tau)+c_4]\mathbb{P}\left\{N_\tau \leq x-1\right\}\end{aligned}$$

Moreover, it is possible to see that

$$\sum_{k=0}^{x-1}\mathbb{P}\left\{N_\tau \leq k\right\} \leq x \quad \text{and} \quad \tilde{c}_2(\tau) \leq \tilde{c}_2(0) \tag{3.4.2}$$

since the function $\tilde{c}_2(.)$ is non-increasing. Therefore,

$$
\begin{aligned}
&\frac{\partial C(x,\tau)}{\partial \tau} \leq 0 \\
\iff & e^{-\delta\tau}\lambda(\tau)\tilde{c}_2(\tau) + e^{-\delta\tau}(c_1 - \delta c_4)\sum_{k=0}^{x-1}\mathbb{P}\left\{N_\tau \leq k\right\} \\
&\quad \leq e^{-\delta\tau}\lambda(\tau)[c_2(\tau) + c_4]\mathbb{P}\left\{N_\tau \leq x-1\right\} \\
\Longleftarrow & \lambda(\tau)\tilde{c}_2(0) + (c_1 - \delta c_4)x \\
&\quad \leq \lambda(\tau)[c_2(\tau) + c_4]\mathbb{P}\left\{N_\tau \leq x-1\right\} \quad \text{(Eqn. (3.4.2))} \\
\Longleftarrow & \lambda(\tau)\tilde{c}_2(0) + \lambda(\tau)(c_1 - \delta c_4)x \\
&\quad \leq \lambda(\tau)[c_2(\tau) + c_4]\mathbb{P}\left\{N_\tau \leq x-1\right\} \quad \text{(Since } \lambda \geq 1) \\
\iff & \tilde{c}_2(0) + (c_1 - \delta c_4)x \\
&\quad \leq [c_2(\tau) + c_4]\mathbb{P}\left\{N_\tau \leq x-1\right\}
\end{aligned}
$$

□

The following Corollary 3.4.5 shows a lower bound on the best switching time τ^*. The first condition of the corollary uses the assumption that λ is non-increasing.

Corollary 3.4.5. *For a fixed x, let τ^{lb} be the largest τ value satisfying $\lambda(\tau) \geq 1$ and*

$$
\mathbb{P}\left\{N_\tau \leq x-1\right\}[c_2(\tau) + c_4] \geq x(c_1 - \delta c_4) + \tilde{c}_2(0) \tag{3.4.3}
$$

If the inequality does not hold for any $\tau \in [0,T]$, let $\tau^{lb} = 0$. Then, τ^{lb} is a lower bound on the best switching time τ^*.

Proof. It suffices to show that any τ value less than τ^{lb} satisfy the above inequality as well, thereby making $\partial C(x,\tau)/\partial\tau$ negative. To achieve this, we first note that λ is non-increasing, therefore, any $\tau \leq \tau^{ub}$ satisfies $\lambda(\tau) \geq 1$. Next, we show that the function

$$
g(\tau) := \mathbb{P}\left\{N_\tau \leq x-1\right\}[c_2(\tau) + c_4]
$$

is a non-increasing function. Taking derivative of $g(\tau)$ yields that

$$\begin{aligned}\frac{\partial g(\tau)}{\partial \tau} &= -\lambda(\tau)\mathbb{P}\left\{N_\tau \leq x-1\right\}\left[c_2(\tau)+c_4\right] \\ &\quad + \mathbb{P}\left\{N_\tau \leq x-1\right\} c_2'(\tau)\end{aligned}$$

The first term is negative due to the assumptions that $c_2(\tau) + c_4 \geq 0$ and that $\lambda(\tau) \geq 0$. The second term is negative since both $c_3(\tau)$ and $\tilde{c}_2(\tau)$ is non-increasing. □

3.5 Summary of the Results

This section summarizes the analytical results presented in this Chapter 3. Recall that Chapter 3 focuses on the problem $S/1/Z$ presented in (3.1.2). This problem is solved by [25] when setup cost $K = 0$ and initial inventory $x_0 = 0$. By proceeding as in [45], we provide a rigorous proof in Section 3.2 that (s, S) policy is optimal for a fixed switching time τ. That is, $S - x_0$ amount is ordered if $x_0 < s$, and an order is not placed if $x_0 \geq s$. This result reduces the computations by enabling us to characterize the order amount for any initial inventory level, instead of calculating this order amount from scratch whenever the initial inventory changes. Therefore, it suffices to solve the problem

$$\inf_{0\leq\tau\leq T}\left\{c(\delta^{s(\tau),S(\tau)}(x_0)) + C(\delta^{s(\tau),S(\tau)}(x_0),\tau)\right\}, \quad x_0 \in \mathbb{Z}_+$$

where $\delta^{s(\tau),S(\tau)}$ denotes $(s(\tau), S(\tau))$ policy for a fixed $\tau \in [0, T]$. Also the optimality of (s, S) policy can be extended to solve the benchmark model $S/\infty/F$ (multiple orders and switching time). For each switching time τ, one can use [45] to show that time-dependent (s, S) policy characterizes the best order policy. In such a case, s and S values will depend on the switching time as well as the period that we consider placing an order. Also see [44, Chapter 4] for a proof of the optimality of (s, S)-type policies in a general setting involving the lost sales case.

Section 3.3 shows that for any switching time $\tau \in [0, T]$, the order-up-to-level $S(\tau)$ is increasing in τ as long as demand rate and the cost of outside/alternative source are high. This result can be used to reduce computations by enabling us to use previous order-up-to levels whenever we delay the switching time τ. Moreover, it shows an expected result that if the time of disposing inventory increases, so does the order amount that will be used to satisfy the demand until this time. Finally, in case there is a sufficient initial inventory level and the manufacturer does not place an order at time zero, Section 3.4 finds upper and lower bounds on the best switching time. Those bounds reduce the computations since it suffices to solve the problem

$$\inf_{\tau \in \left[\tau^{lb}, \tau^{ub}\right]} \Big\{ C(x_0, \tau) \Big\}, \quad x_0 \in \mathbb{Z}_+$$

where τ^{lb} and τ^{ub} denote the lower and upper bounds respectively.

Chapter 4

Numerical Analyses

This chapter provides numerical results regarding the output of dynamic programming algorithms and the analytical results presented earlier. For numerical calculations, we further assume that $c_3(u) = \bar{c}_3 e^{-\gamma u}$ where $\bar{c}_3 \in \mathbb{R}_+$ is constant and $\gamma \in \mathbb{R}_+$ is decline rate. Moreover, $c_2(u) = \bar{c}_2 + c_3(u)$ for $\bar{c}_2 \in \mathbb{R}_+$. Subsection 4.1 shows the results while using the parameters and demand structure presented by [20]. In Subsection 4.2, we provide new demand structures and sensitivity analysis to understand the value generated by the differences in benchmark models as well as the system parameters.

4.1 Verification of Code and Comparison of Models with the Literature

Subsection 4.1 verifies our code and compares our main model as well as the selected benchmarks models by using the numerical values of a case study presented by [20]. We use the parameters to motivate our approach by showing the improvement achieved by implementing various extensions. Table 4.1 shows the numerical values. In this subsection, we assume that the intensity function $\lambda(u)$ is a piece-wise constant function with values $\lambda, \beta\lambda, \beta^2\lambda$ on intervals

$\left[0, \frac{T}{3}\right], \left[\frac{T}{3}, \frac{2T}{3}\right], \left[\frac{2T}{3}, T\right]$ respectively, as in [20]. We extend the numerical analyses with new intensity functions in Subsection 4.2.

Name	Values
Unit Procurement Cost	$\bar{c} = 225$
Holding Cost	$c_1 = 2.25$
Penalty Cost	$\bar{c}_2 = 280$
$c_3(0)$ at time zero	$\bar{c}_3 = 645$
Discount of c_3	$\gamma = 0.03$
Scrapping Cost	$c_4 = 35$
Service Cost	$c_{se} = 20$
Planning Horizon	$T = 66$
Time Discount	$\delta = 0.0035$
Decline rate of intensity function	$\beta = 0.5$
Total Expected Demand in $[0, T]$	$\int_0^T \lambda(u)\, du = 10T$

Table 4.1: Numerical values of the case study.

Table 4.2 verify our code by comparing $V^{S/1/Z}(0) + A$ and $V^{D/1/Z}(0) + A$ with the numerical results presented in [20]. The percent differences in optimal total costs are 2.55×10^{-5} and 2.05×10^{-4} for $S/1/Z$ and $D/1/Z$ respectively, since the time of stopping holding inventory does not have to be in $\{0, 1, \ldots, T\}$ in [20].

Comparison of $S/1/Z$ in Subsection 2.3.3 with Table 1 Column $P2$ Row $q = 0$ in [20]			
	Best Switching Time (t^*)	Order Amount at Time 0	Expected Cost
[20]	23.69	390	156704
$S/1/Z$	24	393	156708

Comparison of $D/1/Z$ in Subsection 2.3.2 with Table 1 Column $P3$ Row $q = 0$ in [20]			
	Order Amount at Time 0	Expected Cost	
[20]	395	153212	
$D/1/Z$	395	153243	

Table 4.2: Verification of our code for dynamic programming algorithms. The percent differences in expected costs are 2.55×10^{-5} and 2.05×10^{-4} for $S/1/Z$ and $D/1/Z$ respectively, since the time of stopping holding inventory has to be in $\{0, 1, \ldots, T\}$ in our model.

After calculating the value function $\widetilde{V}(t, x)$ in (2.2.3) for $t \in \{0, 1, \ldots, 66\}$ and $x \in \{0, 1, \ldots, 1000\}$, we depict the ordering and stopping regions in Figure 4.1. For each time t and inventory x, a point (t, x) is in the stopping region (black color) if the best action is to stop. Likewise, (t, x) is in the ordering region (gray color) if the best action is to place an order. (t, x) is in the continuation region (white color) if the best action is neither to stop nor to order. Our first observation is that if the initial inventory is higher than 25 (3.8% of the expected total demand), the manufacturer does not place an order at time zero. Moreover, the stopping region has two sides. We stop in the region that $t \in (60, 66)$ and $x \in (225, 250)$ to scrap excessive inventory, reducing holding costs. On the other hand, we stop in the region $t \in (25, 66)$ and $x \in (0, 5)$ to prevent stock-outs, reducing penalty costs. Furthermore, the regions are neither convex nor monotone increasing/decreasing, since the problem parameters are non-stationary.

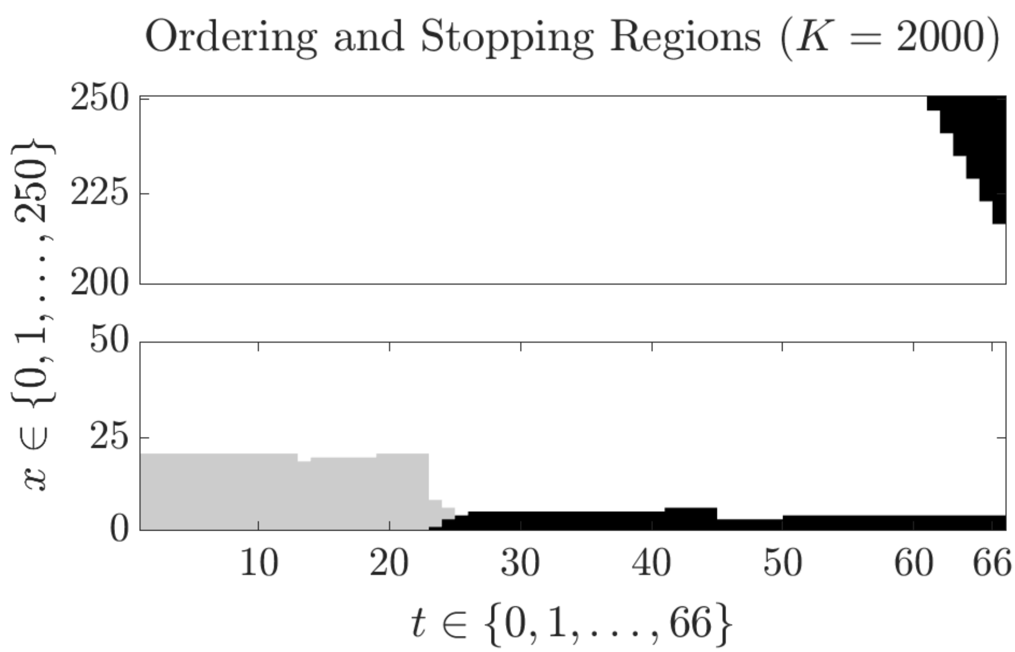

Figure 4.1: Ordering and stopping regions for each time t and inventory level x as output of DP algorithm $\widetilde{V}(t, x)$ in Subsection 2.2.1. Black region corresponds to the points where stopping the process is optimal. An order is placed in the gray region. In the white region, process continues without any action. We break the figure when $x \in \{51, 52 \ldots, 199\}$. The parameters in Table 4.1 are used and setup cost $K = 2000$.

Figure 4.2 shows the percent differences in the optimal total cost of models $D/1/F$ and $D/1/Z$, that is, $100\% \times (V^{D/1/Z}(x) - V^{D/1/F}(0, x, 1))/(V^{D/1/F}(0, x, 1) + A)$. If initial inventory is too low ($x = 0$), the manufacturer places a single order at time zero although having the opportunity to delay, hence the difference is 0%. Medium level initial inventory ($x = 250$) enables the manufacturer to utilize it before placing a delayed order, explaining the 7%-8% improvement of $D/1/F$ over $D/1/Z$. Excessive initial inventory ($x = 400$) may prevent ordering at all, so the difference is 0% again. Table 4.3 shows the % differences for selected initial inventory levels.

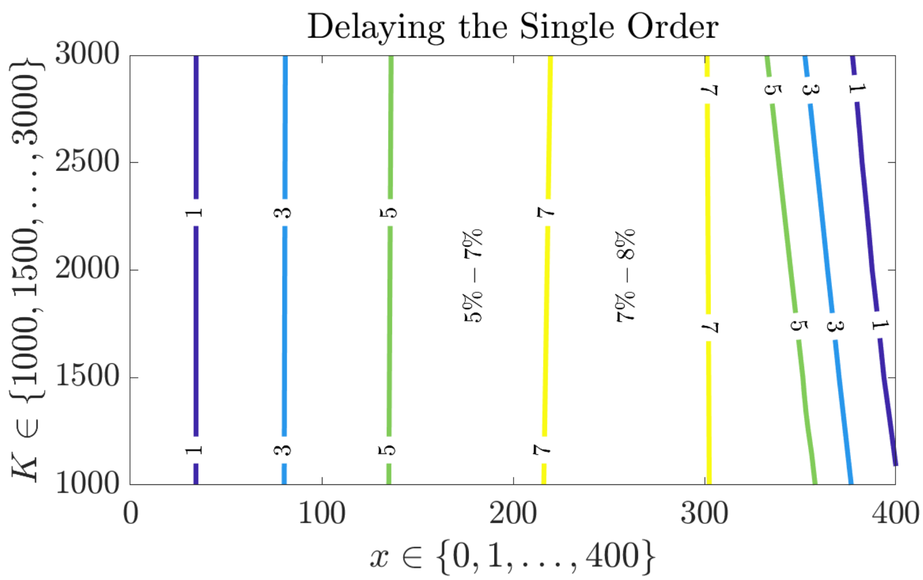

Figure 4.2: Contour lines show the percent increase in the optimal total cost $100\% \times (V^{D/1/Z}(x) - V^{D/1/F}(0, x, 1))/(V^{D/1/F}(0, x, 1) + A)$ while comparing single order at any time $V^{D/1/F}(0, x, 1)$ in Subsection 2.3.1 and single order at time zero $V^{D/1/Z}(x)$ in Subsection 2.3.2. The parameters in Table 4.1 are used.

$K \setminus x$	0	21	44	69	93	121	151	189	323	324	350	385	398
0	0%	1%	2%	3%	4%	5%	6%	7%	7%	6%	5%	3%	2%
1000	0%	1%	2%	3%	3%	5%	6%	6%	7%	6%	6%	2%	1%
2000	0%	1%	2%	3%	3%	5%	5%	6%	7%	6%	4%	1%	0%
3000	0%	1%	1%	3%	3%	4%	5%	6%	6%	6%	3%	1%	0%
5000	0%	1%	1%	2%	3%	4%	5%	6%	4%	3%	1%	0%	0%

Table 4.3: Percent increase in the optimal total cost $100\% \times (V^{D/1/Z}(x) - V^{D/1/F}(0, x, 1))/(V^{D/1/F}(0, x, 1) + A)$ while comparing single order at any time $V^{D/1/F}(0, x, 1)$ in Subsection 2.3.1 and single order at time zero $V^{D/1/Z}(x)$ in Subsection 2.3.2. The parameters in Table 4.1 are used.

Remark 4.1.1. In case the manufacturer is given the opportunity to order at any time, the cut-off initial inventory level which prevents ordering at time zero can be quite low. Therefore, the prevalent assumption that a final order is to be placed at time zero can be a very strong assumption.

Next, Table 4.4 shows the percent differences in the optimal total cost of models $D/1/Z$ and $S/1/Z$, that is, $100\% \times (V^{S/1/Z}(x) - V^{D/1/Z}(x))/(V^{D/1/Z}(x) + A)$. We first observe that the % differences are same for all setup cost K values, since a single order can be placed at time zero in both cases. Moreover, the % difference is highest when initial inventory is high ($x = 596$), yet it is below the expected total demand ($\Lambda(T) = 10T = 660$). In this region of x, it is likely that you utilize the initial inventory and then stop instead of placing an order. Therefore, using a stopping time can be considerably valuable.

$K \setminus x$	0	66	242	339	396	458	596	597	629	648	660	670	679
0	2%	3%	4%	5%	6%	7%	7%	6%	5%	4%	3%	2%	1%
500	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
1000	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
1500	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
2000	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
2500	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
3000	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
3500	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
4000	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
4500	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%
5000	2%	2%	3%	4%	6%	7%	7%	6%	5%	4%	3%	2%	1%

Table 4.4: Percent increase in the optimal total cost $100\% \times (V^{S/1/Z}(x) - V^{D/1/Z}(x))/(V^{D/1/Z}(x) + A)$ while comparing single order at time zero with stopping time $V^{D/1/Z}(x)$ in Subsection 2.3.2 and single order at time zero with deterministic switching time $V^{S/1/Z}(x)$ in Subsection 2.3.3. The parameters in Table 4.1 are used.

Remark 4.1.2. Stopping time notion seems to be critical as for some (t, x) combinations, disposing the available inventory is a correct decision. Moreover, the dynamic selection of the time to stop (via stopping time) can be valuable in case the manufacturer has such flexibility.

Figure 4.3 shows the order-up-to levels $x_t + \mu_t^*(x_t) \in \mathbb{Z}_+$ at the border of stopping region in Figure 4.1, that is, highest x_t such that ordering is the best action. The order-up-to levels are decreasing over time except at two time points we observe positive jumps. It seems natural to see decreasing order up-to-levels, since expected demand rate is decreasing over time as well as time horizon is finite. The positive jumps are most likely due to the setup cost. At the time point being right before the jump, you may choose to place a small order to utilize the multiple orders opportunity. At the time point being right after the jump, you may want to place a larger order to avoid future setup costs, considering that future demand is expected to be lower. Indeed, we do not observe these positive jumps when setup cost K equals 0.

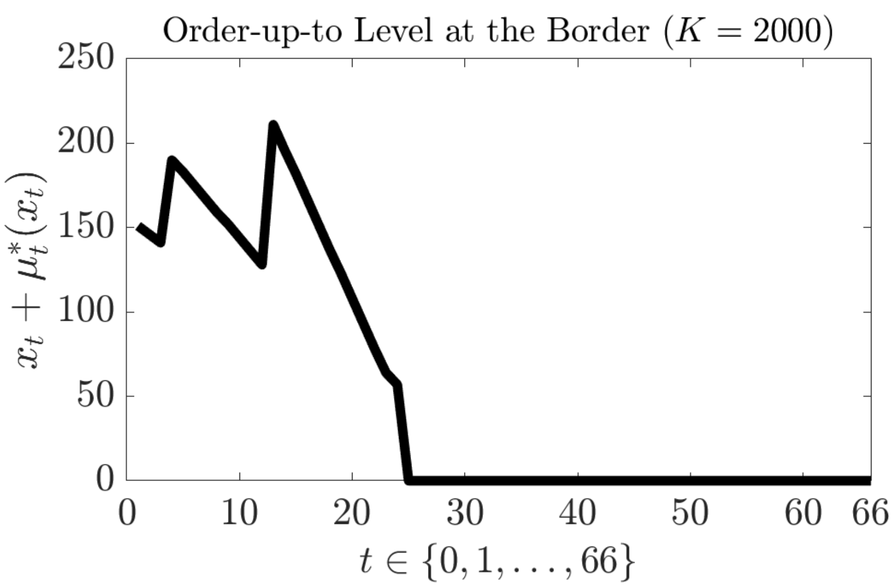

Figure 4.3: Order-up-to level $x_t + \mu_t^*(x_t) \in \mathbb{Z}_+$ at the border of stopping region in Figure 4.1, that is, highest x_t such that ordering is the best action. The parameters in Table 4.1 are used and setup cost K is equal to 2000.

Contour lines in Figure 4.4 show the percent differences in the optimal total cost of models $D/\infty/F$ and $D/1/F$, that is, $100\% \times (V^{D/1/F}(0, x, 1) - \widetilde{V}(0, x))/(\widetilde{V}(0, x) + A)$. The results are quite natural. Higher setup cost K increases the cost of placing multiple orders, thereby reducing the value of $D/\infty/F$ gained over $D/1/F$. Similarly, higher initial inventory x obviates the need to place multiple orders, again reducing the value. However, we see that such gain can be considerably high (4%-5%) under realistic setup cost and initial inventory values. Table 4.5 shows the % differences for selected initial inventory levels.

Remark 4.1.3. Allowing multiple orders is important for systems with reasonable fixed ordering cost.

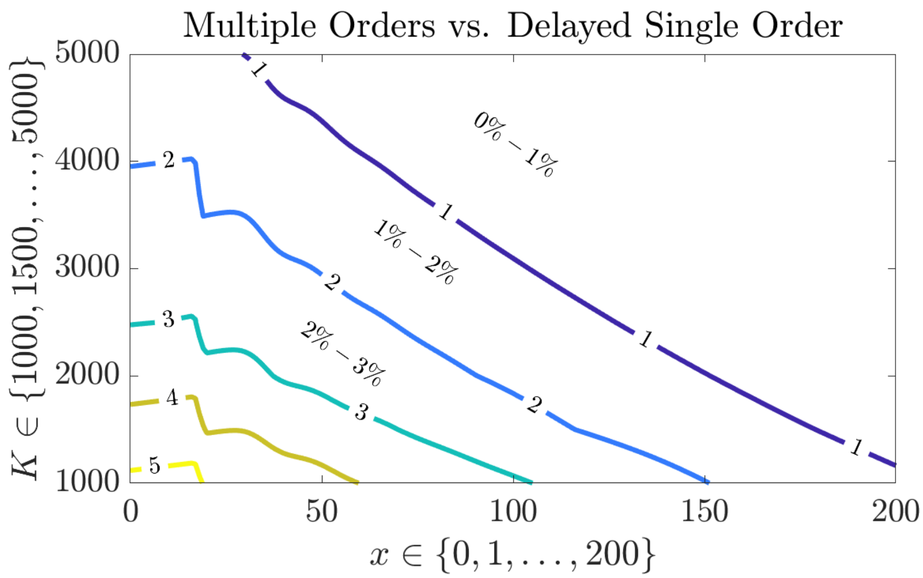

Figure 4.4: Contour lines show the percent increase in the optimal total cost $100\% \times (V^{D/1/F}(0, x, 1) - \widetilde{V}(0, x))/(\widetilde{V}(0, x) + A)$ while comparing multiple orders $\widetilde{V}(0, x)$ in Subsection 2.2.1 and single order at any time $V^{D/1/F}(0, x, 1)$ in Subsection 2.3.1. This difference is calculated for various setup cost K and initial inventory x values. The parameters in Table 4.1 are used.

