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Ramanujan's circular summation, t-cores and twisted partition identities [☆]

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ABSTRACT

In this paper, we give new evaluations for Ramanujan's circular summation function. We also provide simpler proofs for known evaluations and give some generalizations. We discover modular relations among circular summation function partition function and give uniform proof of Ramanujan's partition congruences for the moduli 5, 7 and 11. We also prove several interesting congruence relations.

1. Introduction

On page 54 in his lost notebook [10], Ramanujan claims that

Entry 1.1. *For each positive integer n and $|ab| < 1$,*

$$\sum_{-n/2<r\leq n/2}\left(\sum_{\substack{k=-\infty\\ k\equiv r \,(\mathrm{mod}\, n)}}^{\infty} a^{k(k+1)/(2n)}b^{k(k-1)/(2n)}\right)^n = F_n(ab)\sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2}, \tag{1.1}$$

where

$$F_n(q) := 1 + 2nq^{(n-1)/2} + \cdots, \qquad n \geq 3. \tag{1.2}$$

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Recall that the classical theta functions are defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} \tag{1.3}$$

and

$$E(q) := (q;q)_\infty =: q^{-1/24}\eta(\tau), \tag{1.4}$$

where $q = \exp(2\pi i\tau)$, Im $\tau > 0$, and η denotes the Dedekind eta–function.

Ramanujan also gave the following evaluations of $F_n(q)$ for $n = 2, 3, 4, 5$ and 7.

$$F_2(q) = \varphi(\sqrt{q}). \tag{1.5}$$

$$F_3(q) = \left(\frac{E^9(q)}{E^3(q^3)} + 27q\frac{E^9(q^3)}{E^3(q)}\right)^{1/3} = \frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)}. \tag{1.6}$$

$$F_4(q) = \varphi^3(q^2) + (2\sqrt{q})^3\psi^3(q^4). \tag{1.7}$$

$$F_5(q) = \frac{E^5(q)}{E(q^5)} + 5q\frac{E^5(q^5)}{E(q)}. \tag{1.8}$$

$$F_7(q) = \frac{E^7(q)}{E(q^7)} + 7qE^3(q)E^3(q^7) + 7q^2\frac{E^7(q^7)}{E(q)}. \tag{1.9}$$

All five of the foregoing identities were established by Rangachari [13] and Son [14]. Several authors have determined the identification of $F_n(q)$ in further special cases. S. Ahlgren [1] considered the cases $n = 6, 8, 9$ and 10. K. Ono established $F_{11}(q)$, while Chua [6] derived the corresponding result for $F_{13}(q)$. A summary of all known identifications of $F_n(q)$ can be found in Son's paper [15].

In the next section will give simpler proofs of (1.5)–(1.8). Some of these evaluations will follow from general theta function identities established in this section. New evaluations for $R_n(q)$ for $n = 6, 8, 9, 10, 11, 12, 16$ will also be given. In the last section, we find identities that relates $F_n(q)$ to the generating function for the partitions. These identities permit us to give uniform proof of Ramanujan's partition congruences for the modulus 5, 7 and 11 along with new congruence relations for the coefficients of $F_n(q)$.

2. Definitions and preliminary results

We first recall Ramanujan's definitions for a general theta function and some of its important special cases. Set

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab| < 1. \tag{2.1}$$

If k is an integer, [3, p. 34, Entry 18]

$$f(a,b) = a^{k(k+1)/2}b^{k(k-1)/2}f(a(ab)^k, b(ab)^{-k}). \tag{2.2}$$

The function $f(a,b)$ also satisfies the well-known Jacobi triple product identity [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.3}$$

The three most important special cases of (2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2.4}$$
$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$E(q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = q^{-1/24}\eta(\tau). \tag{2.5}$$

The product representations in (2.4)–(2.5) are special cases of (2.3). The function $f(a,b)$ also satisfies a useful addition formula. For each integer n, let

$$U_n := a^{n(n+1)/2} b^{n(n-1)/2} \qquad \text{and} \qquad V_n := a^{n(n-1)/2} b^{n(n+1)/2}.$$

Then [3, p. 48, Entry 31]

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.6}$$

With $a=b=q$ and $n=2$, we find from (2.6) that

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{2.7}$$

Similarly, with $a=q$, $b=q^3$, and $n=2$, we find that

$$\psi(q) = f(q^6,q^{10}) + qf(q^2,q^{14}). \tag{2.8}$$

Define

$$R_n(z,q) = f^n(zq, z^{-1}q). \tag{2.9}$$

Throughout the manuscript, $[z^k]M(z,q)$ will denote the coefficient of $[z^k]$ in the expansion of $M(z,q)$. Let

$$R_n(q) := [z^0]R_n(z,q) = \sum_{\substack{l_1,l_2,\ldots,l_n=-\infty \\ l_1+l_2+\ldots+l_n=0}}^{\infty} q^{l_1^2+l_2^2+\ldots+l_n^2}. \tag{2.10}$$

From (2.2) with $a=zq$, $b=z^{-1}q$ and $k=1$, we find that

$$R_n(zq^2,q) = (z^{-1}q^{-1})^n R_n(z,q). \tag{2.11}$$

Clearly,

$$R_n(z,q) = R_n(z^{-1},q). \tag{2.12}$$

Let

$$R_n(z,q) = \sum_{l=-\infty}^{\infty} A_l(q) z^l. \tag{2.13}$$

Using (2.11) and (2.12), we deduce that

$$A_{nk+j}(q) = q^{nk^2+2kj}A_j(q) \ \text{ and } \ A_k(q) = A_{-k}(q). \tag{2.14}$$

Therefore,

$$R_n(z,q) = \sum_{j=0}^{n-1}\sum_{k=-\infty}^{\infty} A_j(q)q^{nk^2+2kj}z^{nk+j} = \sum_{j=0}^{n-1} z^j A_j(q) f(z^n q^{n+2j}, z^{-n}q^{n-2j}). \tag{2.15}$$

