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BETTI NUMBERS FOR CERTAIN COHEN–MACAULAY TANGENT CONES

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Abstract

We compute Betti numbers for a Cohen–Macaulay tangent cone of a monomial curve in the affine 4-space corresponding to a pseudo-symmetric numerical semigroup. As a byproduct, we also show that for these semigroups, being of homogeneous type and homogeneous are equivalent properties.

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1. Introduction

Let $S = \langle n_1, \dots, n_k \rangle = \{u_1 n_1 + \cdots + u_k n_k \mid u_i \in \mathbb{N}\}$ be a numerical semigroup generated by the positive integers $n_1, \dots, n_k$ with $\gcd(n_1, \dots, n_k) = 1$. For a field K, let $A = K[X_1, X_2, \dots, X_k]$ and let $K[S]$ be the semigroup ring $K[t^{n_1}, t^{n_2}, \dots, t^{n_k}]$ of S. Then $K[S] \simeq A/I_S$, where I_S is the kernel of the surjection $\phi_0 : A \to K[S]$, associating X_i to t^{n_i}. If C_S is the affine curve with parameterisation

$$X_1 = t^{n_1}, X_2 = t^{n_2}, \dots, X_k = t^{n_k}$$

corresponding to S and $1 \notin S$, then the curve is singular at the origin. The smallest minimal generator of S is called the *multiplicity* of C_S. To understand this singularity, it is natural to study algebraic properties of the local ring $R_S = K[[t^{n_1}, \dots, t^{n_k}]]$ with the maximal ideal $\mathfrak{m} = \langle t^{n_1}, \dots, t^{n_k} \rangle$ and its associated graded ring

$$gr_{\mathfrak{m}}(R_S) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1} \cong A/I_S^*,$$

where $I_S^* = \langle f^* \mid f \in I_S \rangle$ with f^* denoting the least homogeneous summand of f. When K is algebraically closed, $K[S]$ is the coordinate ring of the monomial curve C_S and

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$gr_{\mathfrak{m}}(R_S)$ is the coordinate ring of its tangent cone. A natural set of invariants for these coordinate rings is the Betti sequence. We refer to Stamate's survey [12] for a comprehensive literature on this subject. The Betti sequence $\beta(M) = (\beta_0, \ldots, \beta_{k-1})$ of an A-module M is the sequence consisting of the ranks of the free modules in a minimal free resolution $\mathbf{F}$ of M, where

$$\mathbf{F} : 0 \longrightarrow A^{\beta_{k-1}} \longrightarrow \cdots \longrightarrow A^{\beta_1} \longrightarrow A^{\beta_0}.$$

When $\beta(A/I_S^*) = \beta(K[S])$, the semigroup S is said to be of homogeneous type as defined in [6]. In particular, if a semigroup is of homogeneous type then the Betti sequence of its Cohen–Macaulay tangent cone can be obtained from a minimal free resolution of $K[S]$. To take advantage of this idea, Jafari and Zarzuela Armengou introduced the concept of a *homogeneous* semigroup in [8]. When the multiplicity of a monomial curve corresponding to a homogeneous semigroup is n_i, homogeneity guarantees the existence of a minimal generating set for I_S whose image under the map

$$\pi_i : A \to \bar{A} = K[X_1, \ldots, \bar{X}_i, \ldots, X_k]$$

is homogeneous, where $\pi(X_i) = \bar{X}_i = 0$ and $\pi(X_j) = X_j$ for $i \neq j$. Together with the assumption of a Cohen–Macaulay tangent cone, this property is inherited by a standard basis of I_S and the authors of [8] were able to prove that S is of homogeneous type. The converse is not true in general: there exists a 3-generated numerical semigroup with a complete intersection tangent cone which is of homogeneous type but not homogeneous; see [8, Example 3.19]. They also ask in [8, Question 4.22] if there are 4-generated semigroups of homogeneous type which are not homogeneous having noncomplete intersection tangent cones. Since homogeneous-type semigroups have Cohen–Macaulay tangent cones, we restrict our attention to monomial curves having Cohen–Macaulay tangent cones in this article.