Table 4.6 completes this Subsection 4.1 by presenting the order amounts at time $t = 0$ with initial inventory $x = 0$. As expected, order amount in $D/\infty/F$ is an increasing function of setup cost K. The order amount in models $D/1/F$ and $D/1/Z$ are same since the manufacturer places an order although having opportunity to delay. Order amount in $D/1/Z$ is slightly more than $S/1/Z$ since the manufacturer can scrap the inventory in case of high holding cost.

Remark 4.1.4. Considering the joint effect of stopping time and order quantity is important.

$K \setminus x$	0	20	40	60	80	100	120	140	160	180	200
0	10%	10%	10%	9%	9%	8%	7%	7%	6%	6%	5%
500	7%	6%	6%	5%	5%	4%	4%	3%	3%	2%	2%
1000	5%	5%	4%	4%	4%	3%	3%	2%	2%	1%	1%
1500	4%	4%	4%	3%	3%	2%	2%	2%	1%	1%	1%
2000	4%	3%	3%	3%	2%	2%	2%	1%	1%	1%	0%
2500	3%	3%	2%	2%	2%	1%	1%	1%	0%	0%	0%
3000	3%	2%	2%	2%	1%	1%	1%	0%	0%	0%	0%
3500	2%	2%	2%	1%	1%	1%	0%	0%	0%	0%	0%
4000	2%	2%	1%	1%	1%	0%	0%	0%	0%	0%	0%
4500	2%	1%	1%	1%	0%	0%	0%	0%	0%	0%	0%
5000	1%	1%	1%	0%	0%	0%	0%	0%	0%	0%	0%

Table 4.5: Percent increase in the optimal total cost $100\% \times (V^{D/1/F}(0,x,1) - \widetilde{V}(0,x))/\widetilde{V}(0,x)$ while comparing multiple orders $\widetilde{V}(0,x)$ in Subsection 2.2.1 and single order at any time $V^{D/1/F}(0,x,1)$ in Subsection 2.3.1. This difference is calculated for various setup cost K and initial inventory x values. The parameters in Table 4.1 are used.

$K \setminus$ Model	$D/\infty/F$	$D/1/F$	$D/1/Z$	$S/1/Z$
0	29	395	395	393
500	90	395	395	393
1000	117	395	395	393
1500	150	395	395	393
2000	151	395	395	393
2500	152	395	395	393
3000	214	395	395	393
3500	215	395	395	393
4000	215	395	395	393
4500	215	395	395	393
5000	216	395	395	393

Table 4.6: Order amount at time $t = 0$ with initial inventory $x = 0$. The parameters presented in Table 4.1 are used.

4.2 D/∞/F - Multiple Orders Opportunity at Any Time and Stopping Time - Sensitivity of Results to the Problem Parameters

In Section 4.2, we use more extensive parameter settings as well as demand structures to provide a thorough analysis. These settings are used to provide sensitivity analyses for the problem $D/\infty/F$ as well as other benchmark models for comparison. The set of parameter values are presented in Table 4.7. To understand the effect of demand rate, we use 4 different rate structures, all having equal expected demand values over the horizon considered: linear decrease of demand rate over time (we name it linear); rate decreasing following a convex function (convex), rate decreasing following a concave function (concave) and finally constant rate to use as a benchmark. The intensity functions are depicted in Figure 4.5. The result of DP algorithms when $x = 0$ can be found in the following link: All Results $x = 0$.

Name	Values
Unit Procurement Cost	$\bar{c} = 100$
Setup Cost	$K \in \{10\bar{c}, 50\bar{c}\}$
Holding Cost	$c_1 = 0.01\bar{c}$
Penalty Cost	$\bar{c}_2 = \{2\bar{c}, 10\bar{c}\}$
$c_3(0)$ at time zero	$\bar{c}_3 = 2\bar{c}$
Discount of c_3	$\gamma \in \{10^{-6}, 0.01\}$
Scrapping Cost	$c_4 \in \{\bar{c}/4, -\bar{c}/4\}$
Planning Horizon	$T \in \{50, 100\}$
Time Discount	$\delta \in \{10^{-6}, 0.005\}$
Demand Structures	Convex, Concave, Linear, Constant See Figure 4.5

Table 4.7: Numerical values used in sensitivity analysis. Total number of parameter settings is 256. We run 4 different models. Total number of runs: 1024.

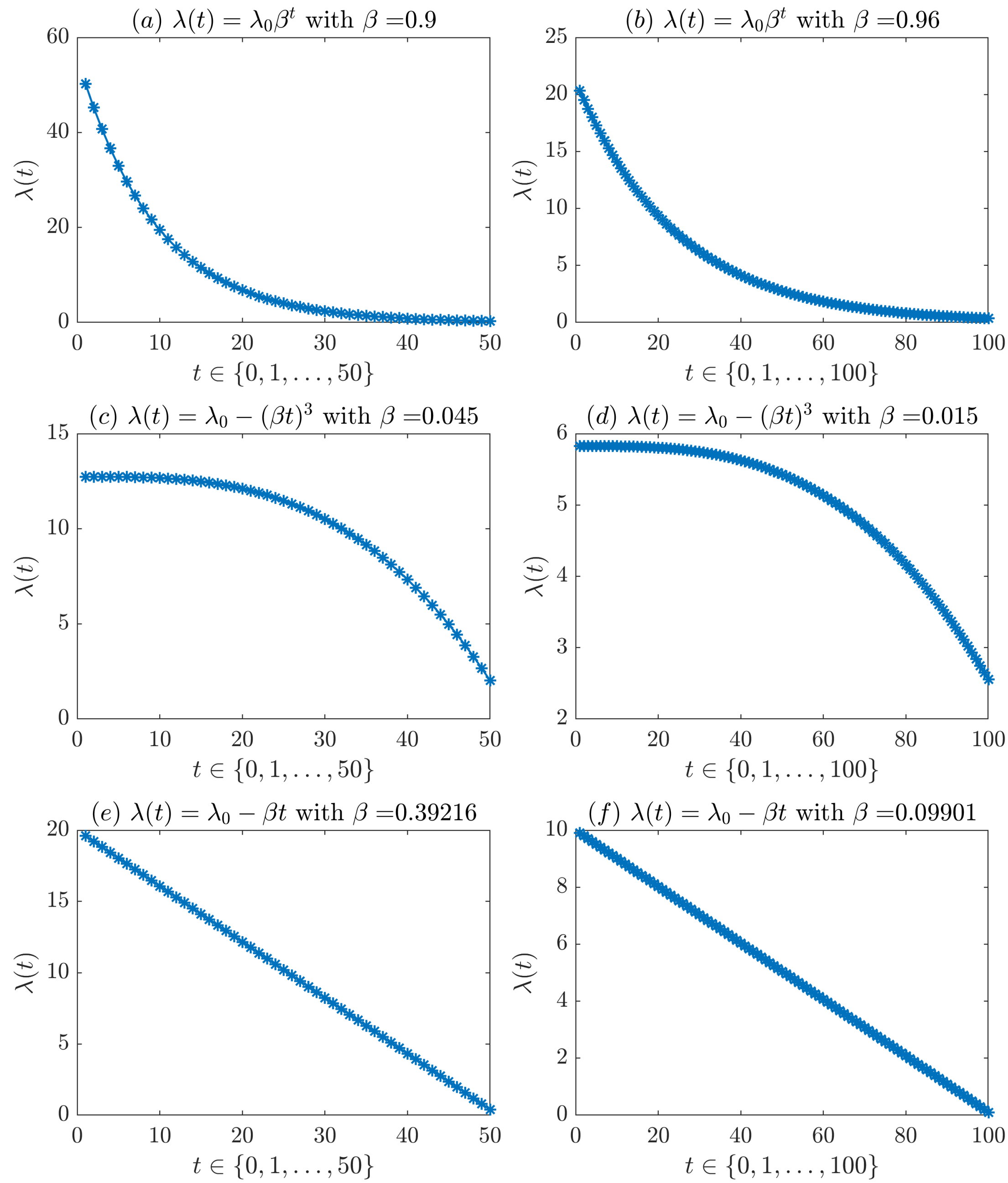

Figure 4.5: Piece-wise constant intensity functions λ used in sensitivity analyses. The value of $\lambda(t)$ changes in every period $t \in \{0, 1, \dots, T\}$ and it is constant during $[t, t+1]$. Initial point λ_0 is selected such that expected total demand $\int_0^T \lambda(t)\, dt$ is equal to 500. Left and right panels shows λ when $T = 50$ and $T = 100$ respectively.

4.2.1 Effect of Demand Structure

Table 4.8 shows the effect of demand structure on $\widetilde{V}(0,x)+A$ defined in Subsection 2.2.1, by showing the % difference when intensity is concave and convex. If both initial inventory and setup cost are low ($x=0, K=0$), the cost under convex demand is higher, since more demand is satisfied earlier, meaning that more procument cost is paid earlier. If initial inventory is low yet setup cost is high ($x=0, K=5000$), the cost under concave demand is higher, since more setup might be needed throughout the horizon under concave demand, as the decline rate of demand is lower. If you start with a very large initial inventory ($x=500$) meaning that you may not need much ordering, then holding cost component dominates and hence you have a much higher cost for the concave case as you deplete the inventory much slower.

Table 4.9 shows the results when $T=100$. The trend is similar to what was said for $T=50$. However, as the horizon is longer and total expected demand is the same for both time horizons, the expected drop in on-hand inventory for the convex case relative to concave is less, and hence % differences for large x values are not as large as for $T=50$ case.

Remark 4.2.1. If you have either high initial inventory level x or high setup cost K, it might be wise to encourage (even give incentives) customers to come earlier - hence make your demand rate look like convex compared to the original one. On the other hand, with small x and K combinations (north west part of the Table 4.8), you may look for strategies making customers come later (e.g. deliberate backlogging [46]).

$K \setminus x$	0	30	60	90	120	150	180	210	240	270	300	500
0	-5%	-5%	-6%	-6%	-5%	-5%	-4%	-3%	-1%	1%	4%	94%
1000	-1%	-2%	-2%	-3%	-3%	-3%	-2%	-2%	-1%	1%	3%	88%
5000	4%	3%	2%	1%	0%	0%	-1%	-1%	0%	1%	2%	87%

Table 4.8: 100% $(V_{Concave} - V_{Convex})$ / V_{Convex}: Comparison of $\widetilde{V}(0,x)+A$ when demand is convex (Figure 4.5 a) and concave (Figure 4.5 c). The relevant parameters are $T=50, c_2=2\bar{c}, \gamma=0.01, c_4=c/4, \delta=0.005$.

$K \setminus x$	0	50	100	150	200	250	300	350	400	450	500
1000	-11%	-13%	-14%	-15%	-14%	-11%	-6%	6%	28%	63%	80%
5000	-11%	-14%	-18%	-21%	-23%	-22%	-16%	0%	27%	63%	80%

Table 4.9: 100% $(V_{Concave} - V_{Convex})$ / V_{Convex}: Comparison of $\widetilde{V}(0,x) + A$ when demand is convex (Figure 4.5 a) and concave (Figure 4.5 c). The relevant parameters are $T = 100, c_2 = 2\bar{c}, \gamma = 0.01, c_4 = c/4, \delta = 0.005$.

4.2.2 Effect of Outside Source/Alternative Policy

The %'s in Table 4.10 show the effect of an outside/alternative source by presenting $\widetilde{V}(0,x) + A$ with a decreasing cost of this source ($\gamma = 0.01$) versus and a nearly constant cost ($\gamma = 10^{-6}$) over time (both discounted with the same rate of time discount). The benefit of decreasing unit cost of the outside source of satisfying demand (call it alternative policy) is observed, as the case where the cost is constant over the horizon yield higher total expected cost for every x and K.

When $x = 0, K = 0$, the manufacturer may not use alternative policy at all since the cost of procurement can be sufficiently low. When $x = 0$ and $K = 5000$ however, the manufacturer would prefer placing a large order at time zero, and then using alternative policy if needed. Hence, as K increases the value of having decreasing unit cost in the alternative policy also increases.

When x is in the region (350, 450) for any K, it is likely that the manufacturer utilizes initial inventory and then switches to alternative policy, instead of placing an order. Hence, given the cost structure of the alternative policy cost one can observe the highest % values in the expected total cost differential in this region of x.

When $x = 550$ or higher, the manufacturer may not use alternative policy at all until the stopping time, since the initial inventory seems sufficiently high to cover the demand. Note that for $x = 550$ it is likely that optimal stopping time realizes closer to T. Therefore, any change in the unit cost of alternative policy over time has practically no impact on the expected total cost. Of course, as x goes higher (which might not be very reasonable for the problem structure), we see that optimal solution may prefer to stop before the end of the horizon (almost at the same time for all K values) and start to use alternative source for the remaining part of the horizon. The fact that %'s are higher simply reflect the unit cost difference in the alternative policy considered in the cases compared.

Remark 4.2.2. Larger percentages in Table 4.10 support the fact that we need to support/encourage/give incentives for the alternative source to become more cost efficient (hence cheaper) over time (like developing technologies to lower the price).

$K \setminus x$	0	100	250	300	350	400	450	481	500	550	600	700
0	1%	1%	1%	2%	2%	2%	3%	2%	1%	0%	2%	5%
500	3%	3%	5%	6%	8%	10%	11%	7%	4%	0%	2%	5%
1000	4%	5%	7%	9%	11%	13%	15%	10%	5%	0%	2%	5%
1500	4%	6%	9%	11%	13%	17%	19%	11%	5%	0%	2%	5%
2000	5%	7%	10%	12%	14%	19%	21%	13%	6%	0%	2%	5%
2500	6%	7%	12%	13%	15%	22%	24%	14%	6%	0%	2%	5%
3000	6%	8%	13%	13%	17%	25%	26%	14%	6%	0%	2%	5%
3500	6%	8%	13%	13%	19%	28%	28%	14%	6%	0%	2%	5%
4000	7%	9%	12%	14%	21%	31%	29%	14%	6%	0%	2%	5%
4500	8%	10%	12%	15%	24%	33%	30%	14%	6%	0%	2%	5%
5000	8%	11%	12%	16%	26%	36%	30%	14%	6%	0%	2%	5%

Table 4.10: $100\%(V_{\gamma=10^{-6}} - V_{\gamma=0.01})/V_{\gamma=0.01}$: Comparison of $\widetilde{V}(0,x) + A$ when $\gamma = 10^{-6}$ and $\gamma = 0.01$ to show the effect of alternative policy. The relevant parameters are $T = 100, c_2 = 2\bar{c}, c_4 = \bar{c}/4, \delta = 0.005$, convex intensity (Figure 4.5 b).

4.2.3 Effect of Time Horizon

Table 4.11 shows the effect of time horizon T by presenting $\widetilde{V}(0,x) + A$ when $T = 50$ and $T = 100$. When $x = 0, K = 0$, expected total cost under $T = 100$ is lower, since the manufacturer places small orders later in time, utilizing discount. On the other hand, when $x = 0, K = 5000$, a sufficiently large order is placed at time zero. Since this purchasing cost occurs at time zero in both cases, the costs are similar.

For a relatively small range for x - (300-330)(For instance when $x = 331$) and large K, the manufacturer does not place an order and uses alternative policy. If $T = 50$, this policy is used earlier at a time when unit cost of alternative policy cost is relatively higher and discount has less effect. This results in a 13% difference in relative total expected costs. On the other hand, for $x = 331, K = 0$, the manufacturer can place small orders instead of using alternative policy or facing penalty. This explains a very small % difference observed. When x is larger than the range given above, for smaller K values we start the observe the negative effects of longer horizon, as longer horizon brings more carrying cost over time and hence greater expected costs for the case with $T = 100$.

Remark 4.2.3. If you have moderate to high K values, and relatively low x value (South middle-east part of the Table 4.11) it might be wiser to extend the horizon to $T = 100$ under the knowledge that in that case demand will be flatter thru 100 periods.

$K \backslash x$	0	100	150	250	331	332	340	400	402	433	435
0	7%	8%	8%	7%	1%	0%	-1%	-12%	-13%	-23%	-24%
500	4%	6%	6%	6%	2%	2%	1%	-8%	-8%	-17%	-18%
1000	3%	5%	5%	6%	3%	3%	2%	-6%	-6%	-13%	-14%
1500	3%	5%	5%	6%	3%	3%	3%	-3%	-3%	-10%	-11%
2000	3%	5%	5%	6%	3%	3%	3%	0%	0%	-7%	-8%
2500	2%	4%	5%	6%	4%	4%	4%	3%	3%	-4%	-5%
3000	2%	4%	5%	6%	5%	5%	5%	6%	6%	-2%	-3%
3500	1%	4%	5%	6%	7%	7%	7%	9%	9%	-1%	-2%
4000	1%	4%	5%	6%	9%	9%	10%	11%	11%	0%	-2%
4500	1%	4%	5%	6%	11%	11%	12%	13%	13%	0%	-1%
5000	1%	4%	5%	6%	13%	13%	14%	15%	14%	0%	-1%

Table 4.11: $100\%(V_{T=50} - V_{T=100})/V_{T=100}$: Comparison of $\widetilde{V}(0, x) + A$ when $T = 50$ and $T = 100$ to show the effect of time horizon. The relevant parameters are $c_2 = 2\bar{c}, c_4 = \bar{c}/4, \gamma = 0.01, \delta = 0.005$, convex intensity (Figure 4.5 a and Figure 4.5 b).

4.2.4 Effect of Penalty Cost

Tables 4.12 and 4.13 show the effect of penalty $\bar{c}_2$ by presenting $\widetilde{V}(0, x)$ when $\bar{c}_2 = 2\bar{c}$ and $\bar{c}_2 = 10\bar{c}$. Note that for x values which are in between 0 and expected total demand (for instance $x = 300$) change in unit penalty cost is expected to have its highest impact, since the firm takes the risk of penalty for not placing an order. Nevertheless, even if we change penalty cost by a factor of 5, increase in the optimal value of the expected total discounted cost is negligible (less than 2% in all cases). The reason is that the manufacturer can stop holding inventory and use alternative policy to avoid penalty cost.

Remark 4.2.4. As we have an existing alternative (which is much cheaper than the larger penalty cost in Table 4.7) practically there is no significant difference observed after changing the penalty cost. Hence, with the existence of such an alternative, the firm might announce to pay very large penalties for not satisfying demand to attract more demand to begin with. – shows the importance of creating such an alternative. Additionally, if the cost of alternative decreases over time (the periods where the risk of paying the penalty is more) it will be even better for decreasing expected costs.

$K\backslash x$	0	40	80	200	280	440	480	520	560	600
0	0.4%	0.5%	0.5%	0.6%	0.7%	0.8%	0.6%	0.1%	0.0%	0%
500	0.5%	0.6%	0.7%	0.8%	1.0%	1.3%	0.8%	0.1%	0.0%	0%
1000	0.5%	0.6%	0.7%	0.9%	1.0%	1.4%	0.8%	0.1%	0.0%	0%
1500	0.5%	0.7%	0.7%	0.9%	1.1%	1.4%	0.8%	0.1%	0.0%	0%
2000	0.5%	0.7%	0.7%	1.0%	1.1%	1.4%	0.8%	0.1%	0.0%	0%
2500	0.5%	0.7%	0.7%	1.0%	1.1%	1.4%	0.8%	0.1%	0.0%	0%
3000	0.5%	0.7%	0.8%	1.0%	1.1%	1.4%	0.8%	0.1%	0.0%	0%
3500	0.6%	0.8%	0.8%	0.9%	1.1%	1.4%	0.8%	0.1%	0.0%	0%
4000	0.6%	0.8%	0.8%	0.9%	1.1%	1.4%	0.8%	0.1%	0.0%	0%
4500	0.6%	0.8%	0.8%	0.9%	1.2%	1.4%	0.8%	0.1%	0.0%	0%
5000	0.6%	0.8%	0.8%	0.9%	1.4%	1.4%	0.8%	0.1%	0.0%	0%

Table 4.12: $100\%(V_{\bar{c}_2=10\bar{c}} - V_{\bar{c}_2=2\bar{c}})/V_{\bar{c}_2=2\bar{c}}$: Comparison of $\widetilde{V}(0,x) + A$ when $\bar{c}_2 = 2\bar{c}$ and $\bar{c}_2 = 10\bar{c}$ to show the effect of penalty. The relevant parameters are $T = 100, c_4 = \bar{c}/4, \gamma = 0.01, \delta = 0.005$, convex intensity (Figure 4.5 b).

$K\backslash x$	0	40	80	200	280	440	480	520	560	600
0	0.4%	0.4%	0.4%	0.5%	0.6%	0.7%	0.5%	0.1%	0.0%	0.0%
500	0.4%	0.5%	0.5%	0.6%	0.7%	0.9%	0.7%	0.1%	0.0%	0.0%
1000	0.4%	0.5%	0.5%	0.6%	0.7%	0.9%	0.7%	0.2%	0.0%	0.0%
1500	0.4%	0.5%	0.5%	0.6%	0.7%	0.9%	0.8%	0.2%	0.0%	0.0%
2000	0.4%	0.5%	0.5%	0.6%	0.7%	0.9%	0.9%	0.2%	0.0%	0.0%
2500	0.3%	0.5%	0.5%	0.6%	0.7%	0.9%	0.9%	0.2%	0.0%	0.0%
3000	0.3%	0.5%	0.5%	0.6%	0.7%	1.0%	1.0%	0.2%	0.0%	0.0%
3500	0.3%	0.5%	0.5%	0.6%	0.7%	1.0%	1.0%	0.2%	0.0%	0.0%
4000	0.3%	0.5%	0.5%	0.6%	0.7%	1.1%	1.0%	0.2%	0.0%	0.0%
4500	0.3%	0.5%	0.5%	0.6%	0.7%	1.2%	1.0%	0.2%	0.0%	0.0%
5000	0.3%	0.5%	0.5%	0.6%	0.7%	1.3%	1.0%	0.2%	0.0%	0.0%

Table 4.13: $100\%(V_{\bar{c}_2=10\bar{c}} - V_{\bar{c}_2=2\bar{c}})/V_{\bar{c}_2=2\bar{c}}$: Comparison of $\widetilde{V}(0,x) + A$ when $\bar{c}_2 = 2\bar{c}$ and $\bar{c}_2 = 10\bar{c}$ to show the effect of penalty. The relevant parameters are $T = 100, c_4 = \bar{c}/4, \gamma = 10^{-6}, \delta = 0.005$, convex intensity (Figure 4.5 b).

4.2.5 Effect of Time Discount

Table 4.14 shows the effect of time discount δ by presenting $\widetilde{V}(0,x) + A$ when $\delta = 0.005$ and $\delta = 10^{-6}$. The expected total cost is always higher when time discount is close to zero, as can be predicted. The effect of discount decreases in K, for smaller x values. However, for intermediate x values (350-450) the effect is reversed or disappears as it is likely an order is needed later in the horizon and hence the value of K becomes critical.

Remark 4.2.5. The time-discount value seems to be effective for different x values rather than K; and hence reiterating the importance of initial inventory.