Moreover,

$$A_{n-j} = q^{n-2j}A_j. \tag{2.16}$$

Let w_n denote a primitive nth root of unity. For a fixed integer l, $0 \le l \le n-1$, multiply both sides of (2.15) by z^{-l}. Then by replacing z by w_n^r and summing over r, $0 \le r \le n-1$, we find that

$$nA_l(q)f(q^{n+2l}, q^{n-2l}) = \sum_{r=0}^{n-1} w_n^{-lr} R_n(w_n^r, q). \tag{2.17}$$

Clearly, $R_n(q) = [z^0]R_n(z,q) = A_0(q)$ and for this particular case, we have

$$n\varphi(q^n)R_n(q) = \sum_{r=0}^{n-1} R_n(w_n^r, q). \tag{2.18}$$

Now assume that n is prime. We use the definition (2.9) to expand the sum on the right hand side of (2.18). After inverting the order of summation, we find that

$$\begin{aligned}
n\varphi(q^n)R_n(q) &= \sum_{r=0}^{n-1}\left(\sum_{l_1,l_2,\ldots l_n=\infty}^{\infty} w_n^{r(l_1+l_2+\ldots+l_n)} q^{l_1^2+l_2^2+\ldots+l_n^2}\right)\\
&= \sum_{l_1,l_2,\ldots l_n=\infty}^{\infty}\left(\sum_{r=0}^{n-1} w_n^{r(l_1+l_2+\ldots+l_n)}\right) q^{l_1^2+l_2^2+\ldots+l_n^2}\\
&= n \sum_{\substack{l_1,l_2,\ldots,l_n=-\infty \\ l_1+l_2+\ldots+l_n\equiv 0 \pmod n}} q^{l_1^2+l_2^2+\ldots+l_n^2}.
\end{aligned}$$

That is when n is prime

$$\varphi(q^n)R_n(q) = \sum_{\substack{l_1,l_2,\ldots,l_n=-\infty \\ l_1+l_2+\ldots+l_n\equiv 0 \pmod n}} q^{l_1^2+l_2^2+\ldots+l_n^2}. \tag{2.19}$$

It was observed in [5] (see Theorem 3.1 and equation (3.6)) that

$$R_n(\tau/2) = \frac{1}{\sqrt{n}}(-i\tau)^{\frac{1-n}{2}} F_n(-\frac{1}{n\tau}). \tag{2.20}$$

Therefore, we can state Ramanujan's claims in their equivalent forms as follows

Theorem 2.1.

$$R_2(q) = \varphi(q^2), \tag{2.21}$$

$$R_3(q) = \frac{1}{4}\left(\frac{\varphi^3(-q)}{\varphi(-q^3)} + 3\frac{\varphi^3(-q^3)}{\varphi(-q)}\right), \tag{2.22}$$

$$R_4(q) = \frac{1}{2}\left(\varphi^3(q) + \varphi^3(-q)\right), \tag{2.23}$$

$$R_5(q) = \frac{E^5(q^2)}{E(q^{10})} + 25q^2\frac{E^5(q^{10})}{E(q^2)}, \tag{2.24}$$

$$R_7(q) = 7^3q^4\frac{E^7(q^{14})}{E(q^2)} + \frac{E^7(q^2)}{E(q^{14})} + 7^2q^2E^3(q^2)E^3(q^{14}). \tag{2.25}$$

Proof. The equation (2.21) is immediate from [3, p. 45, Entry 29]

$$R_2(z,q) = \varphi(q^2)f(z^2q^2, z^{-2}q^2) + 2z^{-1}q\psi(q^4)f(z^2, z^{-2}q^4). \tag{2.26}$$

Next, we prove (2.22). Recall that Ramanujan's cubic theta function is defined by

$$a(q) = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6). \tag{2.27}$$

From [4, eq. (2.26), eq. (2.27)], we deduce that

$$a(q^2) = \frac{1}{4}\left(\frac{\varphi^3(-q)}{\varphi(-q^3)} + 3\frac{\varphi^3(-q^3)}{\varphi(-q)}\right). \tag{2.28}$$

Therefore, by (2.28), it suffices to prove that $R_3(q) = a(q^2)$. By (2.26), we find that

$$R_3(q) = [z^0](R_3(z,q)) = [z^0]\left(f(zq, z^{-1}q)\left(\varphi(q^2)f(z^2q^2, z^{-2}q^2) + 2z^{-1}q\psi(q^4)f(z^2, z^{-2}q^4)\right)\right). \tag{2.29}$$

Observe that

$$\begin{aligned}[z^0]\left(f(zq, z^{-1}q)f(z^2q^2, z^{-2}q^2)\right) &= [z^0]\sum_{n,m=-\infty}^{\infty} z^{n+2m}q^{n^2+2m^2} \\ &= \sum_{\substack{n,m=-\infty \\ n+2m=0}}^{\infty} q^{n^2+2m^2} = \sum_{m=-\infty}^{\infty} q^{6m^2} = \varphi(q^6).\end{aligned}$$

Similarly,

$$[z^1]\left(f(zq, z^{-1}q)f(z^2, z^{-2}q^4)\right) = 2q\psi(q^{12}). \tag{2.30}$$

Hence,

$$R_3(q) = \varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12}) \tag{2.31}$$

as desired.

