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Longest increasing subsequences in involutions avoiding patterns of length three

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Abstract: We study the longest increasing subsequences in random involutions that avoid the patterns of length three under the uniform probability distribution. We determine the exact and asymptotic formulas for the average length of the longest increasing subsequences for such permutation classes.

Key words: Pattern-avoidance, involutions, longest increasing subsequences, Chebyshev polynomials, generating functions

1. Introduction

A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ on the set $[n] := \{1, 2, \cdots, n\}$ is any arrangement of the elements of $[n]$, which can also be considered as a bijection on $[n]$ where $i \to \sigma_i$. A permutation σ is called an involution if $\sigma = \sigma^{-1}$ where $\sigma_i^{-1} = j$ if and only if $\sigma_j = i$. We use S_n and $\mathbf{Inv}_\mathrm{n}$ to denote the set of all permutations and involutions of length n, respectively. For $\tau = \tau_1\tau_2\cdots\tau_k \in S_k$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, it is said that τ appears as a pattern in σ if there exists a subset of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\sigma_{i_s} < \sigma_{i_t}$ if and only if $\tau_s < \tau_t$ for all $1 \leq s, t \leq k$. If τ does not appear as a pattern in σ, then σ is called a τ-avoiding permutation. For example, $132 \in S_3$ appears as a pattern in 246513 because it has the subsequences $24---3$, $2-6--3$, $2-65--$, $2--5-3$, or $-465--$. On the other hand $4213 \in S_4$ does not appear as a pattern in 246513. We denote by $S_n(\tau)$ the set of all τ-avoiding permutations of length n. More generally, for a set T of patterns, we use the notation $S_n(T) = \bigcap_{\tau\in T} S_n(\tau)$. We denote the corresponding pattern-avoiding involution classes by $\mathbf{Inv}_\mathrm{n}(\tau)$ and $\mathbf{Inv}_\mathrm{n}(T)$. A nice introduction to the subject is provided in the fourth and fifth chapters of [4]. Specifically for more on the pattern-avoiding involutions, see [5, 10–12, 15–17, 20].

In this paper, we shall study the longest increasing subsequence problem on $\mathbf{Inv}_\mathrm{n}(T)$ under the uniform probability distribution for subsets $T \subseteq S_3$. We use $L_n(\sigma)$ to denote the length of the longest increasing subsequence in σ, i.e.

$$L_n(\sigma) = \max\{k \in [n] : \text{there exist } 1 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ and } \sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}\}.$$

Note that, in general, there might be more than one such subsequence. The problem of determining the asymptotic behavior and limiting distribution of L_n on S_n under the uniform probability distribution has led to very interesting and important research in the last fifty years, which made some unexpected connections

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among different fields of mathematics and physics; see [1–3, 7, 8, 13, 18, 26] and references therein. Probabilistic study of pattern-avoiding permutation classes has recently become an active area of research; for some recent works in this direction, see [9, 19, 22–24].

The symmetries reverse and complement defined on S_n as $\sigma_i^r = \sigma_{n+1-i}$, $\sigma_i^c = n+1-\sigma_i$, respectively, reduce the number of cases needed to be studied. Note that $\mathbf{Inv}_{\mathrm{n}}(\tau) = \mathbf{Inv}_{\mathrm{n}}(\tau^{-1})$, and if $\sigma \in \mathbf{Inv}_{\mathrm{n}}(\tau)$, then $\sigma^{rc} \in \mathbf{Inv}_{\mathrm{n}}(\tau)$. Note also that for any $\sigma \in \mathbf{Inv}_{\mathrm{n}}$, we have $L_n(\sigma) = L_n(\sigma^{rc})$.

We will make use of some Dyck path arguments in our proofs. Recall that a Dyck path L of length $2n$ is made up of n up steps (U), $(x,y) \nearrow (x+1,y+1)$, and n down steps (D), $(x,y) \searrow (x+1,y-1)$, where the path starts at $(0,0)$, ends at $(2n,0)$, and never falls below the x-axis. The left-factor of a Dyck path is made up of all the steps that precede the last up step.

