STABILITY OF THIRD ORDER CONEWISE LINEAR SYSTEMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF
MASTER OF SCIENCE
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

By
Muhammad Zakwan
July 2019

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

STABILITY OF THIRD ORDER CONEWISE LINEAR SYSTEMS

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A conewise linear, time-invariant system is a piecewise linear system in which the state-space is a union of polyhedral cones. Each cone has its own dynamics so that a multi-modal system results. We focus our attention to global asymptotic stability so that each mode (or subsystem) is autonomous. i.e., driven only by initial states. Conewise linear systems are of great relevance from both practical and theoretical point of views as they represent an immediate extension of linear, time-invariant systems. A clean and complete necessary and sufficient condition for stability of this class of systems has been obtained only when the cones are planar, that is only when the state space is $\mathbf{R}^2$. This thesis is devoted to the case of state-space being $\mathbf{R}^3$, although occasionally we also consider the general case $\mathbf{R}^n$.

We aim to determine conditions for stability exploring the geometry of the modes. Thus our results do not make use of a Lyapunov function based approach for stability analysis. We first consider an individual mode and determine whether a cone with a given dynamics can be classified as a sink, source, or transitive from one or two borders. It turns out that the classification not only depends on the geometry of the eigenvectors and the geometry of the cone but also on entries of the A-matrix that defines the dynamics. Under suitable assumptions on the configuration of the eigenvectors relative to the cone, we manage to obtain relatively clean charecterizations for transitive modes. Combining this with a complete characterization of sinks and sources, we use some tools from graph theory and obtain an interesting sufficient condition for stability of a conewise system composed of transitive modes, sources, and sinks. Finally, we apply our results to study the stability of a linear RC electrical network containing diodes.

Keywords: Conewise linear systems, stability analysis, switched systems.

ÖZET

DOĞRUSAL, ZAMANLA DEĞİŞMEYEN KONİ-UZAYLI SİSTEMLERİN KARARLILIĞI

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Temmuz 2019

Doğrusal, zamanla değişmeyen koni-uzaylı bir sistem, durum-uzayı çok-yüzlü konilerin birleşiminden oluşan, parçalı-doğrusal bir sistemdir. Her bir koni içindeki devinim farklı olabileceği için, bu tür sistemler çok-rejimli (çok-halli) sistemlerdir. Burada ilgimiz toplam asimtotik kararlılık olduğundan, her bir rejimin otonom olduğu, yani sadece başlangıç durum değerlerine dayandığı, varsayılacaktır. Koni-uzaylı sistemler doğrusal zamanla değişmeyen sistemlerin en basit genellemesiyle elde edildiği için bu tür sistemler hem teorik hem de pratik açılardan büyük bir ilgi odağıdır. Koni-uzaylı sistemlerin kararlılığı için temiz ve eksiksiz bir gerek ve yeter koşul bugüne kadar yalnızca düzlemsel koniler durumunda, yani durum-uzayının $\mathbf{R}^2$ olduğu durumda bulunabilmiştir. Bu tezde, zaman zaman en genel $\mathbf{R}^n$ durumu dikkate alınsa da, asıl ilgimiz durum-uzayının $\mathbf{R}^3$ olduğu durumdur.

Amacımız kararlılık için koşulları rejimlerin geometrik yapısını öne çıkararak bulmaktır. Dolayısıyla, kararlılık analizimizde Lyapunov fonksiyonları yöntemini kullanmayacağız. Önce her bir rejimi kendi başına inceleyip, onun (bir veya iki yüzünden) geçişken mi, kaynak mı veya çukur mu olduğu tasnifini yapacağız. Bu inceleme bize tasnifin sadece özvektörlerin ve koninin geometrisine değil aynı zamanda rejim içindeki devinimi belirleyen A-matrisinin elemanlarının değerlerine de bağlı olduğunu gösterecek. Ancak, özvektörlerin koniye göre olan pozisyonları ile ilgili bazı makul varsayımlar yaparak, geçişken rejimlerin yeterince temiz bir tasnifini elde etmeyi başaracağız. Bu tasnifi kaynak ve çukurların kapsamlı tasnifleriyle birleştirerek ve grafik teorisinden bazı araçlar kullanarak, sadece geçişken, kaynak veya çukur karakterli rejimleri içeren koni-uzaylı bir sistemin kararlılığı için ilginç bir yeter koşul vereceğiz. Son olarak, bu sonuçları diyotlar içeren doğrusal bir RC devresine uygulayarak, bu parçalı devrenin kararlılık analizini

yapacağız.

Anahtar sözcükler: Koni-uzaylı sistemler, parçalı-doğrusal sistemler, kararlılık, anahtarlı sistemler.

Acknowledgement

First of all, I would like to thank my professor A. Bülent Özgüler who supported me throughout my Master's studies. His valuable suggestions and comments improved the quality of this significantly. He always supported me both technically and morally throughout my Masters. I appreciate his help for teaching me how to conduct fundamental research.

I would also like to thank the faculty members of our department to equip me with valuable knowledge about electrical engineering. Especially, I would like to thank Prof. Hitay Özbay and Prof. Ömer Morgül for helping me in solving my research problems and reviewing my research articles. I would like to express my gratitude to our department's chair Prof. Orhan Arikan for his motivational words that always inspired me.

I would like to say a few words for people who contributed to my thesis indirectly. My first thanks go to my parents and siblings who supported me throughout my stay at university. They motivated me to carry out my higher education out of my home country. I am very grateful to Dr. Saeed Ahmed for our collaborations and discussions on several research problems. He supported me morally and technically throughout my coursework and thesis. A special thanks to Dr. Corentin Briat for always being there for me whenever I need him. He always appreciated my ideas, reviewed my work, and provided his valuable suggestions. I would also like to mention Dr. Nil Şahin and my friends Mr. Aras Yurtman, Mr. Abdul Waheed, Mr. Muhammad Nabi, Mr. Suleman Aijaz Memon, Mr. Sina Gholizade, Mr. Talha Masood Khan, Mr. and Mrs. Naveed Mehmood, Mr. Muhammad Waqas Akbar, and Mr. and Mrs. Anjum Qureshi for their moral support during my Master's studies. Last but not least I would like to mention Ezgi Güzel for her never-ending support and for being around during my bad times.

This work is supported in full by the Science and Research Council of Turkey (TÜBİTAK) under project EEEAG-117E948.