$K \backslash x$	0	50	100	150	200	300	350	400	450	500	550
0	11%	12%	13%	15%	16%	19%	20%	20%	17%	13%	17%
500	9%	10%	11%	13%	14%	18%	19%	20%	18%	13%	17%
1000	8%	9%	11%	12%	14%	17%	19%	20%	18%	13%	17%
1500	8%	9%	10%	12%	13%	17%	19%	21%	18%	13%	17%
2000	7%	9%	10%	11%	12%	17%	19%	21%	18%	13%	17%
2500	7%	8%	9%	11%	12%	17%	20%	21%	18%	13%	17%
3000	7%	7%	9%	10%	12%	17%	20%	21%	18%	13%	17%
3500	6%	7%	9%	10%	12%	17%	21%	21%	18%	13%	17%
4000	6%	7%	9%	10%	12%	17%	21%	21%	18%	13%	17%
4500	6%	7%	9%	10%	12%	18%	21%	21%	18%	13%	17%
5000	6%	7%	8%	10%	12%	19%	21%	21%	18%	13%	17%

Table 4.14: $100\%(V_{\delta=10^{-6}} - V_{\delta=0.005})/V_{\delta=0.005}$: Comparison of $\widetilde{V}(0,x) + A$ when $\delta = 0.005$ and $\delta = 10^{-6}$ to show the effect of discount. The relevant parameters are $T = 100, c_2 = 2\bar{c}, c_4 = \bar{c}/4, \gamma = 0.01$, convex intensity (Figure 4.5 a and Figure 4.5 b).

4.2.6 Effect of Scrapping Cost

Table 4.15 shows the impact of scrapping cost c_4 by presenting $\widetilde{V}(0,x) + A$ when $c_4 = \bar{c}/4$ and $c_4 = -\bar{c}/4$. If $x < 450$, most likely the inventory is used to satisfy the demand, therefore, we may not need to scrap inventory. In this region of x, we stop holding inventory only if the inventory level is about to hit zero and the risk of penalty arises. In such a case, only a negligible amount of inventory is scrapped (for instance, see Figure 4.1), meaning that the scrapping cost has a negligible effect on the expected total cost. On the other hand, if $x > 450$, the excess inventory may need to be scrapped. Therefore, the scrapping cost can have an impact on the expected total cost.

K / x	0	100	200	300	400	450	500	550	600	700
0	0.0%	0.0%	0.0%	0.0%	0.1%	0.1%	2.8%	15.1%	29.0%	52.5%
500	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
1000	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
1500	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
2000	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
2500	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
3000	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
3500	0.1%	0.1%	0.1%	0.1%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
4000	0.1%	0.1%	0.1%	0.2%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
4500	0.1%	0.1%	0.1%	0.2%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%
5000	0.1%	0.1%	0.1%	0.2%	0.2%	0.2%	2.8%	15.1%	29.0%	52.5%

Table 4.15: $100\%(V_{c_4=\bar{c}/4} - V_{c_4=-\bar{c}/4})/V_{c_4=-\bar{c}/4}$: Comparison of $\widetilde{V}(0,x) + A$ when $c_4 = \bar{c}/4$ and $c_4 = -\bar{c}/4$ to show the effect of scrapping cost. The relevant parameters are $T = 100, c_2 = 2\bar{c}, \gamma = 0.01$, $\delta = 0.005$, convex intensity (Figure 4.5 a and Figure 4.5 b).

4.3 Expected Penalty of a Misspecified Intensity Function

We analyze the impact of an error in selecting the intensity function of non-homogeneous Poisson process. Table 4.16 shows the % difference in expected total cost that is calculated under convex intensity, but the best decision variables are found by using convex and linear intensities. Those intensities are depicted in Figure 4.5(a) and Figure 4.5(e) respectively. If $x = 0, K = 0$ and intensity is linearly decreasing, the manufacturer places small orders whenever needed in almost all periods. On the other hand, if intensity is decreasing in a convex manner, the manufacturer tends to place larger orders at the beginning and smaller orders towards the end. Hence, by presuming a linearly decreasing intensity function and taking actions based on this assumption, the manufacturer can observe excess penalty cost at the beginning and excess holding cost towards the end, resulting in a significant loss (above 30%). We believe that this is a motivation to study the problem with random intensity. Also note that for other x and K combinations the loss can be as high as 130%.

The largest % loss can be observed when there is an initial inventory level which is close to the expected demand throughout the horizon. For instance when $x = 450$, it is likely that you wait for an amount of time and then place an order. If intensity is linearly decreasing, this future order can be large. Hence, if the manufacturer places a large order in the future, most likely excess holding and procurement costs incur as the arrival rate under decreasing-convex case is much lower towards the end of horizon.

When x is large (above expected demand), one may be less willing to stop towards the end under the presumption that intensity rate is decreasing linearly. Hence, this difference in the stopping region create small, but meaningful % difference in expected total costs indicating the importance of the stopping time.

Overall, high % differences in the expected total cost motivates us to approach the problem of selecting the intensity function of a non-homogeneous Poisson process. Chapter 5 tackles this issue by allowing the intensity rate to be a stochastic process.

$K \setminus x$	0	50	150	250	350	450	500	550	600	650	750
0	36%	29%	15%	6%	5%	10%	7%	0%	1%	3%	5%
500	14%	12%	17%	21%	27%	54%	49%	1%	1%	3%	5%
1000	11%	10%	17%	16%	18%	85%	64%	1%	1%	3%	5%
1500	20%	30%	18%	47%	16%	135%	64%	1%	1%	3%	5%
2000	23%	14%	29%	48%	19%	137%	59%	1%	1%	3%	5%
2500	22%	15%	44%	47%	77%	136%	51%	0%	1%	3%	5%
3000	25%	30%	45%	48%	80%	136%	44%	0%	1%	3%	5%
3500	24%	30%	36%	38%	78%	135%	34%	0%	1%	3%	5%
4000	25%	31%	11%	37%	76%	132%	25%	0%	1%	3%	5%
4500	29%	20%	11%	37%	74%	124%	16%	0%	1%	3%	5%
5000	29%	18%	11%	36%	73%	116%	11%	0%	1%	3%	5%

Table 4.16: $100\%(V_{linear-decvar} - V_{convex-decvar}) \;/\; V_{convex-decvar}$: The cost $V_{linear-decvar}$ denotes the expected total cost of $\widetilde{V}(0,x) + A$ under convex intensity, while using the best decision variables of $\widetilde{V}(0,x) + A$ under linear intensity. Moreover, $V_{Convex-decvar}$ denotes the expected total cost of $\widetilde{V}(0,x) + A$ under convex intensity, while using the best decision variables of $\widetilde{V}(0,x) + A$ under convex intensity. The decision variables are stop-continue-order decisions and order amount for each time and inventory level. The relevant parameters are $T = 50, c_2 = 2\bar{c}, \gamma = 0.01, c_4 = c/4, \delta = 0.005$.

Chapter 5

The End-of-Life Inventory Problem under Stochastic Intensity - Analytical Results

5.1 Setting and Problem Definition

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space. For any subset A of Euclidean space with usual topology, let $\mathcal{B}(A)$ denote the Borel sigma algebra on A. Let $\lambda : \Omega \times [0, T] \mapsto \mathbb{R}_+$ be a stochastic process. Assume that λ is right-continuous with left-limits, bounded and positive. Let $M : \Omega \times \mathcal{B}([0, T] \times \mathbb{R}_+) \mapsto \mathbb{R}_+$ be a Poisson random measure with mean measure $\mu = \text{Leb} \times \text{Leb}$. Assume that λ and M are independent. Define the random measure $N : \Omega \times \mathcal{B}([0, T]) \mapsto \mathbb{R}_+$ by

$$N(\omega, B) = \int_{[0,T]\times\mathbb{R}_+} M(\omega; du, dz)\, 1_B(u) 1_{[0,\lambda_u(\omega)]}(z), \quad B \in \mathcal{B}([0, T]) \qquad (5.1.1)$$

and denote $N_t(\omega) = N(\omega, [0, t])$. We show the properties of the random measure N and the process $\{N_t, t \in [0, T]\}$ in the next sections. We first claim that $\{N_t, t \in [0, T]\}$ is a conditional Poisson process, or a doubly stochastic Poisson process, or a Cox process. The subsequent results reveal the benefit of defining

$\{N_t, t \in [0,T]\}$ by using the random measure N and a closed-form expression in Equation (5.1.1). We further define the filtrations $\mathcal{F}_t^\lambda = \sigma(\{\lambda_s, s \le t\})$, $\mathcal{F}_t^M = \sigma(\{M([0,s] \times \mathbb{R}_+), s \le t\})$, $\mathcal{F}_t^N = \sigma(\{N_s, s \le t\})$ and $\mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_t^\lambda$.

Consider a firm which owns $x \in \mathbb{Z}_+$ items in the inventory and the demand for the item is described by the conditional Poisson process $\{N_t : t \in [0,T]\}$ with intensity rate being the stochastic process λ. The selling period for this item is finite and is denoted by $[0,T]$, where $T \in \mathbb{R}_+$ is fixed and known by the firm. At any time t, (1) the firm can stay in the market if it has positive amount of inventory, (2) it can decide to exit the market, or (3) it is forced to exit the market if the inventory level hits zero. Any rule to describe the firm's decision of exiting the market is given by an $(\mathcal{F}_t)_{t\in[0,T]}$-stopping time τ and the set of all such stopping times are denoted by $\mathcal{T}$. Moreover, the time σ_x when the inventory level hits zero is an $(\mathcal{F}_t)_{t\in[0,T]}$-stopping time as well. To make our model more realistic, we further assume that there is a delay $L \in [0,T]$ between when the firm decides to exit the market and when it actually exits the market. Therefore, there are three possible times when the firm ends its operations. It may serve customers until T, it may exit the market at some $\tau + L \in [0,T]$, or it may be forced to exit the market at σ_x. An optimal stopping problem with L-delay and without forcing the exit is introduced by [42] and its extensions are studied by [47].

The running costs depend on the amount of items in the inventory and the arrival times. Those are described by

$$c_1\mathbb{E}\int_0^{\tau+L} du\, e^{-\delta u}(x - N_u)^+ + c_5\mathbb{E}\int_0^{(\tau+L)\wedge\sigma_x} dN_u\, e^{-\delta u}$$

where $c_1, c_5 \in \mathbb{R}_+$. Moreover, the firm is penalized for willingly exiting the market before T and the cost is given by

$$\mathbb{E}\int_{\tau+L}^{T} dN_u\, e^{-\delta u} c_3(u) + c_4\mathbb{E}[e^{-\delta(\tau+L)}(x - N_{\tau+L})^+]$$

where $c_3 : [0,T] \mapsto \mathbb{R}_+$ and $c_4 \in \mathbb{R}_+$. Lastly, if the firm is forced to exit the market, an additional cost is defined in terms of each lost demand until $\tau + L$.

$$\mathbb{E}\int_{(\tau+L)\wedge\sigma_x}^{\tau+L} dN_u\, e^{-\delta u}c_2(u)$$

where $c_2 : [0,T] \mapsto \mathbb{R}_+$. We assume $c_2(u) \geq c_3(u)$ for every $u \in [0,T]$ so that the firm can be forced to exit the market. Therefore, the cost functional $C : \mathbb{Z}_+ \times \mathcal{T} \mapsto \mathbb{R}_+$ is

$$\begin{aligned} C(x,\tau) :=& c_1\mathbb{E}\int_0^{\tau+L} du\, e^{-\delta u}(x-N_u)^+ + c_5\mathbb{E}\int_0^{(\tau+L)\wedge\sigma_x} dN_u\, e^{-\delta u} \\ &+ \mathbb{E}\int_{\tau+L}^{T} dN_u\, e^{-\delta u}c_3(u) + c_4\mathbb{E}[e^{-\delta(\tau+L)}(x-N_{\tau+L})^+] \\ &+ \mathbb{E}\int_{(\tau+L)\wedge\sigma_x}^{\tau+L} dN_u\, e^{-\delta u}c_2(u) \end{aligned} \tag{5.1.2}$$

where we use the notation $(\tau+L)\wedge\sigma_x = \inf\{\tau+L, \sigma_x\}$ and define the stopping time $\sigma_x := \inf\{t > 0 : N_t \geq x\}$. We study the following optimal stopping problem with conditional Poisson processes and L-delay.

$$\nu(x) := \inf_{\tau\in\mathcal{T}}\{C(x,\tau)\}$$

5.2 Construction of Conditional Poisson Process

In this section, we construct the random measure N and the conditional Poisson process $\{N_t : t \in [0,T]\}$ defined by Equation (5.1.1). In Theorem 5.2.1, we show that the conditional Laplace functional of N given λ is that of a Poisson random measure. Therefore, it will follow in Corollary 5.2.2 that the conditional Poisson process $\{N_t : t \in [0,T]\}$ defined by Equation (5.1.1) is a generalization of non-homogeneous Poisson processes in the sense that intensity rate λ is a stochastic process. Moreover, Corollary 5.2.3 shows a direction in which our construction extends the existing studies in the literature.

It is known that since λ is assumed to be right-continuous with left-limits, the path $\lambda(\omega) = (t \mapsto \lambda_t(\omega))$ is a random variable as a mapping from $(\Omega, \mathcal{H})$ to $(G, \mathcal{G})$ where G is the space of real functions on $[0, T]$ that are right-continuous and have left-limits, and $\mathcal{G}$ is the sigma-algebra on G. The measurable space $(G, \mathcal{G})$ is called the Skorokhod space and it is described in detail by [48, p. 121]. In particular, G is a Polish space [48, p. 127], implying that $(G, \mathcal{G})$ is a standard measurable space [49, p. 11]. Therefore, a regular version of the conditional probabilities exists [49, p. 151, II.2.7]. For each measurable function $f : G \times [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, we define the measurable function $u_f \colon \Omega \times G \to \bar{\mathbb{R}}_+$ by

$$u_f(\omega, a) \coloneqq u_f(a)(\omega) \coloneqq \int_{[0,T]\times\mathbb{R}_+} M(\omega; du, dz)\, f(a, u, z), \quad (\omega, a) \in \Omega \times G,$$

and the random variable $U_f \colon \Omega \to \bar{\mathbb{R}}_+$ by

$$U_f(\omega) \coloneqq u_f(\omega, \lambda(\omega)) = \int_{[0,T]\times\mathbb{R}_+} M(\omega; du, dz)\, f(\lambda(\omega), u, z), \quad \omega \in \Omega.$$

When $f = 1_D$ for some $D \in \mathcal{G} \otimes \mathcal{B}([0, T] \times \mathbb{R}_+)$, we write $u_D = u_{1_D}$ and $U_D = U_{1_D}$. In Appendix A.1, Theorem A.1.1, Corollary A.1.2 and Theorem A.1.3 create the relation between $U_f(\omega)$ and $u_f(\omega, a)$ in a more general setting. We use Theorem A.1.3 throughout the document to provide rigorous proofs on the heuristic argument that conditional on λ, the random measure N behaves like a Poisson random measure. The following Theorem 5.2.1 characterizes the probability law of conditional Poisson process by means of Laplace functional.

Theorem 5.2.1. *The conditional Laplace functional of N given λ is that of a Poisson random measure on $[0, T]$ with intensity function λ. Therefore, for every Borel measurable function $\phi : [0, T] \mapsto \mathbb{R}_+$, the Laplace functional of N is*

$$\mathbb{E}[e^{-N\phi}] = \mathbb{E} \exp_{-} \int_{[0,T]} dt\, \lambda_t (1 - e^{-\phi(t)}) \tag{5.2.1}$$

where $\exp_{-} x \coloneqq e^{-x}$, $(N\phi)(\omega) \coloneqq \int_{[0,T]} N(\omega; dt)\, \phi(t)$, *and the integral on the right-hand-side is understood as* $\int_{[0,T]} dt\, \lambda_t \psi(t)$ *with* $\psi(t) = 1 - e^{-\phi(t)}$.

Proof. The proof essentially extends ϕ from indicator functions to simple positive functions to positive measurable functions. At each step, we first see $\mathbb{E}\left[e^{-N\phi} \mid \mathcal{F}_T^\lambda\right]$ as $\mathbb{E}\left[e^{-U_f} \mid \mathcal{F}_T^\lambda\right]$ for some $f(\lambda, u, z)$. Then, we convert $\mathbb{E}\left[e^{-U_f} \mid \mathcal{F}_T^\lambda\right]$ into $\mathbb{E}\left[e^{-u_f(a)}\right]\big|_{a=\lambda}$ by using Theorem A.1.3. Next, we use the Laplace functional of Poisson random measure M with deterministic function $f(a, u, z)$. Finally, we plug in λ and obtain the right-hand-side of 5.2.1.

First, assume that $\phi(t) = 1_B(t)$ for some $B \in \mathcal{B}([0, T])$. Then,

$$(N\phi)(\omega) = N(\omega, B) = \int_{[0,T]\times\mathbb{R}_+} M(\omega; dt, dz)\, 1_B(t) 1_{[0,\lambda_t(\omega)]}(z) = U_D(\omega)$$

where $D = \{(a, t, z) \in G \times [0, T] \times \mathbb{R}_+ \mid a \in A, t \in B, z \in [0, a_t]\}$ with A being the set of positive functions in G. It is possible to see that $D \in \mathcal{G} \otimes \mathcal{B}([0, T] \times \mathbb{R}_+)$ since $A \in \mathcal{G}$. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[e^{-N\phi} \mid \mathcal{F}_T^\lambda\right] \\
=&\mathbb{E}\left[e^{-U_D} \mid \mathcal{F}_T^\lambda\right] \\
=&\mathbb{E}\left[e^{-U_D} \mid \lambda\right] \qquad \text{(Remark A.2.2)} \\
=&\mathbb{E}\left[e^{-u_D(a)}\right]\Bigg|_{a=\lambda} \quad \text{(Theorem A.1.3)} \\
=&\mathbb{E}\left[\exp_-\left(\int_{[0,T]\times\mathbb{R}_+} M(dt, dz)\, 1_B(t) 1_{[0,a_t]}(z)\right)\right]\Bigg|_{a=\lambda} \quad \text{(Definition of } u_D(a)\text{)} \\
=&\left[\exp_-\int_{[0,T]\times\mathbb{R}_+} \mu(dt, dz)\, (1 - e^{-1_B(t)1_{[0,a_t]}(z)})\right]\Bigg|_{a=\lambda} \\
=&\left[\exp_-\int_B dt\, a_t(1 - e^{-1})\right]\Bigg|_{a=\lambda} \quad \text{(Integration wrto } \mu\text{=Leb}\times\text{Leb)} \\
=&\exp_-\left(\int_{[0,T]} dt\, \lambda_t(1 - e^{-\phi(t)})\right) \quad \text{(Since } \phi(t) = 1_B(t)\text{)}
\end{aligned}$$

and fourth equality is due to $(t, z) \mapsto 1_B(t)1_{[0,a_t]}(z)$ being a Borel measurable function, and due to the Laplace functional of M [49, p. 252, VI.2.9]. Next, assume that $\phi(t) = \sum_{i=1}^n b_i 1_{B_i}(t)$ where $b_1, \ldots, b_n$ are in $\mathbb{R}_+$ and $B_1, \ldots, B_n$ are disjoint subsets of $\mathcal{B}([0,T])$. Then,

$$\begin{aligned} N\phi(\omega) &= \sum_{i=1}^{n} b_i N(\omega, B_i) = \int_{[0,T]\times\mathbb{R}_+} M(\omega; dt, dz) \left(\sum_{i=1}^{n} b_i 1_{B_i}(t) 1_{[0,\lambda_t(\omega)]}(z) \right) \\ &= U_f(\omega) \end{aligned}$$

with $f : G \times [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $f(\lambda(\omega), t, z) = \sum_{i=1}^{n} b_i 1_{B_i}(t) 1_{[0,\lambda_t(\omega)]}(z)$. It is possible to see that f is positive and measurable with respect to $\mathcal{G} \otimes \mathcal{B}([0,T] \times \mathbb{R}_+)$ and $\mathcal{B}(\mathbb{R}_+)$. Therefore,

$$\begin{aligned} &\mathbb{E}\left[e^{-N\phi} \mid \mathcal{F}_T^{\lambda}\right] \\ =&\mathbb{E}\left[e^{-U_f} \mid \mathcal{F}_T^{\lambda}\right] \\ =&\mathbb{E}\left[e^{-U_f} \mid \lambda\right] \qquad \text{(Remark A.2.2)} \\ =&\mathbb{E}\left[e^{-u_f(a)}\right]\Big|_{a=\lambda} \qquad \text{(Theorem A.1.3)} \\ =&\mathbb{E}\left[\exp_{-}\int_{[0,T]\times\mathbb{R}_+} M(dt,dz) \left(\sum_{i=1}^{n} b_i 1_{B_i}(t) 1_{[0,a_t]}(z) \right)\right]\Bigg|_{a=\lambda} \quad \text{(Definition of } u_f(a)\text{)} \\ =&\left[\exp_{-}\int_{[0,T]\times\mathbb{R}_+} \mu(dt,dz) \left(1 - \exp_{-}\sum_{i=1}^{n} b_i 1_{B_i}(t) 1_{[0,a_t]}(z) \right)\right]\Bigg|_{a=\lambda} \\ =&\left[\exp_{-}\left(\sum_{i=1}^{n} \int_{B_i} dt\, a_t (1 - e^{-b_i}) \right)\right]\Bigg|_{a=\lambda} \quad \text{(Since } B_1, \ldots, B_n \text{ are disjoint)} \\ =&\exp_{-}\left(\int_{[0,T]} dt\, \lambda_t (1 - e^{-\phi(t)}) \right) \quad \text{(Since } \phi(t) = \textstyle\sum_{i=1}^{n} b_i 1_{B_i}(t)\text{)} \end{aligned}$$

and fourth equality follows from the Laplace functional of M [49, p. 252, VI.2.9]. Lastly, assume that ϕ is a positive measurable function. Choose $(\phi_n)_{n\in\mathbb{N}}$ such that ϕ_n's are simple positive functions and $\phi_n \uparrow \phi$ pointwise. It holds from the previous step that

$$\mathbb{E}\left[e^{-N\phi_n} \mid \mathcal{F}_T^{\lambda}\right] = \exp_{-}\left(\int_{[0,T]} dt\, \lambda_t (1 - e^{-\phi_n(t)}) \right), \quad n \in \mathbb{N}$$

Therefore, it suffices to show that

$$(i)\quad \mathbb{E}\left[e^{-N\phi_n} \,\middle|\, \mathcal{F}_T^\lambda\right] \to \mathbb{E}\left[e^{-N\phi} \,\middle|\, \mathcal{F}_T^\lambda\right] \text{ a.s.}$$

$$(ii)\quad \exp_-\left(\int_{[0,T]} dt\, \lambda_t(1-e^{-\phi_n(t)})\right) \to \exp_-\left(\int_{[0,T]} dt\, \lambda_t(1-e^{-\phi(t)})\right) \text{ a.s.}$$

(i) The assumption $\phi_n(t) \uparrow \phi(t)$ implies $(N\phi_n)(\omega) \uparrow (N\phi)(\omega)$ by the monotone convergence theorem (MON) with random measure N [49, p. 244, VI.1.5]. Therefore, it follows from the (conditional) bounded convergence theorem that $\mathbb{E}\left[e^{-N\phi_n}|\mathcal{F}_T^\lambda\right] \to \mathbb{E}\left[e^{-N\phi}|\mathcal{F}_T^\lambda\right]$.