In some of our arguments, we will also use the well-known Robinson–Schensted correspondence. Recall that a standard Young tableau is a Ferrers shape with n boxes such that each box contains an entry from the set $[n]$ where the columns and rows have entries appearing in increasing order. The Robinson–Schensted correspondence is a mapping that takes a permutation $\sigma \in S_n$ and returns a unique ordered pair (P_σ, Q_σ) of standard Young tableaux of the same shape and of size n [27]. The length of the first row in these two standard Young tableaux corresponds to the length of the longest increasing subsequence in σ. Moreover, if the correspondence maps σ to (P_σ, Q_σ), then it maps the inverse permutation σ^{-1} to the pair (Q_σ, P_σ). Therefore, any involution $\sigma \in \mathbf{Inv}_{\mathrm{n}}$ is identified with a unique single standard Young tableau P_σ.

We mainly use generating functions to prove our results, which are defined by

$$F_{T;m}(x) = \sum_{n\geq 0} \sum_{\substack{\sigma\in\mathbf{Inv}_{\mathrm{n}}(T)\\ L_n(\sigma)=m}} x^n \text{ and } F_T(x,q) = \sum_{m\geq 0} F_{T;m}(x) q^m \tag{1.1}$$

where $T \subseteq S_3$. For simplicity, we write F_τ for $\{\tau\} \subseteq S_3$ and $F_{\tau,\tau'}$ for $\{\tau,\tau'\} \subseteq S_3$. The coefficient of x^n in a generating function $G(x)$ is represented by $[x^n]G$.

In the rest of this paper, we use generic $\mathbf{P}$ to denote the uniform probability distribution on the sets $\mathbf{Inv}_{\mathrm{n}}(T)$ with $T \subseteq S_3$. That is, for any subset $A \subseteq \mathbf{Inv}_{\mathrm{n}}(T)$,

$$\mathbf{P}(A) = \frac{|A|}{|\,\mathbf{Inv}_{\mathrm{n}}(T)|}.$$

The cardinality of a set A is denoted by $|A|$. The expected value $\mathbf{E}(L_n)$ of the random variable L_n on $\mathbf{Inv}_{\mathrm{n}}(T)$ under $\mathbf{P}$ can be calculated by

$$\mathbf{E}(L_n) = \frac{1}{|\,\mathbf{Inv}_{\mathrm{n}}(T)|}[x^n]\frac{\partial}{\partial q} F_T(x,q)\Big|_{q=1} \tag{1.2}$$

where $|\,\mathbf{Inv}_{\mathrm{n}}(T)| = [x^n]F_T(x,1)$.

It is a well-known fact that the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ is given by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. Recall also that for any $\tau \in S_3$, $|S_n(\tau)| = \frac{1}{n+1}\binom{2n}{n}$. For involutions, we have $|\,\mathbf{Inv}_{\mathrm{n}}(\tau)| = \binom{n}{\lfloor n/2 \rfloor}$ for $\tau \in \{123, 132, 213, 321\}$ and $|\,\mathbf{Inv}_{\mathrm{n}}(\tau)| = 2^{n-1}$ for $\tau \in \{231, 312\}$ [28].

For two sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

2. One-pattern case: $\mathbf{E}(L_n)$ on $\mathbf{Inv}_\mathrm{n}(\tau)$ with $\tau \in S_3$

2.1. $\tau = 123$

The generating function for the number of involutions in $\mathbf{Inv}_\mathrm{n}(123)$ is given by $\frac{1}{1-x-x^2C(x^2)}$ [28]. Note that $F_{123;m}(x) = 0$ for all $m \geq 3$. Clearly, $F_{123;0}(x) = 1$ because the only involution σ with $L_n(\sigma) = 0$ is the empty involution. If $L_n(\sigma) = 1$ and $\sigma \in \mathbf{Inv}_\mathrm{n}(123)$, then $\sigma = n(n-1)\cdots 1$ and hence $F_{123;1}(x) = \frac{x}{1-x}$. Therefore, $F_{123;2}(x) = \frac{1}{1-x-x^2C(x^2)} - 1 - \frac{x}{1-x} = \frac{1}{1-x-x^2C(x^2)} - \frac{1}{1-x}$. Since the coefficient of x^n in $\frac{1}{1-x-x^2C(x^2)}$ is given by $\binom{n}{\lfloor n/2 \rfloor}$, we have

$$\mathbf{E}(L_n) = \frac{1 + 2\left(\binom{n}{\lfloor n/2 \rfloor} - 1\right)}{\binom{n}{\lfloor n/2 \rfloor}} \sim 2.$$

