

## ALMOST UNIT-CLEAN RINGS

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ABSTRACT. A ring  $R$  is almost unit-clean provided that every element in  $R$  is equivalent to the sum of an idempotent and a regular element. We investigate conditions under which a ring is almost unit-clean. We prove that every ring in which every zero-divisor is strongly  $\pi$ -regular is almost unit-clean and every matrix ring of elementary divisor domains is almost unit-clean. Furthermore, it is shown that the trivial extension  $R(M)$  of a commutative ring  $R$  and an  $R$ -module  $M$  is almost unit-clean if and only if each  $x \in R$  can be written in the form  $ux = r + e$  where  $u \in U(R)$ ,  $r \in R - (Z(R) \cup Z(M))$  and  $e \in Id(R)$ . We thereby construct many examples of such rings.

Throughout, all rings are associative with an identity. An element  $x \in R$  is regular if  $xy = 0 \implies y = 0$  and  $yx = 0 \implies y = 0$ , i.e.,  $x \in R$  is neither right nor left zero-divisor. An element  $a \in R$  is almost clean provided that it is the sum of an idempotent and a regular element. A ring  $R$  is almost clean provided that every element in  $R$  is almost clean. The subject of almost clean rings (in particular, in commutative case) is interested for so many mathematicians, e.g., [1, 3, 6] and [8, 9], as they are related to the well-studied clean rings of Nicholson [5]. Though almost clean rings are popular, the conditions a bit restrictive. In the current paper, we shall seek to remedy this by looking at an interesting generalization of almost clean rings. This new class enjoys many interesting properties and examples. Recall that two elements  $a$  and  $b$  in a ring  $R$  are equivalent if there exist invertible  $u, v \in R$  such that  $uav = b$ . A ring  $R$  is called almost unit-clean if every element in  $R$  is equivalent to an almost clean element. An element  $a \in R$  is strongly  $\pi$ -regular

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if  $a^n \in a^{n+1}R$  for some  $n \in \mathbb{N}$ . We prove that every ring in which each zero-divisor is strongly  $\pi$ -regular is almost unit-clean and every matrix ring of elementary divisor domains is almost unit-clean. Furthermore, we prove that the trivial extension  $R(M)$  of a commutative ring  $R$  and an  $R$ -module  $M$  is almost unit-clean if and only if each  $x \in R$  can be written in the form  $uxv = r + e$  where  $u, v \in U(R)$ ,  $r \in R - (Z(R) \cup Z(M))$  and  $e \in Id(R)$ . Many kind of such rings are thereby provided.

The notations  $Id(R), U(R), reg(R)$  and  $J(R)$  stand for the set of all idempotents, all units, all regular elements and the Jacobson radical of a ring  $R$ , respectively. The set of all natural numbers is denoted by  $\mathbb{N}$ . We begin with an example which provides various types of such rings.

**Example 1.**

- (1) Boolean rings and local rings (i.e., ring whose Jacobson radical is the unique maximal ideal) are almost unit-clean.
- (2) Every unit-regular ring  $R$  ( every element  $x$  in  $R$  is unit-regular, i.e, there exists a  $u \in U(R)$  such that  $x = xux$ ) is almost unit-clean.
- (3) Every almost clean ring is almost unit-clean.
- (4) Every 2-good ring (i.e., every element is the sum of two units) is almost unit-clean.

**Theorem 2.** *Let  $R$  be a ring. Then the following are equivalent:*

- (1)  *$R$  is almost unit-clean.*
- (2) *Every element in  $R$  is the sum of a unit-regular element and a regular element.*

*Proof.* (1)  $\Rightarrow$  (2) For any  $a \in R$ , there exist units  $u, v \in R$  such that  $uav = b+c$ , where  $b^2 = b$  and  $c$  is regular. So  $a = u^{-1}bv^{-1}+u^{-1}cv^{-1}$ . Clearly,  $u^{-1}bv^{-1} = (u^{-1}bv^{-1})(vu)(u^{-1}bv^{-1}) \in R$  is unit-regular and  $u^{-1}cv^{-1} \in reg(R)$ . This proving (1).