(ii) If $\phi_n(t) \uparrow \phi(t)$, then $1-e^{-\phi_n(t)} \uparrow 1-e^{-\phi(t)}$ as well. Since $\lambda_t(\omega)$ is positive by assumption, we obtain $\lambda_t(\omega)(1-e^{-\phi_n(t)}) \uparrow \lambda_t(\omega)(1-e^{-\phi(t)})$. Therefore, it follows from MON with Lebesgue measure that

$$\exp_-\int_{[0,T]} dt\, \lambda_t(\omega)(1-e^{-\phi_n(t)}) \to \exp_-\int_{[0,T]} dt\, \lambda_t(\omega)(1-e^{-\phi(t)})$$

which completes the first claim of Theorem 5.2.1. The second claim of Theorem 5.2.1 follows from the first claim since

$$\mathbb{E}\left[e^{-N\phi}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-N\phi} \,\middle|\, \mathcal{F}_T^\lambda\right]\right] = \mathbb{E}\left[\exp_-\int_{[0,T]} dt\, \lambda_t(1-e^{-\phi(t)})\right]$$

□

Several studies define the conditional Poisson processes by assuming the form of conditional Laplace transform [38, p. 211, Sec. 3.14], [39], or conditional probability [40, p. 169, Sec. 6.2.], and argue their equivalence. These constructions are widely used in the literature. On the other hand, we construct the conditional Poisson processes by using a closed-form expression given by Equation (5.1.1), and provide a rigorous proof on their Laplace functionals in the preceding Theorem 5.2.1. The following results in Corollary 5.2.3 and Section 5.3 reveal the benefits of this construction. To the best of our knowledge, the closest such construction is mentioned by [50] and [49, p. 262, Exercise VI.2.36], yet the Laplace functional is given as a comment. The following Corollary 5.2.2 completes our claim on $\{N_t, t \in [0,T]\}$ being a conditional Poisson process.

Corollary 5.2.2. *If λ is a deterministic function as a mapping from $[0,T]$ to $\mathbb{R}_+$, then N is a Poisson random measure with mean measure $\Lambda(B) = \int_B dt\, \lambda_t$ for each $B \in \mathcal{B}([0,T])$. Moreover, $\{N_t : t \in [0,T]\}$ is a non-homogeneous Poisson process with intensity function λ.*

Next Corollary 5.2.3 enables us to calculate the conditional probabilities of conditional Poisson process.

Corollary 5.2.3. *For every $t \leq T$ and $B \in \mathcal{B}([0,t])$,*

$$\mathbb{P}\left\{ N(B) = k \mid \mathcal{F}_T^\lambda \right\} = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}, \quad k \in \mathbb{N} \tag{5.2.2}$$

and

$$\mathbb{P}\left\{ N(B) = k \mid \mathcal{F}_t^\lambda \right\} = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}, \quad k \in \mathbb{N} \tag{5.2.3}$$

where $\Lambda(B) = \int_B du\, \lambda_u$.

Proof. The conditional Laplace functional of N given λ is that of a Poisson random measure by Theorem 5.2.1. Therefore, it is known that

$$\mathbb{P}\left\{ N(B) = k \mid \mathcal{F}_T^\lambda \right\} = \mathbb{P}\left\{ N(B) = k \mid \lambda \right\} = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}, \quad B \in \mathcal{B}([0,t]), k \in \mathbb{N}$$

See [39] and the references therein for the relation between conditional Laplace functionals and conditional probabilities, and [38, p. 211] for conditional Poisson processes. This shows equality (5.2.2). To see equality (5.2.3), we change the time horizon from $[0,T]$ to $[0,t]$ so that $\mathcal{F}_t^\lambda$ contains the necessary information on λ. To that end, let $M_{[0,t]}$ be the restriction of M on $[0,t] \times \mathbb{R}_+$. $M_{[0,t]}$ is a Poisson random measure on $([0,t] \times \mathbb{R}_+, \mathcal{B}([0,t] \times \mathbb{R}_+))$. Construct the random measure $N_{[0,t]}$ with $M_{[0,t]}$ and λ. By Theorem 5.2.1, the conditional Laplace functional of $N_{[0,t]}$ given $\mathcal{F}_t^\lambda$ is that of a Poisson random measure on $[0,t]$. Therefore,

$$\mathbb{P}\left\{ N_{[0,t]}(B) = k \,\middle|\, \mathcal{F}_t^\lambda \right\} = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}, \quad B \in \mathcal{B}([0,t]), k \in \mathbb{N}$$

Moreover, it is possible to see that $N_{[0,t]}$ is the restriction of N on $[0,t]$ since for any $B \in \mathcal{B}([0,t])$,

$$\begin{aligned}
N(B \cap [0,t]) &= \int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{B\cap[0,t]}(u) 1_{[0,\lambda_u]}(z) \\
&= \int_{[0,t]\times\mathbb{R}_+} M(du,dz) 1_B(u) 1_{[0,\lambda_u]}(z) \\
&= \int_{[0,t]\times\mathbb{R}_+} M_{[0,t]}(du,dz) 1_B(u) 1_{[0,\lambda_u]}(z) \\
&= N_{[0,t]}(B)
\end{aligned}$$

Hence,

$$\mathbb{P}\left\{ N(B) = k \mid \mathcal{F}_t^\lambda \right\} = \mathbb{P}\left\{ N_{[0,t]}(B) = k \,\middle|\, \mathcal{F}_t^\lambda \right\} = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}, \quad B \in \mathcal{B}([0,t]), k \in \mathbb{N}$$

□

The difference between Equation (5.2.2) and Equation (5.2.3) shows a direction in which our construction extends the existing studies. We do not require the whole filtration of λ, $\mathcal{F}_T^\lambda$, to calculate the conditional probabilities of N and $\{N_t, t \in [0,T]\}$. On the other hand, an assumption being prevalent among existing studies is that $\mathcal{F}_T^\lambda$ is known at time zero. For instance, see [41, p. 61, Thm. 1.8.2]. The next section reveals more benefits of our closed-form expression in Equation (5.1.1).

5.3 The Optimal Stopping Problem

In this section, after proving the necessary properties of N and $\{N_t, t \in [0,T]\}$, we reduce the optimal stopping problem with L-delay to a classical optimal stopping problem. The main idea is to represent the expected costs incurring in $[\tau, \tau + L]$ as a function of $(L, \tau, \lambda_\tau, N_\tau)$ by proving and using the strong Markov property of the conditional Poisson process $\{N_t, t \in [0,T]\}$. To that end, Proposition 5.3.1 and Corollary 5.3.2 show the Markov property of $\{N_t, t \in [0,T]\}$ and provide two functions to calculate the conditional expectation given $\mathcal{F}_t$. Example 5.3.3 illustrates a calculation. Corollary 5.3.4 shows that $\{N_t, t \in [0,T]\}$ admits the $(\mathcal{F}_t)_{t\in[0,T]}$-intensity λ in the sense that $N_t - \int_{[0,t]} du\, \lambda_u$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. This result also help us deal with the integrals taken with respect to the conditional Poisson processes. In Proposition 5.3.6 and Corollary 5.3.7, we prove the strong Markov property of $\{N_t, t \in [0,t]\}$. Our main Theorem 5.3.11 reduces the optimal stopping problem with L-delay to a classical optimal stopping problem. Subsection 5.3.4 shows the necessary adjustments in our main Theorem 5.3.11 in case the intensity λ is unobservable.

We start by introducing the following definitions and notations. Define the measurable space $(G_{[t,T]}, \mathcal{G}_{[t,T]})$ where $G_{[t,T]}$ is the space of real functions on $[t,T]$ that are right-continuous and have left-limits, and $\mathcal{G}_{[t,T]}$ is the sigma-algebra on $G_{[t,T]}$, similar to the measurable space $(G, \mathcal{G})$. Let $\sigma_y^t := \inf\{u > 0 : N_{u+t} - N_t \geq y\}$ be the arrival time of the y^{th} item when the counting starts at time t. Moreover, we use the notation $\lambda_u^t = \lambda_u - \lambda_t$ and the following assumptions when needed.

INDEP-INC : λ has independent increments, that is, $\lambda_{t+s} - \lambda_t$ is independent of $\mathcal{F}_t^\lambda$ for every s and t in $[0,T]$.
sMARKOV : λ has the strong Markov property, that is, $\lambda_{\tau+s}$ is conditionally independent of $\mathcal{F}_\tau^\lambda$ given λ_τ for every $s \in [0,T]$ and $\tau \in \mathcal{T}$.

5.3.1 Markov Property and Martingale of Conditional Poisson Process

Proposition 5.3.1. *(Markov Property) Assume* ***INDEP-INC****. Then, N_{t+s} is conditionally independent of $\mathcal{F}_t$ given λ_t and N_t for every s and t in $[0,T]$. Moreover, for every bounded continuous function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$,*

$$\mathbb{E}[g \circ (N_{t+s} - N_t)|\mathcal{F}_t] = \hat{g} \circ (s, t, \lambda_t) \tag{5.3.1}$$

where $\hat{g} : [0,T] \times [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is defined by*

$$\hat{g}(s,t,p) = \mathbb{E}\left[\widetilde{g} \circ (s,t,p,\{\lambda_u^t, u \in (t,T]\})\right] \tag{5.3.2}$$

and $\widetilde{g} : [0,T] \times [0,T] \times \mathbb{R}_+ \times G_{[t,T]} \mapsto \mathbb{R}_+$ *is defined by*

$$\widetilde{g}(s,t,p,\{a_u^t, u \in [t,T]\}) = \mathbb{E}\left[g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du,dz)\, 1_{[t,t+s]}(u) 1_{[0,p+a_u^t]}(z)\right)\right] \tag{5.3.3}$$

and $\lambda_u^t = \lambda_u - \lambda_t$.

Proof. (Proof of First Claim) The idea behind the proof is that since M and λ are conditionally independent from the past given the present, same property should hold for N as well. Below we show for every s and t in $[0,T]$ that

$$\left[\sigma(N([t,t+s]))\right] \subset \left[\sigma(M([t,t+s]\times\mathbb{R}_+)) \vee \sigma(\{\lambda_u : u \in [t,t+s]\})\right] \overset{\lambda_t}{\perp\!\!\!\perp} \left[\mathcal{F}_t^M \vee \mathcal{F}_t^\lambda\right] = \mathcal{F}_t$$

To begin with, we can see by definition of N that

$$\begin{aligned}
N_{t+s} =& N([0,t+s]) \\
=& \int_{[0,T]\times\mathbb{R}_+} M(du,dz)\, 1_{[0,t+s]}(u) 1_{[0,\lambda_u]}(z) \\
=& \int_{[0,T]\times\mathbb{R}_+} M(du,dz) \left(1_{[0,t]}(u) + 1_{[t,t+s]}(u)\right) 1_{[0,\lambda_u]}(z) \\
=& \int_{[0,T]\times\mathbb{R}_+} M(du,dz)\, 1_{[0,t]}(u) 1_{[0,\lambda_u]}(z) + \int_{[0,T]\times\mathbb{R}_+} M(du,dz)\, 1_{[t,t+s]}(u) 1_{[0,\lambda_u]}(z) \\
=& N([0,t]) + N([t,t+s])
\end{aligned}$$

Therefore, it suffices to show that

$$N([t,t+s]) = \int_{[0,T]\times\mathbb{R}_+} M(du,dz)\, 1_{[t,t+s]}(u) 1_{[0,\lambda_u]}(z)$$

is conditionally independent of $\mathcal{F}_t$ given λ_t. To that end, we proceed with four steps. First, M being a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ implies that

$$\sigma(M([t, t+s] \times \mathbb{R}_+)) \perp\!\!\!\perp \mathcal{F}_t^M \tag{5.3.4}$$

The assumption $M \perp\!\!\!\perp \lambda$ implies that

$$\sigma(M([t, t+s] \times \mathbb{R}_+)) \perp\!\!\!\perp \mathcal{F}_t^\lambda \tag{5.3.5}$$

$$\mathcal{F}_t^M \perp\!\!\!\perp \mathcal{F}_t^\lambda \tag{5.3.6}$$

Combining relations (5.3.4), (5.3.5), (5.3.6) and using Lemma A.2.1 a., we have

$$\sigma(M([t, t+s] \times \mathbb{R}_+)) \perp\!\!\!\perp \mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_t^\lambda \tag{5.3.7}$$

Observe that $\sigma(\lambda_t) \subset \mathcal{F}_t^\lambda \subset \mathcal{F}_t$ and apply Lemma A.2.1 c. to relations (5.3.6) and (5.3.7) separately to obtain

$$\mathcal{F}_t^M \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t^\lambda \tag{5.3.8}$$

$$\sigma(M([t, t+s] \times \mathbb{R}_+)) \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_t^\lambda \tag{5.3.9}$$

Second, λ has the Markov property by Lemma A.2.1 e., meaning that

$$\sigma(\{\lambda_u : u \in [t, t+s]\}) \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t^\lambda \tag{5.3.10}$$

Moreover, note the assumption $M \perp\!\!\!\perp \lambda$ and use Lemma A.2.1 c. with the observation $\sigma(\lambda_t) \subset \sigma(\{\lambda_u : u \in [t, t+s]\})$ to write

$$\sigma(\{\lambda_u : u \in [t, t+s]\}) \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t^M \tag{5.3.11}$$

It follows from relations (5.3.8),(5.3.10), (5.3.11) and Lemma A.2.1 b. that

$$\sigma(\{\lambda_u : u \in [t, t+s]\}) \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_t^\lambda \tag{5.3.12}$$

Third, the assumption $M \perp\!\!\!\perp \lambda$ and the observation $\sigma(\lambda_t) \subset \sigma(\{\lambda_u : u \in [t, t+s]\})$

and Lemma A.2.1.c. implies that

$$\sigma(M([t, t+s] \times \mathbb{R}_+)) \overset{\lambda_t}{\perp\!\!\!\perp} \sigma(\{\lambda_u : u \in [t, t+s]\}) \tag{5.3.13}$$

Hence, using the relations (5.3.9), (5.3.12), (5.3.13) as well as Lemma A.2.1 b., we obtain

$$\sigma(M([t, t+s] \times \mathbb{R}_+)) \vee \sigma(\{\lambda_u : u \in [t, t+s]\}) \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t$$

Finally, by definition of N, $\mathcal{F}_t^N \subset \mathcal{F}_t^M \vee \mathcal{F}_t^\lambda$ for every t in $[0, T]$. Therefore, $\sigma(N([t, t+s])) \subset \sigma(M([t, t+s] \times \mathbb{R}_+)) \vee \sigma(\{\lambda_u : u \in [t, t+s]\})$. Hence,

$$\sigma(N([t, t+s])) \overset{\lambda_t}{\perp\!\!\!\perp} \mathcal{F}_t$$

(Proof of Second Claim) We first note that

$$N_{t+s} - N_s = N([t, t+s]) = \int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[t,t+s]}(u) 1_{[0,\lambda_t + (\lambda_u - \lambda_t)]}(z) = U_f$$

where $f : G \times [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by $f(\lambda, u, z) = 1_{[t,t+s]}(u) 1_{[0,\lambda_t + (\lambda_u - \lambda_t)]}(z)$.

Therefore, we can write

$$
\begin{aligned}
&\mathbb{E}[g \circ (N_{t+s} - N_t) \mid \mathcal{F}_t] \\
=&\mathbb{E}\left[g \circ (N_{t+s} - N_t) \mid \lambda_t\right] \qquad \text{(First claim)} \\
=&\mathbb{E}\left[\mathbb{E}\left[g \circ (N_{t+s} - N_t) \mid \mathcal{F}_T^{\lambda}\right] \bigm| \lambda_t\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[t,t+s]}(u) 1_{[0,\lambda_t+(\lambda_u-\lambda_t)]}(z)\right) \Bigm| \mathcal{F}_T^{\lambda}\right] \Bigm| \lambda_t\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[g \circ U_f \mid \mathcal{F}_T^{\lambda}\right] \bigm| \lambda_t\right] \qquad \text{(Definition of } f \text{ and } U_f\text{)} \\
=&\mathbb{E}\left[\mathbb{E}\left[g \circ U_f \mid \lambda\right] \mid \lambda_t\right] \qquad \text{(Remark A.2.2)} \\
=&\mathbb{E}\left[\mathbb{E}\left[g \circ u_f(a)\right] |_{a=\lambda} \mid \lambda_t\right] \qquad \text{(Theorem A.1.3)} \\
=&\mathbb{E}\left[\mathbb{E}\left[g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[t,t+s]}(u) 1_{[0,a_t+(a_u-a_t)]}(z)\right)\right]\Bigg|_{a=\lambda} \Bigm| \lambda_t\right] \quad \text{(Definition of } u_f(a)\text{)} \\
=&\mathbb{E}\left[\widetilde{g}\left(s, t, a_t, \{a_u - a_t, u \in (t, t+s]\}\right) |_{a=\lambda} \Bigm| \lambda_t\right] \qquad \text{(Definition of } \widetilde{g} \text{ in (5.3.3))} \\
=&\mathbb{E}\left[\widetilde{g} \circ (s, t, \lambda_t, \{\lambda_u - \lambda_t, u \in (t, t+s]\}) \mid \lambda_t\right] \\
=&\hat{g} \circ (s, t, \lambda_t) \qquad \text{(Since } \lambda_t \perp\!\!\!\perp \{\lambda_u - \lambda_t, u \in (t, t+s]\} \text{ from } \textbf{INDEP-INC}\text{)}
\end{aligned}
$$

with $\hat{g} : [0, T] \times [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by (5.3.2) and $\widetilde{g} : [0, T] \times [0, T] \times \mathbb{R}_+ \times G_{[t,T]} \mapsto \mathbb{R}_+$ defined by (5.3.3). □

Proposition 5.3.1 holds when the assumption M being Poisson random measure is replaced with a weaker assumption that M is a σ-finite random measure and has the Markov property in the sense of relation (5.3.4). Therefore, we speculate that the construction given in Equation (5.1.1) might be used for more general (conditional) random measures and point processes. Moreover, the assumption g being bounded can be dropped by using a cut-off argument, as shown in the next Corollary 5.3.2.

Corollary 5.3.2. *Assume* ***INDEP-INC****. Then, for every continuous function* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$,

$$\mathbb{E}[g \circ (N_{t+s} - N_t) | \mathcal{F}_t] = \hat{g} \circ (s, t, \lambda_t)$$

where the function $\hat{g}$ *is defined together with* $\widetilde{g}$ *by relations* (5.3.2) *and* (5.3.3) *respectively.*

Proof. Let g be given. Define $g_n(x) = g(x) \wedge n$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$ so that $g_n(x) \uparrow g(x)$ as $n \to \infty$. Moreover, since g_n is a bounded continuous function, it follows from Proposition 5.3.1 that

$$\mathbb{E}\left[g_n \circ (N_{t+s} - N_t) | \mathcal{F}_t\right] = \hat{g}_n \circ (s, t, \lambda_t), \quad n \in \mathbb{N}$$

where the function $\hat{g}_n$ is defined together with $\widetilde{g}_n$ similar to relations (5.3.2) and (5.3.3) respectively. It suffices to show that as $n \to \infty$,

$$\begin{aligned}
&(i) \quad \mathbb{E}\left[g_n \circ (N_{t+s} - N_t) | \mathcal{F}_t\right] \to \mathbb{E}\left[g \circ (N_{t+s} - N_t) | \mathcal{F}_t\right] \text{ a.s.} \\
&(ii) \quad \hat{g}_n \circ (s, t, \lambda_t) \to \hat{g} \circ (s, t, \lambda_t) \text{ a.s.}
\end{aligned}$$

(i) $g_n(x) \uparrow g(x)$ implies $g_n \circ (N_{t+s} - N_t) \uparrow g \circ (N_{t+s} - N_t)$ a.s. Using MON, we obtain (i).
(ii) $g_n(x) \uparrow g(x)$ implies

$$\begin{aligned}
g_n \circ &\left(\int_{[0,T]\times\mathbb{R}_+} M(du, dz)\, 1_{[t,t+s]}(u) 1_{[0,p+a_u^t]}(z) \right) \\
&\Big\uparrow g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du, dz)\, 1_{[t,t+s]}(u) 1_{[0,p+a_u^t]}(z) \right) \text{ a.s.}
\end{aligned}$$

Then, using MON, we obtain

$$\widetilde{g}_n(s, t, p, \{a_u^t, u \in [t, T]\}) \uparrow \widetilde{g}(s, t, p, \{a_u^t, u \in [t, T]\})$$

which implies that

$$\widetilde{g}_n \circ (s, t, p, \{\lambda_u^t, u \in (t, T]\}) \uparrow \widetilde{g} \circ (s, t, p, \{\lambda_u^t, u \in (t, T]\}) \text{ a.s.}$$

Again, using MON, we obtain $\hat{g}_n(s, t, p) \uparrow \hat{g}(s, t, p)$ and $\hat{g}_n \circ (s, t, \lambda_t) \uparrow \hat{g} \circ (s, t, \lambda_t)$ a.s. $\square$

Informally speaking, the relation between the functions $\widetilde{g}$ and $\hat{g}$ is as follows. By using Theorem A.1.3, we first take expectation with respect to M as if λ is a deterministic function a in Equation (5.3.3). Next, we plug in λ. After that, by using independent increments of λ, we take expectation with respect to $\{\lambda_u, t \in (t, t+s]\}$ in Equation (5.3.2). Finally, we plug in λ_t and obtain an $\mathcal{F}_t$ measurable random variable. Example 5.3.3 illustrates the usage of $\widetilde{g}$ and $\hat{g}$.

Example 5.3.3. If $g(x) = x$, then

$$\begin{aligned}
&\widetilde{g}(s, t, p, \{a_u^t, u \in [t, T]\}) \\
=&\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_+} M(du, dz)\, 1_{[t,t+s]}(u) 1_{[0,p+a_u^t]}(z)\right] \quad \text{(Definition of } \widetilde{g}) \\
=&\int_{[0,T]\times\mathbb{R}_+} \mu(du, dz)\, 1_{[t,t+s]}(u) 1_{[0,p+a_u^t]}(z) \quad (\mu \text{ is mean measure of } M) \\
=&\int_{[t,t+s]} du\,(p + a_u^t) \quad (\mu = \text{Leb} \times \text{Leb.})
\end{aligned}$$

Hence, $\hat{g}(s, t, p) = \mathbb{E}\left[\widetilde{g} \circ (s, t, p, \{\lambda_u^t, u \in (t, t+s]\})\right] = \mathbb{E}\left[\int_{[t,t+s]} du\,(p + \lambda_u^t)\right]$ and

$$\begin{aligned}
\mathbb{E}\left[N_{t+s} - N_t \mid \mathcal{F}_t\right] = \hat{g} \circ (s, t, \lambda_t) =&\mathbb{E}\left[\int_{[t,t+s]} du\,(p + \lambda_u^t)\right]\Bigg|_{p=\lambda_t} \\
=&\mathbb{E}\left[\int_{[t,t+s]} du\,\lambda_u \,\middle|\, \lambda_t\right]
\end{aligned}$$

This example can also be solved by using conditional law of N given λ being that of a Poisson random measure. That is,

$$\begin{aligned}
\underbrace{\mathbb{E}\left[N_{t+s} - N_t \mid \mathcal{F}_t\right] = \mathbb{E}\left[N_{t+s} - N_t \mid \lambda_t\right]}_{\text{First claim of Proposition 5.3.1.}} =&\mathbb{E}\left[\mathbb{E}\left[N_{t+s} - N_t \mid \lambda\right] \mid \lambda_t\right] \\
=&\mathbb{E}\left[\int_{[t,t+s]} du\,\lambda_u \,\middle|\, \lambda_t\right]
\end{aligned}$$

Indeed, combining the first claim of Proposition 5.3.1 with Theorem 5.2.1 is an alternative to calculate

$$\mathbb{E}\left[g \circ (N_{t+s} - N_t) \mid \mathcal{F}_t\right] = \mathbb{E}\left[g \circ (N_{t+s} - N_t) \mid \lambda_t\right] = \mathbb{E}\left[\mathbb{E}\left[g \circ (N_{t+s} - N_t) \mid \lambda\right] \mid \lambda_t\right]$$

for any continuous bounded $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$, since the conditional law of N given λ is that of a Poisson random measure.