2.2. $\tau = 132, 213$

Recall that the Chebyshev polynomials of the second kind [25] are defined by the following recurrence relation:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \text{ with } U_0(x) = 1 \text{ and } U_1(x) = 2x.$$

By symmetries under reverse and complement, we have that $F_{132}(x, q) = F_{213}(x, q)$.

Note that $|\{\sigma \in \mathbf{Inv}_\mathrm{n}(132) : L_n(\sigma) = k\}| = |\,\mathbf{Inv}_\mathrm{n}(132, 12\cdots k+1) \setminus \mathbf{Inv}_\mathrm{n}(132, 12\cdots k)|$. The generating function for the number of involutions in $\mathbf{Inv}_\mathrm{n}(132, 12\cdots k)$ is given by $G_k(x) = \frac{1}{xU_k(1/2x)} \sum_{j=0}^{k-1} U_j(1/2x)$ [16]. Hence,

$$F_{132;m}(x) = \frac{1}{xU_{m+1}(1/2x)} \sum_{j=0}^{m} U_j(1/2x) - \frac{1}{xU_m(1/2x)} \sum_{j=0}^{m-1} U_j(1/2x).$$

Moreover, we know that $F_{132;m}(x) = G_{m+1}(x) - G_m(x)$ is the generating function for the number of left factors of Dyck paths of length n that have height m [16]; see also Sequence A132890 in [29]. Thus, by Theorem 3 in [21], and also Sequence A132891 in [29], we get

$$\mathbf{E}(L_n) \sim \ln(2)\sqrt{2\pi n}.$$

2.3. $\tau = 231, 312$

By symmetries under reverse and complement, we have that $F_{231}(x, q) = F_{312}(x, q)$. Note that any nonempty involution $\sigma \in \mathbf{Inv}_\mathrm{n}(231)$ can be written as $\sigma = j \cdots 1\sigma'$ where σ' is an involution on $\{j+1, j+2, \ldots, n\}$ that avoids 231 and $j \geq 1$ [11]. Thus,

$$F_{231}(x, q) = 1 + \frac{xq}{1-x} F_{231}(x, q),$$

where 1 counts the empty involution, $\frac{xq}{1-x}$ counts the nonempty decreasing sequence $j(j-1)\cdots 1$, and $F_{231}(x, q)$ counts the involutions σ'. Note that the coefficient of x^n in $\frac{\partial}{\partial q} F_{231}(x, q)\mid_{q=1}$ is given by $(n+1)2^{n-2}$ for all $n \geq 1$. Thus,

$$\mathbf{E}(L_n) = \frac{n+1}{2}.$$

2.4. $\tau = 321$

By the Robinson–Schensted correspondence, we see that each involution σ in $\mathbf{Inv}_{\rm n}(321)$ with $L_n(\sigma) = m$ has at most two rows and exactly m columns (in its corresponding Young tableau). By distinguishing between the cases where σ has one row or two rows, we obtain

$$F_{321}(x,q) = 1 + \sum_{n\geq 1} x^n q^n + \sum_{n\geq 2}\sum_{m=\lceil n/2\rceil}^{n-1} \frac{2m+1-n}{m+1}\binom{n}{m} q^m x^n.$$