We will also make use of the following identity for $a(q)$ [4, Lemma 2.1, Prop. 2.2]

$$a(q^3) = 3q\frac{f^3(-q^9)}{f(-q^3)} + \frac{f^3(-q)}{f(-q^3)}. \tag{2.32}$$

To prove (2.23), we prove the more general statement that

Lemma 2.2.

$$2R_4(z,q) = \varphi^3(q)f(z^2q, z^{-2}q) + \varphi^3(-q)f(-z^2q, -z^{-2}q) + 8z^{-1}q\psi^3(q^2)f(z^2, z^{-2}q^2). \tag{2.33}$$

Proof. To prove (2.33) we need three more identities. From [3, p. 46, Entry 30 (iv), (v), (vi)] with $a = zq$ and $b = z^{-1}q$, we find that

$$f(zq, z^{-1}q)f(-zq, -z^{-1}q) = f(-z^2q^2, -z^{-2}q^2)\varphi(-q^2) \tag{2.34}$$

and

$$2f^2(zq^2, z^{-1}q^2) = \varphi(q)f(zq, z^{-1}q) + \varphi(-q)f(-zq, -z^{-1}q). \tag{2.35}$$

Setting $z = 1$ in (2.26) and (2.34), we also find that

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{2.36}$$

$$\varphi^2(-q^2) = \varphi(q)\varphi(-q). \tag{2.37}$$

Next we square both sides of (2.35) and employ (2.34), (2.37), (2.26) and (2.36), we find that

$$\begin{aligned}
4f^4(zq^2, z^{-1}q^2) &= \varphi^2(q)f^2(zq, z^{-1}q) + \varphi^2(-q)f^2(-zq, -z^{-1}q) \\
&\quad + 2\varphi^3(-q^2)f(-z^2q^2, -z^{-2}q^2) \\
&= \varphi^2(q)\left(\varphi(q^2)f(z^2q^2, z^{-2}q^2) + 2z^{-1}q\psi(q^4)f(z^2, z^{-2}q^4)\right) \\
&\quad + \varphi^2(-q)\left(\varphi(q^2)f(z^2q^2, z^{-2}q^2) - 2z^{-1}q\psi(q^4)f(z^2, z^{-2}q^4)\right) \\
&\quad + 2\varphi^3(-q^2)f(-z^2q^2, -z^{-2}q^2) \\
&= \varphi(q^2)\left(\varphi^2(q) + \varphi^2(-q)\right) f(z^2q^2, z^{-2}q^2) \\
&\quad + 2z^{-1}q\psi(q^4)\left(\varphi^2(q) - \varphi^2(-q)\right) f(z^2, z^{-2}q^4) \\
&\quad + 2\varphi^3(-q^2)f(-z^2q^2, -z^{-2}q^2) \\
&= 2\varphi^3(q^2)f(z^2q^2, z^{-2}q^2) + 16z^{-1}q^2\psi^3(q^4)f(z^2, z^{-2}q^4) \\
&\quad + 2\varphi^3(-q^2)f(-z^2q^2, -z^{-2}q^2),
\end{aligned}$$

which is (2.33) with q replaced by q^2. $\square$

Lastly, we prove (2.24). From (2.33), we find that

$$\begin{aligned}
2R_5(q) &= 2[z^0]\left(f(zq, z^{-1}q)R_4(z,q)\right) \\
&= \varphi^3(q)[z^0]\left(f(zq, z^{-1}q)f(z^2q, z^{-2}q)\right) + \varphi^3(-q)[z^0]\left(f(zq, z^{-1}q)f(-z^2q, -z^{-2}q)\right) \\
&\quad + 8q\psi^3(q^2)[z^4]\left(f(zq, z^{-1}q)f(z^2, z^{-2}q^2)\right).
\end{aligned}$$

Observe that

$$[z^0]\left(f(zq,z^{-1}q)f(z^2q,z^{-2}q)\right)=[z^0]\sum_{n,m=-\infty}^{\infty}z^{n+2m}q^{n^2+m^2}$$
$$=\sum_{\substack{n,m=-\infty\\ n+2m=0}}^{\infty}q^{n^2+m^2}=\sum_{m=-\infty}^{\infty}q^{5m^2}=\varphi(q^5).$$

Arguing similarly we find that

$$[z^0]\left(f(zq,z^{-1}q)f(-z^2q,-z^{-2}q)\right)=\varphi(-q^5)$$

and

$$[z^4]\left(f(zq,z^{-1}q)f(z^2,z^{-2}q^2)\right)=2q\psi(q^{10}).$$

Therefore, we conclude that

$$R_5(q)=\frac{1}{2}\left(\varphi^3(q)\varphi(q^5)+\varphi^3(-q)\varphi(-q^5)\right)+8q^2\psi^3(q^2)\psi(q^{10}). \tag{2.38}$$

Ramanujan gave Lambert series representation for $\dfrac{f^5(-q^5)}{f(-q)}$ and $\dfrac{f^5(-q)}{f(-q^5)}$. In [2], authors employed Ramanujan's representations to give explicit formulas for the coefficients of $\varphi^3(q)\varphi(q^5)$ and $4q\psi^3(q)\psi(q^5)$. Using these formulas we will show that the equation (2.38) and (2.24) are equivalent to each other. From Ramanujan's Lambert series representations, we deduce (see [2, p. 22] for details)

$$\alpha(n):=[q^n]\frac{E^5(q)}{E(q^5)}=-5\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{2.39}$$

and

$$\beta(n):=[q^n]\frac{qE^5(q^5)}{E(q)}=5^d\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}(-1)^{w_j}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{2.40}$$

where $n>0$ has the prime factorization

$$n=5^d\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

with $p_i\equiv\pm1\pmod 5$ and $q_i\equiv\pm2\pmod 5$. Clearly, we can write (2.39) and (2.40) in their equivalent form

$$\alpha(n)=-5\frac{1-(-2)^{g+1}}{3}u(n) \tag{2.41}$$

and

$$\beta(n)=5^d(-1)^{t+g}\frac{1-(-2)^{g+1}}{3}u(n), \tag{2.42}$$

where

$$u(n)=\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j} \tag{2.43}$$

and $n > 0$ has the prime factorization

$$n = 2^g 5^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

with $p_i \equiv \pm 1 \pmod 5$ and $q_i \equiv \pm 2 \pmod 5$, q_i is odd and t is the number of odd prime divisors of n, counting multiplicities, that are congruent to $\pm 2 \pmod 5$. It was shown in [2, Th. 7.1] that

$$[q^n](\varphi^3(q)\varphi(q^5)) = b(n) = (-1)^{n-1}(1 + 5^{d+1}(-1)^{g+t})\frac{(5 + (-2)^{g+1})}{3}u(n),$$
$$[q^n](4q\psi^3(q)\psi(q^5)) = c(n) = (-2)^g(-1 + 5^{d+1}(-1)^{g+t})u(n).$$

