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ORIGINAL PAPER

Operator theory-based discrete fractional Fourier transform

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Abstract

The fractional Fourier transform is of importance in several areas of signal processing with many applications including optical signal processing. Deploying it in practical applications requires discrete implementations, and therefore defining a discrete fractional Fourier transform (DFRT) is of considerable interest. We propose an operator theory-based approach to defining the DFRT. By deploying hyperdifferential operators, a DFRT matrix can be defined compatible with the theory of the discrete Fourier transform. The proposed DFRT only uses the ordinary Fourier transform and the coordinate multiplication and differentiation operations. We also propose and compare several alternative discrete definitions of coordinate multiplication and differentiation operations, each of which leads to an alternative DFRT definition. Unitarity and approximation to the continuous transform properties are also investigated in detail. The proposed DFRT is highly accurate in approximating the continuous transform.

Keywords Fractional Fourier transform (FRT) · Operator theory · Discrete transforms · Hyperdifferential operators

1 Introduction

The fractional Fourier transform (FRT) is the generalization of Fourier transform (FT) [1–5]. While the ordinary FT takes a signal from the time (or space) domain to the frequency (or spatial frequency) domain, FRT allows transformations to any intermediate domain in between. Therefore, FRT also generalizes the concept of *frequency domain* [4]. FRT of order a, denoted by $\mathcal{F}^a$ where a is a real number, is the a'th power of the ordinary Fourier transform (FT). The case when $a = 1$ reduces to FT and $a = 0$ reduces to the identity operation.

FRT is of importance for signal processing [6–13], time/space–frequency representations [5,14–16], image processing [17–20], video processing [21,22], pattern recognition [23], radar/sonar signal processing [24,25] and beamforming [26,27]. FRT finds applications in wave and beam propagations, diffraction and generally in Fourier optics [1,28,29]. Being one of the most important special cases of linear canonical transforms (LCTs) [30], FRT also plays a central role in LCT-related contexts and applications. Applications in radar signal processing [31], speech processing [32] as well as image encryption and watermarking [33–37] can be listed to name a few examples. More on this literature can be found in [4,30].

To deploy FRT in the above application areas, discretization and digital computation of FRT is of prominent importance. To this end, several discrete fractional Fourier transform (DFRT) definitions have been proposed and studied in detail. Some of these works approach the problem from a computational and sampling point of view, [38,39], while several others use eigenvector decompositions to define a DFRT [40–47]. Very recently, a very detailed and thorough analysis and comparisons on this large literature about DFRTs have been presented in [48].

Although there exist several DFRT definitions, no single definition distinguishes itself among others and research is being conducted to develop new approaches. In doing so, the main objectives can be listed as follows:

(i) to mimic as many properties of the continuous FT as possible,
(ii) being a unitary definition,
(iii) providing a good numerical approximation to the continuous transform,
(iv) working for all possible fractional orders with similar performance,

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(v) being consistent with the general discrete Fourier transform (DFT) definition and its circulant structure,
(vi) satisfying index additivity/reversibility properties.

Recently, a new approach based on hyperdifferential operators has been proposed to define a discrete linear canonical transform (DLCT) [49]. By utilizing hyperdifferential operators, this definition uses discrete versions of the very simple building blocks of coordinate multiplication, differentiation and FT in a way that is totally consistent and highly analogous with the established definition of the DFT. The approach presented in [49], where the emphasis is given to preserve the general structural symmetry between coordinate multiplication and differentiation operations, defines the general class of discrete LCTs without special focus on the most important special case of FRT. Indeed, a version of DFRT can be defined as a special case of the approach given in [49], which we denote by DFRT1. However, this definition is not the only possible way of using operator theory in defining DFRT and its accuracy of the approximation to the continuous FRT is open to improvement. In this paper, we further study the special case of FRT in more detail and propose two more different definitions of DFRT, namely DFRT2 and DFRT3, in which we use different discrete definitions of U and D. We also study and compare several alternative discrete definitions of coordinate multiplication and differentiation in detail in order to obtain utmost computational accuracy. The hyperdifferential formulation provided here constitutes not only a theoretically pure approach to defining the DFRT, but also serves as a framework for high-accuracy numerical computations. We also compare our proposed DFRT definition with the widely noted definition of Candan et al. [42]. Our basis of comparison is a highly inefficient but accurate brute force numerical integration-based method, which is taken as the ultimate reference for accuracy.