(2)  $\Rightarrow$  (1) Let  $a \in R$ . Then  $a = b + d, b \in R$  is unit-regular,  $d \in reg(R)$ . Hence, we easily find a  $c \in U(R)$  such that  $b = bcb$ . Then  $ca = cb + cd$ , where  $cb \in Id(R)$  and  $cd \in reg(R)$ . So  $R$  is almost unit-clean.  $\square$

Obviously, an element in a ring is unit-regular if and only if it is the product of an idempotent and a unit. From this observation, we easily derive

**Corollary 3.** *Let  $R$  be a ring. Then the following are equivalent:*

- (1)  $R$  is almost unit-clean.
- (2) For any  $a \in R$ , there exists an element  $u \in U(R)$  such that  $ua \in R$  is almost clean.
- (3) For any  $a \in R$ , there exists an element  $u \in U(R)$  such that  $au \in R$  is almost clean.

Thus, every almost unit-clean ring is right-left symmetric. That is, a ring  $R$  is almost unit-clean if and only if the opposite ring  $R^{op}$  is almost unit-clean. Let  $R[[x]]$  denote the ring of all power series over a ring  $R$ . We have

**Corollary 4.** *Let  $R$  be almost unit-clean. Then  $R[[x]]$  is almost unit-clean.*

*Proof.* Let  $f(x) = \sum_{i=0}^{\infty} r_i x^i \in R[[x]]$ , then  $f(x) = r_0 + r_1 x + \dots$ . Since  $R$  is an almost unit-clean ring, then  $r_0 = eu + r$  where  $e \in Id(R)$ ,  $u \in U(R)$  and  $r \in reg(R)$ . So

$$\begin{aligned} f(x) &= r_0 + r_1 x + r_2 x^2 + \dots \\ &= eu + r + r_1 x + r_2 x^2 + \dots \end{aligned}$$

where  $g(x) = r + r_1 x + r_2 x^2 + \dots$ . If  $g(x) \notin reg(R[[x]])$ , then we may assume that there exists  $h(x) = \sum_{i=0}^{\infty} h_i x^i \neq 0$  satisfying  $g(x)h(x) = 0$ , thus  $rh(x) = 0$ . Hence, each  $rh_i = 0$ . This implies that each  $h_i = 0$ , which is a contradiction. So  $g(x) \in reg(R[[x]])$  and  $e \in Id(R) \subseteq Id(R[[x]])$ . Therefore  $R[[x]]$  is almost unit-clean.  $\square$

If  $R$  is almost unit-clean, analogously, we prove that the ring  $R[x]$  of all polynomials over  $R$  is almost unit-clean. By induction, we easily obtain the following result.

**Proposition 5.** *Let  $R$  be almost unit-clean. Then  $R[[X_1, \dots, X_n]]$  and  $R[X_1, \dots, X_n]$  are almost unit-clean.*

**Example 6.** *Let  $R$  be an algebra over an infinite field. If every zero-divisor in  $R$  is algebraic, then  $R$  is almost unit-clean.*

*Proof.* Let  $R$  be an algebra over an infinite field  $F$ , and let  $a \in R$ . If  $a \in R$  is regular, then  $a = 0 + a$ , as desired. We now assume that  $a \notin \text{reg}(R)$ . By hypothesis, there exists a non-constant polynomial  $p(x) \in F[x]$  such that  $p(a) = 0$ . Let  $n = \text{deg}p(x)$ , and let  $S$  be the set of all roots of  $p(x)$  in  $F[x]$ . Then  $|S| \leq n$ . Let  $\alpha \in F - S$ . Then  $p(\alpha) \neq 0$ . Set  $q(x) = p(x + \alpha) = \sum_{i=0}^n q_i x^i$  with each  $q_i \in F$ . Then 0 is not a root of  $q(x)$ , and so  $q_0 \neq 0$ . Thus,  $0 = q(a - \alpha \cdot 1_R) = q_0 \cdot 1_R + \sum_{i=1}^n q_i (a - \alpha \cdot 1_R)^i$ . Therefore,  $a - \alpha \cdot 1_R \in U(R)$ , as required.  $\square$