Clearly the equivalence (2.38) and (2.24) is given by the identity

$$b(2n) + 2c(n) = \alpha(n) + 25\beta(n).$$

Observe that

$$\begin{aligned}
&b(2n) + 2c(n)\\
&= (-1)^{2n-1}(1 + 5^{d+1}(-1)^{g+t+1})\frac{(5 + (-2)^{g+2})}{3}u(n) + 2(-2)^g(-1 + 5^{d+1}(-1)^{g+t})u(n)\\
&= -(1 + 5^{d+1}(-1)^{g+t+1})\frac{(5 + (-2)^{g+2})}{3}u(n) + (-2)^{g+1}(1 + 5^{d+1}(-1)^{g+t+1})u(n)\\
&= (1 + 5^{d+1}(-1)^{g+t+1})u(n)\left((-2)^{g+1} - \frac{(5 + (-2)^{g+2})}{3}\right)\\
&= (1 + 5^{d+1}(-1)^{g+t+1})\left(\frac{5(-2)^{g+1} - 5}{3}\right)u(n)\\
&= (5^{d+2}(-1)^{t+g} - 5)\left(\frac{1 - (-2)^{g+1}}{3}\right)u(n)\\
&= \alpha(n) + 25\beta(n) \quad \square
\end{aligned}$$

In the next lemma, we give a generalization of (2.22).

Lemma 2.3.

$$R_3(z, q) = a(q^2)f(z^3q^3, z^{-3}q^3) + 3qz^{-1}\frac{f^3(-q^6)}{f(-q^2)}\left(f(z^3q, z^{-3}q^5) + z^2 f(z^{-3}q, z^3q^5)\right), \tag{2.44}$$

and

$$R_3(z, q^3) = \frac{f^3(-q^2)}{f(-q^6)}f(z^3q^9, z^{-3}q^9) + 3q^2\frac{f^3(-q^{18})}{f(-q^6)}f(zq, z^{-1}q). \tag{2.45}$$

Proof. It is easy to check that (2.45) is equivalent to [7, p. 6, eq. (3.10)]. To prove (2.44), we employ (2.2) with $a = zq$, $b = z^{-1}q$ and $n = 3$ and find that

$$f(zq, z^{-1}q) = f(z^3q^9, z^{-3}q^9) + qzf(z^{-3}q^3, z^3q^{15}) + qz^{-1}f(z^3q^3, z^{-3}q^{15}). \tag{2.46}$$

By using (2.46) in (2.45), we find that

$$
\begin{aligned}
R_3(z,q^3) &= \frac{f^3(-q^2)}{f(-q^6)} f(z^3q^9, z^{-3}q^9) \\
&\quad + 3q^2 \frac{f^3(-q^{18})}{f(-q^6)} \left(f(z^3q^9, z^{-3}q^9) + qzf(z^{-3}q^3, z^3q^{15}) + qz^{-1}f(z^3q^3, z^{-3}q^{15}) \right) \\
&= \left(\frac{f^3(-q^2)}{f(-q^6)} + 3q^2 \frac{f^3(-q^{18})}{f(-q^6)} \right) f(z^3q^9, z^{-3}q^9) \\
&\quad + 3q^3 \frac{f^3(-q^{18})}{f(-q^6)} \left(zf(z^{-3}q^3, z^3q^{15}) + z^{-1}f(z^3q^3, z^{-3}q^{15}) \right) \\
&= a(q^3) f(z^3q^9, z^{-3}q^9) + 3q^3 \frac{f^3(-q^{18})}{f(-q^6)} \left(zf(z^{-3}q^3, z^3q^{15}) + z^{-1}f(z^3q^3, z^{-3}q^{15}) \right),
\end{aligned}
$$

where in the last step we used (2.32). Next, we replace q^3 by q and arrive at (2.44). □

Next, we determine $R_n(q)$ for $n = 6, 8, 9, 10, 11, 12$, and 16.

Lemma 2.4.

$$2R_6(q) = \varphi(q^2) \left(\varphi(q^3)\varphi^3(q) + \varphi(-q^3)\varphi^3(-q) \right) + 32q^2\psi(q^4)\psi(q^6)\psi^3(q^2). \tag{2.47}$$

Proof. We have

$$2R_6(q) = [z^0]R_6(z,q) = [z^0] \left(2R_4(z,q)R_2(z,q) \right). \tag{2.48}$$

By employing (2.33) and (2.26), we see that the contributing terms are

$$\varphi(q^2)\varphi^3(q)[z^0] \left(f(z^2q, z^{-2}q) f(z^2q^2, z^{-2}q^2) \right) = \varphi(q^2)\varphi^3(q)\varphi(q^3), \tag{2.49}$$

$$\varphi(q^2)\varphi^3(-q)[z^0] \left(f(-z^2q, -z^{-2}q) f(z^2q^2, z^{-2}q^2) \right) = \varphi(q^2)\varphi^3(-q)\varphi(-q^3), \tag{2.50}$$

$$16q^2\psi^3(q^2)\psi(q^4)[z^0] \left(z^{-2}f(z^2, z^{-2}q^2) f(z^2, z^{-2}q^4) \right) = 32q^2\psi(q^4)\psi(q^6)\psi^3(q^2). \tag{2.51}$$

Hence, (2.47) follows. □

Next, we give an expansion for $R_8(z,q)$.