The paper is organized as follows. In Sect. 2, we will give the formal definition and details of FRT. In Sect. 3, our proposed operator theory-based DFRT definition with several alternative discrete coordinate multiplication and differentiation methods will be presented. Unitarity of the proposed DFRT definitions will be proved in Sect. 4. Numerical examples and comparisons among several alternatives will be presented in Sect. 5, and the paper will conclude in Sect. 6.

2 Fractional Fourier transform

Fractional Fourier transform (FRT) is the generalized version of Fourier transform (FT). It has the following parameter matrix:

$$\mathbf{F}^a = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}, \tag{1}$$

where $\theta = \pi a/2$ and a is the fractional order. When $a = 1$, FRT reduces to FT. Also, we note the phase difference that occurs when LCT reduces to FRT:

$$\mathcal{F}^a_{\mathrm{lc}} = \mathcal{F}^a \exp(-i\theta/2). \tag{2}$$

This inconsequential discrepancy comes from the fact that there is a slight difference between the FRT thus defined ($\mathcal{F}^a_{\mathrm{lc}}$) and the more commonly used definition of the FRT ($\mathcal{F}^a$) [4]. The ath-order fractional Fourier transform $\mathcal{F}^a$ of the function $f(u)$ may be defined for $0 < |a| < 2$ as

$$\begin{aligned} \mathcal{F}^a f(u) &= \int_{-\infty}^{\infty} K_a(u, u') f(u')\,\mathrm{d}u', \\ K_a(u, u') &= A_\theta \exp\left[i\pi(u^2\cot\theta - 2uu'\csc\theta + u'^2\cot\theta)\right], \\ A_\theta &= \frac{\exp(-i\pi\,\mathrm{sgn}(\sin\theta)/4 + i\theta/2)}{|\sin\theta|^{1/2}}. \end{aligned} \tag{3}$$

3 Operator theory-based DFRT definition and main results

In [49], the authors introduced hyperdifferential operators for the first time to the process of defining discrete transforms. The development of this approach takes the continuous operator forms of the transforms as a starting point and then tries to generate the continuous and discrete manifestations from this common abstract operator framework. In other words, both continuous and discrete transforms come from the same abstract core structure. In doing so, the aim is to put forth a definition of the DLCT in a manner that preserves structural similarity with the continuous DLCT. This preservation makes the maintenance of strict structural analogy between the continuous and discrete worlds, which makes further theoretical manipulations possible.

In this chapter, we first use this approach to set the framework to define DFRT. We start with recalling what hyperdifferential forms mean. The term hyperdifferential refers to having differential operators in an exponent. There is correspondence among the integral transforms, hyperdifferential operators and the 2×2 parameter matrices that are given in the preliminaries section. More details can be found in [50]. In the DFRT context, there are only second-order coordinate multiplication and differentiation operators in the exponent.

Continuous FRT operator $\mathcal{F}^a_{\mathrm{lc}}$ given in Eq. 2 can be written in the hyperdifferential form as the following [4,50]:

$$\mathcal{F}^a_{\mathrm{lc}} = \exp\left(-ia\pi^2\frac{\mathcal{U}^2 + \mathcal{D}^2}{2}\right), \tag{4}$$

where $\mathcal{U}$ and $\mathcal{D}$ are the coordinate multiplication and differentiation operators, respectively. $\mathcal{U}$ and $\mathcal{D}$ operators are defined in continuous forms as:

$$\mathcal{U} f(u) = u f(u), \quad \mathcal{D} f(u) = \frac{1}{i2\pi} \frac{\mathrm{d} f(u)}{\mathrm{d}u}, \tag{5}$$

where the $(i2\pi)^{-1}$ is included so that $\mathcal{U}$ and $\mathcal{D}$ are precisely Fourier duals (the effect of either in one domain is its dual in the Fourier domain). This duality can be expressed as follows:

$$\mathcal{U} = \mathcal{F}\mathcal{D}\mathcal{F}^{-1}. \tag{6}$$

Operator theory approach is based on defining the DFRT as the discrete manifestation of Eq. 4, with the abstract operators being replaced by matrix operators. This can be written as follows:

$$\mathbf{F}_{\mathrm{lc}}^{a} = \exp\left(-ia\pi^2 \frac{\mathbf{U}^2 + \mathbf{D}^2}{2}\right). \tag{7}$$