The problem of determining the Betti sequence for the tangent cone (see [12, Problem 9.9]) was studied for 4-generated symmetric monomial curves by Mete and Zengin [10]. In this paper, we focus on the next interesting case of 4-generated pseudo-symmetric monomial curves. Using the standard bases we obtained in [11], we determine the Betti sequence for the tangent cone, addressing [12, Problem 9.9] for 4-generated pseudo-symmetric monomial curves having Cohen–Macaulay tangent cones, and prove that being homogeneous and being of homogeneous type are equivalent, answering [8, Question 4.22]. So, in most cases, there is no 4-generated pseudo-symmetric numerical semigroup of homogeneous type which is not homogeneous. Before we state our main result, let us recall from [9] that a 4-generated semigroup $S = \langle n_1, n_2, n_3, n_4 \rangle$ is pseudo-symmetric if and only if there are integers $\alpha_i > 1$, for $1 \leq i \leq 4$, and $\alpha_{21} > 0$ with $\alpha_{21} < \alpha_1 - 1$ such that

$$\begin{aligned}
n_1 &= \alpha_2\alpha_3(\alpha_4 - 1) + 1,\\
n_2 &= \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3,\\
n_3 &= \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1,\\
n_4 &= \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2.
\end{aligned}$$

TABLE 1. Examples of each case.

α_{21}	α_1	α_2	α_3	α_4	n_1	n_2	n_3	n_4	β_0	β_1	β_2	β_3
2	5	3	2	2	7	12	13	22	1	5	6	2
2	4	4	2	4	25	19	22	26	1	5	6	2
2	4	4	2	5	33	23	28	26	1	5	7	3
2	5	4	2	4	25	20	35	30	1	6	9	4
1	3	2	3	3	13	14	9	15	1	5	6	2
3	6	3	4	6	61	82	51	63	1	6	8	3
1	3	2	2	4	13	11	12	9	1	5	6	2
1	4	2	2	4	13	12	19	11	1	5	7	3

Then the toric ideal I_S is given by $I_S = \langle f_1, f_2, f_3, f_4, f_5\rangle$ with

$$f_1 = X_1^{\alpha_1} - X_3X_4^{\alpha_4-1}, \quad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}}X_4, \quad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2,$$
$$f_4 = X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, \quad f_5 = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1} - X_2X_4^{\alpha_4-1}.$$

The Betti sequence of $K[S]$ for a 4-generated pseudo-symmetric semigroup is $\beta(K[S]) = (1, 5, 6, 2)$ by [1]. Hence, S is of homogeneous type if and only if the Betti sequence of the tangent cone is also $\beta(A/I_S^*) = (1, 5, 6, 2)$. We refer the reader to [3] for the Betti sequence of $K[S]$ for 4-generated almost-symmetric semigroups.

Our main result is as follows.

THEOREM 1.1. *Let S be a 4-generated pseudo-symmetric semigroup with a Cohen–Macaulay tangent cone. Then the Betti sequence $\beta(A/I_S^*)$ of the tangent cone is:*

- $\beta(A/I_S^*) = (1, 5, 6, 2)$ *if n_1 is the multiplicity;*
- $\beta(A/I_S^*) = (1, 5, 6, 2)$ *if n_2 is the multiplicity and $\alpha_1 = \alpha_4$;*
  $\beta(A/I_S^*) = (1, 5, 7, 3)$ *if n_2 is the multiplicity and $\alpha_1 < \alpha_4$;*
  $\beta(A/I_S^*) = (1, 6, 9, 4)$ *if n_2 is the multiplicity and $\alpha_1 > \alpha_4$;*
- $\beta(A/I_S^*) = (1, 5, 6, 2)$ *if n_3 is the multiplicity and $\alpha_2 = \alpha_{21} + 1$;*
  $\beta(A/I_S^*) = (1, 6, 8, 3)$ *if n_3 is the multiplicity and $\alpha_2 < \alpha_{21} + 1$;*
- $\beta(A/I_S^*) = (1, 5, 6, 2)$ *if n_4 is the multiplicity and $\alpha_3 = \alpha_1 - \alpha_{21}$;*
  $\beta(A/I_S^*) = (1, 5, 7, 3)$ *if n_4 is the multiplicity and $\alpha_3 < \alpha_1 - \alpha_{21}$.*

We illustrate in Table 1 that there are pseudo-symmetric monomial curves with Cohen–Macaulay tangent cones in all of these cases.

We make repeated use of the following effective result as in [7, 8, 12] in order to reduce the number of cases for determining the Betti numbers of the tangent cones.