Recall that an element in a ring is weakly clean if it is the sum or difference of a unit and an idempotent. A ring is weakly clean if every element is weakly clean [1, 7]. For instance, every weakly nil-clean ring is weakly clean [4].

**Example 7.** *Every ring in which every zero-divisor is weakly clean is almost unit-clean, but the converse is not true.*

*Proof.* Let  $R$  be a ring in which every zero-divisor is weakly clean, and let  $a \in R$ . If  $a \in \text{reg}(R)$ , then  $a = 0 + a$ , as desired. If  $a \notin \text{reg}(R)$ , then  $a$  is weakly clean. Thus, we can find an idempotent  $e \in R$  and a unit  $u \in R$  such that  $a = e + u$  or  $-e + u$ . Hence,  $a \in R$  is almost unit-clean, and we are through.  $\square$

Recall that a ring  $R$  is connected provided that it has only trivial idempotents.

**Proposition 8.** *Let  $R$  be connected. Then the following conditions are equivalent:*

- (1)  $R$  is almost unit-clean.
- (2) For any zero divisor  $x \in R$ ,  $x - u \in \text{reg}(R)$  for some  $u \in U(R)$ .

*Proof.* (1)  $\Rightarrow$  (2) As  $R$  is connected,  $\text{Id}(R) = \{0, 1\}$ . For any zero-divisor  $x \in R$ , since  $R$  is almost unit-clean, there exist  $u \in U(R)$ ,  $e^2 = e$  and a regular  $y \in R$  such that  $ux = e + y$ . If  $e = 1$  then  $x - v \in \text{reg}(R)$ , where  $v := u^{-1}$ . If  $e = 0$ , then  $ux = y$ ; hence,

$x = u^{-1}y \in R$  is regular. This gives a

contradiction. This proving (2).

(2)  $\Rightarrow$  (1) Let  $x$  be a zero-divisor of  $R$  such that  $x \in R$  with  $x - u \in \text{reg}(R)$ . Then  $u^{-1}x - 1 \in \text{reg}(R)$ ; otherwise,  $x \in R$  is regular. So  $x$  is almost unit-clean. This completes the proof.  $\square$

**Theorem 9.** *Every ring in which every zero-divisor is strongly  $\pi$ -regular is almost unit-clean.*

*Proof.* Let  $R$  be a ring in which every zero-divisor is strongly  $\pi$ -regular, and let  $a \in R$ . If  $a \in \text{reg}(R)$ , then  $a$  is the sum of 0 and a regular element, as desired. If  $a \notin \text{reg}(R)$ , by hypothesis,  $a \in R$  is strongly  $\pi$ -regular. Then there exist  $n \in \mathbb{N}, b \in R$  such that  $a^n = a^{n+1}b, b = b^2a$ . Let  $c = b^n a^n b^n$ . Then  $a^n = a^{2n}c$  and  $c = c^2 a^n$ . Set  $v = c + 1 - a^n c$ . Then  $v^{-1} = a^n + 1 - a^n c$ . Set  $f = a^n c$ . Then  $f = f^2 \in R$ . Choose  $w = v^{-1}$ . Then  $a^n = fw$ . Let  $e = 1 - f$  and  $u = a - e$ . Choose  $z = a^{n-1}w^{-1}f - (1 + a + \cdots + a^{n-1})e$ . Then

$$\begin{aligned} uz &= (a - e)(a^{n-1}w^{-1}f - (1 + a + \cdots + a^{n-1})e) \\ &= a^n w^{-1}f + (1 - a)(1 + a + \cdots + a^{n-1})e \\ &= f + (1 - a^n)e \\ &= f + e \\ &= 1. \end{aligned}$$