Lemma 2.5.

$$
\begin{aligned}
2R_8(z,q) &= A(q)f(z^4q^2, z^{-4}q^2) + z^{-2}B(q)f(z^4, z^{-4}q^4) + \varphi^7(-q^2)f(-z^4q^2, -z^{-4}q^2) \\
&\quad + 8qz^{-1}\psi^7(q)f(z^2, z^{-2}q) + 8qz^{-1}\psi^7(-q)f(z^2, -z^{-2}q),
\end{aligned} \tag{2.52}
$$

where

$$A(q) = \frac{1}{2} \left(\varphi(q^2)\varphi^6(q) + \varphi(q^2)\varphi^6(-q) + 128q^2\psi(q^4)\psi^6(q^2) \right) \tag{2.53}$$

$$= \varphi^7(q^2) + 112q^2\psi^6(q^2)\psi(q^4). \tag{2.54}$$

$$B(q) = q \left(\psi(q^4)\varphi^6(q) - \psi(q^4)\varphi^6(-q) + 32q\varphi(q^2)\psi^6(q^2) \right) \tag{2.55}$$

$$= 56q^2\varphi(q^2)\psi^6(q^2) + 128q^4\psi^7(q^4). \tag{2.56}$$

Proof. By squaring both sides of (2.33), we find that

$$\begin{aligned}4R_8(z,q) =&\varphi^6(q)f^2(z^2q,z^{-2}q)+\varphi^6(-q)f^2(-z^2q,-z^{-2}q)+64z^{-2}q^2\psi^6(q^2)f^2(z^2,z^{-2}q^2)\\&+2\varphi^3(q)\varphi^3(-q)f(z^2q,z^{-2}q)f(-z^2q,-z^{-2}q)\\&+16z^{-1}q\varphi^3(q)\psi^3(q^2)f(z^2q,z^{-2}q)f(z^2,z^{-2}q^2)\\&+16z^{-1}q\varphi^3(-q)\psi^3(q^2)f(-z^2q,-z^{-2}q)f(z^2,z^{-2}q^2).\end{aligned}\tag{2.57}$$

Next, we linearize each term in (2.57). By three application of (2.26), with z replaced by z^2, $-z^2$ and z^2q in each case, we have that

$$\begin{aligned}&\varphi^6(q)f^2(z^2q,z^{-2}q)+\varphi^6(-q)f^2(-z^2q,-z^{-2}q)+64z^{-2}q^2\psi^6(q^2)f^2(z^2,z^{-2}q^2)\\&=\varphi^6(q)\left(\varphi(q^2)f(z^4q^2,z^{-4}q^2)+2z^{-2}q\psi(q^4)f(z^4,z^{-4}q^4)\right)\\&\quad+\varphi^6(-q)\left(\varphi(q^2)f(z^4q^2,z^{-4}q^2)-2z^{-2}q\psi(q^4)f(z^4,z^{-4}q^4)\right)\\&\quad+64z^{-2}q^2\psi^6(q^2)\left(\varphi(q^2)f(z^4,z^{-4}q^4)+2z^2\psi(q^4)f(z^4q^2,z^{-4}q^2)\right)\\&=\left(\varphi(q^2)\varphi^6(q)+\varphi(q^2)\varphi^6(-q)+128q^2\psi(q^4)\psi^6(q^2)\right)f(z^4q^2,z^{-2}q^2)\\&\quad+2z^{-2}q\left(\psi(q^4)\varphi^6(q)-\psi(q^4)\varphi^6(-q)+32\varphi(q^2)\psi^6(q^2)\right)f(z^4,z^{-4}q^2)\\&=2A(q)f(z^4q^2,z^{-2}q^2)+2z^{-2}B(q)f(z^4,z^{-4}q^2).\end{aligned}\tag{2.58}$$

From (2.37), and (2.34), we observe that

$$2\varphi^3(q)\varphi^3(-q)f(z^2q,z^{-2}q)f(z^2q,z^{-2}q)=2\varphi^7(-q^2)f(-z^4q^4,-z^{-4}q^4).\tag{2.59}$$

From [3, p. 46, Entry 30 (i)] with $a=z^2$ and $b=z^{-1}q$, we find that

$$f(z^2,z^{-2}q^2)f(z^2q,z^{-2}q)=\psi(q)f(z^2,z^{-2}q).\tag{2.60}$$

By (2.60), we deduce that

$$\begin{aligned}&16z^{-1}q\varphi^3(q)\psi^3(q^2)f(z^2q,z^{-2}q)f(z^2,z^{-2}q^2)\\&=16z^{-1}q\varphi^3(q)\psi^3(q^2)\psi(q)f(z^2,z^{-2}q)=16z^{-1}q\psi^7(q)f(z^2,z^{-2}q),\end{aligned}\tag{2.61}$$

where in the last step we used (2.60) with $z=1$. By (2.57), (2.58), (2.59), and (2.61), we arrive at (2.52). Since the theta function identities implied by (2.53)–(2.56) are modular equations of degree 2, they can be easily verified. We, therefore, skip the proofs of (2.54), and (2.56). □

Recall that $A(q)$ and $B(q)$ defined by (2.53) and (2.55). Now by employing (2.26), (2.44), (2.33), and (2.52) and by arguing as before we can prove

Corollary 2.6.

$$2R_8(q)=A(q)+\varphi^7(-q^2),\tag{2.62}$$

$$2R_9(q)=\varphi(q^{18})A(q)+\varphi(-q^{18})\varphi^7(-q^2)+2q^4\psi(q^{36})B(q)+16q^2\psi(q^9)\psi^7(q)\tag{2.63}$$

$$+16q^2\psi(-q^9)\psi^7(-q),\tag{2.64}$$

$$\begin{aligned}2R_{10}(q)=&\varphi(q^2)\varphi(q^{10})A(q)+\varphi(q^2)\varphi(-q^{10})\varphi^7(-q^2)+2q^2\varphi(q^2)\psi(q^{20})B(q)\\&+32q^2\psi(q^4)\psi(q^5)\psi^7(q)+32q^2\psi(q^4)\psi(-q^5)\psi^7(-q),\end{aligned}\tag{2.65}$$