LEMMA 1.2. *Assume that the multiplicity of the monomial curve C_S is n_i. Suppose that the K-algebra homomorphism $\pi_i : A \to \bar{A} = K[X_1, \ldots, \bar{X}_i, \ldots, X_k]$ is defined by $\pi_i(X_i) = \bar{X}_i = 0$ and $\pi_i(X_j) = X_j$ for $i \neq j$, and set $\bar{I} = \pi_i(I_S^*)$. If the tangent cone $gr_{\mathfrak{m}}(R_S)$ is Cohen–Macaulay, then the Betti sequences of $gr_{\mathfrak{m}}(R_S)$ and of $\bar{A}/\bar{I}$ are the same.*

Proof. If the tangent cone $gr_{\mathfrak{m}}(R_S)$ is Cohen–Macaulay, then X_i is regular on A/I_S^*. The result follows from the well-known fact that Betti sequences are the same up to a regular sequence. □

Therefore, the problem of determining the Betti sequence of the tangent cone is reduced to computing the Betti sequence of the ring $\bar{A}/\bar{I}$. In all proofs about the minimal free resolution of $\bar{A}/\bar{I}$ we use the following criterion by Buchsbaum–Eisenbud to confirm the exactness, leaving the not so difficult task of checking if it is a complex to the reader.

Theorem 1.3 [2, Corollary 2]. *Let*

$$0 \longrightarrow F_{k-1} \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free modules over a Noetherian ring A. Let $\operatorname{rank}(\phi_i)$ *be the size of the largest nonzero minor of the matrix describing ϕ_i and let $I(\phi_i)$ be the ideal generated by the minors of maximal rank. Then the complex is exact if and only if:*

(a) $\operatorname{rank}(\phi_{i+1}) + \operatorname{rank}(\phi_i) = \operatorname{rank}(F_i)$*; and*
(b) *$I(\phi_i)$ contains an A-sequence of length i*

for $1 \le i \le k-1$.

The structure of the paper is as follows. We treat the cases where S is homogeneous in the next section and, when S is not homogeneous, we find the minimal free resolution of the ring $\bar{A}/\bar{I}$ in each subsequent section, completing the proof of Theorem 1.1 by virtue of Lemma 1.2. We refer the reader to [4] for the basics of commutative algebra as we use Singular [5] in our computations.

2. Homogeneous cases

In this section, we characterise which pseudo-symmetric 4-generated semigroups are homogeneous. We start by recalling basic definitions from [8]. The Apéry set of S with respect to $s \in S$ is defined to be $AP(S, s) = \{x \in S \mid x - s \notin S\}$ and the set of lengths of s in S is

$$L(s) = \left\{ \sum_{i=1}^{k} u_i \,\middle|\, s = \sum_{i=1}^{k} u_i n_i, u_i \ge 0 \right\}.$$

Note that $L(s)$ is the set of standard degrees of monomials $X_1^{u_1} \cdots X_k^{u_k}$ of S-degree $\deg_S(X_1^{u_1} \cdots X_k^{u_k}) = s$. A subset $T \subset S$ is said to be homogeneous if either it is empty or $L(s)$ is a singleton for all s with $0 \neq s \in T$. If n_i is the smallest among $n_1, n_2, \ldots, n_k$, the semigroup S is said to be *homogeneous* if the Apéry set $AP(S, n_i)$ is homogeneous.

Proposition 2.1. *Let S be a 4-generated pseudo-symmetric numerical semigroup. Then S is homogeneous if and only if:*

- *n_1 is the multiplicity; or*
- *n_2 is the multiplicity and $\alpha_1 = \alpha_4$; or*

- *n_3 is the multiplicity and $\alpha_2 = \alpha_{21} + 1$; or*
- *n_4 is the multiplicity and $\alpha_3 = \alpha_1 - \alpha_{21}$.*

Proof. By [8, Corollary 3.10], S is homogeneous if and only if there exists a set E of minimal generators for I_S such that every nonhomogeneous element of E has a term that is divisible by X_i when n_i is the multiplicity. Şahin and Şahin [11, Corollary 2.4] states that indispensable binomials of I_S are $\{f_1, f_2, f_3, f_4, f_5\}$ if $\alpha_1 - \alpha_{21} > 2$ and are $\{f_1, f_2, f_3, f_5\}$ if $\alpha_1 - \alpha_{21} = 2$. Therefore, they must appear in every minimal generating set. Let us take $E = \{f_1, \ldots, f_5\}$ in order to prove sufficiency of the conditions.