In Eq. 7, the definition of the DFRT, $\mathbf{F}_{\mathrm{lc}}^{a}$, is presented in the form of a matrix of size $N \times N$ which, upon multiplication, produces the DFRT of a discrete and finite signal of length N, expressed as a column vector. However, this DFRT definition relies on proper definitions of discrete manifestations $\mathbf{U}$ and $\mathbf{D}$, which are matrices of size $N \times N$, of coordinate multiplication and differentiation, respectively. Since everything rests on these two operators, how well they are defined indeed determines how well the hyperdifferential operator theory-based DFRT is defined.

At the heart of the hyperdifferential operator-based discrete transform lie the definitions of $\mathbf{U}$ and $\mathbf{D}$ matrices. The duality between them under the ordinary discrete Fourier transform as given below is also instrumental:

$$\mathbf{D} = \mathbf{F}^{-1}\mathbf{U}\mathbf{F}, \tag{8}$$

where $\mathbf{F}$ is the unitary discrete Fourier transform (DFT) matrix whose elements F_{mn} can be written in terms of $W_N = \exp(-j2\pi/N)$ as $F_{mn} = \frac{1}{\sqrt{N}} W_N^{mn}$.

In what follows, we will analyze several alternatives for $\mathbf{U}$ and $\mathbf{D}$ definitions leading to different DFRT definitions. We will present three different pairs of $\mathbf{U}$ and $\mathbf{D}$ definitions that can be used to define three alternative DFRT definitions by simply replacing these alternative $\mathbf{U}$ and $\mathbf{D}$ matrices into Eq. 7. First one uses the strict structurally analogous $\mathbf{U}$ and $\mathbf{D}$ matrices developed in [49]. The remaining two are our proposed definitions.

3.1 Strict structurally analogous U and D matrices

In [49], the main concern is, by the use of operator theory, to define a discrete transform that is fully compatible with the theory of the discrete Fourier transform (DFT) and its dual and circulant structure. Care is taken to maintain the structural analogy between continuous and discrete domains by treating the time and frequency domains symmetrically. To do this, the abstract differentiation operator is taken as a starting point. This manifests itself as the common derivative in continuous time. In discrete time, the finite difference operation is the closest possible discrete counterpart [49]:

$$\tilde{\mathcal{D}}_h f(u) = \frac{1}{i2\pi} \frac{f(u + h/2) - f(u - h/2)}{h}. \tag{9}$$

If $h \to 0$, then $\tilde{\mathcal{D}}_h \to \mathcal{D}$, since in this case the right-hand side approaches $(i2\pi)^{-1}\mathrm{d} f(u)/\mathrm{d}u$. Therefore, $\tilde{\mathcal{D}}_h$ can be interpreted as a finite difference operator. Then, by using $f(u+h) = \exp(i2\pi h\mathcal{D}) f(u)$, which is an established result in operator theory [4,50], Eq. 9 is expressed in hyperdifferential form:

$$\begin{aligned}\tilde{\mathcal{D}}_h &= \frac{1}{i2\pi} \frac{\mathrm{e}^{i\pi h\mathcal{D}} - \mathrm{e}^{-i\pi h\mathcal{D}}}{h} \\ &= \frac{1}{i2\pi} \frac{2i \sin(\pi h\mathcal{D})}{h} = \mathrm{sinc}(h\mathcal{D})\,\mathcal{D}.\end{aligned} \tag{10}$$