To my parents

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Chapter 1

Introduction

Switched systems comprise of a family of continuous-time subsystems and a rule that defines the switching among them, see [1], [2], and [3] for motivation of switching in systems and control. Mathematically, we can represent a switched system as

$$\dot{x}(t) = f_\sigma x(t). \tag{1.1}$$

The above system consists of a family of sufficiently regular functions $\{f_p \mid p \in \mathcal{P}\}$, defined from $\mathbf{R}^n$ to $\mathbf{R}^n$ and parameterized by an index set $\mathcal{P}$. The function $\sigma : [0, \infty) \to \mathcal{P}$ denotes the piecewise constant switching signal. We call the system (1.1) *non-autonomous switched system* if value of the switching signal σ at a given instant depends on t. On the other hand, if the value of the function σ at time t depends on $x(t)$, then the system (1.1) is referred to as an *autonomous switched system*, [4].

There are three main topics of stability for switched systems [5], which are summarized below.

1. Find necessary and sufficient conditions that guarantee the asymptotic stability of the switched system (1.1) for an arbitrary switching signal.

2. Assuming all the subsystems in (1.1) are asymptotically stable, classify the

set of switching signals for which the switched system (1.1) is asymptotically stable.

3. Compose a particular set of switching signals that make the switched system (1.1) asymptotically stable. We consider this problem in case all of the subsystems are unstable; otherwise, it will be trivial to design a stabilizing switching law.

Piecewise linear systems (PLS) is a class of hybrid systems characterized by a partition of the state-space into multiple regions. The dynamics in each region is linear, [6]. We consider the class of piecewise linear systems where the state partition consists of convex polyhedral cones and in each cone, the dynamics is linear time-invariant. We call such systems as conewise linear systems (CLS), [7]. These systems can be viewed as a particularization of piecewise linear differential inclusions [8], [9], state-dependent switched linear systems [10], or linear parameter varying systems [11]. In this thesis, we study the stability problem for third order multi-modal conewise linear systems with autonomous conic switching. Here conic switching refers to the fact that the switching boundaries are assumed to be two-dimensional subspaces. Lyapunov function based stability approach seems a natural choice to study stability of our class of systems but finding Lyapunov function for switched systems is a challenging problem [12]. Therefore, we aim to provide necessary and sufficient conditions for a given third order conewise linear system to be globally asymptotically stable by exploring the geometry of the eigenvectors of the system.

1.1 Motivation

In this section, we provide motivation for studying conewise linear systems from practical and theoretical points of view.

1.1.1 Application Aspect

Many real-world applications are inherently hybrid and multi-modal in nature (see [13], [14], and [15]). Many electrical networks comprising of non-linear elements can be modeled as piecewise linear systems. Diodes, transistors, thyristors, and switches naturally segregate the topology of the circuits into different modes; see [16] and the references therein. Piecewise linear systems can capture the non-linearity of relays and saturation, [4], [17], [18], and see also [14] for modelling of a DC-DC series resonant converter as a piecewise linear system. Apart from applications in electrical engineering, conewise linear systems also find their application in several mechanical systems; e.g., all-wheel drive clutch system, mechanical cart, nonlinear dynamical systems; see [19] and [20]. These switched systems are also ubiquitous in the field of social sciences; see [21] and [22]. Another important application of conewise linear systems is that it can model a multi-agent flocking problem, [4].

1.1.2 Theoretical Aspect

A converse Lyapunov theorem for the stability of switched linear systems has recently been proposed in [10]. It has been shown in [10] that if there exists an asymptotically stabilizing switching law for a switched linear system then there must exist a polyhedral Lyapunov function with a conic partition based stabilizing switching law. This result also applies to switched linear systems with time-variant parametric uncertainties. Thus, the stability problem of switched linear systems reduces to the controller synthesis problem of a specific switched linear system subject to conic switching. Therefore, finding stability conditions for conewise linear systems leads to characterizing the stability of other switched linear systems subject to specially designed switching laws.

Determining necessary and sufficient condition for the stability of conewise linear systems via non-Lyapunov approach is highly desirable. Although the literature provides many results on stability in the sense of Lyapunov function based approach, the method becomes cumbersome with the requirement of concocting multiple Lyapunov functions for a set of systems; see [23], [24], [25], and [26]. It is worth mentioning that there are many examples where switching among stable subsystems leads to an unstable trajectory which makes the study of PLS more intriguing [4]. A complete set of conditions for the case of planar conewise linear systems is given in [27] (also see [28] and [29]) which motivates us for determining constructive necessary and sufficient conditions for higher order conewise linear systems. A CLS can model various families of switched systems: fuzzy control systems [30]; mixed logical dynamical systems [31]; linear complementarity systems [32], relay systems [33], systems with actuator saturation [34], which motivates the study of CLS.

Although various results that are discussed in next section have been developed pertaining to CLS, the stability issues require further exploration. Finding a necessary and sufficient condition in case of autonomous switching is still an open problem which is the main topic of this thesis.

1.2 Literature Review

Many articles are devoted to the stability of switched systems subject to arbitrary switching. We provide a summary of these results in this section. The most common approach to tackle the stability problem of switched systems is employing a common quadratic Lyapunov function; see [35], [36], [37], and [38]. However as stated earlier, the task becomes cumbersome as the number of subsystem increases. For this particular case, the choice of non-conventional Lyapunov functionals (concatenation of piecewise continuous and piecewise differentiable Lyapunov functions) as discussed in [39], [40], [5], and [41] can be an alternative solution. Many contributions in the literature discuss stability of general switched systems. For instance, the stability analysis of a wide range of switched systems by exploring different switching schemes is discussed in [4] and [42]. Moreover, Lyapunov based stability analysis results are also provided in [4] and [42]. In [43], the notions of observability, controllability and feedback stabilization for switched linear systems are discussed. Some useful results on switched linear systems and a class of linear differential inclusions are summarized in [44]. Moreover, a list of open reseach problems pertaining to this class of systems is also provided [44].

Although conewise linear systems inherit a simple structure, their stability analysis is a challenging task due to their hybrid nature. Moreover, the extension of controllability, observability, and well-posedness results developed for linear time invariant (LTI) systems to the case of piecewise linear systems is not straightforward. In what follows in this section, we summarize some important results available in the literature for piecewise linear systems.

In [45], the authors define the controllability and observability of piecewise affine systems by using mixed-integer linear programming based numerical tests. In [46], algebraic necessary and sufficient conditions for the controllability of conewise linear systems by employing the concepts from geometric control theory and mathematical programming are formulated. In [47], directional derivative and positive invariance techniques are used to characterize two different notions of local observability (finite-time and long-time) for conewise linear systems.