- Since each f_j ($j = 1, \ldots, 5$) has a term that is divisible by X_1, when n_1 is the multiplicity, S is always homogeneous.
- The only binomial in E that has no monomial term divisible by X_2 is f_1. Hence, when n_2 is the multiplicity and $\alpha_1 = \alpha_4$, it follows that f_1 and thus S is homogeneous.
- The only binomial in E that has no monomial term divisible by X_3 is f_2. Hence, when n_3 is the multiplicity and $\alpha_2 = \alpha_{21} + 1$, f_2 and thus S is homogeneous.
- Similarly, only f_3 has no monomial term that is divisible by X_4 and it is homogeneous when $\alpha_3 = \alpha_1 - \alpha_{21}$. Hence, S is homogeneous if n_4 is the multiplicity.

For the necessity of these conditions, recall that f_1, f_2 and f_3 are indispensable, so they must be homogeneous when the multiplicity is n_2, n_3 and n_4, respectively. □

3. The proof when the multiplicity is n_1

If the tangent cone is Cohen–Macaulay and the semigroup is homogeneous, it is known that the semigroup is of homogeneous type. When n_1 is the multiplicity, the pseudo-symmetric semigroup is always homogeneous by Proposition 2.1 and hence the Betti sequence is $(1, 5, 6, 2)$ in this case.

4. The proof when the multiplicity is n_2

Let n_2 be the multiplicity and suppose that the tangent cone is Cohen–Macaulay. If $\alpha_1 = \alpha_4$, then the Betti sequence is $(1, 5, 6, 2)$ by Proposition 2.1. We treat the cases $\alpha_1 < \alpha_4$ and $\alpha_1 > \alpha_4$ separately.

4.1. The proof in the case $\alpha_1 < \alpha_4$. In this case, $\{f_1, f_2, f_3, f_4, f_5\}$ is a standard basis of I_S by [11, Lemma 3.8]. Since $\bar{I}$ is the image of I_S^* under the map π_2 sending only X_2 to 0, it follows that $\bar{I}$ is generated by

$$G_* = \{X_1^{\alpha_1}, X_1^{\alpha_{21}} X_4, X_3^{\alpha_3}, X_4^{\alpha_4}, X_1^{\alpha_{21}+1} X_3^{\alpha_3 - 1}\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^3 \xrightarrow{\phi_3} A^7 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of $\bar{A}/\bar{I}$ by virtue of Lemma 1.2, where

$$\phi_1 = \begin{bmatrix} X_1^{\alpha_1} & X_1^{\alpha_{21}} X_4 & X_3^{\alpha_3} & X_4^{\alpha_4} & X_1^{\alpha_{21}+1} X_3^{\alpha_3-1} \end{bmatrix},$$

$$\phi_2 = \begin{bmatrix} 0 & X_4 & 0 & 0 & X_3^{\alpha_3-1} & 0 & 0 \\ 0 & -X_1^{\alpha_1-\alpha_{21}} & X_1 X_3^{\alpha_3-1} & X_4^{\alpha_4-1} & 0 & 0 & -X_3^{\alpha_3} \\ X_1^{\alpha_{21}+1} & 0 & 0 & 0 & 0 & X_4^{\alpha_4} & X_1^{\alpha_{21}} X_4 \\ 0 & 0 & 0 & -X_1^{\alpha_{21}} & 0 & -X_3^{\alpha_3} & 0 \\ -X_3 & 0 & -X_4 & 0 & -X_1^{\alpha_1-\alpha_{21}-1} & 0 & 0 \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} -X_4 & 0 & 0 \\ 0 & X_3^{\alpha_3-1} & 0 \\ X_3 & X_1^{\alpha_1-\alpha_{21}-1} & 0 \\ 0 & 0 & -X_3^{\alpha_3} \\ 0 & -X_4 & 0 \\ 0 & 0 & X_1^{\alpha_{21}} \\ X_1 & 0 & -X_4^{\alpha_4-1} \end{bmatrix}.$$