To preserve the structural symmetry between $\mathcal{U}$ and $\mathcal{D}$, one needs to define $\tilde{\mathcal{U}}_h$ such that it is related to $\mathcal{U}$, in exactly the same way as $\tilde{\mathcal{D}}_h$ is related to $\mathcal{D}$. In other words, $\tilde{\mathcal{U}}_h$ is defined as [49]:

$$\tilde{\mathcal{U}}_h = \mathrm{sinc}(h\mathcal{U})\,\mathcal{U}, \tag{11}$$

from which it can be observed that as $h \to 0$, we have $\tilde{\mathcal{U}}_h \to \mathcal{U}$, as should be. Then, if $\tilde{\mathcal{U}}_h$ operates on a continuous signal $f(u)$, one gets:

$$\tilde{\mathcal{U}}_h f(u) = \frac{1}{\pi} \frac{\sin(\pi hu)}{h} f(u). \tag{12}$$

It should be observed that the effect is not merely multiplying with the coordinate variable. By sampling Eq. 12, the matrix operator to act on finite discrete signals can be obtained. The sample points will be taken as $u = nh$ to finally yield the $\mathbf{U}$ matrix defined as:

$$U_{mn} = \begin{cases} \frac{\sqrt{N}}{\pi} \sin\left(\frac{\pi}{N} n\right), & \text{for } m = n \\ 0, & \text{for } m \neq n \end{cases} \tag{13}$$

where $m, n = 0, 1, \ldots, N-1$ and N is the number of samples. Finally, the matrix $\mathbf{D}$ can be calculated in terms of $\mathbf{U}$ by using the discrete version of the duality relation given in Eq. 8. This hyperdifferential operator theory-based DFRT definition will be denoted by DFRT1 in our numerical experiments.

As we summarized above, in this approach everything can be traced back to the finite difference operation, which is the closest operation in discrete domain to the continuous differentiation, and to the DFT. Therefore, both the continuous and discrete time cases are built in the same hyperdifferential form and share the same operational structure. Indeed, this is the main distinguishing feature of this approach.

3.2 Simplest forms of U and D matrices

While aiming to maintain maximum theoretical and structural uniformity, the primary purpose of the approach outlined in the previous subsection is not improved accuracy. Accuracy can always be increased by increasing N but this also comes with a cost of increased computational burden. Since everything rests on defining $\mathbf{U}$ and $\mathbf{D}$, their accuracy is also what defines the accuracy of the resulting DFRT. So a search for alternative ways of discretizing $\mathbf{U}$ and $\mathbf{D}$ is helpful to obtain more accurate DFRT definitions.

In this subsection, we propose to use the simplest possible forms of $\mathbf{U}$ and $\mathbf{D}$ matrices. Since $\mathbf{U}$ is the simple coordinate multiplication operation, one can discretize it as the following: Let us have N samples, and we sample over an extent $\sqrt{N}$ with sampling interval $h = 1/\sqrt{N}$. Then, $\mathcal{U} f(u) = u f(u)$ can be discretized as $nhf(nh) = n/\sqrt{N} f[n]$ where $u = nh$ and $n = 0, 1, \ldots, N-1$. Then, $\mathbf{U}$ can be written as

$$U_{mn} = \begin{cases} \frac{n}{\sqrt{N}}, & \text{for } m = n \\ 0, & \text{for } m \neq n \end{cases} \tag{14}$$

where $m, n = 0, 1, \ldots, N-1$.

This discrete definition of the coordinate multiplication is indeed the most accurate approximation of the continuous coordinate multiplication. Again, since we want $\mathbf{U}$ and $\mathbf{D}$ to be Fourier duals of each other in order to preserve the dual structure of the DFRT definition, we use the discrete version of the duality relation given in Eq. 8 to obtain $\mathbf{D}$ matrix. We will denote this approach with DFRT2.

3.3 Numerical forms of U and D matrices

As aforementioned, since defining $\mathbf{U}$ and $\mathbf{D}$ matrices lies at the heart of the DFRT definition, utilizing highly accurate numerical forms is another alternative that needs to be studied. In this paper, we propose to use spectral methods in defining discrete transform through hyperdifferential operator theory approach. Spectral methods [51] are advanced numerical techniques used in scientific computing primarily for solving certain differential equations numerically.