The stability and controllability of planer bimodal linear complementarity systems by employing the trajectory based analysis is discussed in [29]. Another trajectory based analysis for the stability of piecewise linear system is given in [27] where the stabilization of bimodal systems based on an explicit stability test for the planar systems is proposed. The stability test provided in [27] requires the coefficients of transfer functions of subsystems. Moreover, a necessary stability condition and a sufficient stability condition for higher-order and bimodal systems is also provided in [27]. These conditions are given in terms of the eigenvalue loci and the observability of subsystems. Exponential stability of planar systems is discussed in [48]. In [48], necessary and sufficient condition is derived using the integral function of continuous-time planar system. An algorithm to compute this particular integral function is also provided in [48]. A necessary and sufficient condition for a second-order conewise linear system is proposed in [49]. In [49], the geometry of the eigenvectors and their determinants is exploited to determine the stability. The work in this thesis is an extention of [49] by adopting a similar geometeric approach. Asymptotic stability of bimodal PLS is discussed in [50] and [51] using a trajectory based approach. In [50] and [51], a distinction between transitive and non-transitive trajectories is provided, and a special structure of state-matrices is assumed to formulate stability conditions.

An alternative approach using the surface Lyapunov functionals and impact maps (maps from one switching surface to the next switching surface) for the stabilty of piecewise linear systems is proposed in [23]. Recently, cone-copositive piecewise quadratic Lyapunov functions (PWQ-LFs) for the stability analysis of conewise linear systems are proposed in [52]. In [52], linear matrix inequalities (LMIs) are used as tool to formulate the existence of PWQ-LF as a feasibility of a cone-copositive programming problem.

Various results related to controller synthesis and observer design for PLS have been studied in the literature. For instance, the controller design using LMIs for uncertain piecewise linear system is given in [53]. Similarly, the design of a static output-feedback controller for discrete piecewise linear systems using LMIs is given in [54], whereas Luenberger type observers for the bimodal piecewise linear systems in both continuous and discrete time domains are discussed in [55].

1.3 Problem Statement and Thesis Contributions

We define our problem statement as to find useful conditions for a given third order (spatial) conewise linear system to be globally asymptotically stable. To solve this problem we employ a two-step approach. In the first step, by exploring the geometry of associated eigenvectors relative to the cone; we characterize a given mode as a source, sink or transitive. In the second step, using tools from graph theory, we provide a sufficient conditions for global asymptotic stability.

The contributions of this thesis are then summarized as

1. Characterization of a given mode as a source, sink or transitive by employing the geometry of the subsystem.

2. Providing a new sufficient condition for global asymptotic stability for a class of conewise linear systems.

3. Modeling a non-linear electrical network as a conewise linear system and checking its global asymptotic stability.

1.4 Outline

Chapter 2 provides a mathematical model of conewise linear systems, and classical results from the geometry and graph theory. Chapter 3 discusses the main results of the thesis. In Chapter 4, we provide modeling of a non-linear electrical network as a conewise linear system and perform stability analysis by using our proposed approach. Finally, we conclude the thesis by giving conclusions and future directions in Chapter 5.

1.5 Notation

We denote the real numbers, n-dimensional real vector space, and the set of real $n \times m$ matrices by $\mathbf{R}$, $\mathbf{R}^n$, and $\mathbf{R}^{n\times m}$, respectively. The natural basis vectors of $\mathbf{R}^n$ are $\mathbf{e}_1, \ldots, \mathbf{e}_k$ with $\mathbf{e}_j$ having its only nonzero element 1 at its j-th position. The norm of a vector $\mathbf{v} \in \mathbf{R}^n$ will be denoted by $|\mathbf{v}|$. If $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$, then $\mathbf{v} \times \mathbf{w}$ will denote the cross product of the vectors and $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$, their dot product, where 'T' denotes 'transpose.' An arbitrary permutation of $(1, 2, 3)$ will be denoted by (k, l, m). $tr(\cdot)$ represents the standard trace of a square matrix.

We will adopt the following convention to visualize a "tri-hedral" (three-faced) cone and vectors in 3D, i.e., in space. Consider Figure 1 that depicts a bird's view of a cone $\mathbf{S}$ that belongs to a single mode with associated vectors $\mathbf{v}_i, i = 1, 2, 3$

(that will actually be, below, the eigenvectors associated with the A-matrix that define the linear dynamics in the cone). The origin of the space is somewhere inside the triangle down in the page and the tip of the border vectors $\mathbf{s}_i, i = 1, 2, 3$ are dots at the corners of the triangle. Each border $\mathsf{B}_k = cone\{\mathbf{s}_l, \mathbf{s}_m\}$ is a planar convex cone. In case each plane defined by two distinct vectors $\mathbf{v}_i, \mathbf{v}_j$ intersects $\mathbf{S}$, **sectors** in $\mathbf{S}$ are obtained. The four sectors $\mathsf{S}_i, i = 1, 2, 3, 4$ in the figure are obtained by the intersections of $\mathbf{v}_1\mathbf{v}_2$-plane and $\mathbf{v}_3\mathbf{v}_1$-plane with $\mathbf{S}$. Each sector is a polyhedral (multi-faced) convex cone with three to five faces, which are themselves planar convex cones. In Figure 1, S_1 and S_3 for example have 3 and 4 borders, respectively. If shown, there would be two more sectors that come from a partition of S_3 and that result from the intersection of $\mathbf{v}_2\mathbf{v}_3$-plane with $\mathbf{S}$.

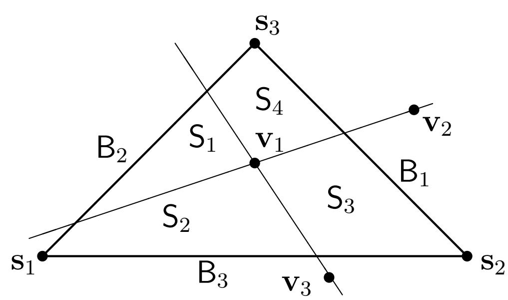


Figure 1.1: Four sectors when $\mathbf{v}_1 \in \mathbf{S}$ and $\mathbf{v}_1, \mathbf{v}_2 \notin \mathbf{S}$.