The benefit of next Corollary 5.3.4 is twofold. It shows that the process $\{N_t, t \in [0,T]\}$ constructed by Equation (5.1.1) admits the stochastic intensity λ [49, p. 309, VI.6.42] and it is a conditional Markov process [51] with countably infinite state space. Moreover, it helps us see that an integral taken with respect to the conditional Poisson process can be converted into a Lebesgue integral.

Corollary 5.3.4. *(Martingale of Conditional Poisson Process) Assume* ***INDEP-INC****. The process $\widetilde{N} : \Omega \times [0,T] \mapsto \mathbb{R}$ defined by*

$$\widetilde{N}_t = N_t - \int_{[0,t]} du\, \lambda_u$$

is an L^2-bounded $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. Moreover, for every $(\mathcal{F}_t)_{t\in[0,T]}$-stopping time τ bounded by T and every non-negative $(\mathcal{F}_t)_{t\in[0,T]}$-predictable process $H : \Omega \times [0,T] \mapsto \mathbb{R}_+$,

$$\mathbb{E}\left[\int_0^\tau dN_u\, H_u\right] = \mathbb{E}\left[\int_0^\tau du\, \lambda_u H_u\right] \tag{5.3.14}$$

Proof. (Proof of first claim) It is possible to see that for some $C \in \mathbb{R}_+$

$$\sup_{t\in[0,T]} \mathbb{E}\left[\widetilde{N}_t^2\right] \le \mathbb{E}\left[N_T^2\right] + 2\mathbb{E}\left[N_T \int_{[0,T]} du\, \lambda_u\right] + \mathbb{E}\left[\left(\int_{[0,T]} du\, \lambda_u\right)^2\right] = C < \infty$$

since λ is positive and bounded. Moreover, for every s and t in $[0,T]$, we have

$$\begin{aligned}
&\mathbb{E}\left[\widetilde{N}_{t+s} - \widetilde{N}_t \,\middle|\, \mathcal{F}_t\right] \\
=&\mathbb{E}\left[N_{t+s} - N_t \mid \mathcal{F}_t\right] - \mathbb{E}\left[\int_{[t,t+s]} du\, \lambda_u \,\middle|\, \mathcal{F}_t\right] \qquad \text{(Definition of } \widetilde{N}) \\
=&\mathbb{E}\left[N_{t+s} - N_t \mid \mathcal{F}_t\right] - \mathbb{E}\left[\int_{[t,t+s]} du\, \lambda_u \,\middle|\, \lambda_t\right] \qquad \text{(Relation (5.3.12))} \\
=&\mathbb{E}\left[\int_{[t,t+s]} du\, \lambda_u \,\middle|\, \lambda_t\right] - \mathbb{E}\left[\int_{[t,t+s]} du\, \lambda_u \,\middle|\, \lambda_t\right] \qquad \text{(Example 5.3.3)} \\
=&0
\end{aligned}$$

(Proof of second claim) We first note that the integral on the left-hand-side exists as a Lebesgue-Stieltjes integral since $t \mapsto N_t(\omega)$ is increasing and right-continuous. For every $n \in \mathbb{N}$, define $H^n : \Omega \times [0,T] \mapsto \mathbb{R}_+$ by $H^n_u(\omega) = H_u(\omega) 1_{\{|H_u(\omega)| \le n\}}(\omega)$.

The process $\{H_u^n, u \in [0,T]\}$ is bounded and predictable. Therefore, the process $L^n : \Omega \times [0,T] \mapsto \mathbb{R}_+$ defined by

$$L_t^n = \int_{[0,t]} d\widetilde{N}_u \, H_u^n = \int_{[0,t]} dN_u \, H_u^n - \int_{[0,t]} du \, \lambda_u H_u^n$$

is an L^2-bounded $(\mathcal{F}_t)_{t\in[0,T]}$-martingale as an Itô integral with respect to $\widetilde{N}$ [52, p. 49, IV.27.9]. Therefore, it follows from Doob's stopping theorem that $(L^n_{t\wedge\tau})_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale as well. Hence,

$$\mathbb{E}\left[L_\tau^n\right] = \mathbb{E}\left[L_{T\wedge\tau}^n\right] = \mathbb{E}\left[L_{0\wedge\tau}^n\right] = \mathbb{E}\left[L_0^n\right] = 0$$

which implies that

$$\mathbb{E}\left[\int_0^\tau dN_u \, H_u^n\right] = \mathbb{E}\left[\int_0^\tau du \, \lambda_u H_u^n\right]$$

Finally, $H_u^n(\omega) \uparrow H_u(\omega)$ as $n \to \infty$ by the definition of H_u^n. Then, using MON twice, we obtain equality (5.3.14). $\square$

Remark 5.3.5. Another proof of the second claim can be the following. Define $H_u^n(\omega) = H_u(\omega) 1_{\{|H_u(\omega)|\le n\}}(\omega)$. It is known that $\widetilde{N}$ is a local martingale since it is a martingale. Therefore, L^n is a local martingale as well [52, p. 3]. Using H^n being bounded, it is possible to show that $\mathbb{E}\left[\sup_{s\le t}(L_s^n)\right] < \infty$ for every $t \in [0,T]$. Hence, L^n is a martingale [53, p. 38, Thm. 51]. Finally, let $H^n \uparrow H$ and obtain equation (5.3.14).

5.3.2 Strong Markov Property of Conditional Poisson Process

By using Proposition 5.3.6, we show the strong Markov property in Corollary 5.3.7. We essentially approach the arbitrary stopping times with discrete ones, and heavily benefit from our closed-form expression in Equation (5.1.1). To the best of our knowledge, the closest discussion on strong Markov property of conditional Poisson processes is given by [41, p. 63]. There, the authors assume that $\mathcal{F}_T^\lambda$ is known at time zero.

Proposition 5.3.6. *Assume **INDEP-INC** and **sMARKOV**. Then, for every $\tau \in \mathcal{T}$ and every bounded continuous function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$*

$$\mathbb{E}\left[g \circ (N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] = \hat{g} \circ (s, \tau, \lambda_\tau)$$

where the function $\hat{g}$ is defined together with $\widetilde{g}$ by relations (5.3.2) *and* (5.3.3) *respectively.*

Proof. Assume for now that τ takes values in a countable subset D of $[0, T]$. Observe that

$$g \circ (N_{\tau+s} - N_\tau) = g \circ N([\tau, \tau + s]) = \sum_{t \in D} 1_{\{\tau = t\}} g \circ N([t, t + s])$$

Therefore, for any $H \in \mathcal{F}_\tau$,

$$\begin{aligned}
&\mathbb{E}\Big[1_H g \circ N([\tau, \tau + s])\Big] \\
=&\mathbb{E}\Big[\sum_{t \in D} 1_{H \cap \{\tau = t\}} g \circ N([t, t + s])\Big] \\
=&\sum_{t \in D} \mathbb{E}\Big[1_{H \cap \{\tau = t\}} g \circ N([t, t + s])\Big] \quad \text{(MON)} \\
=&\sum_{t \in D} \mathbb{E}[1\underbrace{H \cap \{\tau = t\}}_{\in \mathcal{F}_t} \mathbb{E}[g \circ N([t, t + s]) | \mathcal{F}_t]] \\
=&\sum_{t \in D} \mathbb{E}\Big[1_{H \cap \{\tau = t\}} \hat{g} \circ (s, t, \lambda_t)\Big] \quad \text{(Proposition 5.3.1)} \\
=&\mathbb{E}\Big[1_H \hat{g} \circ (s, \tau, \lambda_\tau)\Big] \quad \text{(MON and } \hat{g} \text{ being a function)}
\end{aligned}$$

Next, assume that τ takes values in $[0, T]$ and let $(\tau_n)_{n \in \mathbb{N}}$ be an approximating sequence of τ in the sense of [49, p. 178, V.1.20]. That is, $\tau_n = d_n \circ \tau$ where

$$d_n(t) = \begin{cases} \frac{k+1}{2^n}, & \text{if } \frac{k}{2^n} \le t < \frac{k+1}{2^n} \text{ for some } k \in \mathbb{N} \\ \infty, & \text{if } t = \infty \end{cases}$$

Then, $\tau_n \geq \tau$ a.s. for all $n \in \mathbb{N}$, τ_n is countably valued and $\tau_n \downarrow \tau$ a.s. Moreover, for any $H \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}$,

$$\mathbb{E}\bigg[1_H g \circ N([\tau_n, \tau_n + s])\bigg] = \mathbb{E}\bigg[1_H \hat{g} \circ (s, \tau_n, \lambda_{\tau_n})\bigg], \qquad n \in \mathbb{N}$$

holds by the previous step and the fact that τ_n is countably valued. Hence, it suffices to show that as $n \to \infty$,

$$\begin{aligned} &(i) \quad \mathbb{E}\bigg[1_H g \circ N([\tau_n, \tau_n + s])\bigg] \to \mathbb{E}\bigg[1_H g \circ N([\tau, \tau + s])\bigg] \\ &(ii) \quad \mathbb{E}\bigg[1_H \hat{g} \circ (s, \tau_n, \lambda_{\tau_n})\bigg] \to \mathbb{E}\bigg[1_H \hat{g} \circ (s, \tau, \lambda_\tau)\bigg] \end{aligned}$$

(i) We can see by definition of N that $N([\tau_n, \tau_n + s]) = N([0, \tau_n + s]) - N([0, \tau_n])$ and

$$N([0, \tau_n + s]) = \int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[0,\tau_n+s]}(u) 1_{[0,\lambda_u]}(z)$$

Moreover, the assumption $\tau_n \downarrow \tau$ a.s. implies that $1_{[0,\tau_n+s]}(u) \to 1_{[0,\tau+s]}(u)$ a.s. and for every $u \in [0, T]$. By the bounded convergence theorem with random measure M, we have

$$\begin{aligned} &\int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[0,\tau_n+s]}(u) 1_{[0,\lambda_u]}(z) \\ &\qquad \to \int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[0,\tau+s]}(u) 1_{[0,\lambda_u]}(z) \text{ a.s.} \end{aligned}$$

which is equivalent to $N([0, \tau_n + s]) \to N([0, \tau + s])$ a.s. Similarly, $N([0, \tau_n]) \to N([0, \tau])$ a.s., so $N([\tau_n, \tau_n + s]) \to N([\tau, \tau + s])$ a.s. By g being continuous,

$$g \circ N([\tau_n, \tau_n + s]) \to g \circ N([\tau, \tau + s]) \text{ a.s.}$$

By g being bounded and bounded convergence theorem,

$$\mathbb{E}\left[1_H g \circ N([\tau_n, \tau_n + s])\right] \to \mathbb{E}\left[1_H g \circ N([\tau, \tau + s])\right]$$

(ii) The assumption that $\tau_n \downarrow \tau$ a.s. implies $\lambda_{\tau_n} \to \lambda_\tau$ a.s. by λ being right-continuous and bounded. Likewise, $\lambda_u^{\tau_n} = \lambda_u - \lambda_{\tau_n} \to \lambda_u - \lambda_\tau = \lambda_u^\tau$ a.s. Using

[54, Lemma 1], we obtain

$$\begin{aligned} 1_{[0,\lambda_{\tau_n}+\lambda_u^{\tau_n}]}(z) &\to 1_{[0,\lambda_\tau+\lambda_u^\tau]}(z) \quad \text{a.s. and Leb-a.e. } z \in \mathbb{R}_+ \\ 1_{[\tau_n,\tau_n+s]}(u) &\to 1_{[\tau,\tau+s]}(u) \quad \text{a.s. and every } u \in [0,T] \end{aligned}$$

and it follows from the bounded convergence theorem with random measure M that

$$\begin{aligned} &\int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{[\tau_n,\tau_n+s]}(u) 1_{[0,\lambda_{\tau_n}+\lambda_u^{\tau_n}]}(z) \\ &\quad \to \int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{[\tau,\tau+s]}(u) 1_{[0,\lambda_\tau+\lambda_u^\tau]}(z) \text{ a.s.} \end{aligned}$$

Since g is continuous,

$$\begin{aligned} &g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{[\tau_n,\tau_n+s]}(u) 1_{[0,\lambda_{\tau_n}+\lambda_u^{\tau_n}]}(z) \right) \\ &\quad \to g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{[\tau,\tau+s]}(u) 1_{[0,\lambda_\tau+\lambda_u^\tau]}(z) \right) \text{ a.s.} \end{aligned}$$

By g being bounded and bounded convergence theorem,

$$\begin{aligned} &\mathbb{E}\left[g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{[\tau_n,\tau_n+s]}(u) 1_{[0,\lambda_{\tau_n}+\lambda_u^{\tau_n}]}(z) \right) \right] \\ &\quad \to \mathbb{E}\left[g \circ \left(\int_{[0,T]\times\mathbb{R}_+} M(du,dz) 1_{[\tau,\tau+s]}(u) 1_{[0,\lambda_\tau+\lambda_u^\tau]}(z) \right) \right] \text{ a.s.} \end{aligned}$$

Therefore, it is possible to see that

$$\widetilde{g} \circ (s, \tau_n, \lambda_{\tau_n}, \{\lambda_u^{\tau_n}, u \in [\tau_n, T]\}) \to \widetilde{g} \circ (s, \tau, \lambda_\tau, \{\lambda_u^\tau, u \in [\tau, T]\}) \text{ a.s.}$$

Moreover, $\widetilde{g}$ is bounded since g is bounded. Using bounded convergence theorem, we obtain

$$\mathbb{E}\left[\widetilde{g} \circ (s, \tau_n, \lambda_{\tau_n}, \{\lambda_u^{\tau_n}, u \in [\tau_n, T]\})\right] \to \mathbb{E}\left[\widetilde{g} \circ (s, \tau, \lambda_\tau, \{\lambda_u^\tau, u \in [\tau, T]\})\right]$$

which is equivalent to

$$\hat{g} \circ (s, \tau_n, \lambda_{\tau_n}) \to \hat{g} \circ (s, \tau, \lambda_\tau)$$

Finally, $\hat{g}$ is bounded since $\widetilde{g}$ is bounded. Bounded convergence theorem yields

$$\mathbb{E}\left[1_H\, \hat{g} \circ (s, \tau_n, \lambda_{\tau_n})\right] \to \mathbb{E}\left[1_H\, \hat{g} \circ (s, \tau, \lambda_\tau)\right]$$

$\square$

The next Corollary 5.3.7 shows the strong Markov property of conditional Poisson processes.

Corollary 5.3.7. *(Strong Markov Property) Assume* ***INDEP-INC*** *and* ***sMARKOV****. Then, for every bounded continuous function $h : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$,*

$$\mathbb{E}\left[h \circ (N_\tau, N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] = \hat{h} \circ (s, \tau, \lambda_\tau, N_\tau)$$

where $\hat{h} : [0,T] \times [0,T] \times \mathbb{R}_+ \times \mathbb{Z}_+ \mapsto \mathbb{R}_+$ is defined by

$$\hat{h}(s, t, p, y) = \mathbb{E}\left[\widetilde{h} \circ (s, t, p, y, \{\lambda_u^t, u \in [t,T]\})\right]$$

and $\widetilde{h} : [0,T] \times [0,T] \times \mathbb{R}_+ \times \mathbb{Z}_+ \times G_{[t,T]} \mapsto \mathbb{R}_+$ is defined by

$$\widetilde{h}(s, t, p, y, \{a_u^t, u \in [t,T]\}) = \mathbb{E}\left[h \circ \left(y, \int_{[0,T]\times\mathbb{R}_+} M(du, dz) 1_{[t,t+s]}(u) 1_{[0,p+a_u^t]}(z)\right)\right]$$

Moreover, $N_{\tau+s}$ is conditionally independent of $\mathcal{F}_\tau$ given λ_τ and N_τ for every stopping time $\tau \in \mathcal{T}$ and $s \in [0,T]$.

Proof. It follows from Proposition 5.3.6 that

$$\begin{aligned}
&\mathbb{E}\left[h \circ (N_\tau, N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] \\
=&\sum_{y \in \mathbb{Z}_+} \mathbb{E}\left[h \circ (y, N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] 1_{\{N_\tau = y\}} \\
=&\sum_{y \in \mathbb{Z}_+} \hat{h} \circ (s, \tau, \lambda_\tau, y) 1_{\{N_\tau = y\}} \qquad \text{(Proposition 5.3.6)} \\
=&\hat{h} \circ (s, \tau, \lambda_\tau, N_\tau)
\end{aligned}$$

Let $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a bounded continuous function. With $h(x, y) = g(x + y)$, which is also a bounded continuous function, we obtain

$$\begin{aligned}
\mathbb{E}\left[g \circ N_{\tau+s} \mid \mathcal{F}_\tau\right] =&\mathbb{E}\left[g \circ (N_\tau + N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] \\
=&\mathbb{E}\left[h \circ (N_\tau, N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] \\
=&\hat{h} \circ (s, \tau, \lambda_\tau, N_\tau) \qquad \text{(First claim)} \\
=&\mathbb{E}\left[\hat{h} \circ (s, \tau, \lambda_\tau, N_\tau) \,\middle|\, \sigma(\lambda_\tau, N_\tau)\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[h \circ (N_\tau, N_{\tau+s} - N_\tau) \mid \mathcal{F}_\tau\right] \mid \sigma(\lambda_\tau, N_\tau)\right] \qquad \text{(First claim)} \\
=&\mathbb{E}\left[\mathbb{E}\left[h \circ (N_\tau, N_{\tau+s} - N_\tau) \mid \sigma(\lambda_\tau, N_\tau)\right] \mid \mathcal{F}_\tau\right] \\
=&\mathbb{E}\left[h \circ (N_\tau, N_{\tau+s} - N_\tau) \mid \sigma(\lambda_\tau, N_\tau)\right] \\
=&\mathbb{E}\left[g \circ N_{\tau+s} \mid \sigma(\lambda_\tau, N_\tau)\right]
\end{aligned}$$

which shows the strong Markov property on $\{N_t, t \in [0, T]\}$. □

5.3.3 Reducing the Optimal Stopping Problem with L-Delay

While reducing the optimal stopping problem with L-delay, we essentially represent the expected costs incurring in $[\tau, \tau+L]$ as a function of $(L, \tau, \lambda_\tau, N_\tau)$ by using the strong Markov property of the conditional Poisson process $\{N_t, t \in [0, T]\}$ as well as the other terms in Equation (5.1.2). The previous Subsection 5.3.2 shows the conditional independence of $\{N_t, t \in [0, T]\}$ from the past given the present.

The conditional independence of the remaining terms in Equation (5.1.2) are shown by the following Lemma 5.3.8, Lemma 5.3.9 and Lemma 5.3.10.

Lemma 5.3.8. *Assume* ***sMARKOV****. Then,* $\lambda_{\tau+s}$ *is conditionally independent of* $\mathcal{F}_\tau$ *given* λ_τ *for every* $s \in [0,T]$ *and* $\tau \in \mathcal{T}$*.*

Proof. Let $s \in [0,T]$ and $\tau \in \mathcal{T}$ be given. By assumption,

$$\sigma(\lambda_{\tau+s}) \overset{\lambda_\tau}{\perp\!\!\!\perp} \mathcal{F}^\lambda_\tau$$

Moreover, the assumption $\lambda \perp\!\!\!\perp M$ implies

$$\sigma(\{\lambda_{\tau+u}, u \in [0,T]\}) \perp\!\!\!\perp \mathcal{F}^M_\tau$$

Observing that $\sigma(\lambda_\tau) \subset \sigma(\{\lambda_{\tau+u}, u \in [0,T]\})$ and using Lemma A.2.1 c. we obtain

$$\sigma(\{\lambda_{\tau+u}, u \in [0,T]\}) \overset{\lambda_\tau}{\perp\!\!\!\perp} \mathcal{F}^M_\tau$$

Since $\sigma(\lambda_{\tau+s}) \subset \sigma(\{\lambda_{\tau+u}, u \in [0,T]\})$,

$$\sigma(\lambda_{\tau+s}) \overset{\lambda_\tau}{\perp\!\!\!\perp} \mathcal{F}^M_\tau$$

and observing that $\sigma(\lambda_\tau) \subset \mathcal{F}^\lambda_\tau$ and using Lemma A.2.1 c. we obtain

$$\mathcal{F}^\lambda_\tau \overset{\lambda_\tau}{\perp\!\!\!\perp} \mathcal{F}^M_\tau$$

Finally, it follows from Lemma A.2.1 b. that

$$\sigma(\lambda_{\tau+s}) \overset{\lambda_\tau}{\perp\!\!\!\perp} \mathcal{F}_\tau$$

□

The following Lemma 5.3.9 is intuitive. If we choose to stop at time τ, it is possible to see whether the inventory level hits zero or not before τ.

Lemma 5.3.9. *For every $\tau \in \mathcal{T}$, the set $\{\tau < \sigma_x\}$ is in $\mathcal{F}_\tau^N$. Therefore, $1_{\{\tau<\sigma_x\}}$ is $\mathcal{F}_\tau^N$ measurable.*

Proof. The proof is from [55, p. 33, I.6.8]. Since τ and σ_x are $(\mathcal{F}_t^N)_{t\in[0,T]}$-stopping times, after letting $D_t = (\mathbb{Q} \cap [0,t]) \cup \{t\}$, we obtain

$$\{\tau < \sigma_x\} \cap \{\tau \le t\} = \bigcup_{r\in D_t} \underbrace{\{\tau \le r\}}_{\in\mathcal{F}_t^N} \cap \underbrace{\{r < \sigma_x\}}_{\in\mathcal{F}_t^N} \in \mathcal{F}_t^N$$

□

The following Lemma 5.3.10 shows that if the inventory level does not hit zero by the time τ, the remaining time before it hits zero is conditionally independent of the past given the present.

Lemma 5.3.10. *Assume **INDEP-INC** and **sMARKOV**. Then, for every $\tau \in \mathcal{T}$ and $x \in \mathbb{Z}_+$, on $\{\tau < \sigma_x\}$,*

$$\sigma_x = \sigma^\tau_{x-N_\tau} + \tau$$

where $\sigma^\tau_{x-N_\tau} = \inf\{t > 0 : N_{t+\tau} - N_\tau \ge x - N_\tau\}$. Moreover, $\sigma^\tau_{x-N_\tau}$ is conditionally independent of $\mathcal{F}_\tau$ given λ_τ and N_τ.

Proof. On the set $\{\tau < \sigma_x\}$, we see that $N_\tau < x$. Then, on $\{\tau < \sigma_x\}$, we have

$$\begin{aligned}\sigma_x &= \inf\{t > 0 : N_t \ge x\} = \inf\{t > \tau : N_t \ge x\} \\ &= \inf\{u > 0 : N_{u+\tau} - N_\tau + N_\tau \ge x\} + \tau = \sigma^\tau_{x-N_\tau} + \tau\end{aligned}$$

For the second claim, it follows from Corollary 5.3.7 that $N_{u+\tau}$ is conditionally independent of $\mathcal{F}_\tau$ given λ_τ and N_τ for any $u > 0$. Hence, by definition, $\sigma^\tau_{x-N_\tau}$ is conditionally independent of $\mathcal{F}_\tau$ given λ_τ and N_τ. □

Now, we are ready to state the main result of this section that the optimal stopping problem with L-delay can be reduced to a classical optimal stopping problem. The classical optimal stopping problem $\hat{\nu}(x)$ has the structure that the costs incurring during the delay $[\tau, \tau+L]$ can be represented as a function of $(L, \tau, \lambda_\tau, N_\tau)$ mainly due to the strong Markov property of the conditional Poisson process $\{N_t, t \in [0,T]\}$.