Note that 1 counts the empty involution; $\sum_{n\geq 1} x^n q^n$ counts the involutions with one row, which are $12\cdots n \in \mathbf{Inv}_{\rm n}(321)$; and the last term counts the involutions with two rows. Note also that the number of involutions σ with two rows and m columns, i.e. $L_n(\sigma) = m$, is given by $\frac{2m+1-n}{m+1}\binom{n}{m}$; see [15]. Recall that any standard Young tableau Y with at most two rows and n boxes can be represented as a left factor of a Dyck path L as follows: whenever we read an entry j from the first (second) row of Y, we create an up (down) step in L, for all $j = 1, 2, \ldots, n$. For instance, the involution $2143 \in \mathbf{Inv}_4(321)$ can be represented as a standard Young tableau $Y = \begin{matrix} 13 \\ 24 \end{matrix}$ and as a left factor $UDUD$ of a Dyck path. Hence, by Sequence A014314 in [29], we get that the total number of all up steps in all the left factors of Dyck paths of length n, which is equal to $\sum_{\sigma\in\mathbf{Inv}_{\rm n}(321)} L_n(\sigma)$, is given by the following formulas:

$$\text{if } n = 2k, \text{ then it equals } 2^{2k-1} + (2k-1)\binom{2k-1}{k}, \text{ and } \text{ if } n = 2k+1, \text{ then it equals } 2^{2k} + 2k\binom{2k+1}{k}.$$

Hence, by [28], we have

$$\mathbf{E}(L_{2k}) = \frac{2^{2k-1} + (2k-1)\binom{2k-1}{k}}{\binom{2k}{k}} \sim k \quad \text{and} \quad \mathbf{E}(L_{2k+1}) = \frac{2^{2k} + 2k\binom{2k+1}{k}}{\binom{2k+1}{k}} \sim \frac{2k+1}{2},$$

which leads to

$$\mathbf{E}(L_n) \sim \frac{n}{2}.$$

3. Two-pattern case: $\mathbf{E}(L_n)$ on $\mathbf{Inv}_{\rm n}(\tau, \tau')$ with $\{\tau, \tau'\} \subseteq S_3$

In this section, we determine $\mathbf{E}(L_n)$ on $\mathbf{Inv}_{\rm n}(\tau, \tau')$ for any $\{\tau, \tau'\} \subseteq S_3$. There are 15 possible cases but thanks to the symmetries, $\mathbf{Inv}_{\rm n}(\tau, \tau') = \mathbf{Inv}_{\rm n}(\tau^{-1}, (\tau')^{-1})$ or $L_n(\sigma) = L_n(\sigma^{rc})$, we only need to consider some specific cases. See the Table. Note that $\mathbf{Inv}_{\rm n}(123, 321) = \emptyset$ for $n \geq 5$. Hence, we omit this case.

We will first deal with the case $T = \{132, 213\}$, which requires more work than the other cases considered in Theorem 3.3. Recall that a composition of a nonnegative integer n is any sequence $c = c_1c_2\cdots c_m$ of positive integers such that $c_1 + \cdots + c_m = n$. In this context $c_1, \cdots, c_m$ are called parts of c. A composition is called palindromic if it reads the same from left to right as from right to left. For example, the compositions of 4 are $4, 31, 22, 211, 13, 121, 112, 1111$ and the palindromic compositions of 4 are $4, 22, 121, 1111$. We use P_n to denote the set of all palindromic compositions of n. The following lemma gives a bijection between $\mathbf{Inv}_{\rm n}(132, 213)$ and P_n.

Table. Summary of the two pattern-avoiding cases. For each case, we have some corresponding pairs under symmetries $\mathbf{Inv}_n(\tau,\tau') = \mathbf{Inv}_n(\tau^{-1},(\tau')^{-1})$ or $L_n(\sigma) = L_n(\sigma^{rc})$.