$$\begin{aligned}4R_{12}(q) &= \varphi^3(q)\varphi(q^6)A(q) + \varphi^3(q)\varphi(-q^6)\varphi^7(-q^2) + 2q\varphi^3(q)\psi(q^{12})B(q)\\ &\quad + \varphi^3(-q)\varphi(q^6)A(q) + \varphi^3(-q)\varphi(-q^6)\varphi^7(-q^2) - 2q\varphi^3(-q)\psi(q^{12})B(q)\\ &\quad + 128q^2\psi^3(q^2)\psi(q^6)\psi^7(q) + 128q^2\psi^3(q^2)\psi(-q^6)\psi^7(-q), \quad (2.66)\\ 4R_{16}(q) &= A(q)^2\varphi(q^4) + 2B(q)^2\psi(q^8) + \varphi^{14}(-q^2)\varphi(q^4) + 128q^2\psi(q^2)\psi^{14}(q)\\ &\quad + 128q^2\psi(q^2)\psi^{14}(-q) + 2A(q)\varphi(-q^4)\varphi^7(-q^2) + 256q^2\varphi^7(-q^4)\psi^8(-q^2), \quad (2.67)\\ 2R_{11}(q) &= A(q)\left(a(q)\varphi(q^{66}) + 2q^8c(q)f(q^{22},q^{110})\right)\\ &\quad + \varphi(-q^2)^7\left(a(q)\varphi(-q^{66}) - 2q^8c(q)f(-q^{22},-q^{110})\right)\\ &\quad + 2q^2B(q)(c(q)f(q^{44},q^{88}) + q^{14}a(q)\psi(q^{132}))\\ &\quad + 16q^2\psi^7(q)\left(c(q)f(q^{11},q^{22}) + q^3a(q)\psi(q^{33})\right)\\ &\quad + 16q^2\psi^7(-q)(c(q)f(-q^{11},q^{22}) - q^3a(q)\psi(-q^{33})), \quad (2.68)\end{aligned}$$

where $a(q)$ is defined by (2.27) and $c(q) := 3\dfrac{E(q^6)^3}{E(q^2)}$.

We should remark that all the theta functions appearing in (2.68) including the functions $f(q,q^2)$ and $f(q,q^5)$ are in fact eta-quotients.

3. Applications to Ramanujan's partition congruences

In this section we will give uniform proofs of Ramanujan's partition congruences [11]

$$p(5n-1) \equiv 0 \pmod 5, \quad (3.1)$$

$$p(7n-2) \equiv 0 \pmod 7, \quad (3.2)$$

$$p(11n-5) \equiv 0 \pmod{11}. \quad (3.3)$$

These congruences will follow from the following theorem

Theorem 3.1. *For $t > 3$ a prime let $\delta_t = (t^2-1)/24$. Then, for $t = 5, 7$ and 11, we have*

$$tE^t(q)\sum_{n=1}^{\infty} p(tn-\delta_t)q^n = R_t(q^{1/2}) - \frac{E^t(q)}{E(q^t)}. \quad (3.4)$$

Proof. For $t = 5$ and 7 once the value of $R_t(q)$ is substituted from (2.24) and (2.25), the resulting identities reduces to those of Ramanujan [12], namely,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{E(q^5)}{E^6(q)}, \quad (3.5)$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 49q\frac{E^7(q^7)}{E^8(q)} + 7\frac{E^3(q^7)}{E^4(q)}. \quad (3.6)$$

It follows from the works of [6] and [5] that $R_{11}(q^{1/2})$ is an holomorphic modular form of weight 5 with respect to $\Gamma_0(11)$ with Nebentypus character $\chi := (./11)$. It is well known that all the other functions that appear in (3.4) for $t = 11$ are in the same space. This space has dimension five and verifying the identity is a straightforward exercise. We should remark that identities similar to (3.4) for $t = 11$ are stated both in [8] and [5] but they involve 5 terms. □

To derive Ramanujan's partition congruences from (3.4), we will need

Lemma 3.2. *If t is an odd prime number, then we have*

$$R_t(q^{1/2}) - \frac{E^t(q)}{E(q^t)} \equiv 0 \pmod{t^2}. \tag{3.7}$$

Proof. From the definition of $R_t(q)$ and the corresponding formula for $\frac{E^t(q)}{E(q^t)}$, we have that

$$R_t(q^{1/2}) - \frac{E^t(q)}{E(q^t)} = \sum_{\substack{n_0,n_1,\ldots,n_{t-1}=-\infty \\ n_0+n_1+\ldots n_{t-1}=0}}^{\infty} \left(1 - w^{n_1+2n_2+\ldots+(t-1)n_{t-1}}\right) q^{(n_0^2+n_1^2+\ldots+n_{t-1}^2)/2}, \tag{3.8}$$

where w is a primitive t-th root of unity. Fix $n_0, n_1, \ldots, n_{t-1}$ with $n_0 + n_1 + \ldots n_{t-1} = 0$ and let S be the set of all distinct permutations of $\{0, 1, \ldots, t-1\}$. S permutes $\{n_0, n_1, \ldots n_{t-1}\}$ via $\sigma(n_j) = n_{\sigma(j)}$. For each σ in S define $b(\sigma) := n_{\sigma(1)} + 2n_{\sigma(2)} + \ldots + (t-1)n_{\sigma(t-1)}$. It suffices to show that

$$\sum_{\sigma\in S} (1 - w^{b(\sigma)}) \equiv 0 \pmod{t^2}. \tag{3.9}$$

Let $P(j) := \{\sigma \in S \mid b(\sigma) \equiv j \pmod{t}\}$, $0 \le j \le t-1$. We will show that $|P(1)| = |P(2)| = \ldots = |P(t-1)|$ and $|P(1)| \equiv 0 \pmod{t}$. Once this is shown, the equation (3.9) will follow since

$$\sum_{\sigma\in S} (1 - w^{b(\sigma)}) = \sum_{j=0}^{t-1} |P(j)|(1 - w^j) = \sum_{j=1}^{t-1} |P(j)|(1 - w^j) \tag{3.10}$$

$$= |P(1)| \sum_{j=1}^{t-1} (1 - w^j) = t|P(1)| \equiv 0 \pmod{t^2}. \tag{3.11}$$

