COHOMOLOGY OF INFINITE GROUPS REALIZING FUSION SYSTEMS

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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Given a fusion system $\mathcal{F}$ defined on a p-group S, there exist infinite group models, constructed by Leary and Stancu, and Robinson, that realize $\mathcal{F}$. We study these models when $\mathcal{F}$ is a fusion system of a finite group G. If the fusion system is given by a finite group, then it is known that the cohomology of the fusion system and the $\mathbb{F}_p$-cohomology of the group are the same. However, this is not true in general when the group is infinite. For the fusion system $\mathcal{F}$ given by finite group G, the first main result gives a formula for the difference between the cohomology of an infinite group model π realizing the fusion $\mathcal{F}$ and the cohomology of the fusion system. The second main result gives an infinite family of examples for which the cohomology of the infinite group obtained by using the Robinson model is different from the cohomology of the fusion system. The third main result gives a new method for the realizing fusion system of a finite group acting on a graph. We apply this method to the case where the group has p-rank 2, in which case the cohomology ring of the fusion system is isomorphic to the cohomology of the group.

Keywords: Fusion Systems, Cohomology of Groups, Cohomology of Fusion Systems, Graph of Groups.

ÖZET

FÜZYON SİSTEMLERİNİ GERÇEKLEYEN SONSUZ GRUPLARIN KOHOMOLOJİSİ

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S bir sonlu p-grup ve $\mathcal{F}$ de S üzerinde tanımlı bir füzyon sistemi olsun. Leary-Stancu ve Robinson bu $\mathcal{F}$ füzyonunu gerçekleyen sonsuz grup modelleri vermişlerdir. Biz bu modelleri füzyon sisteminin aslında sonlu bir G grubundan gelmiş olduğu durumlarda çalıştık. Füzyon sistemi bir sonlu grup tarafından verildiğinde, füzyon sisteminin kohomolojisi ile grubun $\mathbb{F}_p$ kohomolojisinin aynı olduğu bilinmektedir. Fakat bu sonsuz gruplar için her zaman doğru değildir. İlk ana sonuç, sonlu füzyonlar için füzyonu gerçekleyen sonsuz grubun kohomolojisi ile füzyonun kohomolojisinin ilişkisini formüle etmek oldu. İkinci ana sonuçta bu formüldeki farkın sıfır olmadığı duruma sonsuz bir aileyi örnek gösterdik. Üçüncü ana sonuçta ise füzyonun p rankı 2 olan sonlu bir gruptan geldiği durumda yeni bir model bulduk. Bu sonsuz grup modeli hem füzyonu gerçekliyor hem de kohomolojisini tam olarak veriyor. Bu bölümde ortaya koyduğumuz yeni yöntem bir sonlu grubun bir altgrup posetine yaptığı etkiyi kullanarak yeni füzyon gerçekleyen sonsuz gruplar bulmak.

Anahtar sözcükler: Füzyon Sistemleri, Grup Kohomolojisi, Grup Grafları, Füzyon Sistemlerinin Kohomolojisi.

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Chapter 1

Introduction

Let p be prime and G be a discrete group. Let S be a finite subgroup of G having order a power of p. We say S is a Sylow p-subgroup of G if any p-subgroup of G is a conjugate to a subgroup of S in G. By Sylow Theorems, if G is finite then it has a Sylow p-subgroup. However, there are some infinite groups that do not have any Sylow p-subgroups. For example, the group $C_3 * C_3$ does not have any Sylow 3-subgroups, where C_3 is the cyclic group of order 3.

For discrete group G with Sylow p-subgroup S, we define *the fusion system on S given by G* as the category with objects as all the subgroups of S and morphisms given by conjugations of elements of G. We denote this by $\mathcal{F}_S(G)$. An abstract fusion system defined on a p-subgroup S is a category with objects as subgroups of S and morphisms that satisfies some conditions explained in Definition 3.1.1. Given a fusion system $\mathcal{F}$ defined on a p-group S, if there exists a group G with Sylow p-subgroup S such that $\mathcal{F} = \mathcal{F}_S(G)$, we say G *realizes the fusion* $\mathcal{F}$. Chapter 3.1 is devoted to the theory of the fusion systems.

Leary-Stancu [1] and Robinson [2] give infinite group models realizing fusion systems. That means given a fusion system S, there are infinite groups realizing the fusion $\mathcal{F}$. However, we may not find a finite group realizing the fusion $\mathcal{F}$. We say $\mathcal{F}$ is a *finite fusion* if there exists a finite group G realizing $\mathcal{F}$.

Leary-Stancu and Robinson uses the method of graphs of groups to construct infinite group models realizing fusion systems. The theory of graph of groups is discussed in Chapter 2 which is the first preliminary chapter of the thesis.

Assume G is a finite group with Sylow p-subgroup S and $\mathcal{F} = \mathcal{F}_S(G)$. Let π be an infinite group realizing the fusion $\mathcal{F}$ constructed via Robinson or Leary-Stancu model. In this case, there is a homomorphism $\chi : \pi \to G$ that satisfies some properties. We call such a homomorphism storing homomorphism (see Definition 3.3.3). This homomorphism is used to understand the relation between their cohomology groups. These infinite group constructions and our new definition of "storing homomorphism" are explained in Section 3.3.

The cohomology of the fusion system $\mathcal{F}$ is defined as the inverse limit

$$H^*(\mathcal{F}) := \lim_{P \in \mathcal{F}} H^*(P; \mathbb{F}_p)$$

or, equivalently, as the $\mathcal{F}$-stable elements in $H^*(S; \mathbb{F}_p)$. For finite fusions, by a theorem of Cartan-Eilenberg we have $H^*(\mathcal{F}) \cong H^*(G, \mathbb{F}_p)$ where G is the finite group realizing $\mathcal{F}$. However, for infinite groups this isomorphism does not hold in general.

Let G be a group with Sylow p-subgroup S. For a fusion system $\mathcal{F}$ defined on S, we say that G *realizes the fusion* $\mathcal{F}$ *and its cohomology* if G realizes the fusion and if $H^*(\mathcal{F}) \cong H^*(G; \mathbb{F}_p)$. The infinite group models of Robinson and Leary-Stancu do not realize the cohomology of the fusion system $\mathcal{F}$, in general. Counterexamples were already known and we give an infinite family of examples in Chapter 4. The question of whether there exists an infinite group model realizing $\mathcal{F}$ and its cohomology given a fusion $\mathcal{F}$ is still open.

In Chapter 4, we present our main results about the cohomology of infinite groups realizing fusion systems. Our first theorem is about explaining the difference between the cohomology of a given finite fusion system and the cohomology of an infinite group model realizing the fusion.

We say H controls p-fusion in G if $H < G$ such that $\mathcal{F}_S(G) = \mathcal{F}_S(H)$. We say G is *p-minimal* if G has no proper subgroup H controlling p-fusion.

Main Theorem 1. *Let $\mathcal{F} = \mathcal{F}_S(G)$ be a fusion system of a finite group G. Assume that G is p-minimal, and let π denote the infinite group realizing $\mathcal{F}$ obtained by either the Leary-Stancu model or the Robinson model. Then there is a group extension $1 \to F \to \pi \to G \to 1$ where F is a free group, and there is an isomorphism of cohomology groups*

$$H^{*-1}(G; \mathrm{Hom}(F_{ab}, \mathbb{F}_p)) \oplus H^*(\mathcal{F}) \cong H^*(\pi; \mathbb{F}_p)$$

where $F_{ab} := F/[F, F]$ denotes the abelianization of F.

As we state in Theorem 6.1.10, Libman and Seeliger show that $H^*(\mathcal{F})$ is a direct summand of $H^*(\pi; \mathbb{F}_p)$ but the difference $\ker(res_S^\pi)$ is not calculated. Here, in the first main theorem, we calculate the difference for finite fusion systems.

We have an example of a fusion system where in the Leary-Stancu model, the difference between the cohomology of the fusion system and the cohomology of the infinite group realizing fusion system is not zero. Our second main theorem gives infinitely many examples for the Robinson model where the difference in the previous theorem is not zero.

Main Theorem 2. *Let $G = GL(n, 2)$ for $n \geqslant 5$. Let S be the Sylow 2-subgroup consisting of upper triangular matrices in G. Let $(\mathcal{G}, Y)$ be the graph of groups constructed according to Robinson model for $\mathcal{F} = \mathcal{F}_S(G)$. Then we have*

$$H^2(\mathcal{F}) \ncong H^2(\pi(\mathcal{G}, Y), \mathbb{F}_2).$$

Since there are examples where Leary-Stancu model or Robinson model do not realize the cohomology of the fusion, we try to find a new model that realizes fusion and its cohomology. In Chapter 5, we give the method of obtaining infinite group models realizing fusion systems by using subgroup posets. By using an action of a group on its subgroup poset, we obtain a graph of groups which has a fundamental group realizing the fusion under certain conditions. By using this method we find a new model that realizes fusion and its cohomology for finite fusion of p-rank 2 groups. Here, we say a group G has p-rank n, if n

is the maximum number such that there exists a subgroup of G isomorphic to $(C_p)^n := \underbrace{C_p \times C_p \times C_p \times \cdots \times C_p}_{n \text{ copies}}$. We denote this by $\mathrm{rank}_p(G) = n$.

Main Theorem 3. *Assume G is a finite group with Sylow p-subgroup S and $\mathrm{rank}_p(G) = 2$. Let X be the poset of elementary abelian subgroups of S. Then $\Gamma := \pi_1(EG \times_G X)$ realizes the fusion of G on S, i.e. $\mathcal{F}_S(\Gamma) = \mathcal{F}_S(G)$. Moreover, there is an isomorphism of $\mathbb{F}_p$-cohomology groups $H^*(\Gamma, \mathbb{F}_p) \cong H^*(G, \mathbb{F}_p)$.*

In Chapter 6, we introduce the theory of the linking systems, and give the proof of the main theorem of the paper [3]. This theorem shows that the cohomology of the fusion system is a direct summand of the $\mathbb{F}_p$-cohomology of the infinite group model realizing the fusion under some conditions on the model. Then, we give a group theoretic proof of the fact that the $\mathbb{F}_p$-cohomology of $\theta(P)$ is zero for dimensions $i \geqslant 2$. This fact is used in our paper [4] to find a long exact sequence from the spectral sequence associated with an extension of a category (see [4, Theorem 1.3]).

Chapter 2

Graph of Groups

The functor π_1 : **Top** → **Grp** sends a topological space X to its fundamental group $\pi_1(X)$. In the first definition of this chapter, we introduce the functor $K(-,1)$: **Grp** → **Top** that sends a group G to a topological space which has fundamental group isomorphic to the group G. These two functors give a relation between the category of groups and the category of topological spaces. In the reference [5], the graph of groups considered as a topological method in group theory where the relationship between the categories of groups and topological spaces used. In this method, we take several groups indexed by a graph, and glue their corresponding topological spaces, then we get a group by taking the fundamental group of the last total space. After introducing this theory from [5], we speak briefly of the algebraic construction of the same theory from [6].

2.1 $K(G,1)$ spaces

Definition 2.1.1. Let Y be a topological space. A covering space of Y is a topological space X such that there is a continuous surjective map $p : X \to Y$ which satisfies that for any $y \in Y$, there exists an open neighborhood U of y, such that the preimage $p^{-1}(U)$ is a union of disjoint open sets in X, each of which is mapped homeomorphically onto U by p.

A covering space is a *universal covering* space if it is simply connected.

Definition 2.1.2. Let G be a discrete group. A topological space Y is called a *K(G,1) space* if it satisfies the following conditions:

(i) Y is connected.

(ii) $\pi_1 Y = G$.

(iii) The universal cover X of Y is contractible.

The circle S^1 is a $K(\mathbb{Z}, 1)$ because the line, the universal cover of S^1, is contractible, and S^1 is connected with $\pi_1(S^1) = \mathbb{Z}$. The infinite dimensional real projective space $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$.

Let G be a group. As shown in [7, page 89], the classifying space construction for one object category G gives a CW-complex which is a $K(G, 1)$. Then, we can always refer to a CW-complex $K(G, 1)$ for any group G. Also, it is shown that the homotopy type of a CW-complex $K(G, 1)$ is uniquely determined by G. Then we state the following result proven in [7].

Theorem 2.1.3. *For any group G, there exists a CW-complex $K(G, 1)$ which is unique up to homotopy.*

Remark 2.1.4. This theorem is crucial for the well-definedness of the fundamental group of a graph of groups. In the construction of the fundamental group of a graph of groups, we glue CW-complex $K(G, 1)$ spaces and take the fundamental group of the glued space. Since a CW-complex $K(G, 1)$ space unique up to homotopy, the total glued space has fundamental group independent of choice of CW-complex $K(G, 1)$'s. These arguments are explained in the next section.

2.2 Graph of Groups

In this section, we introduce the theory of Graph of Groups from the references [7] which has a short but well-explained introduction, and [5] which has a topological approach for graph of groups. Also we have [6] for algebraic approach for the theory that will be discussed later.

Definition 2.2.1. An abstract graph Γ consists of two sets $\mathrm{E}(\Gamma)$ and $\mathrm{V}(\Gamma)$, called the edges and vertices of Γ, an involution on $\mathrm{E}(\Gamma)$ which sends e to $\bar{e}$, where $e \neq \bar{e}$, and a map $\partial_0 : \mathrm{E}(\Gamma) \to \mathrm{V}(\Gamma)$.

We define $\partial_1 e := \partial_0 \bar{e}$ and say that e is an edge from $\partial_0 e$ to $\partial_1 e$.

Definition 2.2.2. A graph of groups $(\mathcal{G}, Y)$ consists of an abstract graph Y (which will always be assumed to be connected) together with a function $\mathcal{G}$ assigning to each vertex v of Y a group G_v and to each edge e a group G_e, with $G_{\bar{e}} = G_e$, and an injective homomorphism $\phi_e : G_e \to G_v$ when $v = \partial_0(e)$.

From now on, we construct the theory of graph of groups topologically as it is done in [5]. Then, we will speak briefly of the algebraic approach in [6].

Definition 2.2.3.

(i) A graph of topological spaces consists of an abstract graph Y together with a function assigning to each vertex v of Y a topological space X_v and to each edge e a topological space X_e , with $X_{\bar{e}} = X_e$, and a continuous map $f_e : X_e \to X_v$, for $v = \partial_0(e)$, which is injective on homotopy groups.

(ii) A total space $X(\mathcal{G}, Y)$ corresponding to above graph of spaces is the quotient of

$$\bigcup_{v \in V(Y)} X_v \cup \bigcup_{e \in E(Y)} (X_e \times [0,1])$$

by the identifications

$$X_e \times [0,1] \to X_{\bar{e}} \times [0,1] \text{ by } (x,t) \mapsto (x, 1-t)$$

$$X_e \times \{0\} \to X_{\partial_0 e} \text{ by } (x,0) \mapsto f_e(x).$$

Here, if we start with CW-complexes and glue them via cellular maps, we will obtain a CW-complex as a glued space.

Definition 2.2.4. Given a graph of groups $(\mathcal{G}, Y)$ with vertex groups G_v for a vertex v and edge groups G_e for an edge e and injective homomorphisms $\phi_e : G_e \to G_v$. We construct the graph of topological spaces by assigning a vertex v to a CW-complex $K(G_v, 1)$ and an edge e to a CW-complex $K(G_e, 1)$ with injective cellular maps f_e on edges so that they induce ϕ_e homomorphisms.

The fundamental group of the total space of this graph of spaces called *the fundamental group of the graph of groups* which we denote by $\pi(\mathcal{G}, Y)$.

Example 2.2.5. (Amalgamation) Consider a graph consisting of one edge with two vertices. Let A and B be the vertex groups and C be the edge group with two monomorphisms $A \hookleftarrow C \hookrightarrow B$. By Van Kampen theorem, the fundamental group of the graph of groups gives the amalgamated product $A *_C B$ which is the quotient of the free product $A * B$ by identifying two images of C under monomorphism.

Example 2.2.6. (HNN product) Consider an abstract graph with one edge with one vertex, i.e. the graph is just a loop. If the vertex group is A and edge group is C and monomorphism the identity embedding $C \hookrightarrow A$ and $\phi : C \hookrightarrow A$, we obtain an HNN product $A *_C$ which is the group defined by $\langle A, t \mid tct^{-1} = \phi(c), \forall c \in C \rangle$ as explained in [5].

The fundamental group of a graph of groups defined algebraically in [6]. Let $(\mathcal{G}, Y)$ be a graph of groups. Take a spanning tree T in Y. For an edge e and $a \in G_e$, we denote the image of a in ϕ_e by a^e. Let E be the free group with generator set as $E(Y)$. Define $F(\mathcal{G}, Y)$ as the quotient group of the free product

$$E * \Big(\mathop{\ast}_{v \in V(Y)} G_v \Big)$$

by the normal subgroup N, where N is the normal closure of the relations

$$ea^e e^{-1} = a^{\bar{e}} \text{ and } \bar{e} = e^{-1}$$

for all $e \in E(Y)$ and $a \in G_e$. The group $\pi(\mathcal{G}, Y, T)$ is defined as the quotient group of $F(\mathcal{G}, Y)$ subject to the relations $e = 1$ if $e \in E(Y)$. It is shown in [6, Proposition 20]), the group $\pi(\mathcal{G}, Y, T)$ is independent of the choice of the spanning tree T. So we write $\pi(\mathcal{G}, Y)$ instead of $\pi(\mathcal{G}, Y, T)$. This definition and the Definition 2.2.4 are equivalent as shown in [8, page 204].

Theorem 2.2.7. *Let $(\mathcal{G}, Y)$ be a graph of groups. The total space of the corresponding graph of spaces has a contractible universal covering. For any vertex group G_v we have an injective homomorphism $G_v \to \pi(\mathcal{G}, Y)$*

Here, we always work with $K(G, 1)$-spaces which are CW-complexes in order to construct theory carefully.

Proof. Consider the corresponding graph of spaces. We have $K(G_v, 1)$ space X_v for a vertex v and $K(G_e, 1)$ space X_e for an edge e. Let X be the total space of the corresponding graph of spaces. We will show that the universal cover $\widetilde{X}$ is contractible.

For any vertex $v \in Y$, define $L_v = X_v \bigcup (\bigcup_{\partial_0 e = v} X_e \times [0, 1])$ where we have intersections $X_v \bigcap (X_e \times [0, 1]) = X_e \times \{0\}$ as we glued in the definition of total space.