In this alternative, we propose to use the opposite avenue in defining $\mathbf{U}$ and $\mathbf{D}$ matrices. In other words, we now first define discrete differentiation and then use the duality relation to obtain discrete coordinate multiplication. Another modification is that, this time, we directly define and use second-order discrete matrices, i.e., $\mathbf{U}^2$ and $\mathbf{D}^2$.

We use the discrete second-order differentiation matrix $\mathbf{D}^2$ from [51], which is obtained using spectral methods, defined as, for even N:

$$\mathbf{D}^2_{mn} = \begin{cases} -\frac{\pi^2}{3h^2} - \frac{1}{6}, & \text{for } m = n \\ -\frac{1}{2}(-1)^{m-n} \csc^2\left(\frac{(m-n)h}{2}\right), & \text{for } m \neq n \end{cases} \tag{15}$$

and for odd N:

$$\mathbf{D}^2_{mn} = \begin{cases} -\frac{\pi^2}{3h^2} - \frac{1}{12}, & \text{for } m = n \\ -\frac{1}{2}(-1)^{m-n} \csc\left(\frac{(m-n)h}{2}\right) \cot\left(\frac{(m-n)h}{2}\right), & \text{for } m \neq n \end{cases} \tag{16}$$

where $m, n = 0, \ldots, N-1$ and $h = 2\pi/N$.

Then, by using the second-order version of the duality relation ($\mathbf{U}^2 = \mathbf{F}\mathbf{D}^2\mathbf{F}^{-1}$), one can easily obtain the second-order discrete coordinate multiplication matrix $\mathbf{U}^2$. Finally, by simply replacing the above $\mathbf{U}^2$ and $\mathbf{D}^2$ matrices in Eq. 7, we obtain another proposed DFRT definition, which we denote with DFRT3.

4 Unitarity

Among the fundamental properties of a discrete transform definition, arguably the most important one is the unitarity. A continuous transform and its discrete counterpart are generally used to model some physical entity, and unitarity corresponds to energy or power conservation in these physical applications. For this reason, since continuous FRT is unitary, it is strongly desired that our proposed DFRT definitions are also unitary. There are two theorems simply stating that if Hermitian $\mathbf{U}$ and $\mathbf{D}$ matrices are used, then the hyperdifferential operator-based definition given in Eq. 7 is unitary [49]. Then, for the DFRT1 and DFRT2 definitions, we use $\mathbf{U}$ matrices given in Eqs. 13 and 14, respectively. Since these two matrices are real symmetric, they are also Hermitian. Since $\mathbf{D}$ counterparts of these two matrices are also derived from $\mathbf{U}$ matrices through the duality relation, corresponding $\mathbf{D}$ matrices are also Hermitian as can be seen from the following:

$$\mathbf{D}^{\mathbf{H}} = (\mathbf{F}^{-1}\mathbf{U}\mathbf{F})^{\mathbf{H}} = \mathbf{F}^{\mathbf{H}}\mathbf{U}^{\mathbf{H}}(\mathbf{F}^{\mathbf{H}})^{\mathbf{H}} = \mathbf{F}^{-1}\mathbf{U}\mathbf{F} = \mathbf{D}. \tag{17}$$

For the DFRT3, we use directly $\mathbf{U}^2$ and $\mathbf{D}^2$ matrices. Upon inspection of Eqs. 15 and 16, we can observe that $\mathbf{D}^2$ is also real symmetric and hence Hermitian. Finally, after proving that $\mathbf{U}^2$ is Hermitian by:

$$\begin{aligned}(\mathbf{U}^2)^{\mathbf{H}} &= (\mathbf{F}\mathbf{D}^2\mathbf{F}^{-1})^{\mathbf{H}} \\ &= (\mathbf{F}^{\mathbf{H}})^{\mathbf{H}}(\mathbf{D}^2)^{\mathbf{H}}\mathbf{F}^{\mathbf{H}} = \mathbf{F}\mathbf{D}^2\mathbf{F}^{-1} = \mathbf{U}^2, \end{aligned} \tag{18}$$

the same theorem from [49] can also be applied to the second-order case, which proves that DFRT3 is also unitary.