Case	τ,τ'	Symmetric pair	$\mathbf{E}(L_n)$	Case	τ,τ'	Symmetric pair	$\mathbf{E}(L_n)$
A	123, 132	123, 213	~ 2	D	132, 231 213, 312	132, 312	$= \frac{n+1}{2}$
B	123, 231	123, 312	$= 2 - \frac{1}{n}$	E	132, 321	213, 321	$\sim \frac{3n}{4}$
C	231, 321	312, 321	$\sim \frac{1+\sqrt{5}}{2\sqrt{5}}n$	F	231, 312	—	$= \frac{n+1}{2}$
Theorem 3.2	132, 213	213, 231	$\sim \log_2 n$				

Lemma 3.1 *There exists a bijection* $f : \mathbf{Inv}_n(132, 213) \to P_n$ *such that the length of the longest increasing subsequence in* σ *equals the maximal part of* $f(\sigma)$ *for all* $\sigma \in \mathbf{Inv}_n(132, 213)$*.*

Proof Note that any involution $\sigma \in \mathbf{Inv}_n(132, 213)$ can be written either as $12\cdots n$ or as $\sigma = (n+1-j)(n+2-j)\cdots n\sigma'12\cdots j$ with $1 \le j \le n/2$ such that σ' is an involution of $\{j+1, j+2, \ldots, n-j\}$ that avoids both 132 and 213. Define $I_{a,b} = a(a+1)\cdots b$ for all $a \le b$. Thus, by induction, there exists $j_1, j_2, \cdots j_m$ of positive integers such that

$$\begin{aligned}\sigma = \;& I_{n+1-j_1,n} I_{n+1-j_1-j_2,n-j_1} \cdots I_{n+1-j_1-\cdots-j_m,n-j_1-\cdots-j_{m-1}} I_{j_1+j_2+\cdots+j_m+1,n-j_1-\cdots-j_m} \\ & I_{j_1+j_2+\cdots+j_{m-1}+1,j_1+\cdots+j_m} \cdots I_{j_1+1,j_1+j_2} I_{1,j_1}.\end{aligned}$$

If we define $f(\sigma) = j_1 j_2 \cdots j_m(n - 2j_1 - \cdots - 2j_m)j_m \cdots j_2 j_1$, then $f(\sigma)$ is a palindromic composition of n. Clearly, f is a bijection between $\mathbf{Inv}_n(132, 213)$ and P_n. Moreover, the length of the longest increasing subsequence in σ is m if and only if the maximal part of the palindromic composition $f(\sigma)$ is m. □

We define the random variable X_n on P_n as the maximal part in a uniformly random palindromic composition of n. By Lemma 3.1, we have $L_n(\sigma) = X_n(f(\sigma))$ for all $\sigma \in \mathbf{Inv}_n(132, 213)$ and hence $\mathbf{E}(L_n) = \mathbf{E}(X_n)$ for all n.

Theorem 3.2 *Consider* $\mathbf{E}(L_n)$ *on* $\mathbf{Inv}_n(132, 213)$ *under the uniform probability distribution. Then we have* $\mathbf{E}(L_n) \sim \log_2 n$*.*

Proof We will show that $\mathbf{E}(X_n) \sim \log_2 n$. Define P_n^e (P_n^o) to be the set of all palindromic compositions of n with even (odd) number of parts, respectively. Let $p^e(n) = |P_n^e|$, $p^o(n) = |P_n^o|$, and $p(n) = p^e(n) + p^o(n)$. Clearly, $p(n) = 2^{\lfloor n/2 \rfloor}$. We define X_n^e (X_n^o) to be a random variable on P_n^e (P_n^o) as the maximal part in a uniformly random palindromic composition of n with even (odd) number of parts, respectively. Thus,

$$\mathbf{E}(X_n) = \frac{p^e(n)}{p(n)}\mathbf{E}(X_n^e) + \frac{p^o(n)}{p(n)}\mathbf{E}(X_n^o).$$