Likewise,  $zu = 1$ . Therefore  $a = e + u, e = e^2, u \in U(R)$ , as desired.  $\square$

**Corollary 10.** *Every ring with finitely many zero-divisors is almost unit-clean.*

*Proof.* Let  $R$  be a ring with finitely many zero-divisors. Let  $a \in R$  is a zero-divisor. Then  $S := \{a, a^2, a^3, \cdots\}$  is a set of zero-divisors. By hypothesis,  $S$  is a finite set. It follows that  $a^m = a^n$  for some distinct  $m, n \in \mathbb{N}$ ; hence,  $a \in R$  is strongly  $\pi$ -regular. Therefore we complete the proof, by Theorem 9.  $\square$

We claim that every finite ring is almost unit-clean. Since every finite ring has finitely many zero-divisors, we are through by Corollary 10. Following Abu-Khuzam and Bell [2], a ring  $R$  is a  $D^*$ -ring

if every zero-divisor in  $R$  is expressible as a sum of a potent element (i.e,  $p = p^n$  for some  $n \geq 2$ ) and a nilpotent. For instance,  $\mathbb{Z}_{p^k}$  ( $p$  is prime,  $k \geq 1$ ).

**Corollary 11.** *Every  $D^*$ -ring is almost unit-clean.*

*Proof.* Let  $R$  be a  $D^*$ -ring, and let  $a \in R$ . If  $a + 1 \in \text{reg}(R)$ , then  $a = (-1) + (a + 1)$  is almost unit-clean, as desired. If  $a + 1 \notin \text{reg}(R)$ , by hypothesis, we have  $a + 1 = p + w$ , where  $p = p^n, w \in N(R)$ . Hence,  $a = p + (w - 1)$ . Clearly,  $w - 1 \in U(R)$ . We see that  $p = p^n = p(p^{n-2})p = p(p^{n-2})p(p^{n-2})p = pqp$ , where  $q = p^{2n-3}$ . Then  $q = qpq$ . Set  $u = p + 1 - pq$ . Then  $u^{-1} = q + 1 - pq$ . Further, we check that  $p = pu^{-1}p \in R$  is unit-regular. Therefore  $a$  is the sum of a unit-regular element and a unit, as desired.  $\square$

**Corollary 12.** *Every strongly  $\pi$ -regular ring is almost unit-clean.*

*Proof.* This is obvious, in terms of Theorem 9.  $\square$

An element  $a$  in a ring  $R$  is  $\pi$ -Rickart if there exist  $n \in \mathbb{N}$  and a central idempotent  $e \in R$  such that  $r(a^n) = r(e)$  and  $\ell(a^n) = \ell(e)$ , where  $r(x)$  and  $\ell(x)$  denote the right and left annihilators of an element  $x \in R$ . An element  $a$  in a ring  $R$  is almost  $\pi$ -Rickart if there exist a  $\pi$ -Rickart  $p \in R$ , a unit  $u \in R$  and a nilpotent  $w \in R$  such that  $a = pu + w$ . A ring  $R$  is a almost  $\pi$ -Rickart ring if every element in  $R$  is almost  $\pi$ -Rickart. Clearly, every abelian  $\pi$ -regular ring are almost  $\pi$ -Rickart. But the converse is not true. We come now to