Fix a vertex v_0 and let Y_0 be the universal cover of L_{v_0}. The universal cover Y_0 is contractible because it is a union of a universal cover $\widetilde{X_v}$ and copies of universal covers $\widetilde{X_e}$ for edges satisfying $\partial_0(e) = v_0$ where we can contract the copies $\widetilde{X_e} \times [0, 1]$ into $\widetilde{X_v}$ which is also contractible. Here, since the maps $G_e \to G_v$ are injective, we have deformation retraction from $\widetilde{X_e} \times [0, 1]$ to $\widetilde{X_e} \times \{0\}$ which is a copy of $\widetilde{X_v}$ due to gluing.

We define X_1 by adding Y_0 to the spaces $\widetilde{X_v}$'s for vertices satisfying $\partial_1(e) = v$ for some edge e we considered in Y_0. We define Y_1 by adding X_1 to spaces $\widetilde{X_e}$ for edges e satisfying $\partial_0(e) = v$ for some vertex v we considered in the last step. We have an obvious deformation retractions $Y_1 \to X_1 \to Y_0 \to *$. Hence, Y_1 is also contractible.

Step by step, we can construct Y_n which is also contractible. The space $Y = \bigcup_{n\geqslant 1} Y_n$ is contractible and evenly covers the total space X. Hence, X has a contractible universal cover.

Take any vertex $v \in Y$. Consider the inclusion $i : X_v \to X$. Take any loop $\gamma : S^1 \to X_v$. Assume the loop $\alpha = i \circ \gamma$ is null-homotopic. Then the lift $\widetilde{\alpha} : [0,1] \to \widetilde{X}$ is also null-homotopic in the universal cover $\widetilde{X}$. The lift is contained in one of the copies of the $\widetilde{X_v}$ in $\widetilde{X}$ (see [5, page 166] for more details). Since $\widetilde{\alpha}$ is null-homotopic in one of the copies of $\widetilde{X_v}$, γ is null-homotopic in X_v. Hence, the map i induces injective homomorphism in homotopy groups. In other words, the induced homomorphism $i^* : G_v \to \pi(\mathcal{G}, Y)$ is injective.

□

2.3 Groups Acting on Graphs

In this section we mention how a group action on a graph gives a graph of groups structure. Here, we only consider group actions without inversions that means if an element of the group fixes a vertex of an edge then it fixes the edge. In other words $g \cdot e = \bar{e}$ is forbidden for $g \in G$ and $e \in E(Y)$. These actions are also called cellular actions. In fact, given a non-cellular group action on a graph, we can obtain a cellular action by applying a barycentric subdivision.

Lemma 2.3.1. *Let Γ be a quotient graph of a graph Z. For any tree T in Γ there exists a lift T' in Z such that T' is also a tree which is isomorphic to T.*

Proof. Take any vertex v_1 in T and any lift of w_1 in Z. Then consider all the incident edges of the vertex v_1. We take the lifts of theses edges so that the lifts are incident to w_1. Then we continue in this way. For any edge e in T, we consider a lift of e such that e' incident to the current construction of the graph. This construction gives a connected lift T' of the tree T. Here, T' must be a tree because otherwise any loop in T' gives an image loop in T. Note that, the construction of the lift T' of T gives an isomorphism between them. □

Theorem 2.3.2 (Scott-Wall [5])**.** *Let G be the fundamental group of a graph of groups $(\mathcal{G}, Y)$. Let $\widetilde{X}$ be the universal cover of the total space X of the graph of groups as we constructed in Theorem 2.2.7. We consider the standard G-action on $\widetilde{X}$. There exists a tree Z with a cellular G-action such that we have an isomorphism of graphs $f : Z/G \to Y$ and a G-equivariant map $h : \widetilde{X} \to Z$.*

Proof. In the proof of Theorem 2.2.7, we constructed the universal cover $\widetilde{X}$ of the total space X of the graph of groups $(\mathcal{G}, Y)$. Since $\pi(\mathcal{G}, Y)$ is the fundamental group of X, by definition, $\pi(\mathcal{G}, Y)$ acts on the universal cover $\widetilde{X}$. The space $\widetilde{X}$ consists of copies of $\widetilde{X_v}$'s and $\widetilde{X_e} \times [0,1]$'s.

Let $F : \widetilde{X} \times [0,1] \to \widetilde{X}$ be the deformation retract obtained by the contractions of $\widetilde{X_v}$'s and $\widetilde{X_e}$'s. The restriction of F to a copy of $\widetilde{X_v}$ for a vertex v is the contraction of $\widetilde{X_v}$ and the restriction of F to a copy of a $\widetilde{X_e} \times [0,1]$ is the deformation retract of $\widetilde{X_e} \times [0,1]$ to $[0,1]$. Hence, we obtain a homotopy from $\widetilde{X}$ to a graph Z where we have vertices in Z for each copy of $\widetilde{X_v}$'s in $\widetilde{X}$ and we have edges in Z for each copy of $\widetilde{X_e}$ in $\widetilde{X}$. Since $\widetilde{X}$ is contractible, Z is also contractible which means it is a tree.

The $\pi(\mathcal{G}, Y)$-action on $\widetilde{X}$ induces $\pi(\mathcal{G}, Y)$-action on the tree Z where the homotopy respects this action. Then we obtain a $\pi(\mathcal{G}, Y)$-equivariant map $h : \widetilde{X} \to Z$. □

In the proof of the last theorem, the construction of an action on a tree from the graph of groups $(\mathcal{G}, Y)$ is called *the corresponding* $\pi(\mathcal{G}, Y)$-action on a tree. The next theorem says that we can restore the graph of groups from its corresponding $(\pi(\mathcal{G}, Y))$-action on a tree up to conjugate monomorphisms.

Theorem 2.3.3 (Scott-Wall [5])**.** *With the notations and hypothesis in Theorem 2.3.2, from G-action on Z, we can obtain a graph of groups $(\mathcal{G}', Z/G)$ such that* *the corresponding vertex and edge groups of the graph of groups are isomorphic and the monomorphisms may differ by a conjugation with an element $g \in G$.*

Proof. Now, we construct a graph of groups from the $G = \pi(\mathcal{G}, Y)$-action on Z. First, we choose a maximal tree T in the quotient graph $\Gamma := Z/G$. From the Lemma 2.3.1, we can take a lift T' of T in Z so that T' is isomorphic to T. Since T and T' are isomorphic trees, we can use stabilizers of lifts of vertices and edges as vertex and edge groups. For a vertex $v \in T$, we assign the stabilizer of the lift of the vertex in T' (i.e. for $v \in T$ we have vertex group G_v which is the stabilizer of $v' \in T$ where v' is the lift of v in T'). Similarly, for an edge $e \in T$, we assign edge group G_e which the stabilizer of $e' \in T'$ where e' is the lift of e in T'. The stabilizer of an edge $e' \in T'$ is a subgroup of the stabilizers of the end points of e'. Then we have obvious monomorphisms from edges to vertices in T.

Now, we have a graph of group structure on T. Then, we need to extend this structure to Γ. We have vertex groups for all vertices $v \in \Gamma$. So we add edge groups and monomorphisms for edges $e \in \Gamma - T$. Take any $e \in \Gamma - T$ with end points v and w. There exists a unique lift e' of the edge e such that e' has end point v' where v' is the lift of v satisfying $v' \in T'$. The other end point of e' is $g \cdot w'$ for some $g \in G$ where w' is the lift of w in T'. Then the stabilizer $G_e := \mathrm{Stab}(e')$ of e' is a subgroup of $\mathrm{Stab}(v') = G_v$ and $\mathrm{Stab}(gw') = g\mathrm{Stab}(w')g^{-1} = gG_wg^{-1}$. Then we assign G_e as edge group for $e \in \Gamma$ and monomorphisms $\phi_{e_1} : G_e \hookrightarrow G_v$ as inclusion and $\phi_{e_2} : G_e \to G_w$ by sending $x \mapsto g^{-1}xg$. By completing this process for all $e \in \Gamma - T$, we obtain a new graph of groups $(\mathcal{G}', \Gamma)$.

For an edge (or vertex) $x \in Y$, we have one G-orbit of $\widetilde{X_x}$ in $\widetilde{X}$ which corresponds one G-orbit in Z. Then, we have exactly one edge (or vertex) in $\Gamma = Z/G$, constructing the desired isomorphism $Y \to \Gamma$. Moreover, for an edge group (or vertex group) G_x in $(\mathcal{G}, Y)$, we have G-orbits of $\widetilde{X_x}$ in $\widetilde{X}$ which corresponds a G-orbit where any point has stabilizer isomorphic to G_x. Then, the map $Y \to \Gamma$ sends x to an edge (or vertex) having edge group (or vertex group) isomorphic to G_x. Since the construction of monomorphisms in $(\mathcal{G}', \Gamma)$ depend on the choice of maximal tree in Γ, they may differ by a conjugation by an element of $g \in G$. □

In the proof of the last theorem, the construction of graph of groups $((G)', Z/G)$ from a G-action on a tree Z is called *the graph of groups obtained from the G-action on the tree Z.*

Example 2.3.4. Let $G = A *_C B$ be as in Example 2.2.5. Then G acts a tree Z induced by the G-action on $\widetilde{X_G}$ as we see in the proof of Theorem 2.3.2. Then the vertices of Z corresponds to $K(A,1)$-complexes and $K(B,1)$-complexes. These spaces having stabilizers isomorphic to A and B respectively under the action of G. This gives that the vertices of Z having stabilizers A or B. Similarly, we can deduce that the edges of Z having stabilizers isomorphic to C.

Now, take any path starting from the reference point of a copy of $K(A,1)$-complex to the reference point of $K(A,1)$-complex in the universal cover $\widetilde{X_G}$ of the total space of the graph of groups. After dividing by G-action this path must become a loop. This shows that all these $K(A,1)$-complexes are in the same orbit under the G-action on $\widetilde{X_G}$. Then, passing to Z, the tree Z has two vertex orbits under the G-action which are those having stabilizer A and those having stabilizer B. Similarly, Z has one edge orbit under the G-action which having stabilizer C.

For the generalization of the construction of a graph of groups for a G-action on a tree to all graphs, we have the following result

Proposition 2.3.5 (page 84 in [9])**.** *Let G acts on a graph X. For the construction of the graph of groups $(\mathcal{G}, Y)$ for this action, we have $\pi(\mathcal{G}, Y) = \pi(EG \times_G X)$.*

Here, we can consider EG as the universal cover of a CW-complex $K(G,1)$ space.

Proof. Here, $Y = X/G$ from the construction. Let U be a CW-complex $K(G,1)$ space with universal cover $\widetilde{U} \cong EG$. For a subgroup $H < G$, we have that $\widetilde{U}/H$ is a $K(H,1)$ space having CW-complex structure. Define the map $f : \widetilde{U} \times X \to X$ by forgetting the first coordinate. We induce the map $\bar{f} : \widetilde{U} \times_G X \to X/G = Y$ in quotient spaces. Here, for any vertex $v \in Y$ we have $\bar{f}^{-1}(v) = \widetilde{U}/G_v$ which is a $K(G_v, 1)$. Here, G_v is the stabilizer of a lift of v, or equivalent the vertex group corresponding to v in the above construction. Similarly, we have $K(G_e, 1)$ spaces for edges glued with vertices. Hence, $\widetilde{U} \times_G X$ is a realization of the graph of groups $(\mathcal{G}, Y)$. That means $\pi(\mathcal{G}, Y) = \pi(\widetilde{U} \times_G X)$. □

From now on, we construct the theory of graph of groups in a topological way as [5] does. This theory can be constructed in an algebraic approach as it is done in [6]. Now we speak briefly of the theory in [6]. We start a group G acting on a graph X. We construct a graph of groups $(\mathcal{G}, Y)$ as we explain in Construction 1. Then we construct the tree $T = \widetilde{X}(\mathcal{G}, Y, T)$ as explained in [6, page 51]. Then we have the following theorem.

Theorem 2.3.6 (Serre, [6])**.** *With the above notation and hypothesis, the following properties are equivalent*

i-) X *is a tree.*

ii-) $\psi : \widetilde{X} \to X$ *is an isomorphism.*

iii-) $\pi(\mathcal{G}, Y, T) \xrightarrow{\phi} G$ *is an isomorphism.*

Proof. See [6, page 55]. □

With our topological notations and hypothesis used in this chapter, the same theorem can be stated. Assume G acts on a connected graph X without inversion. Let $Y := G/X$ and $(\mathcal{G}, Y)$ be the graph of groups constructed from that action. We consider the corresponding action of $\pi(\mathcal{G}, Y)$ on a tree T. We have a surjective map of graphs $\psi : T \to X$ and a surjective homomorphism of groups $\phi : \pi(\mathcal{G}, Y) \to G$ so that the following are equivalent

i-) X is a tree.

ii-) $\psi : T \to X$ is an isomorphism.

iii-) $\pi(\mathcal{G}, Y) \xrightarrow{\phi} G$ is an isomorphism.

Here, we point a topological approach for the proof of Theorem 2.3.6. From Proposition 2.3.5, we have $\pi(\mathcal{G}, Y) = \pi(EG \times_G X)$. Define $f : EG \times X \to EG$ by annihilating X. Since f is G-map, we can induce $\bar{f} : EG \times_G X \to BG$ by dividing via G-action. $\bar{f}$ induces in homotopy groups $\phi : \pi(EG \times_G X) \to G$ or equivalently, $\phi : \pi(\mathcal{G}, Y) \to G$. Since any loop in BG has a non-trivial preimage loop in $EG \times_G X$ under $\bar{f}$. We can say ϕ is surjective. As we explained before, we construct T by using the universal cover of the total space $EG \times_G X$ of the graph of groups $\pi(\mathcal{G}, Y)$. Then, the surjective map from the universal cover to the cover $EG \times_G X$ gives that surjective map $\psi : T \to X$. *For (i)* $\iff$ *(ii)* , since T is a tree and $T \xrightarrow{\psi} X$ induced from a covering, X is tree if and only if ψ is an isomorphism. *For (i)* $\iff$ *(iii)* , from Theorem 2.3.2, we know that $X/G = Y$ is isomorphic to $T/\pi(\mathcal{G}, Y)$. If X is tree then T is isomorphic to X and the surjective homomorphism $\pi(\mathcal{G}, Y) \xrightarrow{\phi} G$ must be an isomorphism. And, if $\pi(\mathcal{G}, Y) \xrightarrow{\phi} G$ is isomorphism then T must be isomorphic to X.

Now we have a corollary of Theorem 2.3.6 on the subgroups of $\pi(\mathcal{G}, \mathcal{Y})$.

Corollary 2.3.7. *Let $(\mathcal{G}, Y)$ be a graph of groups with vertex groups G_v's and edge groups G_e's. If $H < \pi(\mathcal{G}, Y)$, then H is the fundamental group of a graph of groups with vertex groups as subgroups of conjugates G_v's and edge groups as subgroups of conjugates of G_e's.*

Proof. We construct the $\pi(\mathcal{G}, Y)$-action on a tree Z. Since H is a subgroup of $\pi(\mathcal{G}, Y)$, H acts on tree Z with stabilizers as conjugates of subgroups of vertex and edge groups of $(\mathcal{G}, Y)$. From Theorem 2.3.6, H-action on Z gives a graph of groups $(\mathcal{H}, Y_0)$ where vertex groups are subgroups of conjugates of G_v's and edge groups are subgroups of conjugates of G_e's with $\pi(\mathcal{H}, Y_0) \cong H$. □

Then, we have a useful corollary of the previous corollary.

Corollary 2.3.8. *Let H be a subgroup of $\pi(\mathcal{G}, Y)$. If H intersects trivially with all the vertex groups of $(\mathcal{G}, Y)$, then H is free.*

Proof. If H intersects trivially with all the vertex and edge groups of $(\mathcal{G}, Y)$ then the vertex and edge groups of the corresponding graph of groups of H are all trivial. Then, H is the fundamental group of a graph. Hence, H is free. □

2.4 Cohomology of a Graph of Groups

In this section, we obtain homological results using group actions on trees.

Given a CW-complex X, we write $C_*(X)$ for cellular chain complex of X. If X is a graph, then $C_n(X) = 0$ for $n \geqslant 2$. A graph is called a *tree* if it is connected and has no loops.

Lemma 2.4.1. *[6, page 126] For the chain complex of a tree X, we have an exact sequence*

$$0 \to C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Moreover, if a group G acts on X cellulary, the exact sequence above is an exact sequence of $\mathbb{Z}G$-modules.

Proof. Let E and V denote the egde and vertex sets of X, respectively. $C_1(X)$ consists of the elements of the form $\sum_{i=1}^{n} n_i e_i$ where $e_i \in E$.

Now, we fix an orientation for edges of the graph X. In other words, for any edge e the two vertices of it distinguished to be initial and final which are denoted by $\partial_0(e)$ and $\partial_1(e)$, respectively. By the way, we have two functions ∂_0 and ∂_1 from E to V. We assume these maps satisfy $d(e) = \partial_1 e - \partial_0 e$.

Assume d is not injective, then there exists a sum $\sum_{i=1}^{n} n_i e_i \in \ker d$. Then,

$$0 = d(\sum_{i=1}^{n} n_i e_i) = \sum_{i=1}^{n} n_i d(e_i) = \sum_{i=1}^{n} n_i(\partial_1 e_i - \partial_0 e_i)$$

Then there exists e_{i_1} such that $\partial_0 e_{i_1} = \partial_1 e_1$ or $\partial_1 e_{i_1} = \partial_1 e_1$ because of the cancellations on the sum over vertices of these all edges. Without loss of generality, we can assume $\partial_0 e_{i_1} = \partial_1 e_1$. Similarly, without loss of generality, there exists e_{i_2} such that $\partial_0 e_{i_2} = \partial_1 e_{i_1}$... In this process, it is not important whether $\partial_0 e_{i_k} = \partial_1 e_{i_{k-1}}$ or $\partial_1 e_{i_k} = \partial_1 e_{i_{k-1}}$. In any case, at the end our sequence $e_1 e_{i_1} e_{i_2} \dots e_{i_n}$ will give a cycle. Since there are finitely many terms on the sum $\sum_{i=1}^{n} n_i(\partial_1 e_i - \partial_0 e_i)$, the process will end up with a loop at a step we find e_{i_n} such that it ends with the starting of e_1(i.e. $\partial_1 e_{i_n} = \partial_0 e_1$). Then the loop $e_1 e_{i_1} e_{i_2} \dots e_{i_n}$ contradicts with the assumption that X is a tree. To see the surjection of ϵ, take any vertex v and any integer n, ϵ sends nv to n.