Let ϵ be the identity permutation and assume without loss of generality $b(\epsilon) = n_1 + 2n_2 + \ldots (t-1)n_{t-1} \equiv 1 \pmod{t}$. Fix k, $1 \le k \le t-1$. Let k' be the multiplicative inverse of k mod t and let $\sigma(i) := ik'$ reduced mod t, we have

$$b(\sigma) = \sum_{i=1}^{t-1} i n_{\sigma(i)} \equiv k \sum_{i=1}^{t-1} ik' n_{\sigma(i)} \equiv kb(\epsilon) \equiv k \pmod{t}. \tag{3.12}$$

This establishes a bijection between $P(1)$ and $P(k)$ and the first claim follows. Lastly, we show that $|P(1)| \equiv 0 \pmod{t}$. For σ in $P(1)$ consider the map $\sigma \to \bar{\sigma}$ with $\bar{\sigma}(n_0) = n_1, \bar{\sigma}(n_1) = n_2 \ldots, \bar{\sigma}(n_{t-1}) = n_0$. Then, $\bar{\sigma}$ in $P(1)$ since

$$\begin{aligned} b(\bar{\sigma}) &= n_{\sigma(2)} + 2n_{\sigma(3)} + \ldots + (t-1)n_{\sigma 0} \\ &= n_{\sigma(2)} + 2n_{\sigma(3)} + \ldots + (t-1)n_{\sigma(0)} + n_{\sigma(0)} + n_{\sigma(1)} + \ldots, n_{\sigma(t-1)} \\ &= b(\sigma) + tn_{\sigma(0)} \equiv b(\sigma) \pmod{t}. \end{aligned}$$

There are no fixed points since σ in $P(1)$ and $\sigma = \bar{\sigma}$ implies

$$1 \equiv b(\sigma) = \frac{t(t-1)}{2} n_{\sigma(1)} \equiv 0 \pmod{t}.$$

Hence, $|P(1)|$ is divisible by t. $\square$

We should remark that identities similar to (3.4) can be given for primes $t > 11$ but they would involve more terms. For example we can easily verify by using the theory of modular forms that

$$13E^{13}(q)\sum_{n=1}^{\infty} p(13n-7)q^n = R_{13}(q^{1/2}) - \frac{E^{13}(q)}{E(q^{13})} + 143qE^{11}(q)E(q^{13}). \tag{3.13}$$

Since

$$E(q^{13}) \equiv E^{13}(q) \pmod{13},$$

by (3.13) and (3.7), we arrive at Ramanujan's congruence relation

$$p(13n-7) \equiv 11\alpha(n) \pmod{13},$$

where $\sum \alpha(n)q^n := qE^{11}(q)$.

Next, we prove an equivalent form of (3.4).

Theorem 3.3. *For $t = 5, 7$ and 11, we have*

$$F_t(q) = tE^t(q^t)\sum_{n=1}^{\infty}\left(1 + \left(\frac{-2n}{t}\right)\right)p(n-\delta_t)q^n + \frac{E^t(q^t)}{E(q^{t^2})}. \tag{3.14}$$

Proof. We will prove that Theorem 3.3 is equivalent to Theorem 3.1 via the transformation $\tau \mapsto -\frac{1}{n\tau}$. Recall that for $g(\tau) = \sum_{n=0}^{\infty} a(n)e^{2\pi i n\tau}$, we define the U operator by

$$U_t(g(\tau)) := \sum_{n=0}^{\infty} a(tn)e^{2\pi i n\tau} = \frac{1}{t}\sum_{\lambda=0}^{t-1} g\left(\frac{\tau+\lambda}{t}\right).$$

Assume that g is an holomorphic modular form of weight $(t-1)/2$ with respect to $\Gamma_0(t)$ with Nebentypus character $\chi := (./t)$. As before we assume that $t \neq 3$ is an odd prime. Let $T_\lambda := \begin{pmatrix} 1 & \lambda \\ 0 & t \end{pmatrix}$, $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For each λ, $1 \le \lambda \le t-1$ let μ be the multiplicative inverse of $-\lambda$ modulo t. Then, $V = \begin{pmatrix} \lambda & -(1+\lambda\mu)/t \\ t & -\mu \end{pmatrix}$ is in $\Gamma_0(t)$ with $T_\lambda S = VT_\mu$. Observe that

$$g\left(\frac{-\frac{1}{\tau}+\lambda}{t}\right) = g(T_\lambda(S(\tau))) = g(V(T_\mu(\tau))) \tag{3.15}$$

$$= \chi(-\mu)\tau^{(t-1)/2}g(T_\mu(\tau)) = \chi(-\mu)\tau^{(t-1)/2}g\left(\frac{\tau+\mu}{t}\right). \tag{3.16}$$

Therefore, we find that

$$tU_t(g(-1/\tau)) = g(-1/(t\tau)) + \sum_{\mu=1}^{t-1} \chi(-\mu)\tau^{(t-1)/2} g\left(\frac{\tau+\mu}{t}\right). \tag{3.17}$$

Let us now assume that $g(\tau) = \dfrac{\eta^t(t\tau)}{\eta(\tau)} = q^{\delta_t}\dfrac{E^t(q^t)}{E(q)} = E^t(q^t)\sum_{n=0}^{\infty} p(n-\delta_t)q^n$.

$$\begin{aligned}\sum_{\mu=1}^{t-1} \chi(\mu) g\left(\frac{\tau+\mu}{t}\right) &= E^t(q)\sum_{\mu=1}^{t-1}\chi(\mu)\sum_{n=0}^{\infty} p(n-\delta_t)e^{2\pi i n(\tau+\mu)/t}\\ &= E^t(q)\sum_{n=0}^{\infty} p(n-\delta_t)e^{2\pi i n\tau/t}\sum_{\mu=1}^{t-1}\chi(\mu)e^{2\pi i n\mu/t}\\ &= E^t(q)\epsilon_t\sum_{n=0}^{\infty} p(n-\delta_t)\chi(n)e^{2\pi i n\tau/t},\end{aligned} \tag{3.18}$$

where we used the fact that the inner sum is a Gauss sum and $\epsilon_t = \sqrt{t}$ if t is congruent to 1 modulo 4 and $i\sqrt{t}$ otherwise. From (3.17) and (3.18), we deduce that

$$tU_t(g(-1/\tau)) = g(-1/(t\tau)) + \epsilon_t\chi(-1)\tau^{(t-1)/2}E^t(q)\sum_{n=0}^{\infty} p(n-\delta_t)\chi(n)e^{2\pi i n\tau/t}. \tag{3.19}$$