Theorem 5.3.11. *Assume **INDEP-INC** and **sMARKOV**. For every $x \in \mathbb{Z}_+$, there exist functions $\hat{h}_1 : [0,T]\times[0,T]\times\mathbb{R}_+\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ and $\hat{h}_2 : [0,T]\times[0,T]\times\mathbb{R}_+\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ and $\hat{g}_4 : [0,T]\times\mathbb{R}_+\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ such that the two optimal stopping problems*

$$\nu(x) = \inf_{\tau\in\mathcal{T}} \{C(x,\tau)\} \tag{5.3.15}$$

$$\begin{aligned}\hat{\nu}(x) := \inf_{\tau\in\mathcal{T}} \Big\{ & \hat{C}(x,\tau) \\ & + \mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u}\left(\hat{h}_1\circ(u,\tau,\lambda_\tau,N_\tau) + 1_{\{\sigma_x>\tau\}}\hat{h}_2\circ(u,\tau,\lambda_\tau,N_\tau)\right)\right] \\ & + \mathbb{E}\left[e^{-\delta\tau}\hat{g}_4\circ(\tau,\lambda_\tau,N_\tau)\right]\Big\}\end{aligned} \tag{5.3.16}$$

are equal, that is,

$$\nu(x) = \hat{\nu}(x), \quad x \in \mathbb{Z}_+$$

where $\hat{C} : \mathbb{Z}_+ \times \mathcal{T} \mapsto \mathbb{R}_+$ is defined by

$$\hat{C}(x,\tau) = \mathbb{E}\left[\int_0^\tau du\, e^{-\delta u}\left[c_1(x-N_u)^+ - c_3(u)\lambda_u + c_2(u)\lambda_u\right]\right] \tag{5.3.17}$$

$$+\mathbb{E}\left[\int_0^{\tau\wedge\sigma_x} du\, e^{-\delta u}\left[c_5\lambda_u - c_2(u)\lambda_u\right]\right] + A \tag{5.3.18}$$

for $A = \mathbb{E}\left[\int_0^T du\, e^{-\delta u} c_3(u)\lambda_u\right] \in \mathbb{R}_+$.

Proof. We first note that $\tau + L$ is again an $(\mathcal{F}_t^N)_{t\in[0,T]}$-stopping time since

$$\{\omega : \tau(\omega) + L \le t\} \equiv \{\omega : \tau(\omega) \le t - L\} \in \mathcal{F}_{t-L}^N \subset \mathcal{F}_t^N \qquad \forall t \in \mathbb{R}_+$$

Now, let us focus on the first term in equation (5.1.2):

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^{\tau+L} du\, e^{-\delta u}(x-N_u)^+\Big]\\
=&\mathbb{E}\Big[\int_0^{\tau} du\, e^{-\delta u}(x-N_u)^+ + \int_\tau^{\tau+L} du\, e^{-\delta u}(x-N_u)^+\Big]
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_\tau^{\tau+L} du\, e^{-\delta u}(x-N_u)^+\Big]\\
=&\mathbb{E}\Big[\int_0^{L} du\, e^{-\delta(u+\tau)}(x-N_{u+\tau})^+\Big]\\
=&\mathbb{E}\Big[e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\,(x-(N_{u+\tau}-N_\tau+N_\tau))^+\Big]\\
=&\mathbb{E}\Big[e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\Big((x-N_\tau)^+-(N_{u+\tau}-N_\tau)\Big)^+\Big] \qquad \text{(Lemma A.2.1 g.)}\\
=&\mathbb{E}\Big[\underbrace{e^{-\delta\tau}}_{\in\mathcal{F}_\tau^N\subset\mathcal{F}_\tau}\,\mathbb{E}\Big[\int_0^{L} du\, e^{-\delta u}\Big((x-N_\tau)^+-(N_{u+\tau}-N_\tau)\Big)^+\Big|\mathcal{F}_\tau\Big]\Big]\\
=&\mathbb{E}\Big[\underbrace{e^{-\delta\tau}}_{\in\mathcal{F}_\tau^N\subset\mathcal{F}_\tau}\int_0^{L} du\, e^{-\delta u}\mathbb{E}\Big[\Big((x-N_\tau)^+-(N_{u+\tau}-N_\tau)\Big)^+\Big|\mathcal{F}_\tau\Big]\Big] \qquad \text{(Lemma A.2.1 d.)}\\
=&\mathbb{E}\Big[e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\hat{g}_1\circ(u,\tau,\lambda_\tau,N_\tau)\Big] \qquad \text{(Proposition 5.3.6)}
\end{aligned}
$$

where the function $\hat{g}_1 : [0,T]\times[0,T]\times\mathbb{R}_+\times\mathbb{Z}_+\mapsto\mathbb{R}_+$ is defined by

$$
\hat{g}_1(u,t,p,y) = \mathbb{E}\left[\widetilde{g}_1\circ(u,t,p,y,\{\lambda_r^t, r\in(t,T]\})\right] \tag{5.3.19}
$$

and the function $\widetilde{g}_1 : [0,T]\times[0,T]\times\mathbb{R}_+\times\mathbb{Z}_+\times G_{[t,T]}\mapsto\mathbb{R}_+$ is defined by

$$
\begin{aligned}
&\widetilde{g}_1(u,t,p,y,\{a_r^t, r\in(t,T]\})\\
=&\mathbb{E}\left[\left((x-y)^+-\int_{[0,T]\times\mathbb{R}_+} M(dr,dz)1_{[t,t+u]}(r)1_{[0,p+a_r^t]}(z)\right)^+\right]
\end{aligned} \tag{5.3.20}
$$

Now, let us focus on the fifth term in equation (5.1.2).

$$\begin{aligned}
&\mathbb{E}\int_0^{(\tau+L)\wedge\sigma_x} dN_u\, e^{-\delta u}\\
=&\mathbb{E}\left[\int_0^{(\tau+L)\wedge\sigma_x} du\, e^{-\delta u}\lambda_u\right] \quad \text{(Corollary 5.3.4)}\\
=&\mathbb{E}\left[1_{\{\sigma_x\le\tau\}}\int_0^{\sigma_x} du\, e^{-\delta u}\lambda_u\right]+\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_0^{(\tau+L)\wedge\sigma_x} du\, e^{-\delta u}\lambda_u\right]
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_0^{(\tau+L)\wedge\sigma_x} du\, e^{-\delta u}\lambda_u\right]\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_0^{(\tau+L)\wedge(\tau+\sigma^\tau_{x-N_\tau})} du\, e^{-\delta u}\lambda_u\right] \quad \text{(Lemma 5.3.10)}\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_0^{\tau+(L\wedge\sigma^\tau_{x-N_\tau})} du\, e^{-\delta u}\lambda_u\right]\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_0^{\tau} du\, e^{-\delta u}\lambda_u\right]+\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_\tau^{\tau+(L\wedge\sigma^\tau_{x-N_\tau})} du\, e^{-\delta u}\lambda_u\right]
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_\tau^{\tau+(L\wedge\sigma^\tau_{x-N_\tau})} du\, e^{-\delta u}\lambda_u\right]\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_0^{L\wedge\sigma^\tau_{x-N_\tau}} du\, e^{-\delta u}\lambda_{u+\tau}\right]\\
=&\mathbb{E}\left[\underbrace{1_{\{\sigma_x>\tau\}}e^{-\delta\tau}}_{\in\mathcal{F}^N_\tau\subset\mathcal{F}_\tau}\mathbb{E}\left\{\int_0^{L\wedge\sigma^\tau_{x-N_\tau}} du\, e^{-\delta u}\lambda_{u+\tau}\middle|\mathcal{F}_\tau\right\}\right] \quad \text{(Lemma 5.3.9)}\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\mathbb{E}\left\{\int_0^{L} du\, e^{-\delta u}1_{\{u<\sigma^\tau_{x-N_\tau}\}}\lambda_{u+\tau}\middle|\mathcal{F}_\tau\right\}\right]\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\mathbb{E}\left\{1_{\{u<\sigma^\tau_{x-N_\tau}\}}(\lambda_\tau+\lambda_{u+\tau}-\lambda_\tau)\middle|\mathcal{F}_\tau\right\}\right] \quad \text{((Lemma A.2.1 d.))}\\
=&\mathbb{E}\left[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\hat{g}_5\circ(u,\tau,\lambda_\tau,N_\tau)\right] \quad \text{(Lemma 5.3.8 and Lemma 5.3.10)}
\end{aligned}$$

where the function $\hat{g}_5 : [0,T] \times [0,T] \times \mathbb{R}_+ \times \mathbb{Z}_+ \mapsto \mathbb{R}_+$ is defined by

$$\hat{g}_5(u,t,p,y) := \mathbb{E}\left[1_{\{u<\sigma^t_{x-y}\}}(p+\lambda^t_u)\right] \tag{5.3.21}$$

As a result, the fifth term in equation (5.1.2) is

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{(\tau+L)\wedge\sigma_x} du\, e^{-\delta u}\lambda_u\right] \\
=&\mathbb{E}\left[1_{\{\sigma_x\le\tau\}}\int_0^{\sigma_x} du\ e^{-\delta u}\lambda_u\right] + \mathbb{E}\left[1_{\{\sigma_x>\tau\}}\int_0^{\tau} du\, e^{-\delta u}\lambda_u\right] \\
&+\mathbb{E}\left[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\hat{g}_5\circ(u,\tau,\lambda_\tau,N_\tau)\right] \\
=&\mathbb{E}\left[\int_0^{\tau\wedge\sigma_x} du\, e^{-\delta u}\lambda_u\right] + \mathbb{E}\left[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\hat{g}_5\circ(u,\tau,\lambda_\tau,N_\tau)\right]
\end{aligned}$$

Now, let us focus on the third term in equation 5.1.2.

$$\begin{aligned}
&\mathbb{E}\left[\int_{\tau+L}^{T} dN_u e^{-\delta u}c_3(u)\right] \\
=&\mathbb{E}\left[\int_{\tau+L}^{T} du\, e^{-\delta u}c_3(u)\lambda_u\right] \quad \text{(Corollary 5.3.4)} \\
=&\mathbb{E}\left[\int_0^{T} du\, e^{-\delta u}c_3(u)\lambda_u\right] - \mathbb{E}\left[\int_0^{\tau+L} du\, e^{-\delta u}c_3(u)\lambda_u\right] \\
=&A - \mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}c_3(u)\lambda_u\right] - \mathbb{E}\left[\int_\tau^{\tau+L} du\, e^{-\delta u}c_3(u)\lambda_u\right] \\
=&A - \mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}c_3(u)\lambda_u\right] - \mathbb{E}\left[e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}c_3(u+\tau)\lambda_{u+\tau}\right] \\
=&A - \mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}c_3(u)\lambda_u\right] - \mathbb{E}\left[\underbrace{e^{-\delta\tau}}_{\in\mathcal{F}^N_\tau\subset\mathcal{F}_\tau}\ \mathbb{E}\left[\int_0^{L} du\, e^{-\delta u}c_3(u+\tau)\lambda_{u+\tau}\,\middle|\,\mathcal{F}_\tau\right]\right] \\
=&A - \mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}c_3(u)\lambda_u\right] \\
&- \mathbb{E}\left[e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\mathbb{E}\left[c_3(u+\tau)(\lambda_\tau+\lambda_{u+\tau}-\lambda_\tau)\mid\mathcal{F}_\tau\right]\right] \quad \text{(Lemma A.2.1 d.)} \\
=&A - \mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}c_3(u)\lambda_u\right] - \mathbb{E}\left[e^{-\delta\tau}\int_0^{L} du\, e^{-\delta u}\hat{g}_3\circ(u,\tau,\lambda_\tau)\right] \quad \text{(Lemma 5.3.8)}
\end{aligned}$$

where the function $\hat{g}_3 : [0,T] \times [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by

$$\hat{g}_3(u,t,p) = \mathbb{E}\left[c_3(u+t)\,(p+\lambda_u^t)\right] \tag{5.3.22}$$

and the constant $A \in \mathbb{R}_+$ is defined by $A = \mathbb{E}\left[\int_0^T du\, e^{-\delta u} c_3(u)\lambda_u\right]$. Furthermore, the fourth term in the equation (5.1.2) is

$$\begin{aligned}
&\mathbb{E}\Bigg[e^{-\delta(\tau+L)}(x-N_{\tau+L})^+\Bigg]\\
=&\mathbb{E}\Bigg[e^{-\delta\tau}e^{-\delta L}\Bigg((x-N_\tau)^+ - (N_{\tau+L}-N_\tau)\Bigg)^+\Bigg] \quad \text{(Lemma A.2.1 g.)}\\
=&\mathbb{E}\Bigg[\underbrace{e^{-\delta\tau}}_{\in\mathcal{F}_\tau^N\subset\mathcal{F}_\tau}\ \mathbb{E}\Bigg\{e^{-\delta L}\Bigg((x-N_\tau)^+ - (N_{\tau+L}-N_\tau)\Bigg)^+\Bigg|\mathcal{F}_\tau\Bigg\}\Bigg]\\
=&\mathbb{E}\Bigg[e^{-\delta\tau}\hat{g}_4\circ(\tau,\lambda_\tau,N_\tau)\Bigg]
\end{aligned}$$

where the function $\hat{g}_4 : [0,T] \times \mathbb{R}_+ \times \mathbb{Z}_+ \mapsto \mathbb{R}_+$ is defined by

$$\hat{g}_4(t,p,y) = \mathbb{E}\left[\widetilde{g}_4\circ(t,p,y,\{\lambda_u^t, u\in[t,T]\})\right] \tag{5.3.23}$$

and the function $\widetilde{g}_4 : [0,T] \times \mathbb{R}_+ \times \mathbb{Z}_+ \times G_{[t,T]} \mapsto \mathbb{R}_+$ is defined by

$$\widetilde{g}_4(t,p,y,\{a_u^t, u\in[t,T]\}) \tag{5.3.24}$$

$$=\mathbb{E}\left[e^{-\delta L}\left((x-y)^+ - \int_{[0,T]\times\mathbb{R}_+} M(dr,dz)\,1_{[t,t+L]}(r)1_{[0,p+a_r^t]}(z)\right)^+\right] \tag{5.3.25}$$

Finally, the second term in equation (5.1.2) is

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{(\tau+L)\wedge\sigma_x}^{\tau+L} dN_u\, e^{-\delta u} c_2(u)\Big] \\
=&\mathbb{E}\Big[\int_{(\tau+L)\wedge\sigma_x}^{\tau+L} du\, e^{-\delta u} c_2(u)\lambda_u\Big] \quad \text{(Corollary 5.3.4)} \\
=&\mathbb{E}\Big[\int_{0}^{\tau+L} du\, e^{-\delta u} c_2(u)\lambda_u\Big] - \mathbb{E}\Big[\int_{0}^{(\tau+L)\wedge\sigma_x} du\, e^{-\delta u} c_2(u)\lambda_u\Big] \\
=&\mathbb{E}\Big[\int_{0}^{\tau} du\, e^{-\delta u} c_2(u)\lambda_u\Big] + \mathbb{E}\Big[e^{-\delta\tau}\int_{0}^{L} du\, e^{-\delta u} c_2(u+\tau)\lambda_{u+\tau}\Big] \\
-&\mathbb{E}\Big[\int_{0}^{\tau\wedge\sigma_x} du\, e^{-\delta u} c_2(u)\lambda_u\Big] - \mathbb{E}\Big[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_{0}^{L} e^{-\delta u}\hat{g}_{22}\circ(u,\tau,\lambda_\tau,N_\tau)\Big] \\
=&\mathbb{E}\Big[\int_{0}^{\tau} du\, e^{-\delta u} c_2(u)\lambda_u\Big] + \mathbb{E}\Big[e^{-\delta\tau}\int_{0}^{L} du\, e^{-\delta u}\hat{g}_{21}\circ(u,\tau,\lambda_\tau)\Big] \\
-&\mathbb{E}\Big[\int_{0}^{\tau\wedge\sigma_x} du\, e^{-\delta u} c_2(u)\lambda_u\Big] - \mathbb{E}\Big[1_{\{\sigma_x>\tau\}}e^{-\delta\tau}\int_{0}^{L} du\, e^{-\delta u}\hat{g}_{22}\circ(u,\tau,\lambda_\tau,N_\tau)\Big]
\end{aligned}
$$

where the function $\hat{g}_{21} : [0,T]\times[0,T]\times\mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by

$$
\hat{g}_{21}(u,t,p) = \mathbb{E}\left[c_2(u+t)\,(p+\lambda_u^t)\right] \tag{5.3.26}
$$

and the function $\hat{g}_{22} : [0,T]\times[0,T]\times\mathbb{R}_+\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ is defined by

$$
\hat{g}_{22}(u,t,p,y) = \mathbb{E}\left[1_{\{u<\sigma_{x-y}^t\}}c_2(u+t)(p+\lambda_u^t)\right] \tag{5.3.27}
$$

Define the functions $\hat{h}_1$ and $\hat{h}_2$ by

$$
\begin{aligned}
\hat{h}_1(u,t,p,y) =& c_1\hat{g}_1(u,t,p,y) - \hat{g}_3(u,t,p) + \hat{g}_{21}(u,t,p) \\
\hat{h}_2(u,t,p,y) =& c_5\hat{g}_5(u,t,p,y) - \hat{g}_{22}(u,t,p,y)
\end{aligned}
$$

The sum of all the terms concludes the theorem.

□

5.3.4 Reducing the Optimal Stopping Problem with Unobservable Intensity

In case the intensity process λ is unobservable, this subsection reduces the problem $\hat{\nu}(x)$ to one with complete observation. To that end, we first construct a filter process by proceeding as in [56]. Denote by

$$\mathcal{M}(\mathbb{R}_+) = \mathcal{M}(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$$

the set of all probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. The space $\mathcal{M}(\mathbb{R}_+)$ is endowed with the topology of weak convergence of measures and it is a Polish space [57, p. 420, A.37]. Define the measure-valued stochastic process $\pi : \Omega \times [0, T] \mapsto \mathcal{M}(\mathbb{R}_+)$ by

$$\pi_t(f) := \mathbb{E}\left[f(\lambda_t) \mid \mathcal{F}_t^N \right] \tag{5.3.28}$$

for any bounded Borel $f : \mathbb{R}_+ \mapsto \mathbb{R}$. Then, the sample paths $t \mapsto \pi_t(\omega)$ are right-continuous with left limits, since $t \mapsto \lambda_t(\omega)$ is right-continuous with left limits [56, Lemma 1.1.]. We also note that $\mathcal{F}_{t+}^N \equiv \mathcal{F}_t^N$ in this study, so π defined in (5.3.28) satisfies the definition in [56]. Moreover, define the process $\dot{\lambda} : \Omega \times [0, T] \mapsto \mathbb{R}_+$ by

$$\dot{\lambda}_t := \mathbb{E}\left[\lambda_t \mid \mathcal{F}_t^N \right] = \int_{\mathbb{R}_+} \pi_t(dp)\, p.$$

The following Proposition 5.3.12 helps us replace a function of λ_t with a function of π_t, so that we can replace the unobservable intensity process λ with the filter process π in the optimal stopping problem.

Proposition 5.3.12. *For any continuous bounded* $g : [0, T] \times [0, T] \times \mathbb{Z}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ *define the functional* $\dot{g} : [0, T] \times [0, T] \times \mathcal{M}(\mathbb{R}_+) \times \mathbb{Z}_+ \mapsto \mathbb{R}$ *by*

$$\dot{g}(u, t, \mu, y) = \int_{\mathbb{R}_+} \mu(dp) g(u, t, p, y).$$

Then, for any $\tau \in \mathcal{T}$*,*

$$\mathbb{E}\left[g \circ (u, \tau, \lambda_\tau, N_\tau) \mid \mathcal{F}_\tau^N \right] = \dot{g} \circ (u, \tau, \pi_\tau, N_\tau)$$

Proof. Assume for now that τ takes values in a countable subset D of $[0,T]$. Then, for any $H \in \mathcal{F}_\tau^N$,

$$
\begin{aligned}
&\mathbb{E}\left[1_H g \circ (u, \tau, \lambda_\tau, N_\tau)\right] \\
=&\mathbb{E}\left[\sum_{t\in D} 1_{H\cap\{\tau=t\}} g \circ (u, t, \lambda_t, N_t)\right] \\
=&\mathbb{E}\left[\sum_{t\in D} 1_{H\cap\{\tau=t\}} \sum_{y\in\mathbb{Z}_+} 1_{\{N_t=y\}} g \circ (u, t, \lambda_t, y)\right] \\
=&\sum_{t\in D}\sum_{y\in\mathbb{Z}_+} \mathbb{E}\left[1_{H\cap\{\tau=t\}} 1_{\{N_t=y\}} g \circ (u, t, \lambda_t, y)\right] \quad \text{(MON)} \\
=&\sum_{t\in D}\sum_{y\in\mathbb{Z}_+} \mathbb{E}\left[\underbrace{1_{H\cap\{\tau=t\}} 1_{\{N_t=y\}}}_{\in\mathcal{F}_t^N} \mathbb{E}\left[g \circ (u, t, \lambda_t, y) \mid \mathcal{F}_t^N\right]\right] \\
=&\sum_{t\in D}\sum_{y\in\mathbb{Z}_+} \mathbb{E}\left[1_{H\cap\{\tau=t\}} 1_{\{N_t=y\}} \int_{\mathbb{R}_+} \pi_t(dp) g(u, t, p, y)\right] \quad \text{(Definition of } \pi) \\
=&\sum_{t\in D}\sum_{y\in\mathbb{Z}_+} \mathbb{E}\left[1_{H\cap\{\tau=t\}} 1_{\{N_t=y\}} \dot{g} \circ (u, t, \pi_t, y)\right] \quad \text{(Definition of } \dot{g}) \\
=&\mathbb{E}\left[1_H \dot{g} \circ (u, \tau, \pi_\tau, N_\tau)\right] \quad \text{(MON)}
\end{aligned}
$$

Next, assume that τ takes values in $[0,T]$ and let $(\tau_n)_{n\in\mathbb{N}}$ be an approximating sequence of τ in the sense of [49, p. 178, V.1.20]. That is, $\tau_n = d_n \circ \tau$ where

$$
d_n(t) = \begin{cases} \frac{k+1}{2^n}, & \text{if } \frac{k}{2^n} \le t < \frac{k+1}{2^n} \text{ for some } k \in \mathbb{N} \\ \infty, & \text{if } t = \infty \end{cases}
$$

Then, $\tau_n \ge \tau$ a.s. for all $n \in \mathbb{N}$, τ_n is countably valued and $\tau_n \downarrow \tau$ a.s. Moreover, for any $H \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}$,

$$
\mathbb{E}\left[1_H g \circ (u, \tau_n, \lambda_{\tau_n}, N_{\tau_n})\right] = \mathbb{E}\left[1_H \dot{g} \circ (u, \tau_n, \pi_{\tau_n}, N_{\tau_n})\right], \quad n \in \mathbb{N} \tag{5.3.29}
$$

holds from the previous step and the fact that τ_n is countably valued. Left-hand-side of equation (5.3.29) converges to $\mathbb{E}\left[1_H g \circ (u, \tau, \lambda_\tau, N_\tau)\right]$ as $n \to \infty$ from bounded convergence theorem, since λ and N are right-continuous as well as g is continuous and bounded.