By Remark 3 and Theorem 4 in [6], we have $p^e(2n) = p^o(2n) = 2^{n-1}$, $p^e(2n+1) = 0$, and $p^o(2n+1) = 2^n$. Hence,

$$\mathbf{E}(X_{2n}) = \frac{1}{2}\mathbf{E}(X_{2n}^e) + \frac{1}{2}\mathbf{E}(X_{2n}^o), \tag{3.1}$$

$$\mathbf{E}(X_{2n+1}) = \mathbf{E}(X_{2n+1}^o). \tag{3.2}$$

Step 1: We will show that $\mathbf{E}(X_{2n}^e) \sim \log_2 n$. For any palindromic composition $\sigma = \sigma_1 \cdots \sigma_{2m} \in P_{2n}^e$, we define $f_e(\sigma) = 0^{\sigma_1-1}10^{\sigma_2-1}1\cdots 0^{\sigma_m-1}$, where a^s denotes the word $aa\cdots a$ with s occurrences of the letter a. Clearly, f_e is a bijection between P_{2n}^e and the set of binary words of length $n-1$. Moreover, the maximal part of σ is m if and only if the length of the maximal run, maximal s such that 0^s is a subword, in $f_e(\sigma)$ is $m-1$. Thus, by Proposition V.1 in [14], we have

$$\mathbf{E}(X_{2n}^e) \sim \log_2 n.$$

Step 2: We will show that $\mathbf{E}(X_{2n}^o) \sim \log_2 n$. Let $\sigma = \sigma_1 \cdots \sigma_{2m+1} \in P_{2n}^o$, so $\sigma_i = \sigma_{2m+2-i}$ for all $i = 1, 2, \ldots, m$, which implies that σ_{m+1} is an even number. We define $f_o(\sigma) = 0^{\sigma_1-1}10^{\sigma_2-1}1\cdots 0^{\sigma_m-1}10^{\sigma_{m+1}/2-1}$. Clearly, f_o is a bijection between P_{2n}^o and the set of binary words of length $n-1$. Moreover, the maximal size of part in σ is at most the length of the maximal run in $f_o(\sigma) + \sigma_{m+1}/2$. Hence, by Proposition V.1 in [14], we have

$$\mathbf{E}(X_{2n}^o) \leq \log_2 n + \frac{\sum_{\sigma_1\cdots\sigma_{2m+1}\in P_{2n}^o} \sigma_{m+1}}{p^o(2n)}.$$

Note that the generating function for the number of $\sigma \in P_{2n}^o$ according to the size of the middle part in σ is given by

$$\begin{aligned} A(x,q) &= \sum_{m\geq 0} \frac{x^{2m}}{(1-x^2)^m} \sum_{s\geq 1} x^{2s} q^{2s} = \frac{(1-x^2)x^2q^2}{(1-2x^2)(1-x^2q^2)} \\ &= q^2x^2 + (q^4+q^2)x^4 + (q^6+q^4+2q^2)x^6 + (q^8+q^6+2q^4+4q^2)x^8 + \cdots, \end{aligned}$$

which leads to

$$\frac{\sum_{\sigma_1\cdots\sigma_{2m+1}\in P_{2n}^o} \sigma_{m+1}}{p^o(2n)} = \frac{[x^{2n}]\frac{\partial}{\partial q}A(x,q)\mid_{q=1}}{[x^{2n}]A(x,1)} \sim 4.$$

Hence,

$$\mathbf{E}(X_{2n}^o) \leq \log_2 n + 4 \sim \log_2 n.$$

By the mapping $\sigma_1 \cdots \sigma_{2m} \in P_{2n}^e$ to $\sigma_1 \cdots \sigma_{m-1}(2\sigma_m)\sigma_{m+1}\cdots\sigma_{2m} \in P_{2n}^o$ (see Remark 3 and Theorem 4 in [6]), we see that $\mathbf{E}(X_{2n}^e) \leq \mathbf{E}(X_{2n}^o)$. Hence,

$$\mathbf{E}(X_{2n}^e) \leq \mathbf{E}(X_{2n}^o) \leq \log_2 n + 4 \sim \log_2 n,$$

which proves that $\mathbf{E}(X_{2n}^o) \sim \log_2 n$.

By $\mathbf{E}(X_{2n}^o) \sim \log_2 n$, $\mathbf{E}(X_{2n}^e) \sim \log_2 n$, and (3.1), we obtain $\mathbf{E}(X_{2n}) \sim \log_2 n$.