Chapter 2

Preliminaries

In this chapter, we first describe, in Section 2.1, the kind of systems the stability of which we will consider. Next, in Section 2.2, we define the well-posedness of such systems. In Section 2.3, a behavioral chracterization of a single sub-system (a single mode) of the system described in Section 3.1 is presented. This will lead to a classification of sub-systems as a source, sink, transitive, or a half-sink. Finally, in Section 2.4, we provide some basic concepts from graph theory [56] that will be instrumental in deriving our main result on stability in Section 3.6.

2.1 General Conewise Linear Systems

An *nth*-order autonomous piecewise linear system can be described as

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_1, \\ A_2\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_2, \\ \vdots & \vdots \quad \vdots \\ A_N\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_N, \end{cases} \tag{2.1}$$

where $A_i \in \mathbf{R}^{n\times n}$ and the set $\mathbf{S}_i$ is a nonempty subset of $\mathbf{R}^n$ for $i = 1, ..., n$. We assume that V_i is nonsingular and $\det V_i > 0$. Note that the latter requires only an appropriate choice of the eigenvectors. We further assume that the interior of each pairwise intersection $int\,\mathbf{S}_i \cap \mathbf{S}_k,\ i \neq k$ is empty (i.e., does not contain any open subset of $\mathbf{R}^n$) and that $\mathbf{S}_1 \cup ... \cup \mathbf{S}_N = \mathbf{R}^n$ i.e., the whole state-space is filled. These assumptions ensure, in the terminology of [27], that (2.1) is *memoryless.* It is normally assumed that each $\mathbf{S}_i$ is a cone, i.e.,

$$\mathbf{S}_i := \{\mathbf{x} \in \mathbf{R}^n : C_i\mathbf{x} \geq 0\},$$

for some $C_i,\ i = 1, 2, ..., N$, where C_i is a matrix of linearly independent rows. This assumption on C_i implies that (2.1) is truly multi-modal ($N \geq 2$) and that the interior of each $\mathbf{S}_i$, $int\,\mathbf{S}_i$, is nonempty.

A *convex polyhedral cone* (or simply, a *cone*) defined by a finite set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is

$$\mathbf{S} = cone\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} := \left\{ \sum_{j=1}^{k} \alpha_j \mathbf{v}_j : \alpha_j \geq 0 \right\}$$

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called a *frame* of the cone $\mathbf{S}$ if it is positively independent (i.e., no member is a nonnegative linear combination of others) and positively spans $\mathbf{S}$ (i.e., any element of $\mathbf{S}$ can be written as a nonnegative linear combination of the set).

If each $\mathbf{S}_i$ is a polyhedral convex cone, then the system (2.1) is a **conewise linear system (CLS)** of N-modes. The N modal system (2.1) will be assumed to be a CLS throughout the rest of this thesis.

The minimal number N of modes that fill up $\mathbf{R}^{n \times n}$ is $n + 1$. This basic fact follows directly from a result of [57] on "frames," as we now outline.

Fact 2.1.1 *[57] If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ positively spans $\mathbf{R}^n$, then there is $\lambda_j > 0$ such that*

$$\mathbf{0} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k.$$

Proof. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans $\mathbf{R}^n$, then for $\alpha_{ij} \geq 0$

$$\begin{aligned} -\mathbf{v}_1 &= \alpha_{11}\mathbf{v}_1 + \ldots + \alpha_{1k}\mathbf{v}_k \\ -\mathbf{v}_2 &= \alpha_{21}\mathbf{v}_1 + \ldots + \alpha_{2k}\mathbf{v}_k \\ &\vdots \\ -\mathbf{v}_k &= \alpha_{k1}\mathbf{v}_1 + \ldots + \alpha_{kk}\mathbf{v}_k \end{aligned} \tag{2.2}$$

so that

$$\begin{aligned} (1+\alpha_{11})\mathbf{v}_1 + \alpha_{12}\mathbf{v}_2 + \ldots + \alpha_{1k}\mathbf{v}_k &= 0 \\ \alpha_{21}\mathbf{v}_1 + (1+\alpha_{11})\mathbf{v}_2 + \ldots + \alpha_{2k}\mathbf{v}_k &= 0 \\ &\vdots \\ \alpha_{k1}\mathbf{v}_1 + \alpha_{k2}\mathbf{v}_2 + \ldots + (1+\alpha_{kk})\mathbf{v}_k &= 0 \end{aligned} \tag{2.3}$$

Adding these, we have $0 = \lambda_1\mathbf{v}_1 + \ldots + \lambda_k\mathbf{v}_k$ with all $\lambda_j > 0$ □

Theorem 2.1.1 *[57] If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a frame of $\mathbf{R}^n$, then $n+1 \leq k \leq 2n$ and $k = n+1$ is possible.*

Proof. We reproduce the proof of Theorem 3.6 of [57] for $k \geq n+1$ and the existence of a minimal frame here. For $k \leq 2n$, the reader is referred to Theorem 6.7(i) of [57].

If $k < n+1$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is either a basis of $\mathbf{R}^n$ (i.e., $k = n$) or it fails to span $\mathbf{R}^n$ (i.e., $k < n$). In the first case of it being a basis of $\mathbf{R}^n$, 0 can not be expressed as a positive linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, i.e.,

$$0 \neq \lambda_1\mathbf{v}_1 + \ldots + \lambda_k\mathbf{v}_k; \lambda_j > 0$$

for any λ_j's. This, however, is a necessary condition for $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to positively span $\mathbf{R}^n$ by the Fact 2.1.1. In the second case that it fails to span $\mathbf{R}^n$, it also can not positively span $\mathbf{R}^n$. This shows that $k \geq n+1$ if it is a frame.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be the natural basis vectors of $\mathbf{R}^n$ and let $\mathbf{e}_0 := -\sum_{j=1}^n \mathbf{e}_j$. The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \mathbf{e}_0\}$ is a minimal frame of $\mathbf{R}^n$. In fact, the set is easily seen to be positively independent. It also positively spans the space $\mathbf{R}^n$ because given any

$\mathbf{w} = [w_1 ... w_n]^T \in \mathbf{R}^n$, with $m := \min_j\{w_j\}$, we have

$$\mathbf{w} = \begin{cases} \sum_{j=1}^{n} w_j \mathbf{e}_j, & \text{if } m \geq 0, \\ -m\mathbf{e}_0 + \sum_{j=1, j\neq k}^{n} (w_j - m)\mathbf{e}_j & \text{if } m = w_k < 0. \end{cases}$$

□

Remark 2.1.1 *By [58], and imitating the construction of the minimal frame of Theorem 2.1.1, we note that given any basis* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *of* $\mathbf{R}^n$*, the set*