It is easy to check that $\operatorname{rank} \phi_1 = 1$, $\operatorname{rank} \phi_2 = 4$, $\operatorname{rank} \phi_3 = 3$. So, we show that $I(\phi_i)$ contains a regular sequence of length i for all $i = 1, 2, 3$. Since this is obvious for $i = 1$, we only discuss the other cases. For the matrix ϕ_2, the 4-minor corresponding to the rows $1, 2, 4, 5$ and columns $1, 5, 6, 7$ is computed to be $-X_3^{3\alpha_3}$. Similarly, the 4-minor corresponding to the rows $2, 3, 4, 5$ and columns $1, 2, 4, 5$ is $X_1^{2\alpha_1}$. As these minors are relatively prime, the ideal $I(\phi_2)$ contains a regular sequence of length 2. The 3-minor of ϕ_3 corresponding to the rows $1, 5, 7$ is $-X_4^{1+\alpha_4}$, to the rows $2, 3, 4$ is $X_3^{2\alpha_3}$ and to the rows $3, 6, 7$ is $X_1^{\alpha_1}$. As they are powers of different variables, they constitute a regular sequence of length 3.

4.2. The proof in the case $\alpha_1 > \alpha_4$. In this case, a standard basis of I_S is $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2} X_3 X_4^{\alpha_4-2}\}$ by [11, Lemma 3.8]. Since $\bar{I}$ is the image of I_S^* under the map π_2 sending only X_2 to 0, it follows that $\bar{I}$ is generated by

$$G_* = \{X_3 X_4^{\alpha_4-1}, X_1^{\alpha_{21}} X_4, X_3^{\alpha_3}, X_4^{\alpha_4}, X_1^{\alpha_{21}+1} X_3^{\alpha_3-1}, X_1^{\alpha_1+\alpha_{21}}\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^4 \xrightarrow{\phi_3} A^9 \xrightarrow{\phi_2} A^6 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of $\bar{A}/\bar{I}$ by virtue of Lemma 1.2, where

$$\phi_1 = \begin{bmatrix} X_3 X_4^{\alpha_4-1} & X_1^{\alpha_{21}} X_4 & X_3^{\alpha_3} & X_4^{\alpha_4} & X_1^{\alpha_{21}+1} X_3^{\alpha_3-1} & X_1^{\alpha_1+\alpha_{21}} \end{bmatrix},$$

ϕ_2 is given by

$$\begin{bmatrix} -X_4 & 0 & 0 & 0 & 0 & X_1^{\alpha_{21}} & 0 & X_3^{\alpha_3-1} & 0 \\ 0 & 0 & -X_1^{\alpha_1} & -X_1X_3^{\alpha_3-1} & -X_4^{\alpha_4-1} & -X_3X_4^{\alpha_4-2} & 0 & 0 & X_3^{\alpha_3} \\ 0 & -X_1^{\alpha_{21}+1} & 0 & 0 & 0 & 0 & 0 & -X_4^{\alpha_4-1} & -X_1^{\alpha_{21}}X_4 \\ X_3 & 0 & 0 & 0 & X_1^{\alpha_{21}} & 0 & 0 & 0 & 0 \\ 0 & X_3 & 0 & X_4 & 0 & 0 & -X_1^{\alpha_1-1} & 0 & 0 \\ 0 & 0 & X_4 & 0 & 0 & 0 & X_3^{\alpha_3-1} & 0 & 0 \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} 0 & -X_1^{\alpha_{21}} & 0 & 0 \\ X_4 & 0 & 0 & 0 \\ 0 & 0 & -X_3^{\alpha_3-1} & 0 \\ X_3 & 0 & X_1^{\alpha_1-1} & 0 \\ 0 & X_3 & 0 & 0 \\ 0 & -X_4 & 0 & -X_3^{\alpha_3-1} \\ 0 & 0 & X_4 & 0 \\ 0 & 0 & 0 & X_1^{\alpha_{21}} \\ X_1 & 0 & 0 & -X_4^{\alpha_4-2} \end{bmatrix}.$$