**Theorem 13.** *Every almost  $\pi$ -Rickart ring is almost unit-clean.*

*Proof.* Suppose  $R$  is an almost  $\pi$ -Rickart ring. Let  $a \in R$ . Then there exist a  $\pi$ -Rickart  $p \in R$ , a unit  $u \in R$  and a nilpotent  $w \in R$  such that  $a = pu + w$ . Hence,  $p = au^{-1} - wu^{-1}$ . By hypothesis,  $r(p^n) = r(e)$  and  $\ell(p^n) = \ell(e)$  for a central idempotent  $e$  and  $n \in \mathbb{N}$ . Then we define  $p^n R \rightarrow eR, p^n x \mapsto ex$  for any  $x \in R$ . Thus, we have an  $R$ -isomorphism  $\varphi : p^n R \cong eR$ . Write  $p^n = \varphi(ex)$ . Then  $p^n = e\varphi(ex)$ . Let  $r = p^n + 1 - e$ . Then  $p^n = er$ . If  $rz = 0$ , then  $(e\varphi(ex) + 1 - e)z = 0$ , and so  $(1 - e)z = 0$ . Moreover,  $e\varphi(ex)z = 0$ , and so  $p^n z = 0$ . This implies that  $z \in r(p^n) = r(e)$ ,

and so  $ez = 0$ . It follows that  $z = (1 - e)z + ez = 0$ . If  $zr = 0$ , then  $z(e\varphi(ex) + 1 - e) = 0$ , and so  $z(1 - e) = 0$ . Moreover,  $ze\varphi(exe)e = 0$ , and so  $zp^n = 0$ . This implies that  $z \in \ell(p^n) = \ell(e)$ , and so  $ze = 0$ . This shows that  $z = z(1 - e)z + ze = 0$ ; hence,  $r \in \text{reg}(R)$ . Clearly,  $p = (1 - e) + (p - 1 + e)$ . If  $(p - 1 + e)x = 0$ , then  $(p - 1 + e)(p^{n-1} + p^{n-2}(1 - e) + \cdots + p(1 - e) + (1 - e))x = 0$ . That is,  $(p^n - 1 + e)x = 0$ . Hence,  $(1 - e)x = 0$ . Furthermore,  $rex = 0$ , and so  $ex = 0$ , as  $r \in \text{reg}(R)$ . Therefore  $x = ex + (1 - e)x = 0$ . Likewise,  $x(p - 1 + e) = 0$  implies that  $x = 0$ . Hence,  $v := p - 1 + e \in \text{reg}(R)$ . One easily checks that  $(au^{-1} - wu^{-1}) - 1 + e = v$ , and so  $a = (1 - e)u + (v + wu^{-1})u$ . We now show that  $(v + wu^{-1})u \in R$  is a non zero-divisor. If  $(v + wu^{-1})ut = 0$  for some  $t \in R$ , then  $vt = wt$ . Say  $w^m = 0$  ( $m \in \mathbb{N}$ ). Then  $v^m t = w^m t = 0$ . As  $v \in R$  is regular, it follows that  $t = 0$ . This shows that  $(v + wu^{-1})u \in \text{reg}(R)$ . Therefore  $a \in R$  is almost unit-clean, in terms of Theorem 2.  $\square$

A matrix  $A$  (not necessarily square) over a ring  $R$  admits diagonal reduction if there exist invertible matrices  $P$  and  $Q$  such that  $PAQ$  is a diagonal matrix  $(d_{ij})$ , for which  $d_{ii}$  is a full divisor of  $d_{(i+1)(i+1)}$  (i.e.,  $Rd_{(i+1)(i+1)}R \subseteq d_{ii}R \cap Rd_{ii}$ ) for each  $i$ . A ring  $R$  is called an elementary divisor ring provided that every matrix over  $R$  admits a diagonal reduction.