We only left with the exactness at $C_0(X)$. Take any generator $v_2 - v_1$ of $\ker \epsilon$. Since X is connected there exists a path $e_1 e_2 \dots e_n$ starting at v_1 ending at v_2 (i.e. $\partial_0 e_1 = v_1, \partial_1 e_i = \partial_0 e_{i+1}$ and $\partial_1 e_n = v_2$. Hence, $d(\sum_{i=1}^{n} e_i) = \sum_{i=1}^{n}(\partial_1 e_i - \partial_0 e_i) = v_2 - v_1$. So, $\operatorname{Im} d = \ker \epsilon$ concludes the proof of the first part.

The G-action on X induces actions on $C_i(X)$ and make them $\mathbb{Z}G$-modules. The trivial G-action on $\mathbb{Z}$ makes it to be a trivial module. Since the actions on $C_i(X)$ and $\mathbb{Z}$ commutes with the maps d and ϵ, these maps become $\mathbb{Z}G$-module maps. $\square$

Theorem 2.4.2 (Serre, [6])**.** *Let a group G acts on a tree X. Let G_v and $o(v)$ denote the stabilizer and orbit of a vertex v, respectively. Similarly, G_e and $o(e)$ denote the stabilizer and orbit of an edge e. And we denote orbit representative set of vertices and edges by OV and OE respectively. For each G-modulo M, we have a long exact cohomology sequence*

$$\cdots \to H^i(G, \mathbb{Z}) \to \prod_{v \in OV} H^i(G_v, \mathbb{Z}) \to \prod_{e \in OE} H^i(G_e, \mathbb{Z}) \to H^{i+1}(G, \mathbb{Z}) \to \cdots .$$

Proof. We have short exact sequence of $\mathbb{Z}G$-modules,

$$0 \to C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Applying $\operatorname{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$ functor, we obtain long exact sequence in cohomology.

$$0 \to \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}G}(C_0(X), \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}G}(C_1(X), \mathbb{Z}) \to \operatorname{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \to \operatorname{Ext}^1_{\mathbb{Z}G}(C_0(X), \mathbb{Z}) \to \operatorname{Ext}^1_{\mathbb{Z}G}(C_1(X), \mathbb{Z}) \to \cdots$$

Using orbit stabilizer theorem we get,

$$\begin{aligned} C_1(X) &= \prod_{e \in OE} \mathbb{Z}o(e) \\ &= \prod_{e \in OE} \mathbb{Z}[G/G_e]. \end{aligned}$$

Similarly,

$$\begin{aligned}C_0(X) &= \prod_{v\in OV} \mathbb{Z}o(v)\\ &= \prod_{v\in OV} \mathbb{Z}[G/G_v].\end{aligned}$$

In Ext-groups we obtain,

$$\begin{aligned}\mathrm{Ext}^i_{\mathbb{Z}G}(C_0(X),\mathbb{Z}) &= \mathrm{Ext}^i_{\mathbb{Z}G}(\prod_{v\in OV} \mathbb{Z}[G/G_v],\mathbb{Z})\\ &= \prod_{v\in OV} \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}[G/G_v],\mathbb{Z})\\ &= \prod_{v\in OV} H^i(G_v,\mathbb{Z}).\end{aligned}$$

where the last equality comes from the Eckmann-Shapiro Lemma (see [10] pg.47). Similarly,

$$\begin{aligned}\mathrm{Ext}^i_{\mathbb{Z}G}(C_1(X),\mathbb{Z}) &= \mathrm{Ext}^i_{\mathbb{Z}G}(\prod_{e\in OE} \mathbb{Z}[G/G_e],\mathbb{Z})\\ &= \prod_{e\in OE} \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}[G/G_e],\mathbb{Z})\\ &= \prod_{e\in OE} H^i(G_e,\mathbb{Z}).\end{aligned}$$

Substituting these in the long exact sequence and writing $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) = H^i(G,\mathbb{Z})$ gives that

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}G}(C_0(X),\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}G}(C_1(X),\mathbb{Z}) \to$$

$$H^1(G,\mathbb{Z}) \to \prod_{v\in OV} H^1(G_v,\mathbb{Z}) \to \prod_{e\in OE} H^1(G_e,\mathbb{Z}) \to$$

$$H^2(G,\mathbb{Z}) \to \prod_{v\in OV} H^2(G_v,\mathbb{Z}) \to \prod_{e\in OE} H^2(G_e,\mathbb{Z}) \to \cdots$$

Here, we have

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) = H^0(G,\mathbb{Z}) = \mathbb{Z}$$

Since a $\mathbb{Z}G$-module homomorphism from $C_0(X)$ to $\mathbb{Z}$ is determined by G-orbit representatives,

$$\mathrm{Hom}_{\mathbb{Z}G}(C_0(X),\mathbb{Z}) = \prod_{v\in OV} H^0(G_v,\mathbb{Z}) = \prod_{v\in OV} \mathbb{Z}.$$

Similarly,

$$\mathrm{Hom}_{\mathbb{Z}G}(C_1(X),\mathbb{Z}) = \prod_{e\in OE} H^0(G_e,\mathbb{Z}) = \prod_{e\in OE} \mathbb{Z}.$$

□

With this theorem we can find a long exact sequence for the fundamental group of graph of groups by considering the standard action of $\pi(\mathcal{G},Y)$ on the tree Z explained in Theorem 2.3.2. In this action, the G_v groups appear as stabilizer groups of vertices of Z and G_e's appear stabilizer groups of edges of Z. Then the theorem gives a long exact sequence relating $\pi(\mathcal{G},Y)$ with vertex and edge groups homologically. For the simplest case, we can obtain Mayer-Vietoris sequence as shown in the next example.

Example 2.4.3. Let $G = A *_C B$ be an amalgamation of groups as in Example 2.3.4. Then corresponding action on tree has one edge orbit having stabilizer group isomorphic to C and two vertex orbit having stabilizer groups isomorphic to A and B. Then from the Theorem 2.4.2, we have

$0 \to H^0(G,\mathbb{Z}) \to H^0(A,\mathbb{Z})\oplus H^0(B,\mathbb{Z}) \to H^0(C,\mathbb{Z}) \to H^1(G,\mathbb{Z}) \to H^1(A,\mathbb{Z})\oplus H^1(B,\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to H^2(G,\mathbb{Z}) \to H^2(A,\mathbb{Z}) \oplus H^2(B,\mathbb{Z}) \to H^2(C,\mathbb{Z}) \to \cdots$
which is the Mayer-Vietoris sequence for gluing a $K(A,1)$-complex and a $K(B,1)$-complex along a $K(C,1)$-complex to obtain a $K(G,1)$-complex.

Chapter 3

Graph of Groups and Realizing Fusion Systems

In the first section of this chapter we present required theory of the fusion systems mostly from the reference [11]. In the second section of this chapter, we give infinite group models realizing fusion systems due to Robinson and Leary-Stancu. In the third section, we construct these infinite group models for a fusion given by a finite group G. We introduce the notion of storing homomorphism from the fundamental group of a graph of groups to the group G. Later, we will use this homomorphism to relate the cohomologies of these groups.

3.1 Fusion Systems

In this section, we give some needed background for the theory of fusion systems, mostly from the reference [11].

We say S is a *Sylow p-subgroup* of a group G if for any p-subgroup Q of G there exist a $g \in G$ such that $gQg^{-1} \in S$. By Sylow theorems, it is known that any finite group has at least one Sylow p-subgroup. However, it is not true for infinite groups in general. For example, the free product $C_3 * C_3$ has no Sylow 3-subgroup.

Let S be a Sylow p-subgroup of a finite group G. A finite group fusion system $\mathcal{F}_S(G)$ is a category having objects as subgroups of S and morphisms are the conjugations by elements in G. By forgetting G, we can define an abstract fusion system on a finite p-group S with certain properties, as shown in the following definition.

Definition 3.1.1. Let S be a finite p-group. A fusion system $\mathcal{F}$ on S is a category has objects as subgroups of S and the morphism set $\mathcal{F}(P, Q)$ consists of injective homomorphisms with following properties

i-) For any $s \in S$ and $P \leqslant S$, the conjugation map $c_s : P \to S$ is contained in $\mathcal{F}(P, S)$

ii-) For any $\phi : P \to Q$ in $\mathcal{F}$, the corresponding isomorphism $\phi : P \to \phi(P)$ is contained in $\mathcal{F}(P, \phi(P))$

iii-) For any group isomorphism $\beta : P \to Q$ in $\mathcal{F}(P, Q)$, the inverse map β^{-1} is in $\mathcal{F}(Q, P)$.

We say a fusion system $\mathcal{F}$ is *finite* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G.

Most of the theorems and ideas of proofs in the theory fusion systems can be done by mimicking their versions in group theory. For example, assume that we have a group G and a p-subgroup P of G, we take a Sylow p-subgroup of G which contains a Sylow p-subgroup of $N_G(P)$. The corresponding argument in the theory of fusion systems is "assume $\mathcal{F} = \mathcal{F}_S$ and $P < S$, we take an $\mathcal{F}$-conjugate $Q < S$ such that $|N_S(Q)|$ is maximal along $\mathcal{F}$-conjugates of P". Similar arguments in this theory motivates the following definition.

Definition 3.1.2. Let $\mathcal{F}$ be a fusion system on S. A subgroup P of S is said to be *fully $\mathcal{F}$-normalized* if for any Q that is $\mathcal{F}$-conjugate to P, we have

$$|N_S(P)| \geqslant |N_S(Q)|.$$

Obviously, for any $P \in \mathcal{F}$, there exists a fully $\mathcal{F}$-normalized Q which is $\mathcal{F}$-conjugate to P.

Definition 3.1.3. Let $\mathcal{F}$ be a fusion system on S. We say $\mathcal{F}$ is *saturated* if

i-) $\mathrm{Aut}_S(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$

ii-) For any $\phi : P \to S$ in $\mathcal{F}$, if $\phi(P)$ is fully $\mathcal{F}$-normalized, then ϕ extends to a morphism $\bar{\phi} : N_\phi \to S$ where $N_\phi := \{g \in N_S(P) | \exists h \in N_S(\phi(P))$ with $\phi(gpg^{-1}) = h\phi(p)h^{-1} \quad \forall p \in P\}$

It can be easily shown that any finite fusion system is saturated. By a *finite fusion system*, we mean the fusion system can be realized by a finite group (i.e. $\mathcal{F} = \mathcal{F}_S(G)$ for some finite G).

Definition 3.1.4. Let P be a non-trivial p-subgroup of G. Then

i-) P is *p-centric* if $Z(P)$ is Sylow p-subgroup of $C_G(P)$.

ii-) P is *p-radical* if $P = O_p(N_G(P))$.

Here, $O_p(X)$ denotes the largest normal p-subgroup of X.

Definition 3.1.5. Let $\mathcal{F}$ be a fusion system on S. Then

i-) P is *$\mathcal{F}$-centric* if for every Q which is $\mathcal{F}$-conjugate to P, we have $C_S(Q) = Z(Q)$.

ii-) P is *$\mathcal{F}$-radical* if $O_p(Aut_{\mathcal{F}}(P)) = Inn(P)$.

Here, being $\mathcal{F}$-centric is a generalization of being p-centric. Although being p-radical does not imply being $\mathcal{F}$-radical, in general, the next lemma shows that they are equivalent to p-centric groups.

Lemma 3.1.6. *Let $\mathcal{F} = \mathcal{F}_S(G)$ and P be a subgroup of S. Then,* *i-) P is p-centric if and only if it is $\mathcal{F}$-centric.*

ii-) P is $\mathcal{F}$-centric and $\mathcal{F}$-radical then it is p-radical and p-centric.

Proof. For i-), assume P is p-centric. Take any Q with $Q = gPg^{-1}$ for some $g \in G$. The automorphism c_g of G sends P to Q and $g^{-1}Sg$ to S. Then, $|C_{g^{-1}Sg}(P)| = |C_S(Q)|$. Since P is p-centric $|C_{g^{-1}Sg}(P)| \leqslant |Z(P)|$. Then, $|C_S(Q)| \leqslant |Z(Q)|$. Hence, $C_S(Q) = Z(Q)$, proving P is $\mathcal{F}$-centric.

For the converse, assume P is $\mathcal{F}$-centric. Let X be any Sylow p-subgroup of $C_G(P)$. Take $g \in G$ such that X contained in $g^{-1}Sg$. The automorphism c_g sends P to Q, and X to gXg^{-1}, and $C_G(P)$ to $C_G(Q)$, and $g^{-1}Sg$ to S. Since P is $\mathcal{F}$-centric, $C_S(Q) = Z(Q)$. Then,

$$gXg^{-1} < C_S(Q) = Z(Q) = gZ(P)g^{-1}$$

So, X is a subgroup of $Z(P)$. Hence, $X = Z(P)$ because X is a Sylow p-subgroup of $C_G(P)$, completing the first part.

For ii-), assume P is $\mathcal{F}$-centric and $\mathcal{F}$-radical. We have $Aut_{\mathcal{F}}(P) \cong N_G(P)/C_G(P)$ and $Inn_{\mathcal{F}}(P) \cong PC_G(P)/C_G(P)$. $Q = O_p(N_G(P))$. Since P normal in $N_G(P)$, $P < Q$. The subgroup $QC_G(P)$ is normal in $N_G(P)$ because Q and $C_G(P)$ are normal in $N_G(P)$. By correspondence, $QC_G(P)/C_G(P)$ is normal in $N_G(P)/CG(P)$. So we must have $P = Q$ otherwise the maximum normal p-subgroup of $N_G(P)/C_G(P)$ would be greater than $PC_G(P)/C_G(P)$. Hence, P is p-centric and p-radical.

□

From [12], we have an example shows that the converse of the second statement of the last lemma is not true in general. We take the dihedral group

$$G = D_{24} = \langle a, b | a^{12} = b^2 = 1 \text{ and } bab = a^{-1} \rangle$$

and its Sylow 2-subgroup $S = \langle a^3, b \rangle$. Let $\mathcal{F} = \mathcal{F}_S(G)$ and $P = \langle a^3 \rangle$. Then P is p-centric because $Z(P) = \langle a^3 \rangle$ is a Sylow 2-subgroup of $C_G(P) = \langle a \rangle$. P is p-radical as $O_p(N_G(P)) = O_p(G) = P$. However, $Aut_{\mathcal{F}}(P)$ consists of two elements the identity and the conjugation by b whereas $Inn(P)$ has only one element, the identity. Since $O_P(Aut_{\mathcal{F}}(P) \neq Inn(P)$, P is not $\mathcal{F}$-radical.

3.1.1 Alperin Fusion Theorem

Alperin fusion theorem states that automorphisms of some family of subgroups of S generate the whole fusion $\mathcal{F} = \mathcal{F}_S$. We will use this theorem for realizing fusion systems. For example, if a group G contains S as a Sylow p-subgroup and elements that realize the generators of the fusion system $\mathcal{F}$ then we can say $\mathcal{F} \subset \mathcal{F}_S(G)$.

Definition 3.1.7. Let $\mathcal{F} = \mathcal{F}_S$. A subgroup P of S is *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and $Out_{\mathcal{F}}(Q) = Aut_{\mathcal{F}}(P)/Inn(P)$ contains a strongly p-embedded subgroup.

Here, we say M is a *strongly p-embedded subgroup of* G if M contains a Sylow p-subgroup of G and $M \cap M^g$ is a p'-group for all $g \in G\backslash M$. In this case, since for any p-subgroup P of G, there exists $g \in G$ such that $P \cap P^g$ is trivial, G has no normal p-subgroup (i.e. $O_p(G) = 1$). That shows an $\mathcal{F}$-essential subgroup must be $\mathcal{F}$-centric and $\mathcal{F}$-radical.

Definition 3.1.8. Let $\mathcal{F}$ be a fusion system on a finite p-subgroup S. A family F of subgroups of S is a *conjugation family* for $\mathcal{F}$ if $\mathcal{F} = \langle Aut_{\mathcal{F}}(U) | U \in F\rangle$.

Theorem 3.1.9 (Alperin Fusion Theorem)**.** *Let $\mathcal{F} = \mathcal{F}_S$ be a saturated fusion system. Then, $\mathscr{C} = \{P | P$ is fully $\mathcal{F}$-normalized essential subgroup of $S\}$ is a conjugation family.*

Proof. See page 122 in [11]. □

Remark 3.1.10. Obviously, any family containing $\mathscr{C}$ is a conjugation family. Since essential subgroups are $\mathcal{F}$-centric and $\mathcal{F}$-radical, the family $\mathscr{C}^{cr} = \{P | P$ is fully $\mathcal{F}$-normalized $\mathcal{F}$-centric $\mathcal{F}$-radical subgroup of $S\}$ is a conjugation family. Also $\mathscr{C}^p = \{P | P$ is a p-centric p-radical subgroup of $S\}$ is a conjugation family because $\mathscr{C}^p \supset \mathscr{C}^{cr} \supset \mathscr{C}$.

3.1.2 Model Theorem

The model theorem states for some fusion systems there exist a finite model group realizing the fusion which is unique up to some condition. In this case, we will say *"let take the model group of $\mathcal{F}$"* to refer to this model theorem.

Definition 3.1.11. Let $\mathcal{F} = \mathcal{F}_S$ and $P < S$. We say P *is normal in* $\mathcal{F}$ if for any morphism $\phi : Q \to R$ in $\mathcal{F}$ there exists a morphism $\bar{\phi} : QP \to QR$ such that the restriction $\bar{\phi}|_P$ is an automorphism of P and $\bar{\phi}|_Q = \phi$.