Step 3: We will show that $\mathbf{E}(X_{2n+1}^o) \sim \log_2 n$. By using the above arguments, we also have

$$\mathbf{E}(X_{2n}^e) \leq \mathbf{E}(X_{2n+1}^o) \leq \log_2 n + \frac{\sum_{\sigma_1\cdots\sigma_{2m+1}\in P_{2n+1}^o} \sigma_{m+1}}{p^o(2n+1)}.$$

Note that the generating function for the number of $\sigma \in P_{2n+1}^o$ according to the size of the middle part in σ is given by

$$\begin{aligned} B(x,q) &= \sum_{m\geq 0} \frac{x^{2m}}{(1-x^2)^m} \sum_{s\geq 1} x^{2s-1} q^{2s-1} = \frac{(1-x^2)xq}{(1-2x^2)(1-x^2q^2)} \\ &= qx + (q^3+q)x^3 + (q^5+q^3+2q)x^5 + (q^7+q^5+2q^3+4q)x^7 + \cdots, \end{aligned}$$

which leads to

$$\frac{\sum_{\sigma_1\cdots\sigma_{2m+1}\in P^o_{2n+1}}\sigma_{m+1}}{p^o(2n+1)}=\frac{[x^{2n+1}]\frac{\partial}{\partial q}B(x,q)\mid_{q=1}}{[x^{2n+1}]B(x,1)}\sim 3.$$

Thus,

$$\mathbf{E}(X^o_{2n+1})\leq \log_2 n+3\sim\log_2 n.$$

Hence, by Step 1, we have

$$\mathbf{E}(X^e_{2n})\leq\mathbf{E}(X^o_{2n+1})\leq\log_2 n+3\sim\log_2 n,$$

which proves that $\mathbf{E}(X^o_{2n+1})\sim\log_2 n$. Thus, by (3.2), we have $\mathbf{E}(X_{2n+1})\sim\log_2 n$, which completes the proof. □

The following theorem covers the remaining cases.

Theorem 3.3 *Consider $\mathbf{E}(L_n)$ on $\mathbf{Inv}_{\rm n}(T)$ under the uniform probability distribution. Then we have:*

A- If $T=\{123,132\}$, then $\mathbf{E}(L_n)\sim 2$.

B- If $T=\{123,231\}$, then $\mathbf{E}(L_n)=2-\frac{1}{n}$.

C- If $T=\{231,321\}$, then $\mathbf{E}(L_n)\sim\frac{1+\sqrt{5}}{2\sqrt{5}}n$.

D- If $T=\{132,231\}$, then $\mathbf{E}(L_n)=\frac{n+1}{2}$.

E- If $T=\{132,321\}$, then $\mathbf{E}(L_n)\sim\frac{3n}{4}$.

F- If $T=\{231,312\}$, then $\mathbf{E}(L_n)=\frac{n+1}{2}$.

Proof

- Case A. By [16], we see that the generating function for the number of involutions in $\mathbf{Inv}_{\rm n}(123,132)$ is given by $G_3(x)=\frac{1+x}{1-2x^2}$. Note that $F_{T;m}(x)=0$ for all $m\geq 3$, $F_{T;0}(x)=1$, which counts only the empty involution, and $F_{T;1}(x)=\frac{x}{1-x}$, which counts the involutions $n\cdots 21$. Thus, $F_{T;2}(x)=G_3(x)-\frac{x}{1-x}-1$. Hence,

$$F_{123,132}(x,q)=1+\frac{xq}{1-x}+q^2\left(\frac{1+x}{1-2x^2}-\frac{1}{1-x}\right).$$

Therefore, by taking the derivative at $q=1$ and then finding the coefficient of x^n,

$$\mathbf{E}(L_n)=\frac{(2-\sqrt{2})(-\sqrt{2})^{n-2}+(2+\sqrt{2})\sqrt{2}^{n-2}-1}{(2-\sqrt{2})(-\sqrt{2})^{n-4}+(2+\sqrt{2})\sqrt{2}^{n-4}}\sim 2.$$