**Theorem 14.** *Let  $R$  be an elementary divisor ring. If  $R$  is almost unit-clean, then  $M_n(R)$  is almost unit-clean for all  $n \in \mathbb{N}$ .*

*Proof.* Let  $A \in M_n(R)$ . Then we have some invertible  $P, Q \in M_n(R)$  such that

$$PAQ = \text{diag}(d_1, \dots, d_n).$$

Since  $R$  is almost unit-clean, for each  $i$ , we can find  $u_i \in U(R)$  such that  $u_i d_i = e_i + v_i$  by Corollary 3, where  $e_i^2 = e_i \in R$  and  $v_i \in \text{reg}(R)$ . Hence,

$$\text{diag}(u_1, \dots, u_n)PAQ = \text{diag}(e_1, \dots, e_n) + \text{diag}(v_1, \dots, v_n).$$

One easily check that

$$\text{diag}(e_1, \dots, e_n) \in \text{Id}(M_n(R)), \text{diag}(v_1, \dots, v_n) \in \text{reg}(M_n(R)).$$

Therefore  $M_n(R)$  is almost unit-clean, as asserted.  $\square$

**Corollary 15.** *Let  $R$  be an elementary divisor domain. Then  $M_n(R)$  is almost unit-clean for all  $n \in \mathbb{N}$ .*

*Proof.* This is obvious, as every domain is almost unit-clean.  $\square$

Elementary divisor domains have been studied by many authors, we refer the reader for the book [10]. As is well known, every principal ideal domain is an elementary divisor domain, we derive

**Corollary 16.** *Let  $R$  be a principal ideal domain. Then  $M_n(R)$  is almost unit-clean for all  $n \in \mathbb{N}$ .*

Since  $\mathbb{Z}$  is a principal ideal domain, we see that  $M_n(\mathbb{Z})$  is almost unit-clean for any  $n \in \mathbb{N}$ . An element  $c \in R$  is adequate if for any  $a \in R$  there exist some  $r, s \in R$  such that (1)  $c = rs$ ; (2)  $rR + aR = R$ ; (3)  $s'R + aR \neq R$  for each non-invertible divisor  $s'$  of  $s$ . A domain  $R$  is called to be adequate if every element in  $R$  is adequate. By an immediate consequence of Corollary 15, we derive

**Corollary 17.** *Let  $R$  be adequate. Then  $M_n(R)$  is almost unit-clean for all  $n \in \mathbb{N}$ .*

Let  $R$  be a ring and  $M$  be an  $R$ -bimodule. The set of pairs  $(r, m)$  with  $r \in R$  and  $m \in M$ , under coordinate-wise addition and the multiplication defined by  $(r, m)(r', m') = (rr', rm' + r'm)$ , for all  $r, r' \in R, m, m' \in M$ . Then  $R(M)$  is called the trivial extension of  $R$  by  $M$ . Let  $M$  be an  $R$ -module. We set  $Z(M) = \{r \in R \mid \exists 0 \neq m \in M \text{ such that } rm = 0\}$ .

**Lemma 18.** *Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. Then  $(r, m) \in Z(R(M))$  if and only if  $r \in Z(R) \cup Z(M)$ .*

*Proof.* See [1, Theorem 2.11].  $\square$

**Theorem 19.** *Let  $R$  be a commutative ring and  $M$  be an  $R$ -module. Then the trivial extension  $R(M)$  of  $R$  and  $M$  is almost unit-clean if and only if each  $x \in R$  can be written in the form  $ux = r + e$  where  $u \in U(R), r \in R - (Z(R) \cup Z(M))$  and  $e \in Id(R)$ .*

*Proof.* Clearly  $Id(R(M)) = \{(e, 0) \in R(M) \mid e \in Id(R)\}$ . By Lemma 18,  $(r, m) \in Z(R(M))$  if and only if  $r \in Z(R) \cup Z(M)$ .



$\Rightarrow$  Suppose that  $R(M)$  is almost unit-clean. Let  $a \in R$ . Then we can find some  $(u, m) \in U(R(M))$  such that  $(u, m)(a, 0) = (r, ma) + (e, 0)$  where  $(r, ma) \in \text{reg}(R(M))$  and  $(e, 0) \in \text{Id}(R(M))$ . Since  $(r, ma) \in \text{reg}(R(M))$ , it follows by Lemma 18,  $r \in R - (Z(R) \cup Z(M))$ . Therefore  $ua = r + e$  where  $r \in R - (Z(R) \cup Z(M))$  and  $e \in \text{Id}(R)$ .