It is easy to check that rank $\phi_1 = 1$, rank $\phi_2 = 5$, rank $\phi_3 = 4$. So, we show that $I(\phi_i)$ contains a regular sequence of length i for all $i = 1, 2, 3$. Since this is obvious for $i = 1$, we only discuss the other cases. For the matrix ϕ_2, the 5-minor corresponding to the rows $1, 2, 3, 5, 6$ and columns $1, 3, 4, 5, 8$ is computed to be $-X_4^{1+2\alpha_4}$. Similarly, the 5-minor corresponding to the rows $1, 2, 4, 5, 6$ and columns $1, 2, 7, 8, 9$ is $-X_3^{3\alpha_3}$. As these minors are powers of different variables, the ideal $I(\phi_2)$ contains a regular sequence of length 2. The 4-minor of ϕ_3 corresponding to the rows $1, 4, 8, 9$ is $X_1^{2\alpha_{21}+\alpha_1}$, to the rows $3, 4, 5, 6$ is $X_3^{2\alpha_3}$ and to the rows $2, 6, 7, 9$ is $-X_4^{1+\alpha_4}$. As they are powers of different variables, they constitute a regular sequence of length 3.

5. The proof when the multiplicity is n_3

Suppose that the tangent cone is Cohen–Macaulay. If $\alpha_2 = \alpha_{21} + 1$, then the Betti sequence is $(1, 5, 6, 2)$ by Proposition 2.1. If $\alpha_2 < \alpha_{21} + 1$, then by [11, Lemma 3.12] a minimal standard basis for I_S is either $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1-1}X_4 - X_2^{\alpha_2-1}X_3^{\alpha_3}\}$ or $\{f_1, f_2, f_3, f_4' = X_4^{\alpha_4} - X_2^{\alpha_2-2}X_3^{2\alpha_3-1}, f_5, f_6\}$. Since π_3 sends only X_3 to 0, it follows that in both cases the ideal $\bar{I} = \pi_3(I_S^*)$ is generated by

$$G_* = \{X_1^{\alpha_1}, X_2^{\alpha_2}, X_1^{\alpha_1-\alpha_{21}-1}X_2, X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}, X_1^{\alpha_1-1}X_4\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^3 \xrightarrow{\phi_3} A^8 \xrightarrow{\phi_2} A^6 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of $\bar{A}/\bar{I}$ by virtue of Lemma 1.2, where

$$\phi_1 = \begin{bmatrix} X_1^{\alpha_1} & X_2^{\alpha_2} & X_1^{\alpha_1-\alpha_{21}-1}X_2 & X_4^{\alpha_4} & X_2X_4^{\alpha_4-1} & X_1^{\alpha_1-1}X_4 \end{bmatrix},$$

$$\phi_2 = \begin{bmatrix} 0 & -X_4 & 0 & 0 & 0 & 0 & X_2 & 0 \\ 0 & 0 & X_1^{\alpha_1-\alpha_{21}-1} & 0 & -X_4^{\alpha_4-1} & 0 & 0 & 0 \\ -X_4^{\alpha_4-1} & 0 & -X_2^{\alpha_2-1} & 0 & 0 & -X_1^{\alpha_{21}}X_4 & -X_1^{\alpha_{21}+1} & 0 \\ 0 & 0 & 0 & X_2 & 0 & 0 & 0 & X_1^{\alpha_1-1} \\ X_1^{\alpha_1-\alpha_{21}-1} & 0 & 0 & -X_4 & X_2^{\alpha_2-1} & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 & X_2 & 0 & -X_4^{\alpha_4-1} \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} 0 & -X_2^{\alpha_2-1} & -X_1^{\alpha_{21}}X_4 \\ -X_2 & 0 & 0 \\ 0 & X_4^{\alpha_4-1} & 0 \\ 0 & 0 & -X_1^{\alpha_1-1} \\ 0 & X_1^{\alpha_1-\alpha_{21}-1} & 0 \\ X_1 & 0 & X_4^{\alpha_4-1} \\ -X_4 & 0 & 0 \\ 0 & 0 & X_2 \end{bmatrix}.$$