5 Numerical results

In this section, we present the numerical experiments in which we compare four different DFRT definitions in terms of accuracy. DFRT1 stands for the approaches given in [49] modified for DFRT (as explained in Sect. 3.1). DFRT2 and DFRT3 stand for the proposed approaches that use the simplest $\mathbf{D}$ and $\mathbf{U}$ matrices (as explained in Sect. 3.2) and the numerical $\mathbf{D}$ and $\mathbf{U}$ matrices (as explained in Sect. 3.3), respectively. Finally, Candan's method of which details are given in [42] is represented by DFRT4.

A highly computationally intensive numerical integration method to calculate the continuous transform samples is taken as the ultimate baseline reference. All DFRT methods are compared against this reference, and the error is defined as the energy of the difference between the particular DFRT method and the reference, normalized by the energy of the reference, and finally expressed as a percentage MSE value.

We consider five different example signals: the discretized versions of the chirped pulse function $\exp(-\pi u^2 - i\pi u^2)$, denoted F1, the trapezoidal function $1.5\mathrm{tri}(u/3) - 0.5\mathrm{tri}(u)$, denoted F2 ($\mathrm{tri}(u) = \mathrm{rect}(u) * \mathrm{rect}(u)$), the damped sine function $\exp(-2|u|)\sin(3\pi u)$, denoted F3, the signal plotted in Fig. 1, denoted by F4, and the shifted chirped Gaussian signal $\exp(-\pi(u-1)^2 - i\pi(u-1)^2$, denoted by F5, are used. DFRT order a is chosen to be ± 0.2, ± 0.6 and ± 1. The number of samples N is taken as 512, 1024 and 2048 for three sets of experiments. All DFRT definitions are implemented for all the example signals and for all orders, and percentage mean square errors (MSE) are calculated with respect to the reference brute force method. The results are tabulated in

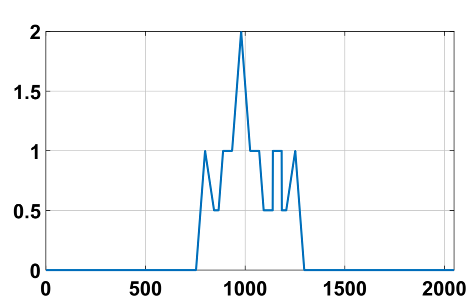


Fig. 1 Example signal F4

Table 1 Percentage MSEs for different DFRT definitions of various fractional orders. Input function: F1 (chirped pulse)

	N	$a = 1$	$a = 0.6$	$a = 0.2$
DFRT1	512	8.1×10^{-3}	3.6×10^{-3}	3.69×10^{-4}
	1024	2.0×10^{-3}	8.9×10^{-4}	9.29×10^{-5}
	2048	5.01×10^{-4}	2.22×10^{-4}	2.33×10^{-5}
DFRT2	512	5.48×10^{-22}	5.35×10^{-22}	5.46×10^{-22}
	1024	5.31×10^{-22}	5.33×10^{-22}	5.46×10^{-22}
	2048	5.71×10^{-22}	5.28×10^{-22}	5.32×10^{-22}
DFRT3	512	5.49×10^{-22}	5.41×10^{-22}	5.44×10^{-22}
	1024	5.52×10^{-22}	5.3×10^{-22}	5.48×10^{-22}
	2048	5.78×10^{-22}	5.16×10^{-22}	5.35×10^{-22}
DFRT4	512	5.46×10^{-22}	1.16×10^{-4}	3.05×10^{-5}
	1024	5.4×10^{-22}	2.87×10^{-5}	7.55×10^{-6}
	2048	5.4×10^{-22}	7.12×10^{-6}	1.88×10^{-6}

Table 2 Percentage MSEs for different DFRT definitions of various fractional orders. Input function: F2 (trapezoid)