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \; -\sum_{j=1}^{n} \lambda_j \mathbf{v}_j\}$$

is a minimal frame for any positive constants $\lambda_j > 0$*. Also, the set*

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \; -\lambda_j \mathbf{v}_1, \ldots, -\lambda_j \mathbf{v}_k\}$$

is a maximal frame of $\mathbf{R}^n$*. Moreover, any minimal and maximal frames of* $\mathbf{R}^n$ *are as above.*

Remark 2.1.2 *In* $\mathbf{R}^3$*,* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, -(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)\}$ *is a minimal frame. The set* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, -\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3\}$ *is a maximal frame (a set of largest cardinality that is positively independent and positively spans* $\mathbf{R}^3$*).*

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$ be a minimal frame of $\mathbf{R}^n$ and, for $j = 1, \ldots, n+1$, define

$$\mathbf{S}_j := \mathit{cone}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{n+1}\}.$$

Then, it is easy to see that

$$\mathbf{R}^n = \bigcup_{j=1}^{n+1} \mathbf{S}_j$$

and that these $n+1$ polyhedral cones are the minimum number of cones that fill up the space $\mathbf{R}^n$. Also note that the pairwise intersection

$$\mathbf{S}_j \cap \mathbf{S}_k = \mathit{cone}\{\mathbf{v}_1, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{n+1}\}$$

for any $j < k$ is itself a cone of dimension $n-1$ (one less dimension than that of $\mathbf{S}_j$'s).

Remark 2.1.3 *Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$ *be a minimal frame of* $\mathbf{R}^n$ *and let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}, \mathbf{w}\}$ *be its extension to* $n+2$ *vectors such that* $\mathbf{w} \neq \mathbf{v}_k$ *for* $k = 1, \ldots, n+1$ *and* $\mathbf{w} \in \mathbf{S}_1$*. Then, span*$\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}, \mathbf{w}\} = \mathbf{R}^n$ *and there are* n *new cones* $\mathscr{C}_1 = cone\{\mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}\}, \ldots, \mathscr{C}_n = cone\{\mathbf{w}, \mathbf{v}_3, \ldots, \mathbf{v}_{n+1}\}$ *in place of* $\mathbf{S}_1$ *and*

$$\left(\bigcup_{j=2}^{n+1} \mathbf{S}_j\right) \cup \left(\bigcup_{k=1}^{n} \mathscr{C}_k\right) = \mathbf{R}^n.$$

By induction, if $\{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$ *are introduced to extend the frame with* $\mathbf{w}_j \in \mathbf{S}_j$*,* $j = 1, \ldots, d$*, then there will be* $(n-1)d$ *new cones created, resulting in a total of* $n + 1 + (n-1)d$ *cones.*

Note that the set of border vectors in this extension is not a frame but they, nevertheless, positively span the space. Also note that in 3D, starting with the minimal number of 4 cones, this procedure will always result in an even number $2(2+d)$ of cones still filling up the space.

2.2 Well-Posedness of a General Conewise Linear System

Consider a single mode

$$\dot{\mathbf{x}} = A\mathbf{x}(t), \mathbf{x} \in \mathbf{S},$$

where $\mathbf{S} = cone\{\mathbf{s}_1, ..., \mathbf{s}_n\}$ is a polyhedral cone of n border vectors. The cone has n faces obtained for $k = 1, ..., n$ by

$$\mathsf{B}_k := cone\{\mathbf{s}_1, ..., \mathbf{s}_{k-1}, \mathbf{s}_{k+1}, ..., \mathbf{s}_n\}.$$

We will refer to a single mode as **well-posed** if, given any $\mathbf{b} \in \mathsf{B}_k$, then for some $\epsilon > 0$ it either holds that $\mathbf{x}(t) \in \mathbf{S}$ for $t \in (0, \epsilon/2)$, $\mathbf{x}(t) \notin \mathbf{S}$ for $t \in (\epsilon/2, \epsilon)$, and $\mathbf{x}(\epsilon/2) = \mathbf{b}$ or it holds that $\mathbf{x}(t) \notin \mathbf{S}$ for $t \in (0, \epsilon/2)$, $\mathbf{x}(t) \in \mathbf{S}$ for $t \in (\epsilon/2, \epsilon)$, and $\mathbf{x}(\epsilon/2) = \mathbf{b}$. This is equivalent to i) not having any sliding modes in B_k and ii) every trajectory having a smooth continuation with respect to the interior and the exterior of $\mathbf{S}$, [59]:

Definition 2.2.1 *Let $\mathcal{S}$ be a subset of $\mathbf{R}^n$. If for every initial state $\mathbf{x}_0$, there exists an $\epsilon > 0$ such that $\mathbf{x}(t) \in \mathcal{S}$ for all $t \in [0, \epsilon]$, then we say that the system has the* **smooth continuation property with respect to $\mathcal{S}$**.

Although there is yet no useful characterization of well-posedness of the general (2.1), we can use the framework of [59] for well-posedness of bimodal ($N = 2$) conewise systems.

Definition 2.2.2 The system (2.1) is well-posed *if, for every $i \neq j \in \{1, ..., N\}$, i) there are no sliding modes in the common face $\mathsf{B}_{ij} := \mathbf{S}_i \cap \mathbf{S}_j$ and ii) for every initial state $\mathbf{b} \in \mathsf{B}_{ij}$, smooth continuation is possible in either with respect to $\mathbf{S}_i$ only or with respect to $\mathbf{S}_j$ only, except when the solutions (trajectories) coincide in both modes.*

We can now recall Lemma 2.1 of [59] that gives a necessary and sufficient condition for well-posedness of a *bimodal* system, which consists of two modes defined on two half-planes separated by a plane B_{ij}.

Let $\mathbf{c}_{ij} \in \mathbf{R}^n$ be a vector that is orthogonal to B_{ij} so that B_{ij} is equal to the null space of $\mathbf{c}_{ij}^T$. Let $\mathcal{O}_i$ and $\mathcal{O}_j$ be the observability matrices associated with $(\mathbf{c}_{ij}, A_i)$ and $(\mathbf{c}_{ij}, A_j)$.