To show that right-hand-side of equation (5.3.29) converges to $\mathbb{E}\left[1_H \dot{g} \circ (u, \tau, \pi_\tau, N_\tau)\right]$, we use [58, Corollary 5.2.] after checking the assumptions. We suppress ω in the sequel for readability. For a.e. $\omega \in \Omega$, we first note that g being continuous implies that $(g(u, \tau_n, p, N_{\tau_n}))_{n\in\mathbb{N}}$ is a family of equicontinuous functions of $p \in \mathbb{R}_+$. Moreover, g being bounded implies that $(g(u, \tau_n, p, N_{\tau_n}))_{n\in\mathbb{N}}$ is uniformly integrable wrto $(\pi_{\tau_n})_{n\in\mathbb{N}}$, that is,

$$\lim_{C\to+\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}_+} \pi_{\tau_n}(dp)\, g \circ (u, \tau_n, p, N_{\tau_n}) 1_{\{p\in\mathbb{R}_+ :\, g\circ(u,\tau_n,p,N_{\tau_n})\geq C\}} = 0.$$

Furthermore, by using bounded convergence theorem, we obtain for a.e. $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \pi_\tau(dp)\, g \circ (u, \tau_n, p, N_{\tau_n}) = \int_{\mathbb{R}_+} \pi_\tau(dp)\, g \circ (u, \tau, p, N_\tau)$$

since $\tau_n \downarrow \tau$, N is right-continuous and g is continuous and bounded. Hence, it follows from [58, Corollary 5.2.] that as $n \to \infty$,

$$\int_{\mathbb{R}_+} \pi_{\tau_n}(dp)\, g \circ (u, \tau_n, p, N_{\tau_n}) \to \int_{\mathbb{R}_+} \pi_\tau(dp)\, g \circ (u, \tau, p, N_\tau), \quad \text{a.s.}$$

Finally, we can write

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[1_H \dot{g} \circ (u, \tau_n, \pi_{\tau_n}, N_{\tau_n})\right] \\
=&\lim_{n\to\infty} \mathbb{E}\left[1_H \int_{\mathbb{R}_+} \pi_{\tau_n}(dp)\, g \circ (u, \tau_n, p, N_{\tau_n})\right] \quad \text{(Definition of } \dot{g}\text{)} \\
=&\mathbb{E}\left[1_H \lim_{n\to\infty} \int_{\mathbb{R}_+} \pi_{\tau_n}(dp)\, g \circ (u, \tau_n, p, N_{\tau_n})\right] \quad \text{(BDD)} \\
=&\mathbb{E}\left[1_H \int_{\mathbb{R}_+} \pi_\tau(dp)\, g \circ (u, \tau, p, N_\tau)\right] \quad \text{[58, Corollary 5.2.]} \\
=&\mathbb{E}\left[1_H \dot{g} \circ (u, \tau, \pi_\tau, N_\tau)\right] \quad \text{(Definition of } \dot{g}\text{)}
\end{aligned}$$

which concludes the proof. □

The following Theorem 5.3.13 finds an equivalent problem to $\hat{\nu}(x)$ in equation

(5.3.16) such that all the processes are adapted to $(\mathcal{F}_t^N)_{t\in[0,T]}$.

Theorem 5.3.13. *Assume* ***INDEP-INC*** *and* ***sMARKOV****. For every* $x \in \mathbb{Z}_+$*, there exist functions* $\dot{h}_1 : [0,T]\times[0,T]\times\mathcal{M}(\mathbb{R}_+)\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ *and* $\dot{h}_2 : [0,T]\times[0,T]\times\mathcal{M}(\mathbb{R}_+)\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ *and* $\dot{g}_4 : [0,T]\times\mathcal{M}(\mathbb{R}_+)\times\mathbb{Z}_+ \mapsto \mathbb{R}_+$ *such that* $\hat{\nu}(x)$ *in equation* (5.3.16) *is equal to* $\dot{\nu}(x)$ *defined by*

$$\begin{aligned}\dot{\nu}(x) := \inf_{\tau\in\mathcal{T}} \Big\{ &\dot{C}(x,\tau) \\ &+ \mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u}\left(\dot{h}_1\circ(u,\tau,\pi_\tau,N_\tau) + 1_{\{\sigma_x>\tau\}}\dot{h}_2\circ(u,\tau,\pi_\tau,N_\tau)\right)\right] \\ &+ \mathbb{E}\left[e^{-\delta\tau}\dot{g}_4\circ(\tau,\pi_\tau,N_\tau)\right]\Big\} \end{aligned} \tag{5.3.30}$$

in the sense that

$$\hat{\nu}(x) = \dot{\nu}(x), \quad x \in \mathbb{Z}_+$$

where $\dot{C} : \mathbb{Z}_+ \times \mathcal{T} \mapsto \mathbb{R}_+$ *is defined by*

$$\begin{aligned}\dot{C}(x,\tau) =& \mathbb{E}\left[\int_0^\tau du\, e^{-\delta u}\left[c_1(x-N_u)^+ - c_3(u)\dot{\lambda}_u + c_2(u)\dot{\lambda}_u\right]\right] \\ &+ \mathbb{E}\left[\int_0^{\tau\wedge\sigma_x} du\, e^{-\delta u}\left[c_5\dot{\lambda}_u - c_2(u)\dot{\lambda}_u\right]\right] + A \end{aligned} \tag{5.3.31}$$

Proof. Let us first convert the running costs. Recall that $\tau \wedge \sigma_x$ is an $(\mathcal{F}_t^N)_{t\in[0,T]}$-stopping time. Hence,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau\wedge\sigma_x} du\, e^{-\delta u}\left[c_5\lambda_u - c_2(u)\lambda_u\right]\right]\\
=&\mathbb{E}\left[\int_0^{T} du\, e^{-\delta u} 1_{\{u<\tau\wedge\sigma_x\}}\left[c_5\lambda_u - c_2(u)\lambda_u\right]\right]\\
=&\int_0^{T} du\, e^{-\delta u}\mathbb{E}\left[1_{\{u<\tau\wedge\sigma_x\}}\left[c_5\lambda_u - c_2(u)\lambda_u\right]\right]\\
=&\int_0^{T} du\, e^{-\delta u}\mathbb{E}\left[\underbrace{1_{\{u<\tau\wedge\sigma_x\}}}_{\in\mathcal{F}_u^N}\mathbb{E}\left[c_5\lambda_u - c_2(u)\lambda_u \mid \mathcal{F}_u^N\right]\right]\\
=&\int_0^{T} du\, e^{-\delta u}\mathbb{E}\left[1_{\{u<\tau\wedge\sigma_x\}}\left[c_5\dot{\lambda}_u - c_2(u)\dot{\lambda}_u\right]\right]\\
=&\mathbb{E}\left[\int_0^{\tau\wedge\sigma_x} du\, e^{-\delta u}\left[c_5\dot{\lambda}_u - c_2(u)\dot{\lambda}_u\right]\right]
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}\left[c_1(x-N_u)^+ - c_3(u)\lambda_u + c_2(u)\lambda_u\right]\right]\\
=&\mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}\mathbb{E}\left[c_1(x-N_u)^+ - c_3(u)\lambda_u + c_2(u)\lambda_u \mid \mathcal{F}_u^N\right]\right]\\
=&\mathbb{E}\left[\int_0^{\tau} du\, e^{-\delta u}\left[c_1(x-N_u)^+ - c_3(u)\dot{\lambda}_u + c_2(u)\dot{\lambda}_u\right]\right]
\end{aligned}
$$

Now, let us focus on the cost of stopping.

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\delta\tau}\hat{g}_4\circ(\tau,\lambda_\tau,N_\tau)\right]\\
=&\mathbb{E}\left[\underbrace{e^{-\delta\tau}}_{\in\mathcal{F}_\tau^N}\mathbb{E}\left[\hat{g}_4\circ(\tau,\lambda_\tau,N_\tau)\mid\mathcal{F}_\tau^N\right]\right]\\
=&\mathbb{E}\left[e^{-\delta\tau}\,\dot{g}_4\circ(\tau,\pi_\tau,N_\tau)\right]\quad\text{(Proposition 5.3.12)}
\end{aligned}
$$

where $\dot{g}_4(t,\mu,y) = \int_{\mathbb{R}_+} \mu(dp)\, \hat{g}_4(t,p,y)$. Moreover,

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u} 1_{\{\sigma_x>\tau\}} \hat{h}_2 \circ (u,\tau,\lambda_\tau,N_\tau)\right] \\
=&\mathbb{E}\left[\underbrace{e^{-\delta\tau}}_{\in\mathcal{F}^N_\tau}\mathbb{E}\left[\int_0^L du\, e^{-\delta u} 1_{\{\sigma_x>\tau\}} \hat{h}_2 \circ (u,\tau,\lambda_\tau,N_\tau)\,\middle|\, \mathcal{F}^N_\tau\right]\right] \\
=&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u}\mathbb{E}\left[1_{\{\sigma_x>\tau\}} \hat{h}_2 \circ (u,\tau,\lambda_\tau,N_\tau)\,\middle|\, \mathcal{F}^N_\tau\right]\right] \quad \text{(Lemma A.2.1 d.)} \\
=&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u} 1_{\{\sigma_x>\tau\}}\mathbb{E}\left[\hat{h}_2 \circ (u,\tau,\lambda_\tau,N_\tau)\,\middle|\, \mathcal{F}^N_\tau\right]\right] \quad \text{(Lemma 5.3.9)} \\
=&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u} 1_{\{\sigma_x>\tau\}} \dot{h}_2 \circ (u,\tau,\pi_\tau,N_\tau)\right] \quad \text{(Proposition 5.3.12)}
\end{aligned}
$$

where $\dot{h}_2(u,t,\mu,y) = \int_{\mathbb{R}_+} \mu(dp)\hat{h}_2(u,t,p,y)$. Similarly,

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u} \hat{h}_1 \circ (u,\tau,\lambda_\tau,N_\tau)\right] \\
=&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u}\mathbb{E}\left[\hat{h}_1 \circ (u,\tau,\lambda_\tau,N_\tau)\,\middle|\, \mathcal{F}^N_\tau\right]\right] \\
=&\mathbb{E}\left[e^{-\delta\tau}\int_0^L du\, e^{-\delta u} \dot{h}_1 \circ (u,\tau,\pi_\tau,N_\tau)\right] \quad \text{(Proposition 5.3.12)}
\end{aligned}
$$

where $\dot{h}_1(u,t,\mu,y) = \int_{\mathbb{R}_+} \mu(dp)\, \hat{h}_1(u,t,p,y)$. □

The optimal stopping problem $\dot{\nu}(x)$ can be solved by means of a continuous-time dynamic programming algorithm. Of course, the derivation of free boundary formulation requires an in-depth analysis and a verification theorem. Therefore, it is left as an extension to this Chapter 5. Among the studies that incorporate unobservable intensity into their models, [59] solves a portfolio optimization problem with terminal utility where the unobservable intensity process is a continuous-time Markov chain. They derive closed-form expressions for several utility functions. Also see [60, 61] for the problems involving unobservable intensity. Moreover, for an optimal stopping problem with markov-modulated dynamics, one may see [62] and the references therein.

Chapter 6

Conclusion and Future Work

In this thesis, we consider several inventory management problems for the manufacturers who strive to properly manage the inventory of spare parts in the end-of-life phase. Background information and motivation of our problems are presented in the introduction (Chapter 1). Here, we summarize the results of this thesis and discuss possible extensions and future research directions.

6.1 Summary

Chapter 2 of this thesis analyzes the value of providing flexibility in placing orders while making use of strategies related to the end-of-life phase. To that end, we consider a manufacturer whose problem is to make one of the three decisions at each period: (1) place an order for spare parts, (2) do nothing and use existing inventory to satisfy demand, or (3) stop holding inventory permanently and use outside/alternative source. This *optimal stopping problem with additional decisions* can be solved by means of a stochastic dynamic programming (DP) algorithm [30]. We provide the DP formulation of our model and the benchmark models which resemble the ideas presented in previous studies.

Chapter 3 of this thesis provides structural properties and analytical insights for the benchmark model $S/1/Z$ in Table 1.2. According to this model, the manufacturer places a single order at time zero and can stop holding inventory, yet the time of stopping is decided at time zero and it is deterministic (rather than a stopping time adapted to the demand filtration). This deterministic time is called switching time. By utilizing the results and expressions in [20], we provide a rigorous proof for the optimality of (s, S)-policy and introduce expressions for the values of the re-order level s and the order-up-to level S. Moreover, we generate an analytical insight that S is a non-decreasing function of the switching time as long as the demand rate and the cost of the outside source are high. Finally, we find conditional upper and lower bounds on the best switching time when the inventory level is fixed.

In Chapter 4, we present our computational results. Section 4.1 utilizes a case study from the literature to verify our code as well as to show the improvements we bring on the solution of the case. In Section 4.2, we expand the parameters and data structures we study to show the benefit of our approach, generating several managerial insights. We mainly generated the analyses and insights while presenting the following remarks. Assuming that a final order must be placed at time zero can be a very strong assumption; the dynamic selection of time to stop (via stopping time) can be valuable; and allowing for multiple orders can be valuable. Moreover, depending on the initial inventory and setup cost values, it might be wise to encourage customers come earlier, invest in outside/alternative source, extend the warranty period, and announce a very large penalty for not satisfying the demand to attract customers.

Chapter 5 of this thesis deals with the problem of selecting the intensity function by allowing intensity rate to be a stochastic process as well. This process is called conditional Poisson process, or doubly stochastic Poisson process, or Cox process. We provide a new construction of this process so that it can be used in an optimal stopping problem related to end-of-life inventory context. We construct this process by using a Poisson random measure and an intensity process that is measurable with respect to the Skorokhod topology. After presenting the necessary definitions, we show the main features of our construction including the

Laplace functional, strong Markov property and its compensated random measure. Then, we use these properties in the solution of continuous-time version of the model $D/1/Z$ (see Table 1.2 for notation), where we further assume that there is a delay between when the stopping decision is made and when the source can be used. We reduce this optimal stopping problem with delay to a classical optimal stopping problem. Finally, in case the intensity process is unobservable, we construct a non-linear filter process taking values in the space of probability measures, reducing the problem to one with complete observation.

6.2 Extensions and Future Work

In Chapter 2 of this thesis, we address the problem of incorporating multiple orders and an outside/alternative source. We assume that the manufacturer can decide to stop and scrap all the inventory. It may also be possible that the manufacturer can partially scrap inventory by paying a fixed and a proportional cost of scrapping. For the existing approaches to this problem, see [63] and the references therein.

Chapter 4 presents numerical analyses and computational results to show the value of our approach. Additional analyses might further show the value of features that we consider in this study. For instance, comparing more benchmark models such as $T/\infty/F, T/1/F, T/1/Z$ and newsvendor model may be useful. Also there may be further runs to analyze the expected total cost in case the period lengths are longer or the setup cost increases over time.

Chapter 5 of this thesis construct a new demand process that can be used while solving an optimal stopping problem related to the end-of-life inventory context. The solution of this problem by means of a continuous-time DP algorithm requires an in-depth analyses and possibly a verification theorem. This can be a natural extension.

In this study, we focus on the trade-off between holding spare parts inventory for non-repairable spare parts and using an outside/alternative source. We do not consider repairable spare parts since it is likely that when a repairable spare part arrives at the system, it will be repaired and given back to the customer. Therefore, such part may not have an effect on the inventory level. On the other hand, it may be possible to design a policy that the manufacturer can use the inventory when a repairable spare part arrives, if doing so reduces the holding cost.

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Appendix A

Supplement for Chapter 5

A.1 Monotone Class Arguments on the Relation between U_f and u_f in a General Setting

Appendix A.1 shows the relation between $U_f(\omega)$ and $u_f(\omega, a)$ in a more general setting. We frequently use this general setting while providing rigorous proofs on the heuristic argument that conditional on λ, N behaves like a Poisson random measure. We essentially prove the following idea. Consider a random measure $M(\omega, dy)$ being independent of a random variable $\lambda(\omega)$, and consider that $M(\omega, dy)$ integrates a random function $f(\lambda(\omega), y)$ whose randomness comes from $\lambda(\omega)$. Then, by conditioning on λ, we first make $M(\omega, dy)$ integrate the deterministic function $f(a, y)$, where a is an arbitrary realization of λ, and then we plug in λ. In words, if a random measure integrates a random function, this random function can be converted into a deterministic function by conditioning.

Such conversion can be of interest to researchers since there is a vast literature on the behavior of $(Mf)(\omega)$, the integral of f with respect to M, when f is a deterministic function. For instance, Laplace transform, mean and variance of Mf are known for a wide range of random measures M.

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space. Let $(E, \mathcal{E}), (G, \mathcal{G})$ be measurable spaces. Let M be a random measure from $(\Omega, \mathcal{H})$ into $(E, \mathcal{E})$. Let $\lambda \colon \Omega \to G$ be a random variable. Suppose that M and λ are independent. For each measurable function $f \colon G \times E \to \bar{\mathbb{R}}_+$, define the measurable function $u_f \colon \Omega \times G \to \bar{\mathbb{R}}_+$ by

$$u_f(\omega, a) \coloneqq u_f(a)(\omega) \coloneqq \int_E M(\omega, dy)\, f(a, y), \quad (\omega, a) \in \Omega \times G,$$

and the random variable $U_f \colon \Omega \to \bar{\mathbb{R}}_+$ by

$$U_f(\omega) \coloneqq u_f(\omega, \lambda(\omega)) = \int_E M(\omega, dy)\, f(\lambda(\omega), y), \quad \omega \in \Omega.$$

When $f = 1_D$ for some $D \in \mathcal{G} \otimes \mathcal{E}$, we write $u_D = u_{1_D}$ and $U_D = U_{1_D}$. Theorem A.1.1 shows the relation between $U_f(\omega)$ and $u_f(\omega, a)$ when M is a finite random measure and f is an indicator function. Corollary A.1.2 extends f to positive measurable functions. Finally, Theorem A.1.3 extends M to a σ-finite random measure. Remark A.1.4 tells how we use this general setting in our study.

Theorem A.1.1. *Suppose that M is finite, that is, $M(\omega, E) < +\infty$ for almost every $\omega \in \Omega$. Then, for every $n \in \mathbb{N}$, every $D_1, \ldots, D_n \in \mathcal{G} \otimes \mathcal{E}$ and every bounded measurable function $h \colon \mathbb{R}_+ \times \bar{\mathbb{R}}_+^n \to \mathbb{R}_+$ that is continuous in its last n arguments, we have*

$$\mathbb{E}\left[h \circ (M(E), U_{D_1}, \ldots, U_{D_n}) | \lambda\right] = \mathbb{E}\left[h \circ (M(E), u_{D_1}(a), \ldots, u_{D_n}(a))\right]|_{a=\lambda}. \tag{A.1.1}$$

In particular, for every bounded continuous function $g \colon \bar{\mathbb{R}}_+^n \to \mathbb{R}_+$.

$$\mathbb{E}\left[g \circ (U_{D_1}, \ldots, U_{D_n}) | \lambda\right] = \mathbb{E}\left[g \circ (u_{D_1}(a), \ldots, u_{D_n}(a))\right]|_{a=\lambda}. \tag{A.1.2}$$

As a further special case, for every $r_1, \ldots, r_n \in \mathbb{R}_+$, we have

$$\mathbb{E}\left[e^{-(r_1 U_{D_1} + \cdots + r_n U_{D_n})} | \lambda\right] = \mathbb{E}\left[e^{-(r_1 u_{D_1}(a) + \cdots + r_n u_{D_n}(a))}\right]\Big|_{a=\lambda}.$$

Proof. Assume for now that $n = 2$. The proof essentially extends D_1 and D_2 from measurable rectangles to arbitrary measurable sets by using monotone class theorem. First, let $D_1 = A_1 \times K_1$ and $D_2 = A_2 \times K_2$ for some $A_1, A_2 \in \mathcal{G}$ and

$K_1, K_2 \in \mathcal{E}$. In this case, we have

$$\begin{aligned} u_{D_1}(\omega, a) = \int_E M(\omega, dy)\, 1_{A_1}(a) 1_{K_1}(y) =& 1_{A_1}(a) \int_E M(\omega, dy)\, 1_{K_1}(y) \\ =& 1_{A_1}(a) M(\omega, K_1), \end{aligned}$$

and similarly,

$$\begin{aligned} U_{D_1}(\omega) =& 1_{A_1}(\lambda(\omega)) M(\omega, K_1). \\ u_{D_2}(\omega, a) =& 1_{A_2}(a) M(\omega, K_2) \\ U_{D_2}(\omega) =& 1_{A_2}(\lambda(\omega)) M(\omega, K_2) \end{aligned}$$

Let h be a bounded measurable function that is continuous in its second and third arguments. Hence,

$$\begin{aligned} & \mathbb{E}\left[h \circ (M(E), (1_{A_1} \circ \lambda) M(K_1), (1_{A_2} \circ \lambda) M(K_2)) | \lambda\right] \\ =& \mathbb{E}\left[h \circ (M(E), 1_{A_1}(a) M(K_1), 1_{A_2}(a) M(K_2))\right] |_{a=\lambda} \text{ (M and λ are independent)} \\ =& \mathbb{E}\left[h \circ (M(E), u_{D_1}(a), u_{D_2}(a))\right] |_{a=\lambda}. \end{aligned}$$

Next, fix $D_1 = A_1 \times K_1$ and let $\mathscr{D}_2$ be the collection of all $D_2 \in \mathcal{G} \otimes \mathcal{E}$ for which (A.1.1) holds for every h as described in the statement in the theorem. We claim that $\mathscr{D}_2$ is a Dynkin system on $G \times E$. By the previous step, $G \times E \in \mathscr{D}_2$. Let $D_2 \in \mathscr{D}_2$. Note that

$$\begin{aligned} u_{D_2^c}(\omega, a) = \int_E M(\omega, dy)\, 1_{D_2^c}(a, y) =& \int_E M(\omega, dy)\, (1 - 1_{D_2}(a, y)) \\ =& M(\omega, E) - u_{D_2}(\omega, a), \end{aligned}$$

and similarly,

$$U_{D_2^c}(\omega) = M(\omega, E) - U_{D_2}(\omega).$$

Since h is a bounded measurable function that is continuous in its second and third arguments, with $\widetilde{h}(x, y, z) = h(x, y, x - z)$, which is also bounded measurable and continuous in its second and third arguments, we obtain

$$\begin{aligned}
&\mathbb{E}\left[h \circ (M(E), U_{D_1}, U_{D_2^c})|\lambda\right] \\
=&\mathbb{E}\left[h \circ (M(E), U_{D_1}, M(E) - U_{D_2})|\lambda\right] \\
=&\mathbb{E}\left[\widetilde{h} \circ (M(E), U_{D_1}, U_{D_2})|\lambda\right] \\
=&\mathbb{E}\left[\widetilde{h} \circ (M(E), u_{D_1}(a), u_{D_2}(a))\right] |_{a=\lambda} \\
=&\mathbb{E}\left[h \circ (M(E), u_{D_1}(a), M(E) - u_{D_2}(a))\right] |_{a=\lambda} \\
=&\mathbb{E}\left[h \circ (M(E), u_{D_1}(a), u_{D_2^c}(a))\right] |_{a=\lambda}.
\end{aligned}$$

Thus, $D_2^c \in \mathscr{D}$. Finally, let $D_2^1 \subseteq D_2^2 \subseteq D_2^3 \subseteq \dots$ be all in $\mathscr{D}_2$. Let $D_2 = \bigcup_{n=1}^{\infty} D_2^n$. By MON for the random measure M, we have

$$u_{D_2}(\omega, a) = \int_E M(\omega, dy) \left(\lim_{n\to\infty} 1_{D_2^n}(a, y)\right) = \lim_{n\to\infty} u_{D_2^n}(\omega, a),$$

and similarly,

$$U_{D_2}(\omega) = \lim_{n\to\infty} U_{D_2^n}(\omega).$$

Let h be a bounded measurable function that is continuous in its second and third arguments. Using bounded convergence theorem (BDD), we obtain

$$\begin{aligned}
&\mathbb{E}\left[h \circ (M(E), U_{D_1}, U_{D_2})|\lambda\right] \\
=&\mathbb{E}\left[\lim_{n\to\infty} h \circ (M(E), U_{D_1}, U_{D_2^n})|\lambda\right] \text{ (continuity of } h) \\
=&\lim_{n\to\infty} \mathbb{E}\left[h \circ (M(E), U_{D_1}, U_{D_2^n})|\lambda\right] \text{ (conditional BDD)} \\
=&\lim_{n\to\infty} \mathbb{E}\left[h \circ (M(E), u_{D_1}(a), u_{D_2^n}(a))\right] |_{a=\lambda} \ (D_2^n \in \mathscr{D}_2 \text{ for each } n \in \mathbb{N}) \\
=&\mathbb{E}\left[\lim_{n\to\infty} h \circ (M(E), u_{D_1}(a), u_{D_2^n}(a))\right] \Big|_{a=\lambda} \text{ (BDD)} \\
=&\mathbb{E}\left[h \circ (M(E), u_{D_1}(a), u_{D_2}(a))\right] |_{a=\lambda} \text{ (continuity of } h).
\end{aligned}$$

So $D_2 \in \mathscr{D}_2$. Hence, $\mathscr{D}_2$ is a Dynkin system containing all measurable rectangles $A_2 \times K_2$ with $A_2 \in \mathcal{G}$, $K_2 \in \mathcal{E}$. By monotone class theorem, $D_2 \in \mathscr{D}_2$ for every $D_2 \in \mathcal{G} \otimes \mathcal{E}$.