Definition 3.1.12. Let $\mathcal{F} = \mathcal{F}_S$ be saturated. If there exists $Q \lhd S$ which is $\mathcal{F}$-centric and normal in $\mathcal{F}$, we say $\mathcal{F}$ *is constrained.*

Theorem 3.1.13 (Broto-Castellana-Grodal-Levi-Oliver, [13])**.** *Let $\mathcal{F} = \mathcal{F}_S$ be saturated and constrained. Then there exists unique finite group G with S as a Sylow p-subgroup so that*

i-) $\mathcal{F} = \mathcal{F}_S(G)$

ii-) $O_{p'}(G) = 1$

iii-) $C_G(O_p(G)) \leqslant O_p(G)$

We say G is the model for *$\mathcal{F}$.*

Corollary 3.1.14. *Let $\mathcal{F} = \mathcal{F}_S$ be a saturated fusion system. If $\mathcal{F} = \langle Aut_{\mathcal{F}}(S) \rangle$, then the finite model group for $\mathcal{F}$ exists.*

Proof. S is $\mathcal{F}$-centric because $C_S(S) = Z(S)$. S is normal in $\mathcal{F}$ because any morphism in $\mathcal{F}$ can be extended to S. Since $\mathcal{F}$ is constrained saturated fusion system the model theorem applies. □

3.2 Realizing Fusion Systems

For an abstract fusion system $\mathcal{F}$ on a p-group S, we say that G realizes the fusion $\mathcal{F}$ if S is a Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_S(G)$. Since there are abstract fusion systems which cannot be realizable by finite groups, the theory of realization of fusion systems includes infinite group models. In this case, the natural question is that can we realize an abstract fusion system by using infinite groups. In 2007, Robinson [2] write an infinite group model realizing an arbitrary abstract fusion system. At the same year, Leary and Stancu [1] published a different infinite group model realizing a given abstract fusion system. These models explained below in terms of graph groups. However, for these models, we lose the property that the $\mathbb{F}_p$ cohomology of the fusion system is the $\mathbb{F}_p$ cohomology of the finite group it realizers. We cannot say this for these models. So finding an infinite group model realizing an abstract fusion system with cohomology fits the fusion systems cohomology is an open problem. Related to this, we quote a theorem from [3] having a relation with the cohomology of the infinite group and the cohomology of fusion for some special infinite group models.

Theorem 3.2.1 (Leary-Stancu, [1])**.** *Let $\mathcal{F}$ be a fusion system on a p-group S generated by morphisms $f_i : P_i \to Q_i$ for $1 \leqslant i \leqslant r$, where P_i's and Q_i's are subgroups of S.*

We define a graph of groups $(\mathcal{G}, Y)$ so that Y is a graph having only one vertex v and edges $e_1, \overline{e_1}, e_2, \overline{e_2}, ..., e_r, \overline{e_r}$. We have vertex group $G_v := S$ and edges groups $G_{e_i} = G_{\overline{e_i}} := P_i$ and the morphisms $\phi_{e_i} : P_i \hookrightarrow S$ are inclusion and the morphisms $\phi_{\overline{e_i}} : P_i \to S$ are f_i composed with inclusion into S monomorphisms.

Then the fundamental group of the graph of groups realizes the fusion system, that is

$$\mathcal{F} = \mathcal{F}_S(\pi(\mathcal{G}, Y)).$$

Example 3.2.2. Let $\mathcal{F} = \mathcal{F}_S(G)$ where $G := S_3$ and $S = C_3$ is the Sylow 3-subgroup of G. The fusion $\mathcal{F}$ can be generated by the nontrivial automorphism of S. According to Leary-Stancu model, our graph of groups has vertex group as S and the edge group S with two monomorphisms the identity and the nontrivial automorphism of S. Then, the infinite group $\pi = \pi(\mathcal{G}, Y) = C_3 \rtimes \mathbb{Z}$ realizes $\mathcal{F}$ (i.e. $\mathcal{F}_S(\pi) = \mathcal{F}$)

Theorem 3.2.3 (Robinson, [2])**.** *Let $\mathcal{F}$ be a fusion system on a p-group S generated by the images $\mathcal{F}_{S_i}(G_i)$ under injective group homomorphisms $f_i : S_i \hookrightarrow S$ for $1 \leqslant i \leqslant r$.*

We define a graph of groups $(\mathcal{G}, Y)$ so that Y has vertices $v_0, v_1, v_2, ..., v_r$ and edges $e_i, \overline{e_i}$ between v_0 and v_i for $1 \leqslant i \leqslant r$. The vertex groups are $G_{v_0} := S$ and $G_{v_i} = G_i$ for $1 \leqslant i \leqslant r$. The edge groups are $G_{e_i} = G_{\overline{e_i}} := S_i$ and monomorphisms $\phi_{e_i} : S_i \hookrightarrow S$, $\phi_{\overline{e_i}} : S_i \hookrightarrow G_i$ are inclusions.

Then the fundamental group of the graph of groups realizes the fusion system that is

$$\mathcal{F} = \mathcal{F}_S(\pi(\mathcal{G}, Y)).$$

Since this construction does not determine the subfusions that generate $\mathcal{F}$ and the realizations of these subfusions are not unique, there are many ways to construct an infinite group realizing $\mathcal{F}$ according to the Robinson model. By using Alperin Fusion theorem, the family of subfusions, where each subfusion is generated by automorphisms of some fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroup of $\mathcal{F}$, generates $\mathcal{F}$. This makes the choice of subfusions $\mathcal{F}_{S_i}(G_i)$ unique. We can also make unique the choice of realizations of these subfusions by using the model theorem. This unique construction stated in the next example which is the most famous way of constructing infinite group for realizing a saturated fusion system according to Robinson model.

Example 3.2.4. Let $\mathcal{F} = \mathcal{F}_S$ be saturated. Let $R_1, R_2, ..., R_k$ be fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S. Let $\mathcal{F}_i = \mathcal{F}_{R_i}$ be the fusion system on R_i generated by the $Aut_{\mathcal{F}}(R_i)$. Then, by Alperin theorem $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k$ generates $\mathcal{F}$. From Corollary 3.1.14, there is a unique model L_i for $\mathcal{F}_i$. Now we construct the Robinson model by taking generators as $\mathcal{F}_{R_i}(L_i)$. Here, L_i's are the vertex groups and R_i's are the edge groups. More explicitly, $\pi = \pi(\mathcal{G}, Y) = S *_{R_1} L_1 *_{R_2} L_2 \cdots *_{R_k} L_k$.

Here, we can also choose R_i's as the fully $\mathcal{F}$-normalized essential subgroups of S.

3.3 Realizing Finite Fusions and Storing Homomorphism

In this section, we focus on finite fusions and their realizations. Now, we mimic Example 3.2.4, by changing L_i's.

Example 3.3.1. Let $\mathcal{F} = \mathcal{F}_S(G)$ where G is finite. We take fully $\mathcal{F}$-normalized F-centric $\mathcal{F}$-radical subgroups $R_1, R_2, ..., R_k$ as we do in Example 3.2.4. We define $N_i = N_G(R_i)$. Since $\mathcal{F}_{R_i}(N_i)$'s generate $\mathcal{F}$ by Alperin Fusion theorem. We construct the Robinson model on these groups. Our infinite group is

$$\pi = \pi(\mathcal{G}, Y) = S *_{R_1} N_1 *_{R_2} N_2, \cdots *_{R_k} N_k$$

realizing $\mathcal{F}$.

In fact, we can make the $\pi = \pi(\mathcal{G}, Y)$ much smaller by changing R_i with the larger subgroups $N_{S_i}(N_i)$. Since $\mathcal{F}_{N_S(R_i)}(N_i)$'s generates $\mathcal{F}$, the infinite group $\pi = \pi(\mathcal{G}, Y) = S *_{N_S(R_1)} N_1 *_{N_S(R_2)} N_2 \cdots *_{N_S(R_k)} N_k$ realizes $\mathcal{F}$ (i.e. $\mathcal{F}_S(\pi) = \mathcal{F}$). The group here is a quotient of the group in previous example.

Now, we state a bit different version of the Robinson model.

Example 3.3.2. Let $\mathcal{F} = \mathcal{F}_S(G)$ where G is finite. We take fully $\mathcal{F}$-normalized F-centric $\mathcal{F}$-radical subgroups $R_1, R_2, ..., R_k$ as we do in previous examples (or we can choose the essential ones from them as we can do in previous examples). We construct a graph of groups $(\mathcal{G}, Y)$ by taking Y as the complete graph with k vertices so that

i-) $G_{v_i} = N_G(R_i)$ are vertex groups ii-) the edge groups between v_i and v_j are equal to $N_G(R_i) \cap N_G(R_j)$.

From proposition 3.3 in [3], we can say S is the Sylow p-subgroup G. By Alperin Fusion theorem, $\mathcal{F}_S(\pi) \supset \mathcal{F}_S(G)$ because $\mathcal{F}_S(\pi)$ contains all fusion of $\mathcal{F}$-normalized essentials which generate $\mathcal{F}_S(G)$. Also $\mathcal{F}_S(\pi) \subset \mathcal{F}_S(G)$ because any fusion in $\mathcal{F}_S(\pi)$ comes from $\mathcal{F}_S(G)$. Hence,

$$\mathcal{F}_S(\pi) = \mathcal{F}.$$

Definition 3.3.3. Let $(\mathcal{G}, Y)$ be a graph of groups and G be a finite group. We say χ *is a storing homomorphism of* $(\mathcal{G}, Y)$ if χ is a homomorphism $\chi : \pi(\mathcal{G}, Y) \to G$ such that for any vertex or edge group G_v and its inclusion map $i_e : G_v \to \pi(\mathcal{G}, Y)$ we have that the composition $\chi \circ i_e : G_v \to G$ is injective.

If the storing homomorphism χ is surjective, we say G *is a store of* $(\mathcal{G}, Y)$.

Note that this definition is more than saying all vertex and edge groups are subgroups of G because it also requires these groups to have the same intersection properties in G as they have in Γ.

Here, the map χ has kernel non-intersecting any vertex or edge groups. Then $\ker \chi$ is a free subgroup of Γ.

Proposition 3.3.4. *For the models constructed in Example 3.3.1, Example 3.3.2 and Theorem 3.2.1 the storing homomorphism always exists. Moreover, the kernel of storing homomorphism is free and when the storing homomorphism is surjective we have an exact sequence of groups*

$$1 \to F \to \pi \xrightarrow{\chi} G \to 1$$

where $F := \ker(\chi)$ *is free.*

Proof. Take any finite group G with Sylow p-subgroup S. Let $\mathcal{F} = \mathcal{F}_S(G)$.

First, we construct the Leary-Stancu model. Let $f_i : P_i \to Q_i$'s generate $\mathcal{F}$. Then, $\pi_{LS} =< S, t_1, t_2, \cdots, t_k | c_{t_i} = f_i >$ is the infinite group realizing $\mathcal{F}$

according to Leary-Stancu. Define $\chi : \pi_R \to G$ by sending $s \mapsto s$ for $s \in S$ and $t_i \mapsto g_i$ where $g_i \in G$ such that $c_{g_i} = f_i$. χ is storing homomorphism because it is identity on the vertex group S.

Second, we construct the Robinson model as in Example 3.3.1. Define $\chi : \pi_R \to G$ by sending the vertex groups $N_G(R_i)$ to their original copies in G. χ is well-defined because for any edge groups, the two different restrictions of χ are the same. χ is storing because it sends each vertex groups injectively.

Third, we consider Example 3.3.2. Define $\chi : \pi_{R'} \to G$ by sending the edge and vertex groups to their original copies in G. Similarly, χ is storing homomorphism.

In each of the cases, $\ker(\chi)$ is a subgroup of $\pi(\mathcal{G}, Y)$ such that it has a trivial intersection with any vertex group of $(\mathcal{G}, Y)$. Then, by Corollary 2.3.8, $F := \ker(\chi)$ is free.

□

Chapter 4

Cohomology of Infinite Groups Realizing Fusion Systems

In the previous chapter, we state several examples of infinite group models realizing fusion system. From now on, we focus on the cohomology of fusion systems. We start with the definition of stable elements from the reference [14].

Let G be a group with subgroup H and A be a coefficient ring. An element a in $H^*(H;A)$ is called *G-stable* if we have $\mathrm{res}^{xHx^{-1}}_{xHx^{-1}\bigcap H}(c_x^*(a)) = \mathrm{res}^{H}_{xHx^{-1}\bigcap H}(a)$ where $c_x^* : H^*(H;A) \to H^*(xHx^{-1})$ is the isomorphism induced by conjugation map $c_x : xHx^{-1} \to H$ defined by $c_x(u) = x^{-1}ux$. We extend this notion to fusion systems. Let $\mathcal{F}$ be a fusion system on S. We say $a \in H^*(S)$ is *$\mathcal{F}$-stable* if for any isomorphism $P \xrightarrow{\phi} Q$ in $\mathcal{F}$, we have $\phi^*(\mathrm{res}^S_Q(a)) = \mathrm{res}^S_P(a)$ where ϕ^* is the isomorphism induced by ϕ.

The cohomology of the fusion system $\mathcal{F} = \mathcal{F}_S$ defined as the inverse limit

$$H^*(\mathcal{F};\mathbb{F}_p) := \lim_{P\in\mathcal{F}} H^*(P;\mathbb{F}_p)$$

or, equivalently, as the $\mathcal{F}$-stable elements of $H^*(S;\mathbb{F}_p)$. Usually, we denote $H^*(\mathcal{F})$ instead of $H^*(\mathcal{F};\mathcal{F}_p)$. By writing commuting diagrams, one can easily show that the condition of being G-stable is the same as the $\mathcal{F}$-stability condition. So we have a version of Cartan-Eilenberg Theorem

Theorem 4.0.1 (Cartan Eilenberg)**.** *Let G be a finite group with Sylow p-subgroup S. If $\mathcal{F} = \mathcal{F}_S(G)$, then*

$$H^*(\mathcal{F}) \cong H^*(G, \mathbb{F}_p).$$

Proof. See [15, Theorem III.10.3]. □

From the previous section, we can realize any fusion by an infinite group. However, this infinite group may not realize the cohomology of the fusion system (in the sense of the last theorem) as the examples in the second section of the next chapter. The open question is

Open Question 4.0.2. *Given a saturated fusion system $\mathcal{F} = \mathcal{F}_S$, is there any infinite group model π realizing $\mathcal{F}$ such that*

$$H^*(\mathcal{F}) = H^*(\pi; \mathbb{F}_p).$$

Although we could not find the answer this question, we study the difference of $H^*(\mathcal{F})$ and $H^*(\pi; \mathbb{F}_p)$. In Theorem 6.1.10, it is shown that $H^*(\mathcal{F})$ is a direct summand of $H^*(\pi; \mathbb{F}_p)$ but the difference were unknown. For finite fusion systems, we calculate the difference in the next section for some infinite group models.

This chapter includes our main theorems. In Section 4.1, we write $H^*(\mathcal{F})$ as a direct summand of $H^*(\pi, \mathbb{F}_p)$ for finite fusion $\mathcal{F}$ and some conditions on the infinite group model realizing $\mathcal{F}$.

For both of the Leary Stancu and Robinson models, we have counterexamples that show that these models do not realize cohomology of the fusion. Moreover, in Section 4.2, we find infinitely many counterexamples for the Robinson model.

4.1 Homology of Graph of Groups Constructed from Subgroups of a Finite Group

Lemma 4.1.1. *Let G be a finite group and $(\mathcal{G}, Y)$ be a graph of groups so that G is a store of $(\mathcal{G}, Y)$. Then the storing homomorphism χ has free kernel F. So it gives an exact sequence $1 \to F \to \pi(\mathcal{G}, Y) \to G \to 1$. From the exact sequence, we have a G-action on the abelianization $F_{ab} = F/[F, F]$.*

Let $\pi(\mathcal{G}, Y)$ acts on a tree T. We consider the induced action of $G \cong \pi(\mathcal{G}, Y)/F$ on the graph $X = T/F$. This gives a G-action on $H_1(X)$.

There is a $\mathbb{Z}G$-module isomorphism between F_{ab} and $H_1(X)$.

Proof. Let $\Gamma := \pi(\mathcal{G}, Y)$ the fundamental group of the graph of groups.

Let $\pi : T \to X$ be the projection map. Fix a vertex $v \in T$. Let $\bar{v} = \pi(v)$.

Define $\phi : F \to \pi_1(X, \bar{v})$ by sending an $f \in F$ to $\pi(p(v, f \cdot v))$ where $f \cdot v$ is the vertex in T obtained by Γ-action on T and $p(v, f \cdot v)$ is the path from v to $f \cdot v$. Here, π projects that path to a loop at $\bar{v}$ (i.e. $\pi(p(v, f \cdot v)) \in \pi_1(X, \bar{v})$).

The map ϕ is well-defined because for any $f \in F$ there is a unique path from v to $f \cdot v$ in the tree and its projection is the loop $\phi(f) \in \pi_1(X, \bar{v})$.

Now, let show ϕ is a homomorphism. Take any $f_1, f_2 \in F$. We have

$$\begin{aligned} \phi(f_1 f_2) &= \pi(p(v, f_1 f_2 v)) \\ &= \pi(p(v, f_1 v) \circ p(f_1 v, f_1 f_2 v)) \\ &= \pi(p(v, f_1 v))\pi(p(f_1 v, f_1 f_2 v)) \\ &= \phi(f_1)\pi(p(v, f_2 v)) \\ &= \phi(f_1)\phi(f_2) \end{aligned}$$

where the notation $\circ$ is for composing paths. Here, $\pi(p(f_1 v, f_1 f_2 v)) = \pi(f_1 p(v, f_2 v)) = \pi(p(v, f_2 v))$ because the projection $\pi : T \to X = T/F$ annihilates the F-action.

For any loop $l \in \pi_1(X, \bar{v})$ there exists a unique lifted path starting at v in the tree T by the path lifting theorem. This path has end point $w \in T$ such that $\pi(w) = \bar{v}$. Then $w = fv$ for some $f \in F$ because v and w has same class in the quotient $X = T/F$. Here, there is a unique $f \in F$ satisfying $w = fv$ because F freely acts on T . So for any loop $l \in \pi_1(X, \bar{v})$, we have a unique $f \in F$ such that $\phi(f) = l$. Then, ϕ is surjective and has no kernel. Hence, ϕ is an isomorphism.

Let $\hat{\phi}$ be induced isomorphism between the abelianization groups F_{ab} and $(\pi_1(X, \bar{v}))_{ab}$. We know that $H_1(X) \cong (\pi_1(X, \bar{v}))_{ab}$. So we have a commutative

$$\begin{array}{ccc} F & \xrightarrow{\phi} & \pi_1(X, \bar{v}) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ F_{ab} & \xrightarrow{\hat{\phi}} & H_1(X) \end{array}$$

where j and k are abelianization maps.