Lemma 2.2.1 *The system (2.1) is well-posed if i) both $\mathcal{O}_i$ and $\mathcal{O}_j$ are nonsingular and ii) the $n \times n$ matrix $\mathcal{O}_i^{-1}\mathcal{O}_j$ is lower triangular with positive diagonal entries.*

We remark here that the observability condition (i) for $(\mathbf{c}_{ij}, A_i)$ and $(\mathbf{c}_{ij}, A_j)$ is equivalent to no eigenvector of A_i or A_j being contained in the common face B_{ij} and eliminates the possible existence of sliding modes [60]. It will be seen below that no eigenvector of any mode can lie in any face of a conewise mode. This means that, for well-posedness of the CLS, it is necessary that each mode need to be observable from each of its faces.

The result of the lemma is, of course, neither necessary nor sufficient even in the case of just two neighboring conewise modes because the common plane does not extend to infinity but is a planar cone and because the modes are defined on spatial cones and not on half-planes.

We will thus assume below that the CLS under consideration is always well-posed without going further into the precise characterization of well-posedness.

2.3 Behavioral Characteristics of Modes

For planar and spatial conewise linear systems, we will provide expressions for trajectories $\mathbf{x}(t, \mathbf{b})$ of (3.2) resulting from an initial condition $\mathbf{b} \in \mathbf{S}$ and determine the conditions under which trajectories may intersect one of the faces (or boundaries). This will lead to a characterization of a given mode as being *transitive*, or a *source*, *sink*, and *half-sink*. Although our main concern is with the 3D case, we would like to provide here definitions that are also valid for the general case of n-dimension.

Definition 2.3.1 *i) A mode (defined in a cone)* $\mathbf{S}$ *is a* **sink** *if for every* $\mathbf{b} \in \mathbf{S}$ *and for all* $t \geq 0$, $\mathbf{x}(t, \mathbf{b}) \in \mathbf{S}$.

ii) A mode $\mathbf{S}$ *is* k**-transitive** *or, equivalently, transitive from its face* B_k *if, (a) for every* $0 \neq \mathbf{b} \in \mathbf{S}$, *there exists a finite* $t_k^* > 0$ *such that* $\mathbf{x}(t_k^*, \mathbf{b}) \in \mathsf{B}_k$ *and* $\mathbf{x}(t, \mathbf{b}) \in \mathbf{S}$ *for all* $t \in (0, t_k^*)$ *and (b) for any* $\mathbf{b} \in \mathsf{B}_j, j \neq k$, *there is* $\epsilon > 0$ *such that* $\mathbf{x}(t, \mathbf{b}) \in \mathsf{S}$ *for all* $t \in (0, \epsilon)$.

iii) A mode $\mathbf{S}$ *is* $k_1 ... k_p$**-transitive** *if (a) holds for all* $k \in \{k_1, ..., k_p\}$ *and (b) holds for all the remaining faces* B_j $(j \in \{1, ..., n\} - \{k_1, ..., k_p\})$.

iv) A mode is a **source** *if, first, for every* $0 \neq \mathbf{b} \in \mathbf{B}_k$, *there exists a finite* $\epsilon > 0$ *such that* $\mathbf{x}(t, \mathbf{b}) \notin \mathbf{S}$ *for all* $t \in (0, \epsilon]$ *and, second, for all* $\mathbf{b} \in \mathbf{S}$, *except those on a cone of dimension one, there exists* $t^* > 0$ *such that,* $\mathbf{x}(t^*, \mathbf{b}) \in \mathsf{B}_k$ *and* $\mathbf{x}(t, \mathbf{b}) \in \mathbf{S}$.

v) A mode is **half-sink** *if it is not one of (i)-(iv).*

In a sink, every trajectory that starts in the cone (interior and the borders) remain in the cone and, in a well-posed conewise system, any trajectory starting sufficiently close to the border with any one of the three neighbor cones will move in to the sink. The trajectories may converge to the origin or diverge to infinity but stay always inside.

By contrast, in a source, any trajectory that starts in the cone will move out of the cone into one of the three neighbor cones with the possible exception of those that start on a ray that extends from the origin to infinity inside the cone. We actually show later that in 2D and 3D cases such a ray must necessarily exist for a cone to be a source.

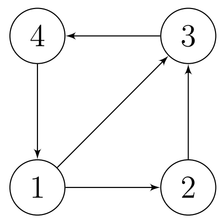
In the transitive cases from one or more faces, the trajectories come in from one or more faces but all eventually go out without converging to the origin or diverging to infinity.

There remains, of course, a whole lot of mixed cases in which the cone is divided into sectors with different behavior in each. These left out cases by (i) to (iv) are labeled "half-sink." In 2D and 3D, such characteristics are a bit easier to identify as we will show below.

2.4 On Cyclic and Acyclic Directed Graphs

We provide here some standard definitions from graph theory, focusing on cyclic directed graphs. These are needed in Chapter 3, where we state our main stability result.

A graph is a mathematical structure used to model pairwise relationship between different objects. It is made up of vertices (nodes) which are connected by edges (lines). The graphs where edges link two vertices symmetrically are called *undirected graphs*; on the other hand the graphs where edges link two vertices asymmetrically are called *directed graphs* (**digraphs**). In this thesis, we will need to consider only digraphs. The *degree* of a node is equal to the number of edges associated with the node. The number of outgoing edges from a node is called **out-degree** and number of incoming edges is called **in-degree**. Readers are referred to [56, Chapter 1] for more formal definitions.



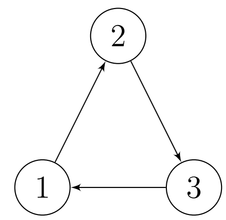


Figure 2.1: Directed cyclic graphs.

Definition 2.4.1 *A* **directed path** *in a directed graph is a sequence of vertices in which there is a (directed) edge pointing from each vertex in the sequence to its successor in the sequence. A simple path is one with no repeated vertices.*

Definition 2.4.2 *A* **directed cycle** *is a directed path (with at least one edge) whose first and last vertices are the same. A simple cycle is a cycle with no repeated edges or vertices (except the requisite repetition of the first and last vertices)*

Consider the graphs given in Figure 3.16. In the graph shown in Figure 2.1 (a), $1 \to 3$, $1 \to 2 \to 3 \to 4$ and $3 \to 4 \to 1$ are simple paths and $1 \to 2 \to 3 \to 4 \to 1$ and $1 \to 3 \to 4 \to 1$ are directed cycles. In the graph shown in Figure 2.1 (b), $1 \to 2 \to 3$ is a directed cycle.

Definition 2.4.3 *A* **directed cyclic graph** *is a graph containing at least one directed cycle.*

Since the graphs given in Figure 3.16 consist of multiple directed cycles, both the graphs are directed cyclic graphs.