As a last step of the case $n = 2$, fix $D_2 \in \mathcal{G}\otimes\mathcal{E}$ and let $\mathscr{D}_1$ be the collection of all

$D_1 \in \mathcal{G} \otimes \mathcal{E}$ for which (A.1.1) holds for every h as described in the theorem. Since h is continuous in its second and third arguments, it is possible to see by using the same steps above that $\mathscr{D}_1$ is a Dynkin system on $G \times E$. Therefore, $\mathscr{D}_1$ is a Dynkin system containing all measurable rectangles $A_1 \times K_1$ with $A_1 \in \mathcal{G}, K_1 \in \mathcal{E}$. By monotone class theorem, $D_1 \in \mathscr{D}_1$ for every $D_1 \in \mathcal{G} \otimes \mathcal{E}$. Hence, we conclude that equation (A.1.1) holds for every $D_1, D_2 \in \mathcal{G} \otimes \mathcal{E}$.

Finally, it is possible to see that the result holds for any $n \in \mathbb{N}$. One can use the same steps above, only difference is that the monotone class theorem is used n times. $\square$

We note that the random variable $M(E)$ inside of $h \circ (M(E), U_{D_1}, \ldots, U_{D_n})$ is an auxiliary random variable. We use the function g in Equation (A.1.2) while using this general setting in our study. The next Corollary A.1.2 extends f from indicators to positive measurable functions.

Corollary A.1.2. *Suppose that M is finite. Then, for every $\mathcal{G} \otimes \mathcal{E}$ measurable and positive f, and every bounded measurable function $h : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ that is continuous in its second argument, we have*

$$\mathbb{E}\left[h \circ (M(E), U_f) | \lambda\right] = \mathbb{E}\left[h \circ (M(E), u_f(a))\right]|_{a=\lambda}. \tag{A.1.3}$$

In particular, for every bounded continuous function $g \colon \bar{\mathbb{R}}_+ \to \mathbb{R}_+$.

$$\mathbb{E}\left[g \circ U_f | \lambda\right] = \mathbb{E}\left[g \circ u_f(a)\right]|_{a=\lambda}.$$

As a further special case, for every $r \in \mathbb{R}_+$, we have

$$\mathbb{E}\left[e^{-rU_f} | \lambda\right] = \mathbb{E}\left[e^{-ru_f(a)}\right]\Big|_{a=\lambda}.$$

Proof. First, assume that $f = 1_D$ for any $D \in \mathcal{G} \otimes \mathcal{E}$. Then, $U_{1_D} = U_D$ and the result follows from the preceding theorem when $n = 1$. Next, assume that

$f = \sum_{i=1}^{n} d_i 1_{D_i}$ where $d_1, \ldots, d_n \in \mathbb{R}_+$ and $D_1, \ldots, D_n \in \mathcal{G} \otimes \mathcal{E}$. Then,

$$\begin{aligned} u_f(\omega, a) = \int_E M(\omega, dy) \left(\sum_{i=1}^{n} d_i 1_{D_i}(a, y) \right) &= \sum_{i=1}^{n} d_i \int_E M(\omega, dy)\, 1_{D_i}(a, y) \\ &= \sum_{i=1}^{n} d_i u_{D_i}(\omega, a) \end{aligned}$$

and similarly,

$$U_f(\omega) = \sum_{i=1}^{n} d_i U_{D_i}(\omega)$$

For any $x, y_1, \ldots, y_n \in \bar{\mathbb{R}}_+$ and bounded measurable function h that is continuous in its second argument, choose $\widetilde{h}(x, y_1, \ldots, y_n) = h(x, \rho(y_1, \ldots, y_n))$ with $\rho(y_1, \ldots, y_n) = d_1 y_1 + \cdots + d_n y_n$. The function $\widetilde{h}$ is bounded and continuous in its last n arguments. Therefore, it follows from the preceding Theorem A.1.1 that

$$\begin{aligned} &\mathbb{E}\left[h \circ (M(E), U_f) | \lambda\right] \\ =&\mathbb{E}\left[h \circ (M(E), d_1 U_{D_1} + \cdots + d_n U_{D_n}) | \lambda\right] \quad \text{(Definition of } U_f\text{)} \\ =&\mathbb{E}\left[\widetilde{h} \circ (M(E), U_{D_1}, \ldots, U_{D_n}) | \lambda\right] \quad \text{(Definition of } \widetilde{h}\text{)} \\ =&\mathbb{E}\left[\widetilde{h} \circ (M(E), u_{D_1}(a), \ldots, u_{D_n}(a))\right] |_{a=\lambda} \quad \text{(Theorem A.1.1)} \\ =&\mathbb{E}\left[h \circ (M(E), d_1 u_{D_1}(a) + \cdots + d_n u_{D_n}(a))\right] |_{a=\lambda} \quad \text{(Definition of } \widetilde{h}\text{)} \\ =&\mathbb{E}\left[h \circ (M(E), u_f(a))\right] |_{a=\lambda} \quad \text{(Definition of } u_f(a)\text{)} \end{aligned}$$

Finally, assume that f is a $\mathcal{G} \otimes \mathcal{E}$ measurable and positive function. Choose a sequence of simple positive functions $(f_n)_{n \in \mathbb{N}}$ such that $f_n \uparrow f$ pointwise. By MON for the random measure M, we have

$$u_f(\omega, a) = \int_E M(\omega, dy) \left(\lim_{n \to \infty} f_n(a, y) \right) = \lim_{n \to \infty} u_{f_n}(\omega, a)$$

and similarly,

$$U_f(\omega) = \lim_{n \to \infty} U_{f_n}(\omega)$$

Let h be a bounded measurable function that is continuous in its second argument.

We have

$$\begin{aligned}\mathbb{E}\left[h\circ(M(E),U_f)|\lambda\right] =&\mathbb{E}\left[\lim_{n\to\infty} h\circ(M(E),U_{f_n})|\lambda\right] \text{ (continuity of } h)\\ =&\lim_{n\to\infty}\mathbb{E}\left[h\circ(M(E),U_{f_n})|\lambda\right] \text{ (conditional BDD)}\\ =&\lim_{n\to\infty}\mathbb{E}\left[h\circ(M(E),u_{f_n}(a))\right]|_{a=\lambda} \text{ (previous step)}\\ =&\mathbb{E}\left[\lim_{n\to\infty} h\circ(M(E),u_{f_n}(a))\right]|_{a=\lambda} \text{ (BDD)}\\ =&\mathbb{E}\left[h\circ(M(E),u_f(a))\right]|_{a=\lambda} \text{ (continuity of h)}\end{aligned}$$

□

Finally, next Theorem A.1.3 extends the results of preceding Corollary A.1.2 when M is a σ-finite random measure and f is a positive measurable function. Here, we stop using the auxiliary random variable $M(E)$ and the function h.

Theorem A.1.3. *Suppose that M is σ-finite, that is, there exists a measurable partition $(E_i)_{i\in\mathbb{N}}$ of E such that for all $i\in\mathbb{N}$, $M(\omega,E_i)<+\infty$ for almost every $\omega\in\Omega$. Then, for every $\mathcal{G}\otimes\mathcal{E}$ measurable and positive function f, and every bounded continuous function $g\colon\bar{\mathbb{R}}_+\to\mathbb{R}_+$.*

$$\mathbb{E}\left[g\circ U_f|\lambda\right]=\mathbb{E}\left[g\circ u_f(a)\right]|_{a=\lambda}.$$

As a further special case, for every $r\in\mathbb{R}_+$, we have

$$\mathbb{E}\left[e^{-rU_f}|\lambda\right]=\mathbb{E}\left[e^{-ru_f(a)}\right]\Big|_{a=\lambda}.$$

Proof. Let $E_1,E_2,\dots\in\mathcal{E}$ be a measurable partition of E such that for all $i\in\mathbb{N}$, $M(\omega,E_i)<\infty$ for almost every $\omega\in\Omega$. Let M_i be the trace of M on E_i. Then, $M_1,M_2,\dots$ are almost everywhere finite random measures on $(E,\mathcal{E})$. Let f be a $\mathcal{G}\otimes\mathcal{E}$ measurable and positive function. For each $i\in\mathbb{N}$, define the measurable functions

$$U_f^{M_i}(\omega) = \int_E M_i(\omega, dy)\, f(\lambda(\omega), y)$$
$$u_f^{M_i}(\omega, a) = \int_E M_i(\omega, dy)\, f(a, y)$$

similar to the preceding definitions, but now the integration is under the random measure M_i. Then, $U_f(\omega) = \sum_{i\in\mathbb{N}} U_f^{M_i}(\omega)$; to see this, we extend f from indicators to simple positive functions to positive measurable functions. First, assume that $f = 1_D$ for some $D \in \mathcal{G} \otimes \mathcal{E}$. Denote $D_i = D \cap (G \times E_i)$. We can write $1_D = \sum_{i\in\mathbb{N}} 1_{D_i}$ since $D_1, D_2, \ldots$ are disjoint. Therefore,

$$\begin{aligned}
U_f(\omega) &= \int_E M(\omega, dy)\, 1_D(\lambda(\omega), y) \\
&= \int_E M(\omega, dy) \left(\sum_{i\in\mathbb{N}} 1_{D_i}(\lambda(\omega), y) \right) \\
&= \sum_{i\in\mathbb{N}} \int_E M_i(\omega, dy)\, 1_{D_i}(\lambda(\omega), y) \qquad \text{(MON)} \\
&= \sum_{i\in\mathbb{N}} \Bigg[\int_E M_i(\omega, dy)\, 1_{D_i}(\lambda(\omega), y) + \underbrace{\sum_{\substack{j\in\mathbb{N} \\ j\neq i}} \int_E M_i(\omega, dy)\, 1_{D_j}(\lambda(\omega), y)}_{=0} \Bigg] \\
&= \sum_{i\in\mathbb{N}} \int_E M_i(\omega, dy)\, 1_{\bigcup_{j\in\mathbb{N}} D_j}(\lambda(\omega), y) \qquad (D_1, D_2, \ldots \text{ are disjoint}) \\
&= \sum_{i\in\mathbb{N}} U_f^{M_i}(\omega)
\end{aligned}$$

where fourth equality is because whenever $i \neq j$,

$$\begin{aligned}
\int_E M_i(\omega, dy)\, 1_{D_j}(\lambda(\omega), y) &= \int_E M(\omega, dy)\, 1_{G\times E_i}(\lambda(\omega), y) 1_{D_j}(\lambda(\omega), y) \\
&= \int_E M(\omega, dy)\, 1_{\emptyset} = 0
\end{aligned}$$

Second, assume that $f = \sum_{j=1}^n d_j 1_{D_j}$ for $d_1, \ldots, d_n \in \mathbb{R}_+$ and $D_1, \ldots, D_n \in \mathcal{G} \otimes \mathcal{E}$.

Then,

$$
\begin{aligned}
U_f(\omega) &= \int_E M(\omega, dy) \left(\sum_{j=1}^{n} d_j 1_{D_j}(\lambda(\omega), y) \right) \\
&= \sum_{j=1}^{n} d_j \int_E M(\omega, dy)\, 1_{D_j}(\lambda(\omega), y) \\
&= \sum_{j=1}^{n} d_j \left[\sum_{i \in \mathbb{N}} \int_E M_i(\omega, dy)\, 1_{D_j}(\lambda(\omega), y) \right] \qquad \text{(By previous step)} \\
&= \sum_{j=1}^{n} d_j \left[\lim_{m \to \infty} \sum_{i=1}^{m} \int_E M_i(\omega, dy)\, 1_{D_j}(\lambda(\omega), y) \right] \\
&= \sum_{i \in \mathbb{N}} \int_E M_i(\omega, dy) \left(\sum_{j=1}^{n} d_j 1_{D_j}(\lambda(\omega), y) \right) \\
&= \sum_{i \in \mathbb{N}} U_f^{M_i}(\omega)
\end{aligned}
$$

Lastly, assume that f is a $\mathcal{G} \otimes \mathcal{E}$ measurable and positive function. Choose a sequence of simple positive functions $(f_n)_{n \in \mathbb{N}}$ such that $f_n \uparrow f$ pointwise. We have

$$
\begin{aligned}
U_f(\omega) &= \int_E M(\omega, dy) \left(\lim_{n \to \infty} f_n(\lambda(\omega), y) \right) \\
&= \lim_{n \to \infty} \int_E M(\omega, dy)\, f_n(\lambda(\omega), y) \qquad \text{(MON)} \\
&= \lim_{n \to \infty} \sum_{i \in \mathbb{N}} \int_E M_i(\omega, dy)\, f_n(\lambda(\omega), y) \qquad \text{(By previous step)} \\
&= \sum_{i \in \mathbb{N}} \int_E M_i(\omega, dy) \left(\lim_{n \to \infty} f_n(\lambda(\omega), y) \right) \qquad \text{(MON)} \\
&= \sum_{i \in \mathbb{N}} U_f^{M_i}(\omega)
\end{aligned}
$$

which shows the equality $U_f(\omega) = \sum_{i \in \mathbb{N}} U_f^{M_i}(\omega)$ for every positive measurable f. Similarly, it is possible to see that

$$
u_f(\omega, a) = \sum_{i \in \mathbb{N}} u_f^{M_i}(\omega, a).
$$

Now, assume for a moment it holds that

$$\mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} U_f^{M_i}\right) \Big| \lambda\right] = \mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} u_f^{M_i}(a)\right)\right]\Big|_{a=\lambda}, \quad n \in \mathbb{N}$$

Then,

$$\begin{aligned}
\mathbb{E}\left[g \circ U_f | \lambda\right] =& \mathbb{E}\left[g \circ \left(\sum_{i=1}^{\infty} U_f^{M_i}\right) \Big| \lambda\right] \\
=& \mathbb{E}\left[\lim_{n\to\infty} g \circ \left(\sum_{i=1}^{n} U_f^{M_i}\right) \Big| \lambda\right] \text{ (Continuity of } g\text{)} \\
=& \lim_{n\to\infty} \mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} U_f^{M_i}\right) \Big| \lambda\right] \text{ (conditional BDD)} \\
=& \lim_{n\to\infty} \mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} u_f^{M_i}(a)\right)\right]\Big|_{a=\lambda} \text{ (assumption above)} \\
=& \mathbb{E}\left[\lim_{n\to\infty} g \circ \left(\sum_{i=1}^{n} u_f^{M_i}(a)\right)\right]\Big|_{a=\lambda} \text{ (BDD)} \\
=& \mathbb{E}\left[g \circ \left(\sum_{i=1}^{\infty} u_f^{M_i}(a)\right)\right]\Big|_{a=\lambda} \text{ (continuity of } g\text{)} \\
=& \mathbb{E}\left[g \circ u_f(a)\right]|_{a=\lambda}
\end{aligned}$$

Therefore, it suffices to show that

$$\mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} U_f^{M_i}\right) \Big| \lambda\right] = \mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} u_f^{M_i}(a)\right)\right]\Big|_{a=\lambda}, \quad n \in \mathbb{N}$$

To that end, for any $K \in \mathcal{E}$, define the random measure M^n by $M^n(\omega, K) = \sum_{i=1}^{n} M_i(\omega, K)$. M^n is a random measure on $(E, \mathcal{E})$ and it is finite. Therefore, it follows from Corollary A.1.2 that

$$\mathbb{E}\left[g \circ U_f^{M^n} | \lambda\right] = \mathbb{E}\left[g \circ u_f^{M^n}(a)\right]|_{a=\lambda}$$

Moreover, one can see that

$$\sum_{i=1}^{n} U_f^{M_i}(\omega) = U_f^{M^n}(\omega) \quad \text{and} \quad \sum_{i=1}^{n} u_f^{M_i}(\omega, a) = u_f^{M^n}(\omega, a)$$

by extending f from indicators to simple positive functions to positive measurable functions, similar to the proof of $U_f(\omega) = \sum_{i\in\mathbb{N}} U_f^{M_i}(\omega)$. We finally have

$$\begin{aligned}\mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} U_f^{M_i}\right) \middle| \lambda\right] =& \mathbb{E}\left[g \circ U_f^{M^n} | \lambda\right] \\ =& \mathbb{E}\left[g \circ u_f^{M^n}(a)\right] |_{a=\lambda} \\ =& \mathbb{E}\left[g \circ \left(\sum_{i=1}^{n} u_f^{M_i}(a)\right)\right] \Big|_{a=\lambda}\end{aligned}$$

which concludes the proof. □

Remark A.1.4. While using Theorem A.1.1, Corollary A.1.2 and Theorem A.1.3, we assume that G is the space of real functions on $[0, T]$ that are right-continuous and have left-limits, and $\mathcal{G}$ is the sigma-algebra on G. The measurable space $(G, \mathcal{G})$ is called the Skorokhod space and it is described in detail by [48, p. 121]. Moreover, we assume that $E = [0, T] \times \mathbb{R}_+$, and that M is a Poisson random measure on $[0, T] \times \mathbb{R}_+$.

A.2 Auxiliary Results

Lemma A.2.1. *(Auxiliary Results) Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space.*

a. The sub-σ-algebras $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of $\mathcal{H}$ are independent if and only if $\mathcal{G}_{\{1,\ldots,n\}} := \bigvee_{i=1}^{n} \mathcal{G}_i$ and $\mathcal{G}_{n+1}$ are independent, for all $n \geq 1$.

b. The sub-σ-algebras $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ of $\mathcal{H}$ are conditionally independent given the sub-σ-algebra $\mathcal{G}$ of $\mathcal{H}$ if and only if $\mathcal{G}_{\{1,\ldots,k\}} := \bigvee_{i=1}^{k} \mathcal{G}_i$ and $\mathcal{G}_{k+1}$ are conditionally independent given $\mathcal{G}$, for $k = 1, 2, \ldots, n-1$.

c. *Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be sub-σ-algebras of $\mathcal{H}$. Assume $\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_2$ and $\mathcal{G}_3 \subset \mathcal{G}_2$. Then, $\mathcal{G}_1$ and $\mathcal{G}_2$ are conditionally independent given $\mathcal{G}_3$.*

d. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{H}$. Let $X : \Omega \times [0,T] \mapsto \mathbb{R}_+$ be a stochastic process. Then,*

$$\mathbb{E}\left[\int_0^L X_u\, du \,\middle|\, \mathcal{G}\right] = \int_0^L \mathbb{E}\left[X_u \mid \mathcal{G}\right] du.$$

e. *Let $X = \{X_t, t \geq 0\}$ be a stochastic process taking values in $(E, \mathcal{E})$. If X has independent increments, then it has the Markov property as well.*

f. *An adapted process with right-continuous paths is progressively measurable.*

g. *For any $x, a, b \in \mathbb{Z}_+$, $(x - a - b)^+ = ((x-a)^+ - b)^+$.*

Proof. a. See [49, p. 84, II.5.5].

b. See [49, p. 159, IV.3.6].

c. Let V_1 and V_2 be $\mathcal{G}_1$ and $\mathcal{G}_2$ measurable random variables respectively. Then,

$$\begin{aligned}
\mathbb{E}\left[V_1V_2|\mathcal{G}_3\right] =& \mathbb{E}\left[\mathbb{E}\left[V_1V_2|\mathcal{G}_2\right]|\mathcal{G}_3\right] \qquad (\mathcal{G}_3 \subset \mathcal{G}_2) \\
=& \mathbb{E}\left[V_2\mathbb{E}\left[V_1|\mathcal{G}_2\right]|\mathcal{G}_3\right] \qquad (V_2 \text{ is } \mathcal{G}_2 \text{ measurable }) \\
=& \mathbb{E}\left[V_2\mathbb{E}\left[V_1\right]|\mathcal{G}_3\right] \qquad (\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_2) \\
=& \mathbb{E}\left[V_1\right]\mathbb{E}\left[V_2|\mathcal{G}_3\right] \qquad (\mathbb{E}\left[V_1\right] \text{ is constant}) \\
=& \mathbb{E}\left[V_1|\mathcal{G}_3\right]\mathbb{E}\left[V_2|\mathcal{G}_3\right] \qquad (\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_3)
\end{aligned}$$

d. Let $G \in \mathcal{G}$ be given. Then,

$$\begin{aligned}
\mathbb{E}\left[1_G \int_0^L X_u\, du\right] &= \int_0^L \mathbb{E}\left[1_G X_u\right] du \\
&= \int_0^L \mathbb{E}\left[1_G \mathbb{E}\left[X_u \mid \mathcal{G}\right]\right] du \\
&= \mathbb{E}\left[1_G \int_0^L \mathbb{E}\left[X_u \mid \mathcal{G}\right] du\right]
\end{aligned}$$

where we use Fubini's Theorem in first and last equations.

e. Let $f : E \mapsto \mathbb{R}$ be a bounded measurable function. Then, for any s and t in $\mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}[f \circ X_{t+s}|\mathcal{F}_t^X] =&\mathbb{E}\left[f \circ \left(X_{t+s} - X_t + X_t\right)\middle|\mathcal{F}_t^X\right]\\ =&\mathbb{E}\left[f \circ \left(X_{t+s} - X_t + X_t\right)\middle|\sigma(X_t)\right]\\ =&\hat{f} \circ X_t\end{aligned}$$

where $\hat{f}(x) = \mathbb{E}[f \circ (X_{t+s} - X_t + x)]$. Second equation is because $X_{t+s} - X_t$ is independent of $\mathcal{F}_t^X$. Last equation is because $X_{t+s} - X_t$ is independent of X_t.

f. See [64, p. 44, I.1.13], or [65, p. 5, I.1.13]

□

Remark A.2.2. Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space and $(G, \mathcal{G})$ be a Skorokhod space. Let $\lambda : \Omega \times [0, T] \to \mathbb{R}_+$ be a right-continuous and left-limited process. Let $\lambda_\bullet : \Omega \mapsto G$ be the path $\lambda_\bullet(\omega) = (t \mapsto \lambda_t(\omega))$. Let $Y_t : G \mapsto \mathbb{R}_+$ be the coordinate mapping, that is, for every $a \in G$, $Y_t(a) = a_t$. We have

$$\mathcal{F}_T^\lambda = \sigma(\lambda_s, s \le T) = \sigma(Y_s(\lambda_\bullet), s \le T) = \sigma(\lambda_\bullet)$$

where second equality is because for every $\omega \in \Omega$ and $s \in [0, T]$, $\lambda_s(\omega) = Y_s(\lambda_\bullet(\omega))$ [49, p. 348, VII.3.24(iii)(a)]. Third equality is because $\mathcal{G} = \sigma(Y_s, s \le T)$ [48, p. 134, 12.5(iii)].