$1 \to F \xrightarrow{\text{i}} \Gamma \xrightarrow{\text{r}} G \to 1$ induces a G-action on F_{ab} by conjugation and the G-action on $H_1(X)$ is induced by the G-action on X.

Then we need to show that given any $[f] \in F_{ab}$ and $g \in G$ we have that

$$\hat{\phi}(g[f]g^{-1}) = g\hat{\phi}([f]).$$

Take $f \in F$ such that $j(f) = [f]$. Take $\gamma \in \Gamma$ such that $r(\gamma) = g$. Then $j(\gamma f \gamma^{-1}) = g[f]g^{-1}$. With the help of commutative diagram, we have

$$\hat{\phi}(g[f]g^{-1}) = \hat{\phi}(j(\gamma f \gamma^{-1})) = k(\phi(\gamma f \gamma^{-1}))$$

and

$$\hat{\phi}([f]) = \hat{\phi}(j(f)) = k(\phi(f)).$$

To finish the proof, we work with ϕ and show that

$$gk(\phi(f)) = k(\phi(\gamma f \gamma^{-1})).$$

We have

$$\begin{aligned}\phi(f) &= \pi(p(v, fv)) \\ \phi(\gamma f \gamma^{-1}) &= \pi(p(v, \gamma f \gamma^{-1} v)) \\ &= \pi(p(v, \gamma v) \circ p(\gamma v, \gamma f v) \circ p(\gamma f v, \gamma f \gamma^{-1} v)) \end{aligned} \tag{4.1}$$

As $F \trianglelefteq \Gamma$, $\gamma f \gamma^{-1} \in F$. Since π annihilates F action, we obtain

$$\pi(p(\gamma f v, \gamma f \gamma^{-1} v)) = \pi(\gamma f \gamma^{-1} p(\gamma v, v)) = \pi(p(\gamma v, v)).$$

Substituting this in equation 4.1, we get

$$\phi(\gamma f \gamma^{-1}) = \pi(p(v, \gamma v) \circ p(\gamma v, \gamma f v) \circ p(\gamma v, v)).$$

Moving to homology,

$$\begin{aligned} k(\phi(\gamma f \gamma^{-1})) &= k(\pi(p(v, \gamma v) \circ p(\gamma v, \gamma f v) \circ p(\gamma v, v)) \\ &= k(\pi(p(v, \gamma v))) + k(\pi(p(\gamma v, \gamma f v))) + k(\pi(p(\gamma v, v))) \\ &= k(\pi(p(\gamma v, \gamma f v))). \end{aligned}$$

where $k(\pi(p(v, \gamma v))) = -k(\pi(p(\gamma v, v))$ as we work in $H_1(X)$.

Here, the path from γv to $\gamma f v$ goes to a loop at $g\bar{v}$ which is g times a loop at $\bar{v}$, working in homology. Writing formally, we have $k(\pi(p(\gamma v, \gamma f v))) = gk(\pi(p(v, fv)))$. Which gives

$$gk(\phi(f)) = k(\phi(\gamma f \gamma^{-1})).$$

That is equivalent to $\hat{\phi}(g[f]g^{-1}) = g\hat{\phi}([f])$, proving $\hat{\phi}$ is G-module isomorphism between F_{ab} and $H_1(X)$. □

Theorem 4.1.2. *Let G be a finite group and $(\mathcal{G}, Y)$ be a graph of groups so that G is a store of $(\mathcal{G}, Y)$. Assume $(\mathcal{G}, Y)$ has a vertex G_v such that the composition $G_v \to \pi(\mathcal{G}, Y) \to G$ sends a Sylow p-subgroup of G_v to a Sylow p-subgroup of G isomorphically. For a field R of characteristic p, there is an isomorphism*

$$H^{*-1}(G; F_{ab} \otimes R) \oplus H^*(G; R) \cong H^*(\pi(\mathcal{G}, Y); R).$$

where F is the kernel of storing homomorphism.

Proof. Let $\Gamma := \pi(\mathcal{G}, Y)$. χ, the store homomorphism, gives an exact sequence $1 \to F \to \Gamma \to G \to 1$ where $F := \ker \chi$ is a free group. We consider the standard Γ-action on the tree T. $G = \Gamma/F$ acts on $X = T/F$, inducing the previous action. Write cellular chain complex for X,

$$0 \to C_1 \to C_0 \to 0.$$

Since X is connected we have an exact sequence of RG-modules $C_1 \to C_0 \to \mathbb{Z} \to 0$, using that G acts on X cellularly. Applying $\mathrm{Hom}_{RG}(-, R)$ functor, we obtain exact sequence $0 \to \mathrm{Hom}_{RG}(\mathbb{Z}, R) \to \mathrm{Hom}_{RG}(C_0, R) \to \mathrm{Hom}_{RG}(C_1, R)$.

From the cochain complex

$$0 \to \mathrm{Hom}_R(\mathbb{Z}, R) \to \mathrm{Hom}_R(C_0, R) \to \mathrm{Hom}_R(C_1, R),$$

we have

$$H^1(X, R) = \mathrm{Hom}_R(C_1, r)/\mathrm{Im}(\mathrm{Hom}_R(C_0, R)).$$

So we complete the exact sequence,

$$0 \to R \to C_0 \to C_1 \to 0$$

Considering G-action on C_i simplices, we have

$$\mathrm{Hom}_R(C_0, R) = \prod_{v \in OV} R[G/G_v], \text{ and}$$

$$\mathrm{Hom}_R(C_1, R) = \prod_{e \in OE} R[G/G_e],$$

where OE and OV are orbit representative sets for edges and vertices respectively. Substituting in the last exact sequence, we get

$$0 \to R \to \prod_{v \in OV} R[G/G_v] \to \prod_{e \in OE} R[G/G_e] \to H^1(X; R) \to 0 \qquad (4.2)$$

Since Γ has Sylow p-subgroup S, there exists G_v containing S. Then the map $R \to \prod_{v \in OV} R[G/G_v]$ splits because we can write splitting over $R[G/G_v]$ as $|[G/G_v]|$ is not divisible by p. We divide the exact sequence in 4.2 by defining

$$K := \mathrm{Im}(\prod_{v \in OV} R[G/G_v] \to \prod_{e \in OE} R[G/G_e]) = \ker(\prod_{e \in OE} R[G/G_e] \to H^1(X; R)).$$

For the four-term exact sequence above, we use the idea stated in [16]. So we have 2 exact sequences

$$0 \to R \to \prod_{v \in OV} R[G/G_v] \to K \to 0 \tag{4.3}$$

$$0 \to K \to \prod_{e \in OE} R[G/G_e] \to H^1(X;R) \to 0. \tag{4.4}$$

From above we have that 4.3 splits, and by Shapiro's lemma, it gives an isomorphism

$$\prod_{v \in OV} H^*(G_v;R) \cong H^*(G;K) \oplus H^*(G;R). \tag{4.5}$$

The exact sequence 4.4 gives a long exact sequence in cohomology

$$\cdots H^{*-1}(G;K) \to H^{*-1}(G, \prod_{e \in OE} R[G/G_e]) \to H^{*-1}(G, H^1(X;R)) \to H^*(G;K) \to H^*(G, \prod_{e \in OE} R[G/G_e]) \to H^*(G, H^1(X;R)) \to \cdots$$

By coninduction and adding $H^*(G;R)$ for consecutive terms, we have

$$\cdots H^{*-1}(G;K) \to \prod_{e \in OE} H^{*-1}(G_e,R) \to H^*(G;R) \oplus H^{*-1}(G, H^1(X;R)) \to H^*(G;R) \oplus H^*(G;K) \to \prod_{e \in OE} H^*(G_e,R) \to H^*(G, H^1(X;R)) \to \cdots.$$

Using Equation 4.5, we have

$$\cdots H^{*-1}(G;K) \to \prod_{e \in OE} H^{*-1}(G_e,R) \to H^*(G;R) \oplus H^{*-1}(G, H^1(X;R)) \to H^*(G; \prod_{v \in OV} R[G/G_v]) \to \prod_{e \in OE} H^*(G_e,R) \to H^*(G, H^1(X;R)) \to \cdots$$

Also we have a long exact sequence for Γ by Theorem 2.4.2,

$$\cdots H^{*-1}(G;K) \to \prod_{e \in OE} H^{*-1}(G_e,R) \to H^*(\Gamma;R) \to H^*(G; \prod_{v \in OV} R[G/G_v]) \to \prod_{e \in OE} H^*(G_e,R) \to H^*(G, H^1(X;R)) \to \cdots$$

By using a five lemma,

$$H^{*-1}(G, H^1(X;R)) \oplus H^*(G,R) \cong H^*(\Gamma,R).$$

By using Lemma 4.1.1, we have

$$H^{*-1}(G, F_{ab} \otimes R) \oplus H^*(G, R) \cong H^*(\Gamma, R).$$

□

Corollary 4.1.3. *Let* $\Gamma := \pi(\mathcal{G}, Y)$*. For the restriction map* $Res^{\Gamma}_{S} : H^*(\Gamma, R) \to H^*(S, R)$ *we have*

$$\ker Res^{\Gamma}_{S} \cong H^{*-1}(G, F_{ab} \otimes R).$$

Proof. From the Theorem 6.1.10 we have

$$H^*(\Gamma, R) \cong H^*(G, R) \oplus \ker Res^{\Gamma}_{S}.$$

Using the Theorem 4.1.2, we obtain

$$\ker Res^{\Gamma}_{S} \cong H^{*-1}(G, F_{ab} \otimes R).$$

□

The next example shows that Leary-Stancu model does not realize cohomology of the fusion, in general.

Example 4.1.4. Let $G = S_3 = \langle a, b | b^3 = a^2 = 1, aba = b^2 \rangle$ with Sylow 3-subgroup $S = \langle b \rangle \cong C_3$ and $\mathcal{F} = \mathcal{F}_S(G)$. The Leary Stancu model for $\mathcal{F}$ is the infinite group

$$\pi = \langle b, t | b^3 = 1, tbt^{-1} = b^2 \cong C_3 \rtimes \mathbb{Z}$$

The storing homomorphism $\chi : \pi \to G$ sends $t \mapsto a$ and $b \mapsto b$. So it is surjective and $F = \ker(\chi) = \langle t^2 \rangle$. Take $R = \mathbb{F}_3$ and use Theorem 4.1.2. Since G acts on F trivially we have

$$H^{n-1}(S_3, \mathbb{F}_3) \oplus H^n(S_3, \mathbb{F}_3) \cong H^n(\pi, \mathbb{F}_3).$$

So, $H^*(\pi; \mathbb{F}_3) \ncong H^*(S_3; \mathbb{F}_3)$ in this case.

4.2 An Infinite Family of Examples

As we mention in Chapter 2, the Robinson model stated in Theorem 3.2.3 realizes fusion system but its cohomology does not fit with the cohomology of the fusion system, in general. As a counter-example, in [17], it is shown that for the fusion system of $\mathcal{F}$ 2-local finite group of $G = C_2^3 \rtimes GL(3,2)$ and the corresponding Robinson model group π_R we have $H^*(\pi_R) \neq H^*(\mathcal{F})$. In this section, we show that, for any fusion system created by $GL(n,2)$, the cohomology of the corresponding Robinson model group does not fits the cohomology of the fusion system for $n > 4$. Then, we have infinitely many examples that realizing fusion system by Robinson model does not give a realization of the cohomology of a given fusion system.

To construct Robinson model on the Sylow 2-subgroup of $GL(n,2)$, we must understand its Sylow 2-subgroup and its $\mathcal{F}$-radical and $\mathcal{F}$-centric subgroups. So we quote some known results.

We have a special case of Borel-Tits theorem having proof in [18] pg. 231.

Theorem 4.2.1 (Borel-Tits)**.** *If $G = GL(n,p)$ then a p-subgroup U is equal to $O_p(N_G(U))$ if and only if $N_G(U)$ is parabolic and U is its unipotent radical.*

Here, we need to understand the parabolics of $GL(n,2)$. A good source is Chapter 6 and Chapter 12 of [19] which are devoted to Borel subgroups and parabolic subgroups. We quote some results for $GL(n,2)$.

Let S be the upper triangular matrices in $G := GL(n,2)$. Since the order of S is $2^{(n-1)(n-2)/2}$, $|G : S|$ is odd. Then S is a Sylow p-subgroup of G. As we see in the proof of Theorem 6.4 in [19], we also have that S is a Borel subgroup of G. That gives $N_G(S) = S$, by using the Theorem 6.12 in [19].

Corollary 4.2.2. *The subgroup of upper triangular matrices S in $G = GL(n,2)$ is a Sylow 2-subgroup. Let $\mathcal{F} = \mathcal{F}_S(G)$. Then a 2-subgroup U is $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully $\mathcal{F}$-normalized if and only if $N_G(U)$ is parabolic containing S and U is its unipotent radical.*

Proof. The first sentence explained above. What is left is the if only if statement. Let us first prove the right direction. Assume 2-subgroup U is $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully $\mathcal{F}$-normalized. From Theorem 4.2.1, $N_G(U)$ is parabolic and U is its unipotent radical. Since $N_G(U)$ is parabolic, $N_G(U) \supset B$ for some Borel subgroup B. Since Borel subgroups are conjugate, there exists $g \in G$ such that $S = gBg^{-1}$. Let $P = gUg^{-1}$. Then $N_G(P) = gN_G(U)g^{-1} \supset gBg^{-1} = S$. Since U is fully $\mathcal{F}$-normalized, we have $|N_S(U)| \geqslant |N_S(P)|$. So $N_S(P) = S$ gives that $N_S(U) = S$ which means $N_G(U)$ contains S as desired.

For the other direction, assume U is 2-subgroup so that $N_G(U)$ is parabolic containing S and U is its unipotent radical. From Theorem 4.2.1, U is p-radical. As it is shown in [20, page 755], we have $C'_G(P) = 1$. So, U is p-radical. Since $N_S(U) = S$, U is fully $\mathcal{F}$-normalized. Since any unipotent radical of a parabolic group is $\mathcal{F}$-centralized as shown in Lemma 4.2.3. □

Lemma 4.2.3. *Let S be the group of upper triangular matrices in $G = GL(n, 2)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Then any unipotent radical U of a parabolic group P containing S is $\mathcal{F}$-centralized.*

Proof. If V is $\mathcal{F}$-centric and $V \subset U$, then U is also $\mathcal{F}$-centric. We know that the maximal parabolics corresponds to the minimal unipotent radicals. Then, it is enough to prove that the statement holds for all maximal parabolic P containing S. Take any maximal parabolic subgroup containing S which is the form (as mentioned in [21])

$$P_m = \begin{bmatrix} GL(m,2) & M_{m,n-m}(\mathbb{F}_2) \\ 0 & GL(m-n,2) \end{bmatrix}$$

with unipotent radical

$$U_m = \begin{bmatrix} I_m & M_{m,n-m}(\mathbb{F}_2) \\ 0 & I_{n-m} \end{bmatrix}.$$

Take any $s \in S$ centralizing U_m, then for any $m \in U_m$, we have $sm = ms$. Let

$$s = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

Then for any $M \in M_{m,n-m}(\mathbb{F}_2)$, we have

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} I_m & M \\ 0 & I_{n-m} \end{bmatrix} = \begin{bmatrix} I_m & M \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

This gives

$$aM = Mc \tag{4.6}$$

for any $M \in M_{m,n-m}(\mathbb{F}_2)$.

Fix any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m-n$. Choosing M having all entries 0 but the (i,j)th entry is 1, the equation 4.6 gives that $c_{j,j} = a_{i,i}$, $c_{j,k} = 0$ for $k \neq j$ and $a_{l,i} = 0$ for $k \neq i$. By doing the argument for all possible (i,j) pairs, we get that a and c are diagonal matrices having all the same diagonal entries. Since $s \in S$, $\det(s) \neq 0$. Then a and b are non-zero matrices. Working in $\mathbb{F}_2$, we must have that $a = I_m$ and $b = I_{n-m}$. That means $s \in U_m$ for any $s \in$ centralizing U_m. Hence,

$$C_S(U_m) = Z(U_m)$$

or equivalently U_m is $\mathcal{F}$-centralized. □

Here, we can mention the Robinson model for the fusion system of $\mathcal{F}_S(GL(n,2))$ because we know what are the $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully $\mathcal{F}$-normalized subgroups of the fusion system. In the following theorem, we construct the Robinson model by using these subgroups.

Theorem 4.2.4. *Let $G = GL(n,2)$ for $n \geqslant 5$. Let S be the Sylow 2-subgroup consisting of upper triangular matrices in G. Let $(\mathcal{G}, Y)$ be the graph of groups of Robinson model constructed for $\mathcal{F} = \mathcal{F}_S(G)$. Then we have*

$$H^2(\mathcal{F}) \neq H^2(\pi(\mathcal{G}, Y), \mathbb{F}_2).$$

Proof. We know that $H^2(\mathcal{F}) = H^2(GL(n,2), \mathbb{F}_2)$. From [22] table 6.1.3, we have that $H^2(GL(n,2)) = 0$ for $n \geqslant 5$. Then it is enough to prove that

$$H^2(\pi(\mathcal{G}, Y), \mathbb{F}_2) \neq 0.$$

From the Alperin Fusion Theorem, the fusion system is generated by the normalizer of S and the normalizers of $\mathcal{F}$-radical, $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized subgroups of S. From Corollary 4.2.2, we say that the fusion system is generated by the fusion systems of $\mathcal{F}_S(P_i)$ for $0 \leqslant i \leqslant k$ where $P_0 = S$ and $P_1, P_2, ..., P_k$ are parabolic subgroups containing S. Note that the Sylow 2-subgroup of P_i's are S because $P_i = N_G(U_i) \geqslant N_S(U_i) = S$ as shown in the proof of Corollary 4.2.2.