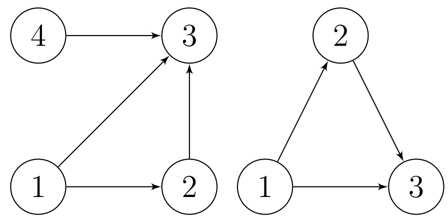


Figure 2.2: Directed acyclic graphs.

Definition 2.4.4 *An* **acyclic digraph** *is a directed graph with no directed cycles.*

Two examples of typical directed acyclic graphs are given in Figure 2.2.

Definition 2.4.5 *A* **cubic digraph** *is a directed graph where each node has a degree equal to three.*

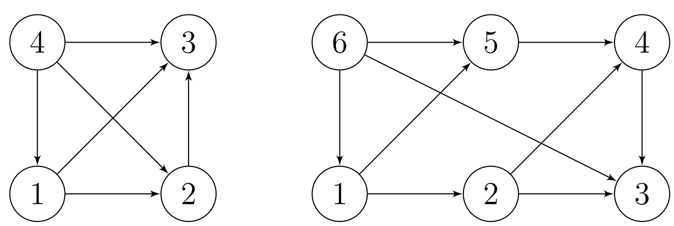


Figure 2.3: Cubic digraphs.

Examples of cubic digraphs are given in Figure 2.3. In Figure 2.3, every node has exactly three edges associated with it. Now we provide a well-known result to check acyclicity of a given directed graph by using the adjacency matrix associated with it. An adjacency matrix P for a directed graph is defined by

$$P_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ is an edge} \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4.1 *Given a directed graph with n vertices represented by an adjacency matrix $P \in \mathbf{R}^n$, the graph is acyclic if and only if one of the following equivalent conditions holds:*

i. $tr(\exp(P) - I) = 0$.

ii. P *is nilpotent.*

iii. *Graph has a topological sort, i.e., there is a linear ordering of its vertices such that for every directed edge* $i \to j$ *from vertex* i *to vertex* j, i *comes before* j *in the ordering.*

Proof.

(i) We reproduce the proof of [61] here. Consider an arbitrary digraph with adjacency matrix P then $P_{ij} = 1$ if and only if we have path from j to i, where i, j are nodes of the graph. Now, consider the product $P_{ik}P_{kj} = 1$ if and only if there is a path of length 2 from j to i via k. Hence, the total number of paths of length $2, 3, ..., r$ from j to i via any vertex is

$$\begin{aligned} R_{ij}^{(2)} &= \textstyle\sum_{k=1}^{n} P_{ik}P_{kj} = [P^2]_{ij} \\ R_{ij}^{(3)} &= \textstyle\sum_{k,l=1}^{n} P_{ik}P_{kl}P_{lj} = [P^3]_{ij} \\ &\vdots \\ R_{ij}^{(r)} &= [P^r]_{ij} \end{aligned} \tag{2.4}$$

For a cyclic case, we have $i = j$, so it leads us to $R_{ii}^{(r)} = [P^r]_{ii},\ i = 1, \ldots, n$. Now consider the following equality

$$\exp(P) - I = P + \frac{P^2}{2!} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k!}P^k \tag{2.5}$$

which can be seen as positive combination of matrices of all possible path lengths i.e., $k = 1, \ldots, \infty$. Since we are concerned with only diagonal entries ii^{th}, if the $diag(\exp(P) - I) = 0$ there exits no path of any length $k = 1, \ldots, \infty$ from node i to i. Thus the digraph is acyclic.

(ii) Consider the Schur decomposition $P = QDQ^T$, where D is strictly upper triangular matrix and $Q^T = Q^{-1}$. Note, P and D has the same eigenvalues. From (i), we have

$$R_{ii}^{(r)} = [P^r]_{ii}. \tag{2.6}$$

The total number of L_r of closed walks of length r in the digraph is equal to the following summation

$$L_r = \sum_{i=1}^{n} [P^r]_{ii} = tr(P^r) \tag{2.7}$$

Using the circularity of $tr(XY) = tr(YX)$, we write the following

$$\begin{aligned} QDQ^T u = Pu = \lambda_P u \\ (Q^T Q) D Q^T u = \lambda_P Q^T u \end{aligned} \tag{2.8}$$

where u is eigenvector and λ_P is eigenvalue of P, hence $Q^T u$ is eigenvector of D with same eigenvalue λ_P. Finally,

$$L_r = tr(P^r) = tr(QD^r Q^T) = tr(QQ^T D^r) = tr(D^r) = \sum_i \lambda_i^r. \tag{2.9}$$

For the acyclic case $tr(P^r) = 0$, thus all the eigenvalues must be equal to zero as a result P is nilpotent.

(iii) The proof is straight forward and can be found in standard books, see [62, Page 551]. □

In a cubic digraph of even number of nodes, one can derive some conditions that need to be satisfied among the number of sources, sinks, and transitive nodes.

Lemma 2.4.2 *Given a cubic digraph with $2M$ nodes, where $M \geq 2$, let p, q, r, s denote the number of nodes with out-degree equal to zero, one, two, and three (or equivalently, with in-degree equal to three, two, one, and zero) respectively. Then, the following holds*

$$\begin{aligned} p = M - \tfrac{1}{3}(2q + r), \\ s = M - \tfrac{1}{3}(q + 2r). \end{aligned} \tag{2.10}$$

Proof. The total number of modes satisfy $p + q + r + s = 2M$. Counting the number of outgoing edges in two ways, we also have $3s + 2r + q = 3M$. These two equalities give (2.10). □

Chapter 3

Main Results

In this chapter, we present some conditions for a given 2D or 3D CLS to be globally asymptotically stable. Section 3.1 illustrates the technique to be used for 3D by restating some known results of stability of a 2D CLS combining geometric characterization of 2D modes with a graph theoretic approach. Section 3.2 and Section 3.3 are devoted to results form [60] on the classification of single 3D modes as transitive, source, or sink. In Section 3.4, we present our main results on global asymptotic stability of a 3D CLS using the tools from graph theory and behavioral characterization of single modes.