By employing the transformation formula $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, we find that $g(-1/(t\tau)) = \dfrac{1}{\sqrt{t}}(-i\tau)^{(t-1)/2} \times \dfrac{\eta^t(\tau)}{\eta(t\tau)}$ and by using this in (3.17), we conclude that

$$tU_t(g(-1/\tau)) = \frac{1}{\sqrt{t}}(-i\tau)^{(t-1)/2}\left(\frac{\eta^t(\tau)}{\eta(t\tau)} + tE^t(q)\sum_{n=0}^{\infty} p(n-\delta_t)\chi(-2n)e^{2\pi i n\tau/t}\right). \tag{3.20}$$

Lastly, replacing τ by $t\tau$, we arrive that

$$tU_t(g(-1/(t\tau))) = \frac{1}{\sqrt{t}}(-it\tau)^{(t-1)/2}\left(\frac{\eta^t(t\tau)}{\eta(t^2\tau)} + tE^t(q^t)\sum_{n=0}^{\infty} p(n-\delta_t)\chi(-2n)e^{2\pi i n\tau}\right). \tag{3.21}$$

Observe that

$$U_t\left(\frac{\eta^t(t\tau)}{\eta(\tau)}\right) = U_t\left(q^{\delta_t}\frac{E^t(q^t)}{E(q)}\right) = E^t(q)U_t\left(\sum_{n=o}^{\infty} p(n)q^{n+\delta_t}\right) = E^t(q)\sum_{n=1}^{\infty} p(tn-\delta_t)q^n. \tag{3.22}$$

We, therefore, completed the transformation of the right hand side of (3.4). For the left hand side we use the transformation of the eta function and (2.20) and arrive at (3.14). □

It follows from the proof of Theorem 3.3 that

Corollary 3.4. *If t is an odd prime, then*

$$tE^t(q^t)\sum_{n=1}^{\infty}\left(\frac{-2n}{t}\right)p(n-\delta_t)q^n + \frac{E^t(q^t)}{E(q^{t^2})} \tag{3.23}$$

is an holomorphic modular form of weight $(t-1)/2$ with respect to $\Gamma_0(t)$ with Nebentypus character $\chi := (./t)$.

It is also clear from Theorem 3.3 that

Corollary 3.5. *Let $a_t(n)$ and $f_p(n)$ be defined by*

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{E^t(q^t)}{E(q)} \quad \text{and} \quad \sum_{n=0}^{\infty} f_t(n)q^n = F_t(q). \tag{3.24}$$

Then, for $t = 5, 7, 11$

$$f_t(n) = \left\{\begin{array}{ll} 0 & \text{if } -2n \text{ is a quadratic non-residue mod } t \\ 2ta_t(n-\delta_t) & \text{if } -2n \text{ is a quadratic residue mod } t \end{array}\right\} \tag{3.25}$$

We should remark that we can find similar identities for higher modulus. For the case of $t = 13$, we can state (3.13) in its equivalent form under the modular substitution $\tau \mapsto -1/13\tau$ as

$$F_{13}(q) = 13E^{13}(q^{13})\sum_{n=1}^{\infty}\left(1+\left(\frac{-2n}{13}\right)\right)p(n-7)q^n + \frac{E^{13}(q^{13})}{E(q^{169})} + 26q^6E(q)E(q^{13})^{11}. \tag{3.26}$$

Observe that $q^6E(q) = \sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2+6}$ and that

$$(3n^2-n)/2+6 \equiv 8(n+2)^2 \pmod{13}$$

is a non-square mod 13. Therefore, we similarly conclude that if n is a quadratic residue mod 13, then $f_{13}(n) = 0$.

As another application of Theorem 3.3, we can provide simpler proofs of some identities of Rademacher [9].

Corollary 3.6.

$$\sum_{n=1}^{\infty} p(n)q^{25n} - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)p(n-1)q^n = \frac{E^5(q)}{E^6(q^5)} \tag{3.27}$$

and

$$\sum_{n=1}^{\infty} p(n)q^{49n} - 7\sum_{n=2}^{\infty}\left(\frac{n}{7}\right)p(n-2)q^n = \frac{E^7(q)}{E^8(q^7)} + 7q\frac{E^3(q)}{E^4(q^7)}. \tag{3.28}$$

Proof. We will prove (3.27). The proof of (3.28) is similar. We find from (3.14) that

$$F_5(q) = 5E^5(q^5)\sum_{n=1}^{\infty}\left(1-\left(\frac{n}{5}\right)\right)p(n-1)q^n + \frac{E^5(q^5)}{E(q^{25})} \tag{3.29}$$

$$= 5q\frac{E^5(q^5)}{E(q^5)} - 5E^5(q^5)\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)p(n-1)q^n + \frac{E^5(q^5)}{E(q^{25})}. \tag{3.30}$$

From (1.8), we also have

$$F_5(q) = \frac{E^5(q)}{E(q^5)} + 5q\frac{E^5(q^5)}{E(q)}. \tag{3.31}$$

Therefore, we conclude that

$$5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)p(n-1)q^n = \frac{1}{E(q^{25})} - \frac{E^5(q)}{E^6(q^5)}. \tag{3.32}$$

That is

$$\sum_{n=1}^{\infty}p(n)q^{25n} - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)p(n-1)q^n = \frac{E^5(q)}{E^6(q^5)}. \quad \square \tag{3.33}$$

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