The graph of groups has vertex groups $P_0, P_1, P_2..., P_k$ and has k many edge groups all are S. From Theorem 2.4.2, we have a long exact sequence

$$\cdots \to \prod_{0 \leqslant i \leqslant k} H^1(P_i; \mathbb{F}_2) \xrightarrow{f} \prod_{1}^{k} H^1(S; \mathbb{F}_2) \xrightarrow{g} H^2(\pi(\mathcal{G}, Y); \mathbb{F}_2) \to \cdots \tag{4.7}$$

For any i, we have $|H^1(P_i; \mathbb{F}_2)| \leqslant |H^1(S; \mathbb{F}_2)|$ because S is a Sylow 2-subgroup of P_i. Without lose of generality, we assume that $P_1, P_2, ..P_{n-1}$ are maximal parabolic subgroups such that, for $1 \leqslant m \leqslant n-1$, we have

$$P_m = \begin{bmatrix} GL(m,2) & M_{m,n-m}(\mathbb{F}_2) \\ 0 & GL(m-n,2) \end{bmatrix}.$$

Then we have that $P_1 \cong P_{m-1} \cong C_2^{n-1} \rtimes GL(n-1,2)$. We have

$$H^1(C_2^{n-1} \rtimes GL(n-1,2); \mathbb{F}_2) = \mathrm{Hom}(C_2^{n-1} \rtimes GL(n-1,2), C_2).$$

Take any $\phi \in \mathrm{Hom}(C_2^{n-1} \rtimes GL(n-1,2), C_2)$. Consider the restriction of ϕ to $GL(n-1,2)$ is a homomorphism from a simple group to C_2. Then ϕ must be zero on $GL(n-1,2)$. If ϕ is non-zero, we have $\phi(a) = 1$ for some $a \in C_2^{n-1}$. Take any $b \in C_2^{n-1} - \{0, -a\}$. Since $GL(n-1,2)$ acts on C_2^{n-1} by conjugation so that it sends any nonzero element to any nonzero element, we have that $\phi(a) = \phi(b) = \phi(a+b)$. Hence a contradiction. So we must have

$$\mathrm{Hom}(C_2^{n-1} \rtimes GL(n-1,2), C_2) = 0.$$

Then we get

$$H^1(P_1; \mathbb{F}_2) = H^1(P_{n-1}; \mathbb{F}_2) = 0 \tag{4.8}$$

In the long exact sequence 4.7, we have that

$$\left|\prod_{0\leqslant i\leqslant k} H^1(P_i;\mathbb{F}_2)\right| \leqslant \left|\prod_1^k H^1(S;\mathbb{F}_2)\right|$$

because in the left-hand side two terms are 0 as shown above and for each other terms in the left we have $|H^1(P_i;\mathbb{F}_2)| \leqslant |H^1(S;\mathbb{F}_2)|$. Then, f cannot be surjective. Since $\ker g = \mathrm{Im} f$, $\ker g$ is not the whole of $\prod_1^k H^1(S;\mathbb{F}_2)$. Then g has a nonzero image. Hence,

$$H^2(\pi(\mathcal{G},Y);\mathbb{F}_2) \neq 0.$$

□

Chapter 5

Using Posets to Generate Infinite Group Models Realizing Fusion Systems

We do not know any infinite group model realizing fusion and its cohomology. Making the open question easier, we try to find an infinite group model realizing fusion and its cohomology for finite fusions. Because of this, we start with the fusion system of a finite group and try to find a desired infinite group realization.

Let G be a finite group acting on a graph X cellularly. As shown in Chapter 2.3, we can obtain a graph of groups $(\mathcal{G}, Y)$ from this action. In the first section of this chapter, we show that, under some conditions we put for the G-action on X, the infinite group $\pi(\mathcal{G}, Y) = \pi(EG \times_G X)$ realizes the fusion and its cohomology.

In the second section of this chapter we give the first example of this theorem. We show that when G has p-rank 2 and X is the realization of the elementary abelian poset of G where G act on by conjugation, the infinite group $\pi(\mathcal{G}, Y) = \pi(EG \times_G X)$ realizes the fusion. At the end of this chapter, we consider a known model as an example of the first theorem of this chapter.

5.1 From Posets to Graph of Groups

We denote the classifying space of G by BG with contractible universal cover EG. We assume all the spaces in this thesis are CW-complexes in order to have well-defined structures as we note in Remark 2.1.4. So BG is a $K(G,1)$-space.

By a G-*graph*, we mean a graph with a G-action on it such that G acts cellularly without inversion. If X is a G-graph, we can talk about the *Borel construction* $EG \times_G X$. For a G-graph X, we say X is G-*connected* if the quotient graph X/G is connected. In this section, we work on 1-dimensional G-connected graphs. When X is a G-poset, by the Borel construction $EG \times_G X$, we mean the 1-dimensional graph realization of X with a G-action. If X is a poset consisting of subgroups of G, then G-action is the conjugation. For example, if G is a finite group and X is the poset of elementary abelian subgroups of G, then we consider the corresponding 1-dimensional G-graph in the notation $EG \times_G X$.

Theorem 5.1.1. *Let G be a finite group with Sylow p-subgroup S. Assume G acts on a connected graph X so that S fixes at least one vertex and $H_1(X;\mathbb{F}_p)$ is projective $\mathbb{F}_pG$-module. If the embedding of S into $\Gamma := \pi_1(EG \times_G X)$ is a Sylow p-subgroup so that $\mathcal{F}_S(\Gamma) = \mathcal{F}_S(G)$ then we determine the cohomology of the fusion system by*

$$H^*(\mathcal{F}) = H^*(\Gamma;\mathbb{F}_p).$$

Proof. Define $f : EG \times X \to X$ by sending $(a,x) \mapsto x$. We consider a G-action on $EG \times X$ by $g(a,x) = (g^{-1}a, gx)$. Then the homotopy equivalence f preserves G-actions. f induces a continuous map

$$g : EG \times_G X \to X/G$$

by dividing G-action. Consider the graph $Y := X/G$. Define $X_v = g^{-1}(v)$ for any vertex of Y. Define $X_e = g^{-1}(e)$ for any edge e in Y.

Fix any edge e and vertex v so that v belongs to e. The deformation retract from X_e to $X_e \cap X_v$ gives a continuous map $f_e : X_e \to X_v$ which is injective on homotopy groups. Define a graph of groups $(\mathcal{G}, Y)$ so that the vertex groups are $G_v : \pi(X_v)$ and the edge groups are $G_e := \pi(X_e)$ and the monomorpisms $\phi_e : G_e \to G_v$ are induced by the f_e maps.

For a vertex $v \in Y$, let $\bar{v}$ be a lift in X. Consider $f^{-1}(orb(\bar{v}))$, the preimage of the G-orbit of $\bar{v}$ under the map f. The space $f^{-1}(orb(\bar{v}))$ contains $|orb(\bar{v})|$ copies of EG. Then $g^{-1}(v) = f^{-1}(orb(\bar{v}))/G$ is homotopic to $EG/stab(\bar{v})$ where $stab(\bar{v})$ is the stabilizer of $\bar{v}$. Since the contractible space EG is the universal cover of $EG/stab(\bar{v})$, X_v is a classifying space of $stab(\bar{v})$. Then the fundamental group of the graph of groups $(\mathcal{G}, Y)$ is $\Gamma = EG \times_G X$. The graph of groups has vertex groups G_v isomorphic to $stab(\bar{v})$ and edge groups isomorphic to stabilizers of their lifts.

Consider the map $s : EG \times X \to EG$ defined by sending $(a, x) \to a$. Similar to above, we divide by G-action. We obtain an induced map $t : EG \times_G X \to BG$ which is surjective. Then we obtain a surjective homomorphism χ from $\pi(EG \times_G X) \cong \pi(\mathcal{G}Y)$ to $\pi(BG) = G$. In fact,

$$\chi : \pi(\mathcal{G}, Y) \to G$$

is a storing homomorphism because t is injective on X_v and X_e spaces. Then from the proof of Theorem 4.1.2, we have

$$H^{*-1}(G, H^1(X; \mathbb{F}_p)) \oplus H^*(G, \mathbb{F}_p) \cong H^*(\Gamma; \mathbb{F}_p)$$

because the graph X can be considered as a T/F appears in the proof where $F := \ker(\chi)$ and T is obtained by developing $\pi(\mathcal{G}, Y)$-action on a tree. Since $H^1(X; \mathbb{F}_p)$ is projective G-module, we have $H^{*-1}(G, H^1(X; \mathbb{F}_p)) = 0$. Also, we have $H^*(\mathcal{F}) \cong H^*(G, \mathbb{F}_p)$ because G is finite and $\mathcal{F} = \mathcal{F}_S(G)$. Hence,

$$H^*(\mathcal{F}) \cong H^*(\Gamma; \mathbb{F}_p).$$

□

Remark 5.1.2. As we see in the proof of the last theorem, for a G-poset X, we consider $EG \times_G X$ as a fundamental group of some graph of groups. By the way, we are able to prove many statements in the borel product language by translating them into the language of the theory of graph of groups. Most of the proof of this chapter has that idea.

Sometimes infinite groups do not have Sylow p-subgroups. For example, the amalgam $\pi = C_2 * C_2$ has no Sylow 2-subgroup. To talk about fusion systems and realization by infinite groups, we first step we need to prove is that infinite group does have Sylow p-subgroup. The next proposition is very useful to prove a given infinite group has a Sylow p-subgroup when the infinite group is the fundamental group of some graph of groups.

Proposition 5.1.3 (Libman-Seeliger [3]). *Let $(\mathcal{G}, Y)$ be a graph of groups and suppose that*

i-) The groups G_v and G_e contain Sylow p-subgroups P_v and P_e for every vertex v in Y and edge e in Y.

ii-) There exists a vertex v_0 such that for any other vertex u of Y there exists a path(directed, without loops) $y_1, y_2, ..., y_n$ from v_0 to u such that for any i the map $G_{y_i} \xrightarrow{a \mapsto a^{y_i}} G_{\partial_1(y_i)}$ carries P_{y_i} onto a Sylow p-subgroup $G_{\partial_1(y_i)}$.

Then, $S := P_{v_0}$ is a Sylow p-subgroup of $\pi = \pi(\mathcal{G}, Y)$.

In the sense of Remark 5.1.2, we translate the last proposition into the language of Borel construction spaces which we need for this chapter.

Corollary 5.1.4. *Let G be finite group with Sylow p-subgroup S and X be a G-connected G-graph. If there exists $v_0 \in X$ such that*

i-) S fixes v_0

ii-) for any vertex $v \in X$ there exists a path(directed without loops) $y_1, y_2, ..., y_n$ from v to gv_0 for some $g \in G$ such that for any $i = 1, 2, ..., n$ the inclusion of the stabilizer $Stab_G(y_i)$ to the stabilizer $Stab_G(\partial_1(y_i))$ carries a Sylow p-subgroup of $Stab_G(y_i)$ onto a Sylow p-subgroup of $Stab_G(\partial_1(y_i))$.

Then, $\pi(EG \times_G X)$ has a Sylow p-subgroup isomorphic to S.

Proof. We consider the corresponding graph of groups $(\mathcal{G}, Y)$ as we do in the proof of Theorem 5.1.1. We have $\pi = \pi(\mathcal{G}, Y)$ where $Y = X/G$ and $(\mathcal{G}, Y)$ has vertex groups G_v for $v \in Y$ so that $G_v = \mathrm{Stab}_G(\widetilde{v})$ where $\widetilde{v} \in X$ is a lift of v. Now, we use Proposition 5.1.3 to conclude the proof. $\square$

5.2 Poset of Elementary Abelian Subgroups

In this section, we prove that the infinite group model $\Gamma := \pi_1(EG \times_G X)$ realizes the fusion of G on S, where X be the poset of elementary abelian subgroups of S. Moreover, it gives exactly the same $\mathbb{F}_p$ cohomology. Here, we work with p-rank 2 groups. By p-rank, we mean the maximum number n so that the group has an elementary abelian p-subgroup $C_p \times C_p \cdots \times C_p$ of rank n. We denote by $\mathtt{rank}_p(G)$. For a finite group G and its Sylow p-subgroup S, we have $\mathtt{rank}_p(G) = \mathtt{rank}_p(S)$.

In this section, we will need the next theorem from theory posets.

Theorem 5.2.1 (Quillen [23])**.** *Let X, Y be posets and $f, g : X \to Y$ be poset maps. If for any $x \in X$ we have $f(x) \leqslant g(x)$, then $|f| \cong |g|$.*

Now we start by writing a theorem for the fusion of Γ and we will continue its homology calculations.

Proposition 5.2.2. *Assume G is a finite group with a Sylow p-subgroup S and $\mathtt{rank}_p(G) = 2$. Let X be the poset of elementary abelian subgroups of S. Then $\Gamma := \pi_1(EG \times_G X)$ realizes the fusion of G on S(i.e. $\mathcal{F}_S(\Gamma) = \mathcal{F}_S(G)$).*

Proof. Let C_i's and E_i's be the elementary abelian subgroups of S of order p and p^2, respectively.

Without loss of generality, we can assume $C_1 \subset Z(S)$ because $Z(S)$ (non-trivial p-group) contains a subgroup of order p.

1-) Let us write Γ as the fundamental group of the graph of groups. From the poset X, we choose one C_i in each G-orbit such that $N_S(C_i)$ is Sylow in $N_G(C_i)$ and one E_i in each G-orbit such that $N_S(E_i)$ is Sylow in $N_G(E_i)$.

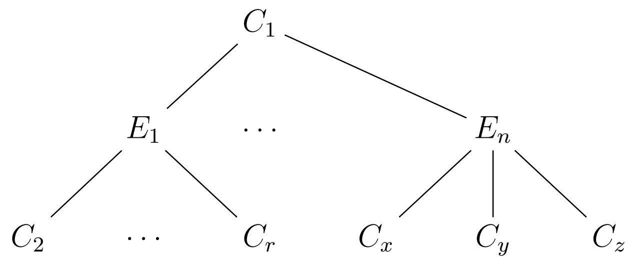


Figure 5.1: Quotient Poset X/G

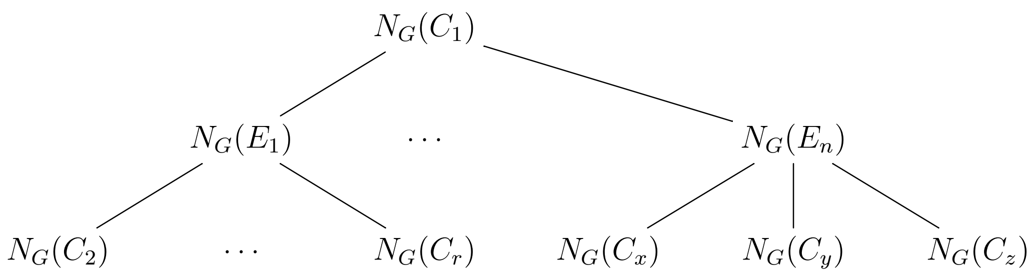


Figure 5.2: Graph of Groups

Now our graph of groups consists of the graph obtained by the quotient X/G and vertex groups $N_G(C_i)$'s and $N_G(E_i)$'s for the indices as chosen above for vertices of X/G, and edge groups formed by the intersection of vertex groups.

Since $C_1 \subset Z(S)$, we have $C_1 \subset E_i$ for all i because otherwise the group C_1E_i would be an elementary abelian group of order p^3, yielding a contradiction with $\texttt{rank}_p(G) = 2$. Also, we have that for $i \neq 1$, any group C_i contained in a unique elementary abelian subgroup $E_j := C_1C_i$ for some j. Hence, the graph of groups has a shape in the figure.

2-) Γ has a Sylow p-subgroup isomorphic to S.

The vertex group associated to C_1 is $N_G(C_1)$. Since $C_1 \subset Z(S)$, we have $S \subset N_G(C_1)$. We argue that the Sylow p-subgroup of this vertex is a Sylow p-subgroup of Γ. Here, we use the Proposition 3.3 in [3] in order to show Γ has a Sylow p-subgroup isomorphic to S. We take that vertex as a reference vertex group mentioned in the proposition.

Now, take a vertex E_j. We have an edge between C_1 and E_j so that the edge group and monomorphisms as follows:

$$N_G(C_1) \hookleftarrow (N_G(C_1) \bigcap N_G(E_j)) \hookrightarrow N_G(E_j).$$

$N_G(E_i)$ has Sylow $N_S(E_i)$ by the choice we have done before. Since $(N_G(C_1) \bigcap N_G(E_j))$ contains $N_S(E_i)$ as a Sylow p-subgroup, we say that the edge carries its Sylow onto Sylow subgroup of $N_G(E_j)$.

Second, we consider the C_i vertices for $i \neq 1$. We have some E_j containing C_i and the reach C_1 from C_i along E_j via two edges. In fact, we have $E_j = C_1 \times C_i$. So the path from C_i to C_1 has shape:

$$N_G(C_1) \hookleftarrow (N_G(C_1) \bigcap N_G(E_j)) \hookrightarrow N_G(E_j) \hookleftarrow (N_G(C_1 \times C_i) \bigcap N_G(C_i)) \hookrightarrow N_G(C_i)$$

Here, from previous paragraph we have that the first edge carries its Sylow subgroup onto Sylow subgroup of $N_G(E_j)$. For second edge we have $N_G(C_1 \times C_i)$ has Sylow $N_S(C_1 \times C_i) = N_S(C_i)$ because $C_1 \subset Z(S)$ implies that S normalizes C_1 . Hence, the second edge also carries its Sylow onto Sylow subgroup of $N_G(C_i)$. By the Proposition 3.3 in [3], we say Γ has a Sylow p-subgroup isomorphic to S.

3-) $\mathcal{F}_S(\Gamma) \supset \mathcal{F}_S(G)$. Clearly, these categories have the same objects. We need to show that for any morphism f in $\mathcal{F}_S(G)$ we have that f is a morphism in $\mathcal{F}_S(\Gamma)$. Take any $P, Q \in obj(\mathcal{F}_S(G))$ and $f \in Mor_{\mathcal{F}_S(G)}(P, Q)$. Since we have finite group fusion, f corresponds a conjugation morphism for some $g \in G$. Take any C conjugation family. Then there exists $P = P_0, P_1, P_2, ..., P_n = Q$ subgroups of S and $Q_1, Q_2, ..., Q_n \in C$ and $g_i \in N_G(Q_i)$ such that

i-) $g_n g_{n-1} ... g_1 = g$

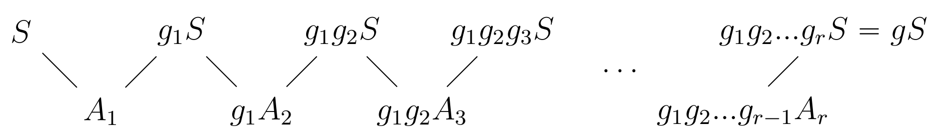
ii-) $g_i(P_{i-1})g_i^{-1} = P_i$ for $i \in \{1, 2, ..., n\}$

iii-) P_{i-1} and P_i are contained in Q_i.