$\Leftarrow$  Let  $a \in R$  and  $m \in M$ . Write  $ua = r + e$  where  $u \in U(R)$ ,  $r \in R - (Z(R) \cup Z(M))$  and  $e \in \text{Id}(R)$ . Then  $(u, 0)(a, m) = (ua, um) = (r, um) + (e, 0)$ . As  $(r, um) \in \text{reg}(R(M))$  and  $(e, 0) \in \text{Id}(R(M))$ , we obtain the result.  $\square$

**Corollary 20.** *Let  $R$  be a ring. Then the ring  $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$  is almost unit-clean if and only if  $R$  is almost unit-clean.*

*Proof.* This is obvious by Theorem 19.  $\square$

**Example 21.** *The ring  $\mathbb{Z}(\mathbb{Z}_6)$  is not almost unit-clean.*

*Proof.* Clearly,  $Z(\mathbb{Z}) = \{0\}$  as  $\mathbb{Z}$  is the domain of all integers. Further,  $Z(\mathbb{Z}_6) = 2\mathbb{Z} \cup 3\mathbb{Z}$ , and so  $\mathbb{Z} - Z(\mathbb{Z}_6) = \{m \in \mathbb{Z} \mid 2, 3 \nmid m\}$ . We easily check that  $3, 3 - 1, 3 + 1, -3, -3 - 1, -3 + 1 \notin \mathbb{Z} - Z(\mathbb{Z}_6)$ . As  $U(\mathbb{Z}) = \{-1, 1\}$ , we conclude that the trivial extension  $\mathbb{Z}(\mathbb{Z}_6)$  is not almost unit-clean, in terms of Theorem 15.  $\square$

**Example 22.** *The ring  $\mathbb{Z}(\mathbb{Z}_{15})$  is almost unit-clean.*

*Proof.* Clearly,  $Z(\mathbb{Z}) = \{0\}$  and  $Z(\mathbb{Z}_{15}) = 3\mathbb{Z} \cup 5\mathbb{Z}$ . Hence,  $\mathbb{Z} - Z(\mathbb{Z}_{15}) = \{m \in \mathbb{Z} \mid 3, 5 \nmid m\}$ . Let  $x \in \mathbb{Z}$ .

Case I.  $3, 5 \nmid x$ . Then  $x = 0 + x$  with  $x \in \mathbb{Z} - Z(\mathbb{Z}_{15})$ .

Case II.  $3 \mid x$ . Then  $x = 3m$  for some  $m \geq 0$ . Then  $3 \nmid 3m - 1, 3m + 1$ . If  $5 \mid 3m - 1, 3m + 1$ , then  $5 \mid 2$ , an absurd. Hence,  $5 \nmid 3m - 1$  or  $3m + 1$ . This implies that  $x - 1$ , or  $x + 1$  in  $\mathbb{Z} - Z(\mathbb{Z}_{15})$ .

Case III.  $5 \mid x$ . Then  $x = 5m$  for some  $m \geq 0$ . Then  $5 \nmid 5m - 1, 5m + 1$ . If  $3 \mid 5m - 1, 5m + 1$ , then  $3 \mid 2$ , an absurd. Thus,  $3 \nmid 5m + 1$  or  $5m - 1$ . This implies that  $x + 1$  or  $x - 1$  is in  $\mathbb{Z} - Z(\mathbb{Z}_{15})$ .

Therefore the trivial extension  $\mathbb{Z}(\mathbb{Z}_{15})$  is almost unit-clean, in terms of Theorem 19. In this case,  $\mathbb{Z}(\mathbb{Z}_{15})$  is not almost clean, in terms of [1, Theorem 2.11].  $\square$

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