	N	$a = 1$	$a = 0.6$	$a = 0.2$
DFRT1	512	1.89	1.02	0.23
	1024	0.51	0.26	5.93×10^{-2}
	2048	0.13	6.69×10^{-2}	1.49×10^{-2}
DFRT2	512	2.06×10^{-6}	5.86×10^{-6}	1.12×10^{-5}
	1024	1.34×10^{-6}	4.09×10^{-6}	7.84×10^{-6}
	2048	1.05×10^{-7}	4.55×10^{-7}	8.41×10^{-7}
DFRT3	512	2.06×10^{-6}	5.86×10^{-6}	1.12×10^{-5}
	1024	1.34×10^{-6}	4.09×10^{-6}	7.84×10^{-6}
	2048	1.05×10^{-7}	4.55×10^{-7}	8.41×10^{-7}
DFRT4	512	1.53×10^{-6}	8.8×10^{-3}	1.03×10^{-2}
	1024	1.3×10^{-6}	2.2×10^{-3}	2.5×10^{-3}
	2048	9.56×10^{-8}	5.25×10^{-4}	6.47×10^{-4}

Tables 1, 2, 3, 4 and 5, and some samples are also plotted in Fig. 2. In these tables, results for only positive a values are given to prevent redundancy since errors for their negative counterparts are almost the same. As can be observed from Tables 1, 2, 3, 4 and 5, errors are quite small for example signal F1.

By inspecting Tables 1, 2, 3, 4 and 5, it can be seen that strict structurally analogous hyperdifferential operator-based DFRT (DFRT1) cannot achieve high performance in terms of accuracy. Although DFRT4 can achieve acceptably low errors values, its performance is still quite lower than that of our proposed definitions of DFRT2 and DFRT3 for most of the DFRT orders. Another observation is that the error values do not depend on the order much. For a particular DFRT definition, and for a given signal, the errors are on the same order of magnitude as we span the DFRT order. As expected, the

Table 3 Percentage MSEs for different DFRT definitions of various fractional orders. Input function: F3 (damped sine)

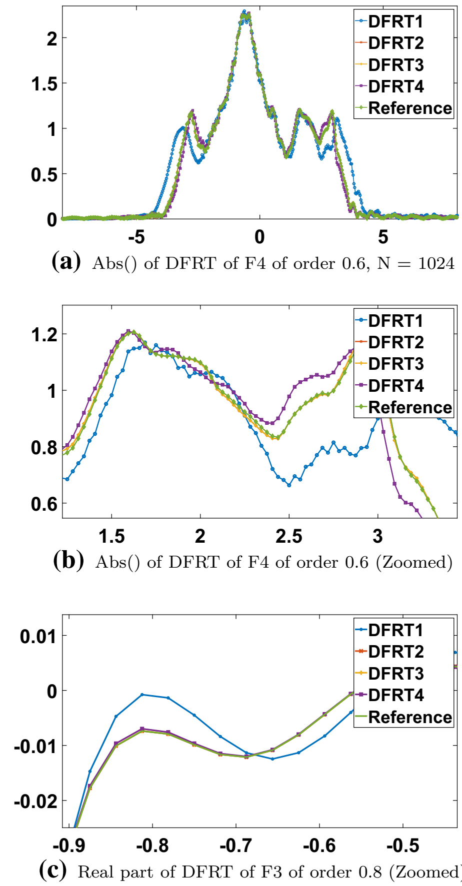
	N	$a = 1$	$a = 0.6$	$a = 0.2$
DFRT1	512	3.43	1.17	0.31
	1024	0.91	0.31	8.76×10^{-2}
	2048	2.37	8.28×10^{-2}	2.51×10^{-2}
DFRT2	512	3.39×10^{-5}	6.18×10^{-5}	9.29×10^{-5}
	1024	4.88×10^{-6}	8.94×10^{-6}	1.35×10^{-5}
	2048	7.14×10^{-7}	1.31×10^{-6}	1.98×10^{-6}
DFRT3	512	3.39×10^{-5}	6.18×10^{-5}	9.29×10^{-5}
	1024	4.88×10^{-6}	8.94×10^{-6}	1.35×10^{-5}
	2048	7.14×10^{-7}	1.31×10^{-6}	1.98×10^{-6}
DFRT4	512	3.37×10^{-5}	1.46×10^{-2}	2.2×10^{-2}
	1024	4.87×10^{-6}	3.8×10^{-3}	6.4×10^{-3}
	2048	7.14×10^{-7}	1.03×10^{-3}	2.01×10^{-3}

Table 4 Percentage MSEs for different DFRT definitions of various fractional orders. Input function: F4