It is enough to show the conjugation $c_{g_1} : P_0 \to P_1$ is contained in $\mathcal{F}_S(\Gamma)$. The others can be done similarly. Since Q_1 is p-group, the center $Z(Q_1)$ is not trivial. Then $\Omega_1(Z(Q_1)) \neq 1$ where, for a p-group A, $\Omega_1(A)$ denotes the elements of A of order p. Since g_1 normalizes Q_1 and $\Omega_1(Z(Q_1))$ characteristic in Q_1, we have that g_1 normalizes $\Omega_1(Z(Q_1))$, that is, $g_1 \in N_G(\Omega_1(Z(Q_1)))$. Also, $Q_1 \subset N_G(\Omega_1(Z(Q_1)))$ because characteristic groups are normal. Hence, $N_G(\Omega_1(Z(Q_1)))$ contains the action $c_{g_1} : P_0 \to P_1$. Since $\Omega_1(Z(Q_1))$ elementary abelian p-subgroup of S, it is included in the poset of elementary abelian p-groups. So $N_G(\Omega_1(Z(Q_1)))$ appears as a vertex group in our graph of groups. Then Γ contains the action $c_{g_1} : P_0 \to P_1$.

4-) $\mathcal{F}_S(\Gamma) \subset \mathcal{F}_S(G)$.

Take any $g \in \Gamma$ such that $P, Q \subset S$ and $gPg^{-1} = Q$. We know that there exists vertex groups $A_1, A_2, ..., A_r$ in Γ and some $g_i \in A_i$ such that $g = g_1g_2g_3...g_r$ is the unique reduced word representation of g. Now, consider the corresponding action of Γ on a tree. We have a path between S and gS which can be deducted by writing $g = g_1g_2...g_r$.



Since $Q \subset S$, Q fixes the vertex S in the tree. Also, $Q = gPg^{-1}$ fixes gS. As Q fixes initial and final vertices of the path, Q fixes all vertices in the path. Here, Q fixes $g_1g_2...g_iS$ implies that $g_1g_2...g_iQ(g_1g_2...g_i)^{-1} \subset S$. Then for each step we have

— $g_1g_2...g_iQ(g_1g_2...g_i)^{-1}$ and $g_1g_2...g_{i+1}Q(g_1g_2...g_{i+1})^{-1}$ contained in S and,

— The conjugation action of g_{i+1} corresponds to conjugation action of some element in G.

Since each step realized by an action in $\mathcal{F}_S(G)$, the total action $c_g : P \to Q$ in $\mathcal{F}_S(\Gamma)$ corresponds an action in $\mathcal{F}_S(G)$. Hence, $\mathcal{F}_S(\Gamma) \subset \mathcal{F}_S(G)$, concluding the proof of $\mathcal{F}_S(\Gamma) = \mathcal{F}_S(G)$.

□

Lemma 5.2.3. *Let G be a finite group and P be a p-subgroup of G. We consider that P acts on $\mathcal{A}_p(G)$, the poset of nontrivial elementary abelian p-subgroups in G. Then the fixed point space $|\mathcal{A}_p(G)|^P$ is contractible.*

Proof. Let $\mathcal{S}_p(G)$ be the poset of all non-trivial p-subgroups of G.

Denote $\mathcal{S}_p(G)^P$ to be the fixed elements of the poset $\mathcal{S}_p(G)$ under the action of P. $\mathcal{A}_p(G)^P$ defined similarly.

Define $f : \mathcal{A}_p(G)^P \to \mathcal{S}_p(G)^P$ by sending $E \in \mathcal{A}_p(G)^P \mapsto E \in \mathcal{S}_p(G)^P$.

For any $Q \in \mathcal{S}_p(G)$, we have $f|_Q = \{E \in \mathcal{A}_p(G)^P \mid E \leqslant Q\} = \mathcal{A}_p(Q)^P$.

To show $|\mathcal{A}_p(Q)^P|$ is contractible, we consider the following poset maps.

$id : \mathcal{A}_p(Q)^P \to \mathcal{A}_p(Q)^P$ by sending $E \mapsto E$,

$g : \mathcal{A}_p(Q)^P \to \mathcal{A}_p(Q)^P$ by sending $E \mapsto EZ$,

$c : \mathcal{A}_p(Q)^P \to \mathcal{A}_p(Q)^P$ by sending $E \mapsto Z$ where Z is an elementary abelian p-subgroup of the center of Q.

By using Theorem 5.2.1, we get $id \cong g$ and $c \cong g$. So the identity is homotopic to a constant map. Hence, $|f|_Q| = |\mathcal{A}_p(Q)^P|$ is contractible.

Since for any $Q \in \mathcal{S}_p(G)$, $|f|_Q|$ is contractible, we say f is homotopy equivalence (i.e. $|\mathcal{A}_p(G)|^P = |\mathcal{S}_p(G)|^P$).

So, the rest is to show $\mathcal{S}_p(G)^P$ is contractible. For $Q \in \mathcal{S}_p(G)^P$, we have that P normalizes Q (i.e. $P \leqslant N_G(Q)$). Then PQ forms a p-group in $\mathcal{S}_p(G)^P$. To show contractibility, we again define homotopic poset maps.

$id_2 : \mathcal{S}_p(G)^P \to \mathcal{S}_p(G)^P$ by sending $Q \mapsto Q$,

$h : \mathcal{S}_p(G)^P \to \mathcal{S}_p(G)^P$ by sending $Q \mapsto PQ$,

$c_2 : \mathcal{S}_p(G)^P \to \mathcal{S}_p(G)^P$ by sending $Q \mapsto P$.

Here, we have $id_2 \cong h \cong c_2$ by Theorem 5.2.1. Since the identity is homotopic to a constant map, we say $|\mathcal{S}_p(G)^P|$ is contractible.

□

Proposition 5.2.4. $\bigcup_{1 \neq H \leqslant S} (|\mathcal{A}_p(G)|)^H$ *is contractible.*

Proof. Let P be the poset with elements $(|\mathcal{A}_p(G)|)^H$ for H non-trivial subgroup of S and relation as the inclusion. Since the poset has a minimal element, we say it is contractible. From Lemma 5.2.3, we have that the sets in the poset are contractible. So the union is contractible. □

Proposition 5.2.5. *Let* $X = |\mathcal{A}_p(G)|$. *Then* $H_1(X, \mathbb{F}_p)$ *is a projective* $\mathbb{F}_pG$*-module.*

Proof. Let $C_1 \to C_0$ be the chain complex of X. Let $D_i \subset C_i$ such that $D_1 \to D_0$ corresponds to the chain complex of $\bigcup_{1 \neq H \leqslant S} X^H$.

Then $C_i = D_i \oplus P_i$ where P_i's are free $\mathbb{F}_pS$-modules because S has a free action on the set $X - \bigcup_{1 \neq H \leqslant S} X^H$.

Now we have a short exact sequence

$$0 \to P_i \to C_i \to D_i \to 0 \qquad \text{for } i \in \{1, 2\}.$$

And the corresponding long exact sequence is

$$0 \to H_1(P) \to H_1(C) \to H_1(D) \to H_0(P) \to H_0(C) \to H_0(D) \to 0.$$

By Proposition 5.2.4, D is acyclic. Writing $H_1(D) = 0$ in the long exact sequence, we get $H_1(P) \cong H_1(C)$. So we have the following short exact sequence

$$0 \to H_1(C) \to P_1 \to P_0 \to 0.$$

The sequence splits because P_i's are free $\mathbb{F}_pS$-module. Hence, $H_1(X, \mathbb{F}_p) = H_1(C)$ is a direct summand of P_1, which means it is a projective $\mathbb{F}_pS$-module. Since S

is a Sylow subgroup of G, we say that projective $\mathbb{F}_pS$-modules are projective $\mathbb{F}_pG$-modules.

$\square$

Theorem 5.2.6. *Assume G is a finite group with a Sylow p-subgroup S and $\mathtt{rank}_p(G) = 2$. Let X be the poset of elementary abelian subgroups of S. Then $\Gamma := \pi_1(EG \times_G X)$ realizes the fusion of G on S(i.e. $\mathcal{F}_S(\Gamma) = \mathcal{F}_S(G)$). Moreover, their $\mathbb{F}_p$-cohomologies also fits, i.e. $H^*(\Gamma, \mathbb{F}_p) = H^*(G, \mathbb{F}_p)$.*

Proof. We have the first part from Proposition 5.2.2. Let us show the cohomology equivalence. We have from the proof of Theorem 4.1.2,

$$H^{*-1}(G, H^1(X; R)) \oplus H^*(G, R) \cong H^*(\Gamma, R).$$

$H_1(X, \mathbb{F}_p)$ is a projective $\mathbb{F}_pG$-module as shown in Proposition 5.2.5. Then

$$H^{*-1}(G, H^1(X; R)) = 0.$$

Hence we get

$$H^*(\Gamma, \mathbb{F}_p) \cong H^*(G, \mathbb{F}_p).$$

$\square$

5.3 Poset of p-radical p-centric p-subgroups of G

Let G be a finite group. In this section we work on the graph

$$X = \{P \subset G \mid P \text{ is } p\text{-radical } p\text{-centric } p\text{-subgroup of } G\}.$$

We will show that $\pi := \pi_1(EG \times_G X)$ realizes the fusion $\mathcal{F} = \mathcal{F}_S(G)$. Let $(\mathcal{G}, Y)$ be the corresponding graph of groups. We take $Y = X/G$ and for $v \in Y$, $G_v = Stab_G(\widetilde{v}) = N_G(P_{\widetilde{v}})$ where $\widetilde{v} \in X$. Then, $(\mathcal{G}, Y)$ is the same graph of groups as we talk about in Example 3.3.2. Then, $\pi(EG \times_G X) = \pi(\mathcal{G}, Y)$ realizes the fusion. By using Theorem 5.1.1, we get the next theorem.

Theorem 5.3.1. *Let G be a finite group with a Sylow p-subgroup S. If X is the poset of p-radical p-centric p-subgroups of G, then $\pi = \pi(EG \times_G X)$ realizes the fusion $\mathcal{F} = \mathcal{F}_S(G)$. Moreover, if $H_1(X; \mathbb{F}_p)$ is projective G-module, then $H^*(\pi; \mathbb{F}_p) = H^*(\mathcal{F})$.*

Chapter 6

On the Signalizer Functors

The theory of signalizer functors and linking systems are developed to understand the fusion systems and its topological properties better. In this Chapter, we give some background on this theory. Then, we will state and prove the main theorem of the paper [3]. This theorem shows that the $\mathbb{F}_p$ cohomology of an infinite group realizing a fusion $\mathcal{F}$ is a direct sum of the cohomology of the fusion system and the kernel of the restriction map under some conditions. Our first main theorem gives a better result which gives a formula for the difference but it was only for finite fusions. The theory and results in this chapter are developed for any saturated fusion. At the end of this chapter, we give a group theoretical proof of the proposition which is used in some results of our paper [4].

6.1 A Theorem of Libman-Seeliger

Definition 6.1.1. Let π be a group with Sylow p-subgroup S and $\mathcal{F} = \mathcal{F}_S(\pi)$. The *transporter system* $\mathcal{T}_S(\pi)$ is a category with objects as the subgroups of S and morphism sets are $\mathcal{T}_S(\pi)(P,Q) := N_\pi(P,Q) = \{g \in \pi | gPg^{-1} \leqslant Q\}$.X

Clearly, we have natural functor $\mathcal{T}_S(\pi) \to \mathcal{F}_S(\pi)$ which is identity on objects and takes quotient by the action of $C_\pi(P)$ on the morphism sets. We usually refer to this standard functor in this chapter by just writing $\mathcal{T}_S(\pi) \to \mathcal{F}_S(\pi)$.

Remark 6.1.2. By $\mathcal{T}_S^c(\pi)$, we mean the full subcategory of $\mathcal{T}_S^c(\pi)$ whose objects are $\mathcal{F}$-centric subgroups of S. Similarly, $\mathcal{F}^c$ is the full subcategory of $\mathcal{F}$ with objects as all the $\mathcal{F}$-centric subgroups of $\mathcal{F}$.

Definition 6.1.3. An associated *centric linking system* $\mathcal{L}$ on a saturated fusion system $\mathcal{F} = \mathcal{F}_S$ is a category such that

i-) $\mathrm{Obj}(\mathcal{L})$ is the set of $\mathcal{F}$-centric subgroups of $\mathcal{F}$

ii-) It is equipped with a surjective functor $\pi : \mathcal{L} \to \mathcal{F}^c$ and an injective functor $\delta : \mathcal{T}_S^c(S) \to \mathcal{L}$ both induce identity on object sets.

iii-) The image of $Z(P)$ under $\delta : N_S(P) \to \mathrm{Aut}_{\mathcal{L}}(P)$ acts freely on $\mathcal{L}(P,Q)$ and $\mathcal{F}(P,Q) \cong \mathcal{L}(P,Q)/Z(P)$

iv-) For any $P, Q \in \mathcal{F}^c$, for any $g \in N_S(P,Q)$ we have $\pi(\delta(g))$ is the conjugation by g on P.

v-) For any $f \in \mathcal{L}(P,Q)$, for any $g \in P \leqslant N_S(P) = \mathrm{Aut}_{\mathcal{T}_S^c}(P)$ the following square commutes

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

Since centric linking systems defined on fusion systems and fusion systems defined on finite p-groups, the triple $(S, \mathcal{F}, \mathcal{L})$ is called *p-local finite group.* More formally,

Definition 6.1.4. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$ of a saturated fusion system on S together with an associated centric linking system.

Definition 6.1.5. Let $\mathcal{F} = \mathcal{F}_S(\pi)$. A *signalizer functor* on π is a functor $\theta : \mathcal{T}_S^c \to \mathrm{Grp}$ sending $P \mapsto \theta(P)$ for $\mathcal{F}$-centric subgroup P such that i-) $\theta(P) \bigcap Z(P) = 1$ and $\theta(P)Z(P) = C_\pi(P)$

ii-) If $gPg^{-1} \leqslant Q$, then $\theta(Q) \leqslant g\theta(P)g^{-1}$.

Signalizer functors play a significant role in the p-local group theory. For example, the next lemma states away of obtaining a centric linking system from a signalizer functor.

Lemma 6.1.6. *Let $\mathcal{F} = \mathcal{F}_S(\pi)$ and θ be a signalizer functor on π. Then, the category $\mathcal{L}_\theta$ defined by $\mathcal{L}_\theta(P,Q) = N_\pi(P,Q)/\theta(P)$ is a centric linking system.*

Proof. We define $Obj(\mathcal{L}) := Obj(\mathcal{F}^c)$. The functor $\pi : \mathcal{L} \to \mathcal{F}^c$ is the identity on objects and for morphisms, π sends $\mathcal{L}(P,Q) = N_\pi(P,Q)/\theta(P)$ to $\mathcal{F}^c(P,Q)$ surjectively.

We define $\delta : \mathcal{T}_S^c(S) \to \mathcal{L}$ so that it sends each object to its copy in $\mathcal{L}$. For $P \in \mathcal{T}_S^c$, δ_P sends $\mathrm{Aut}_{\mathcal{T}_S^c}(P) = N_S(P)$ to $\mathrm{Aut}_{\mathcal{L}}(P) = N_\pi(P)/\theta(P)$ by sending $x \mapsto x\theta(P)/\theta(P)$. We need to show the kernel $\ker(\delta_P) = N_S(P) \bigcap \theta(P)$ is trivial. Since P is $\mathcal{F}$-centric, $N_S(P) \bigcap C_\pi(P) = Z(P)$. Since $\theta(P) \subset C_\pi(P)$, $N_S(P) \bigcap \theta(P) \subset N_S(P) \bigcap C_\pi(P) = Z(P)$ but $\theta(P) \bigcap Z(P) = 1$. Hence, $\ker(\delta_P) = 1$. So we have done with conditions (i) and (ii) in the definition of the centric linking system.

For (iii), $\delta_P(Z(P)) = C_\pi(P)/\theta(P)$ acts on $\mathcal{L}(P,Q) = N_\pi(P,Q)/\theta(P)$ by composition, freely. $\mathcal{L}(P,Q)/\theta_P(Z(P)) \cong N_\pi(P,Q)/C_\pi(P,Q) \cong \mathcal{F}(P,Q)$

For (iv), $\pi(\delta_P(g)) = \pi(x\theta(P)/\theta(P))$ is the conjugation by g on P.

For (v), let $f \in \mathcal{L}(P,Q)$. There exists $n\theta(P) \in N_\pi(P,Q)/\theta(P)$ such that f sends $x \mapsto nxn^{-1}$. We need to show the square commutes

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

where $\delta_P(g)$ is the conjugation by g and $\delta_Q(\pi(f)(g)) = \delta_Q(ngn^{-1})$ is the conjugation by ngn^{-1}.

For $x \in P$, we have $\delta_Q(\pi(f)(g)) \circ f(x) = \delta_Q(\pi(f)(g))(nxn^{-1}) = ngn^{-1}(nxn^{-1})(ngn^{-1})^{-1} = ngxg^{-1}n^{-1} = f(gxg^{-1}) = f \circ \delta_P(g)(x)$ concluding the proof. □

Definition 6.1.7. Let S be a p-group, $F : BS \to X$ be a map. f *gives rise to a fusion system* $\mathcal{F}_S(f)$ on S whose objects are the subgroups of S and a monomorphism $\phi : P \to Q \in \mathcal{F}(f)$ if and only if the composition $BP \xrightarrow{B\phi} BS \xrightarrow{f} X$ is homotopic to the composition $BP \xrightarrow{Bi} BS \xrightarrow{f} X$.

For a map $g : X \to Y$, we have $\mathcal{F}_S(f) \subseteq \mathcal{F}_S(g \circ f)$.