It is easy to check that $\operatorname{rank}\phi_1 = 1$, $\operatorname{rank}\phi_2 = 5$, $\operatorname{rank}\phi_3 = 3$. So, we show that $I(\phi_i)$ contains a regular sequence of length i for all $i = 1, 2, 3$. Since this is obvious for $i = 1$, we only discuss the other cases. For the matrix ϕ_2, the 5-minor corresponding to the rows $1, 2, 3, 5, 6$ and columns $1, 2, 4, 5, 8$ is computed to be $-X_4^{3\alpha_4-1}$. Similarly, the 5-minor corresponding to the rows $2, 3, 4, 5, 6$ and columns $1, 2, 3, 7, 8$ is $-X_1^{3\alpha_1-\alpha_{21}-1}$. As these minors are powers of different variables, the ideal $I(\phi_2)$ contains a regular sequence of length 2. The 3-minor of ϕ_3 corresponding to the rows $1, 2, 8$ is $-X_2^{\alpha_2+1}$, to the rows $3, 6, 7$ is $-X_4^{2\alpha_4-1}$ and to the rows $4, 5, 6$ is $X_1^{2\alpha_1-\alpha_{21}-1}$. As they are powers of different variables, they constitute a regular sequence of length 3.

6. The proof when the multiplicity is n_4

Suppose that the tangent cone is Cohen–Macaulay. If $\alpha_3 = \alpha_1 - \alpha_{21}$, then the Betti sequence is $(1, 5, 6, 2)$ by Proposition 2.1. If $\alpha_3 < \alpha_1 - \alpha_{21}$, then a minimal standard basis for I_S is $\{f_1, f_2, f_3, f_4, f_5\}$ by [11, Lemma 3.17]. Since $\bar{I} = \pi_4(I_S^*)$, under the map π_4 sending only X_4 to 0, it is generated by

$$G_* = \{X_1^{\alpha_1}, X_2^{\alpha_2}, X_3^{\alpha_3}, X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, X_1^{\alpha_{21}+1}X_3^{\alpha_3-1}\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^3 \xrightarrow{\phi_3} A^7 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of $\bar{A}/\bar{I}$ by virtue of Lemma 1.2, where

$$\phi_1 = \begin{bmatrix} X_1^{\alpha_1} & X_2^{\alpha_2} & X_3^{\alpha_3} & X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} & X_1^{\alpha_{21}+1}X_3^{\alpha_3-1} \end{bmatrix},$$

$$\phi_2 = \begin{bmatrix} 0 & X_2^{\alpha_2} & 0 & 0 & X_3^{\alpha_3-1} & 0 & 0 \\ 0 & -X_1^{\alpha_1} & -X_1X_3^{\alpha_3-1} & 0 & 0 & 0 & -X_3^{\alpha_3} \\ -X_1^{\alpha_{21}+1} & 0 & 0 & 0 & 0 & -X_1X_2^{\alpha_2-1} & X_2^{\alpha_2} \\ 0 & 0 & X_2 & -X_1^{\alpha_{21}} & 0 & X_3 & 0 \\ X_3 & 0 & 0 & X_2^{\alpha_2-1} & -X_1^{\alpha_1-\alpha_{21}-1} & 0 & 0 \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} 0 & -X_2^{\alpha_2-1} & 0 \\ 0 & 0 & -X_3^{\alpha_3-1} \\ -X_3 & 0 & X_1^{\alpha_1-1} \\ 0 & X_3 & X_1^{\alpha_1-\alpha_{21}-1}X_2 \\ 0 & 0 & X_2^{\alpha_2} \\ X_2 & X_1^{\alpha_{21}} & 0 \\ X_1 & 0 & 0 \end{bmatrix}.$$

It is easy to check that rank $\phi_1 = 1$, rank $\phi_2 = 4$, rank $\phi_3 = 3$. So, we show that $I(\phi_i)$ contains a regular sequence of length i for all $i = 1, 2, 3$. Since this is obvious for $i = 1$, we only discuss the other cases. For the matrix ϕ_2, the 4-minor corresponding to the rows $1, 3, 4, 5$ and columns $2, 3, 4, 7$ is computed to be $X_2^{3\alpha_2}$. Similarly, the 4-minor corresponding to the rows $2, 3, 4, 5$ and columns $1, 2, 4, 5$ is $-X_1^{2\alpha_1+\alpha_{21}}$. As these minors are relatively prime, the ideal $I(\phi_2)$ contains a regular sequence of length 2. The 3-minor of ϕ_3 corresponding to the rows $1, 5, 6$ is $-X_2^{2\alpha_2}$, to the rows $2, 3, 4$ is $X_3^{1+\alpha_3}$ and to the rows $3, 6, 7$ is $-X_1^{\alpha_1+\alpha_{21}}$. As they are powers of different variables, they constitute a regular sequence of length 3.

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