- Case B. By Section 2.3, we see that any involution σ in $\mathbf{Inv}_{\rm n}(123,231)$ can be written as $\sigma=j\cdots 1n\cdots(j+1)$ with $1\leq j\leq n$. We also know that the generating function for the number of involutions in $\mathbf{Inv}_{\rm n}(123,231)$ is given by $\frac{1-x+x^2}{(1-x)^2}$ [16]. Note that $F_{T;m}(x)=0$ for all $m\geq 3$, $F_{T;0}(x)=1$, $F_{T;1}(x)=\frac{x}{1-x}$, and hence $F_{T;2}(x)=\frac{x^2}{(1-x)^2}$. Therefore,

$$F_{123,231}(x,q)=1+\frac{xq}{1-x}+\frac{q^2x^2}{(1-x)^2}.$$

By taking the derivative at $q = 1$ and then finding the coefficient of x^n, we get $\mathbf{E}(L_n) = 2 - 1/n$.

- Case C. By Subsection 2.3, we see that any involution σ in $\mathbf{Inv}_\mathrm{n}(231, 321)$ can be written as either $\sigma = 1\sigma'$ or $21\sigma''$ where σ', σ'' are 231- and 321-avoiding involutions. Thus,

$$F_{231,321}(x,q) = 1 + xqF_{231,321}(x,q) + x^2qF_{231,321}(x,q),$$

where 1 counts the empty involution, $xqF_{231,321}(x,q)$ counts the involutions of type $1\sigma'$, and $x^2qF_{231,321}(x,q)$ counts the involutions of type $21\sigma''$. Therefore, we obtain

$$F_{231,321}(x,q) = \frac{1}{1 - xq - x^2q}.$$

By taking the derivative at $q = 1$ and then finding the coefficient of x^n, we get

$$\mathbf{E}(L_n) = \frac{\frac{3}{5\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} - \frac{3}{5\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \frac{n+1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \frac{n+1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}},$$

which gives $\mathbf{E}(L_n) \sim \frac{1+\sqrt{5}}{2\sqrt{5}}n$.

- Case D. Any involution σ in $\mathbf{Inv}_\mathrm{n}(132, 231)$ can be written as $\sigma = j(j-1)\cdots 1(j+1)(j+2)\cdots n$ for some $1 \le j \le n$. Thus,

$$F_{132,231}(x,q) = 1 + \frac{xq}{1-x}\left(1 + \frac{xq}{1-xq}\right),$$

where 1 counts the empty involution, $xq/(1-x)$ counts the involutions of type $j = n$, and $x^2q^2/((1-x)(1-xq))$ counts the involutions of type $1 \le j \le n-1$. Hence, $\mathbf{E}(L_n) = \frac{\binom{n+1}{2}}{n} = \frac{n+1}{2}$.

- Case E. Any involution σ in $\mathbf{Inv}_\mathrm{n}(132, 321)$ can be written as $\sigma = (j+1)(j+2)\cdots(2j)12\cdots j(2j+1)(2j+2)\cdots n$ for $0 \le j \le n/2$. Thus,

$$F_{132,321}(x,q) = 1 + \frac{xq}{1-xq} + \frac{x^2q}{(1-xq)(1-x^2q)},$$

where 1 counts the empty involution, $xq/(1-xq)$ counts the involutions of type $j = 0$, and $x^2q/((1-xq)(1-x^2q))$ counts the involutions of type $1 \le j \le n/2$. Hence,

$$\mathbf{E}(L_n) = \frac{\frac{1}{16}(6n^2 + 10n + 1) + \frac{1}{16}(2n-1)(-1)^n}{\frac{1}{4}(2n + 3 + (-1)^n)} \sim \frac{3n}{4}.$$

- Case F. By Section 2.3, we have that $\mathbf{Inv}_\mathrm{n}(231, 312) = \mathbf{Inv}_\mathrm{n}(231)$. Hence, $\mathbf{E}(L_n) = \frac{n+1}{2}$.

□

The cases $T \subseteq S_3$ with $|T| \ge 3$ can readily follow from similar methods we use in this paper.

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