	N	$a = 1$	$a = 0.6$	$a = 0.2$
DFRT1	512	36.04	61.45	92.1
	1024	23.89	75.49	58.9
	2048	27.72	62.72	19.61
DFRT2	512	1.7×10^{-3}	8.9×10^{-3}	2.08×10^{-2}
	1024	3.5×10^{-3}	2.04×10^{-2}	4.03×10^{-2}
	2048	6.87×10^{-4}	4.0×10^{-3}	8.7×10^{-3}
DFRT3	512	1.7×10^{-3}	8.9×10^{-3}	2.08×10^{-2}
	1024	3.5×10^{-3}	2.04×10^{-2}	4.03×10^{-2}
	2048	6.87×10^{-4}	4.0×10^{-3}	8.7×10^{-3}
DFRT4	512	1.3×10^{-3}	8.05	16.31
	1024	3.5×10^{-3}	1.92	3.87
	2048	6.08×10^{-4}	0.47	0.95

Table 5 Percentage MSEs for different DFRT definitions of various fractional orders. Input function: F5 (shifted chirped pulse)

	N	$a = 1$	$a = 0.6$	$a = 0.2$
DFRT1	512	0.47	0.12	1.59×10^{-2}
	1024	0.12	2.93×10^{-2}	4.0×10^{-3}
	2048	2.91×10^{-2}	7.3×10^{-3}	1.0×10^{-3}
DFRT2	512	5.43×10^{-22}	5.36×10^{-22}	5.39×10^{-22}
	1024	5.44×10^{-22}	5.44×10^{-22}	5.49×10^{-22}
	2048	5.79×10^{-22}	5.41×10^{-22}	5.43×10^{-22}
DFRT3	512	5.43×10^{-22}	5.36×10^{-22}	5.38×10^{-22}
	1024	5.44×10^{-22}	5.51×10^{-22}	5.51×10^{-22}
	2048	5.84×10^{-22}	5.54×10^{-22}	5.45×10^{-22}
DFRT4	512	5.43×10^{-22}	3.7×10^{-3}	1.5×10^{-3}
	1024	5.43×10^{-22}	9.12×10^{-4}	3.65×10^{-4}
	2048	5.44×10^{-22}	2.26×10^{-4}	9.04×10^{-5}



(a) Abs() of DFRT of F4 of order 0.6, N = 1024

(b) Abs() of DFRT of F4 of order 0.6 (Zoomed)

(c) Real part of DFRT of F3 of order 0.8 (Zoomed)

Fig. 2 Comparison of the proposed DFRTs

errors depend also on the input signal since, for a particular N, space–bandwidth product of the input signals is important in determining how much energy of the continuous signal is represented by this particular number of samples N. Another expected observation is that as we increase the number of samples N, the error values tend to decrease.

All DFRT definitions mentioned in this manuscript are presented in the form of a matrix of size $N \times N$, upon multiplication, produces the DFRT of a discrete and finite signal of length N. Therefore, the computational costs of all of them involve a matrix multiplication and thus have complexity $O(N^2)$.

6 Conclusions

In this paper, several alternative definitions of the discrete fractional transform (DFRT) based on hyperdifferential operator theory is proposed. For finite-length signals of a discrete variable, a unitary DFRT matrix is obtained so that the DFRT-transformed form of the input signal can be calculated by direct matrix multiplication. Based on a previously proposed operator theory approach for DLCTs, we have proposed and studied DFRT in detail. We have also proposed improved discrete multiplication and discrete differentiation matrices which provide considerably higher accuracy. Several numerical experiments have been done, and comparisons with previous DFRT definitions have been reported.

Another advantage of this approach is that it reduces the problem of defining a DFRT to the problem of simply defining fundamental operations of discrete multiplication and discrete differentiation. The proposed DFRT uses only these two very basic building blocks and ordinary DFT operation.

Moreover, hyperdifferential operator-based approach provides a framework in which the problem of defining DFRT is reduced to only defining discrete multiplication and discrete differentiation. Then, for different purposes, different such discrete multiplication and differentiation definitions can be used so that different alternative DFRTs can be defined. This general framework can open up opportunities for further applications in defining discrete transforms.

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