Definition 6.1.8. Let S be a p-group, X be a space and $f : BS \to X$ be a map. The category $\mathcal{L}_S(f)$ has the same objects as $\mathcal{F} = \mathcal{F}_S(f)$ with morphisms $\mathcal{L}_S(f)(P, Q) := \{(\phi, [H]) | \phi \in \mathcal{F}(P, Q)$ and $[H]$ is the homotopy class of a path in H in $\mathrm{map}(BP, X)$ from $BP \xrightarrow{B\phi} BS \xrightarrow{f} X$ to $BP \xrightarrow{Bi} BS \xrightarrow{f} X\}$.

For inclusion homomorphisms, we use i and Bi always denotes a map inducing inclusion on homotopy. For a homomorphism $\phi : P \to Q$, we use $B\phi : BP \to BQ$ for a map inducing that homomorphism on homotopy. For paths k and l, by kl we mean their composition path.

Lemma 6.1.9. *Let S be a Sylow p-subgroup of π. Assume $Bi : BS \to B\pi$ induces the inclusion $i : S \hookrightarrow \pi$. Then $\mathcal{F}_S(Bi) = \mathcal{F}_S(\pi)$ and $\mathcal{L}_S(Bi) = \mathcal{T}_S(\pi)$.*

Proof. First, let show $\mathcal{F}_S(Bi) \subseteq \mathcal{F}_S(\pi)$. Take any $\phi : P \to Q \in \mathcal{F}_S(Bi)$. Then, the composition $BP \xrightarrow{Bi} BS \xrightarrow{Bi} B\pi$ is homotopic to the composition $BP \xrightarrow{B\phi} BS \xrightarrow{Bi} B\pi$. Let $H : BP \times [0, 1] \to B\pi$ be that homotopy.

Fix $x_0 \in BP$. Let $x_1 := H(x_0, 0)$. Without loss of generality, we can assume $x_1 = B\phi(x_0)$. Then, $H(x_0, 1) = x_1$. Let ρ be the path given by $\rho : [0, 1] \to B\pi$ sending $t \mapsto H(x_0, t)$. Since $\rho(0) = \rho(1) = x_1$, ρ is a loop around x_1 in $B\pi$.

Now, for any loop l in BP, the image of l under $BP \xrightarrow{Bi} B\pi$ is homotopic to the image of the loop l under the map $BP \xrightarrow{B\phi} B\pi$. That homotopy sends l to $\rho^{-1}l\rho$. This means $\phi : P \to Q$ sends $p \mapsto g^{-1}pg$ where $g \in \pi$ corresponds $\rho \in B\pi$. Hence, $\phi \in \mathcal{F}_S(\pi)$. We have done with $\mathcal{F}_S(Bi) \subseteq \mathcal{F}_S(\pi)$.

Second, $\mathcal{F}_S(Bi) \supseteq \mathcal{F}_S(\pi)$ is true for similar reasons. Take any $c_g : P \to Q$ in $\mathcal{F}_S(\pi)$ where $g \in \pi$. Consider the loop ρ corresponding $g \in \pi$. We construct homotopy between the composition $BP \xrightarrow{Bi} BS \xrightarrow{Bi} B\pi$ and the composition $BP \xrightarrow{Bc_g} BS \xrightarrow{Bi} B\pi$ by moving everything around the loop ρ.

We continue with the second isomorphism. By definition, $\mathcal{L}_S(Bi)(P,Q) = \{(\phi,[H]) | \phi \in \mathcal{F}(P,Q)$ and $[H]$ is the homotopy class of a path in H in $\mathrm{map}(BP,X)$ from $BP \xrightarrow{B\phi} BS \xrightarrow{Bi} X$ to $BP \xrightarrow{Bi} BS \xrightarrow{Bi} X\}$

We define $\Psi : \mathcal{T}_S(\pi) \to \mathcal{L}_S(Bi)$ identity on objects and sending $g \in N_\pi(P,Q)$ to $\Psi(g) = (c_g,[H])$ where $c_g : P \to Q$ is a conjugation and H is the homotopy on $B\pi$ shifting BP around the loop l corresponding g. Ψ is an isomorphism and for any $\phi \in \mathcal{F}(P,Q)$, there exist the number of the order of $C_\pi(P)$ pairs of the form $(\phi,[H]) \in L\mathcal{L}_S(Bi)(P,Q)$ where each $[H]$ corresponds a rotation around a loop corresponding an element in $C_\pi(P)$. □

Theorem 6.1.10 (Libman-Seeliger, [3])**.** *Fix a p-local finite group $(S,\mathcal{F},\mathcal{L})$ and let π be a group which contains S as a Sylow p-subgroup. Assume that $\mathcal{F} \subseteq \mathcal{F}_S(\pi)$ and that $\exists$ a map $f : B\pi \to |\mathcal{L}|_p^\wedge$ whose restriction to $BS \subseteq B\pi$ is homotopic to the natural map $\theta : BS \to |\mathcal{L}|_p^\wedge$.*

Then we have

i-) $\mathcal{F} = \mathcal{F}_S(\pi)$

ii-) There exists signalizer functor Θ on π such that $\mathcal{L} = \mathcal{L}_\theta$

iii-) The map $res_S^\pi : H^*(\pi,\mathbb{F}_p) \to H^*(S,\mathbb{F}_p)$ splits and has image isomorphic to $H^*(\mathcal{F};\mathbb{F}_p)$ that gives

$$H^*(\pi;\mathbb{F}_p) \cong H^*(\mathcal{F};\mathbb{F}_p) \oplus \ker(res_S^\pi).$$

Proof. For (i): From [24], we have $\mathcal{F}_S(\theta) = \mathcal{F}$. By the Lemma 6.1.9, we have $\mathcal{F}_S(\pi) = \mathcal{F}(Bi)$. Then, $\mathcal{F}_S(\pi) = \mathcal{F}_S(Bi) \subseteq \mathcal{F}(f \circ Bi) = \mathcal{F}(\theta) = \mathcal{F}$. Hence, $\mathcal{F} = \mathcal{F}_S(\pi)$.

For (ii): From [24], we have $\mathcal{L}^c_S(f \circ Bi) = \mathcal{L}^c_S(\theta) = \mathcal{L}$ because $f \circ Bi \simeq \theta$. From the Lemma 6.1.9, we have $\mathcal{L}^c_S(Bi) = \mathcal{T}^c_S(\pi)$. From $BS \xrightarrow{Bi} B\pi \xrightarrow{f} |\mathcal{L}|^\wedge_p$, f induces a functor $\mathcal{L}^c_S(Bi) = \mathcal{T}^c_S \to \mathcal{L}^c_S(\theta) = \mathcal{L}$ such that the diagram commutes

$$\begin{array}{ccc} \mathcal{T}^c_S(\pi) & \xrightarrow{\rho} & \mathcal{L} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{F}^c & \overset{f}{=\!=\!=} & \mathcal{F}^c. \end{array}$$

We claim that ρ is surjective. Set $\mathcal{T} = \mathcal{T}^c_S(\pi)$. Consider $\mathcal{L}(P,Q)$. Since the composition $\rho \circ \pi$ surjects $\mathcal{F}(P,Q)$, if ρ surjects $Z(P) = \ker(\mathcal{L}(P,Q) \xrightarrow{\pi} \mathcal{F}^c)$ then ρ surjects $\mathcal{L}(P,Q)$. For $Bi^S_P : BP \to BS$, we will show that the composition

$$\mathrm{map}(BP, BS)_{Bi^S_P} \xrightarrow{Bi_*} \mathrm{map}(BP, B\pi)_{Bi^\pi_P} \xrightarrow{f_*} \mathrm{map}(BP, |\mathcal{L}|^\wedge_p) \qquad (6.1)$$

is homotopy equivalence. From page 136 in [12], we have $\mathrm{map}(BP, BS)_{Bi^S_P} \cong BC_S(P)$. Since P is $\mathcal{F}$-centric, we have $C_S(P) = Z(P)$. So, $\mathrm{map}(BP, BS)_{Bi^S_P} \cong BZ(P)$.

From part (c) of Theorem 4.4 in [24], $\mathrm{map}(BP, |\mathcal{L}|^\wedge_p)_{\theta \circ Bi^S_P} \cong BZ(P)$ and we know that $\mathrm{map}(BP, |\mathcal{L}|^\wedge_p)_{\theta \circ Bi^S_P} = \mathrm{map}(BP, |\mathcal{L}|^\wedge_p)_{f \circ Bi^\pi_P}$ from $f \circ Bi = \theta$. So we get $\mathrm{map}(BP, |\mathcal{L}|^\wedge_p)_{f \circ Bi^\pi_P} = BZ(P)$. Hence, we get the first and third ones are homotopic in equation 6.1. So the composition is homotopy equivalence.

Now, we know ρ carries $Z(P)$ in $N_\pi(P) = \mathrm{Aut}_{\mathcal{L}}(P)$ onto $Z(P)$ in $\mathrm{Aut}\,_{\mathcal{L}}(P)$. Define signalizer functor as $\Theta(P) := \ker(\mathrm{Aut}_{\mathcal{T}}(P) \xrightarrow{\rho} \mathrm{Aut}_{\mathcal{L}}(P))$ for $\mathcal{F}$-centric P. Writing $\mathrm{Aut}_{\mathcal{T}}(P) = N_\pi(P)$, we have commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_\pi(P) & \longrightarrow & N_\pi(P) & \xrightarrow{\pi} & \mathrm{Aut}_{\mathcal{F}}(P) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \rho} & & \| & & \\ 1 & \longrightarrow & Z(P) & \longrightarrow & \mathrm{Aut}_{\mathcal{L}}(P) & \xrightarrow{\pi} & \mathrm{Aut}_{\mathcal{F}}(P) & \longrightarrow & 1 \end{array}$$

Since the diagram commutes, there is an isomorphism between the kernel of the second and third rows because the third row is isomorphism. Hence the first condition satisfied $\Theta(P) \to C_\pi(P) \to Z(P)$ is exact.

Now take $g \in \pi$ such that gSg^{-1}. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Theta(P) & \longrightarrow & N_\pi(P) & \longrightarrow & \mathrm{Aut}_{\mathcal{L}}(P) & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle x \mapsto gxg^{-1}} & & \big\downarrow{\scriptstyle \phi \mapsto \rho(g)\phi\rho(g)^{-1}} & & \\ 1 & \longrightarrow & \Theta(gPg^{-1}) & \longrightarrow & N_\pi(gPg^{-1}) & \longrightarrow & \mathrm{Aut}_{\mathcal{L}}(gPg^{-1}) & \longrightarrow & 1 \end{array}$$

Since we have isomorphisms in third and fourth rows, we must have isomorphism in the second row. So, $g\Theta(P)g^{-1} = \Theta(gPg^{-1})$.

To prove Θ is signalizer functor, we need to show that for $P \leqslant Q$, we have $\Theta(Q) \leqslant \Theta(P)$. Fix $P \leqslant Q$. Let $\hat{e} = \delta(e) \in \mathcal{L}(P, Q)$ where $e \in N_S(P, Q)$ gives inclusion. Define $\mathrm{Aut}_{\mathcal{L}}(Q, \downarrow_P) := \{\phi \in \mathrm{Aut}_{\mathcal{L}}(Q) | \exists \phi' \in \mathrm{Aut}_{\mathcal{L}}(P) \text{ such that } \phi \circ \hat{e} = \hat{e} \circ \phi'\}$ and $\mathrm{Aut}_{\mathcal{L}}(P, \uparrow^Q) := \{\phi' \in \mathrm{Aut}_{\mathcal{L}}(Q) | \exists \phi \in \mathrm{Aut}_{\mathcal{L}}(Q) \text{ such that } \phi \circ \hat{e} = \hat{e} \circ \phi'\}$. From Corollary 3.10 in [13], we have cancellation property. More formally we have, $\phi_1 \circ \hat{e} = \phi_2 \circ \hat{e} \Longrightarrow \phi_1 = \phi_2$ and $\hat{e} \circ \phi_1' = \hat{e} \circ \phi_2' \Longrightarrow \phi_1' = \phi_2'$.

So we can send $\phi \in \mathrm{Aut}_{\mathcal{L}}(Q)$ to the unique ϕ and vice versa similarly. So we have an isomorphism

$$\mathrm{Aut}_{\mathcal{L}}(Q, \downarrow_P) \cong \mathrm{Aut}_{\mathcal{L}}(P, \uparrow^Q).$$

Let $N = N_\pi(Q) \bigcap N_\pi(P)$. The preimage of $\mathrm{Aut}_{\mathcal{L}}(Q, \downarrow_P)$ under the map $\rho : N_\pi(Q) \to \mathrm{Aut}_{\mathcal{L}}(Q)$ is N since if g is an element in the kernel we have $g \in N_\pi(Q)$ and $\rho(g) \in \mathrm{Aut}_{\mathcal{L}}(Q, \downarrow_P)$ which implies that $g \in N_\pi(P)$. Hence, $\ker(N \xrightarrow{\rho_Q} \mathrm{Aut}_{\mathcal{L}}(Q, \downarrow_P)) = N \cap \ker(\rho) = N \bigcap \Theta(Q) = \Theta(Q)$. So we obtain a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Aut}_{\mathcal{L}}(Q, \downarrow_P) \\ & \overset{\rho_Q}{\nearrow} & \big\updownarrow \\ N & & \cong \\ & \underset{\rho_P}{\searrow} & \big\updownarrow \\ & & \mathrm{Aut}_{\mathcal{L}}(P, \uparrow^Q). \end{array}$$

Here, $\Theta(P) \bigcap N = \ker(\rho_P) = \ker(\rho_Q) = \Theta(Q)$. Hence, $\Theta(Q) \leqslant \Theta(P)$, concluding the proof of that Θ is signalizer functor.

For (iii): take any $\phi \in \mathcal{F}(P, Q)$. Since $\mathcal{F} = \mathcal{F}_S(\pi)$, $\exists g \in \pi$ such that $c_g|_P = \phi$

(Here, by $c_g|_P$ we mean restriction to P of the conjugation by g map in π). Then $Bc_g : B\pi \to B\pi$ is homotopic to the identity $Bi : B\pi \to B\pi$. Then the diagram commutes up to homotopy

$$\begin{array}{ccccc} BP & \xrightarrow{Bi_P^S} & BS & \xrightarrow{Bi} & B\pi \\ \downarrow{\scriptstyle B\phi} & & & & \|{\scriptstyle Bc_g \cong id} \\ BQ & \xrightarrow{Bi_q^S} & BS & \xrightarrow{Bi} & B\pi. \end{array}$$

Applying $H^*(-;\mathbb{F}_p)$, we get that

$$\begin{array}{ccccc} H^*(B\pi;\mathbb{F}_p) & \xrightarrow{\qquad Bi_S^{\pi}{}^* \qquad} & H^*(BS;\mathbb{F}_p) & & \\ & \swarrow{\scriptstyle Bi_P^S{}^*} & & \searrow{\scriptstyle Bi_Q^S{}^*} & \\ H^*(BP;\mathbb{F}_p) & & \xleftarrow{\qquad \phi^* \qquad} & & H^*(BQ;\mathbb{F}_p) \end{array}$$

commutes for any $\phi \in \mathcal{F}$.

Then the image of the map $res_S^{\pi} = Bi_S^{\pi}{}^*$ contains only $\mathcal{F}$-stable elements of $H^*(BS;\mathbb{F}_p)$ that is

$$Bi^*(H^*(B\pi;\mathbb{F}_p)) \subseteq H^*(\mathcal{F}).$$

For the composition $\theta : BS \xrightarrow{Bi_S^{\pi}} B\pi \xrightarrow{f} |\mathcal{L}|_p^{\wedge}$, we have that $\theta^* : H^*(|\mathcal{L}|_p^{\wedge};\mathbb{F}_p) \to H^*(\mathcal{F}) \subseteq H^*(BS;\mathbb{F}_p)$ is an isomorphism by the Theorem 5.8 in [24]. Here we use f^* to obtain the splitting we need. We have the composition

$$\theta^* : H^*(|\mathcal{L}|_p^{\wedge};\mathbb{F}_p) \xrightarrow{f^*} H^*(B\pi;\mathbb{F}_p) \xrightarrow{Bi^*} H^*(\mathcal{F})$$

is isomorphism. Hence, $res_S^{\pi} : H^*(\pi, \mathbb{F}_p) \to H^*(S, \mathbb{F}_p)$ has image exactly $H^*(\mathcal{F})$ and splits by the map $f^* \circ (\theta^*)^{-1}$. Then, the splitting gives the decomposition

$$H^*(\pi;\mathbb{F}_p) \cong H^*(\mathcal{F};\mathbb{F}_p) \oplus \ker(res_S^{\pi}).$$

$\square$

6.2 Cohomology of Signalizer Groups

For an arbitrary fusion system the cohomology of the associated centric linking system and the cohomology of π can be compared similarly using a spectral sequence (see [15, Theorem VII.6.3]). This gives a long exact sequence described in [4, Theorem 1.3]. The main ingredient for this is the fact that cohomology of $\theta(P)$ is zero for dimensions $i \geqslant 2$ in mod p coefficients. Now, we prove this fact using group theory.

Proposition 6.2.1. *Let $\mathcal{F} = \mathcal{F}_S$ be saturated. Assume π is an infinite group realizing $\mathcal{F}$ obtained by Leary-Stancu model or Robinson model or any other model given by graph of groups. Let θ be a signalizer functor on π such that $\mathcal{L}$ is a quotient of the transporter system $\mathcal{T}_S^c(\pi)$. Then, for any $P \in \mathcal{F}^c$ and for any $i \geqslant 2$, we have $H_i(\theta(P); \mathbb{F}_p) = 0$.*

Proof. We have $\mathcal{F} = \mathcal{F}_S(\pi)$. Take any $\mathcal{F}$-centric subgroup P of S. Assume for a non-trivial finite p-group Q, we have $Q \leqslant \theta(P)$. Since $\theta(P)$ centralizes P, Q centralizes P. So PQ is a p-subgroup of π. Since S is a Sylow p-subgroup of π, there exists $g \in \pi$ such that $gPQg^{-1} \subset S$. For the groups $P' := gPg^{-1}$ and $Q' := gQg^{-1}$, we know that P' and Q' are subgroups of S, and P' is $\mathcal{F}$-centric, and Q' centralizes P'. Then, we say $Q' \subseteq Z(P')$. Hence, $Q \subseteq Z(P)$. However, Q was a subset of $\theta(P)$ which has a trivial intersection with $Z(P)$. So we get a contradiction with the assumption that $\theta(P)$ has non-trivial p-subgroup. The result follows. $\square$

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