3.1 Second Order Conewise Linear Systems

The class of second-order (planar) conewise linear systems is (2.1) with $n = 2$, i.e.,

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_1, \\ A_2\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_2, \\ \vdots & \vdots \quad \vdots \\ A_N\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_N, \end{cases} \tag{3.1}$$

where $A_i \in \mathbf{R}^{2\times 2}$ and, with $C_i \in \mathbf{R}^{2\times 2}$,

$$\mathbf{S}_i := \{\mathbf{x} \in \mathbf{R}^2 : C_i\mathbf{x} \geq 0\},$$

for $i = 1, 2, ..., N$. We assume, as in (2.1), that (3.7) is memoryless and

$$S_i = \left[\begin{array}{cc} \mathbf{s}_{i1} & \mathbf{s}_{i2} \end{array} \right] := C_i^{-1} = \left[\begin{array}{c} \mathbf{c}_{i1}^T \\ \mathbf{c}_{i2}^T \end{array} \right]^{-1}$$

so that $\det S_i > 0$. It is easy to see that, if each $\mathbf{S}_i$, $i = 1, ..., N$ is strictly contained in a half-plane, then $\mathbf{S}_i$ is a convex cone

$$\mathbf{S}_i = cone\{\mathbf{s}_{i1}, \mathbf{s}_{i2}\}.$$

The boundaries of each $\mathbf{S}_i$ are two rays

$$\mathsf{B}_{ik} = cone\{\mathbf{s}_{ik}\},\ k \in \{1, 2\}$$

If, in (3.1), there is a mode defined on a half-plane or a cone larger than a half-plane, then it can be split into two modes having the same dynamics (the same A-matrix) so that each is still defined on a convex cone.

Given a mode i, its eigenvalues will be denoted by $\lambda_{i1}, \lambda_{i2} \in \mathbf{C}$ and, in case of real eigenvalues, they will be indexed so that $\lambda_{i1} \geq \lambda_{i2}$.

We note here without proof, as it is obvious, that, in a memoryless CLS, the minimum number of conewise modes that cover R^2 is three and that the plane can be divided into any given number N of modes provided $N \geq 3$.

3.1.1 Characterization of a Single Mode in $\mathbf{R}^2$

We now summarize the characterization results of [49] for a single planar mode based on the geometry of eigenvectors relative to the cone. We will restrict ourselves to real and distinct eigenvalue case. Consider

$$\dot{\mathbf{x}} = A\mathbf{x},\ \mathbf{x} \in \mathbf{S} = cone\{\mathbf{s}_1, \mathbf{s}_2\}, \tag{3.2}$$

and $S = [\mathbf{s}_1 \;\; \mathbf{s}_2]$ with $\det S > 0$. Suppose the eigenvalues of A are both real and satisfy $\lambda_1 > \lambda_2$ and let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$ be the corresponding eigenvectors so that

$$AV = V\Lambda,\; V = \left[\begin{array}{cc} \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right],$$

where $\Lambda = diag\{\lambda_1, \lambda_2\}$ and

$$W = \left[\begin{array}{c} \mathbf{w}_1^T \\ \mathbf{w}_2^T \end{array}\right] := V^{-1}.$$

We again choose the eigenvectors in such a way that $\det V > 0$, which is equivalent to the pair $(\mathbf{v}_1, \mathbf{v}_2)$ being positively oriented. (In the plane represented by a paper, their cross product comes out of the paper.)

Consider the identity $CV = (WS)^{-1}$ or more explicitly,

$$\left[\begin{array}{cc} \mathbf{c}_1^T\mathbf{v}_1 & \mathbf{c}_1^T\mathbf{v}_2 \\ \mathbf{c}_2^T\mathbf{v}_1 & \mathbf{c}_2^T\mathbf{v}_2 \end{array}\right]\left[\begin{array}{cc} \mathbf{w}_1^T\mathbf{s}_1 & \mathbf{w}_1^T\mathbf{s}_2 \\ \mathbf{w}_2^T\mathbf{s}_1 & \mathbf{w}_2^T\mathbf{s}_2 \end{array}\right] = I. \tag{3.3}$$

Note that, $\mathbf{c}_k^T\mathbf{v}_i = 0$ if and only if $\mathbf{v}_i = \alpha\mathbf{s}_j, j \neq k, \alpha \in \mathbf{R}$ and $\mathbf{w}_k^T\mathbf{s}_i = 0$ if and only if $\mathbf{s}_i = \beta\mathbf{v}_j, j \neq k, \beta \in \mathbf{R}$, for any $i, j, k \in \{1, 2\}$. Both conditions correspond to an eigenvector inside a border, which inplies the existence of a sliding mode. Conversely, if a sliding mode exists at any of the two borders, then an eigenvector must be in that border since the only one dimensional A-invariant subspaces are lines defined by eigenvectors. Thus, it follows that there are no sliding modes at the faces of a single mode if and only if $\mathbf{c}_k^T\mathbf{v}_i \neq 0$ for any $k \neq i \in \{1, 2\}$ (equivalently, $\mathbf{w}_k^T\mathbf{s}_i = 0$ for any $k \neq j \in \{1, 2\}$).

The unique solution of (3.2) for the initial state at $\mathbf{b}$, the *trajectory*, is given by

$$\mathbf{x}(t, \mathbf{b}) = e^{\lambda_1 t}\mathbf{w}_1^T\mathbf{b}\,\mathbf{v}_1 + e^{\lambda_2 t}\,\mathbf{w}_2^T\mathbf{b}\,\mathbf{v}_2 \tag{3.4}$$

This hits the boundary B_i at a finite time $t_i > 0$ if and only if $\mathbf{c}_k^T\mathbf{x}(t_i, \mathbf{b}) = 0$, $i \neq k$, where

$$\mathbf{c}_k^T\mathbf{x}(t, \mathbf{b}) = e^{\lambda_1 t}\,n_{k1} + e^{\lambda_2 t}\,n_{k2},$$

with

$$n_{ki} := \mathbf{c}_k^T\mathbf{v}_i\,\mathbf{w}_i^T\mathbf{b},\; i = 1, 2.$$

The direction of the trajectory is given by

$$\dot{x}(t, \mathbf{b}) = \lambda_1 e^{\lambda_1 t} \mathbf{w}_1^T \mathbf{b} \mathbf{v}_1 + \lambda_2 e^{\lambda_2 t} \mathbf{w}_2^T \mathbf{b} \mathbf{v}_2 \tag{3.5}$$

If $\mathbf{x}(t_i, \mathbf{b}) = 0$ for $\mathbf{b} \in \mathbf{R}^n$, then, by $\mathbf{c}_k^T \mathbf{x}(t_i, \mathbf{b}) = 0$, we can write $e^{(\lambda_1 - \lambda_2)t_i}\, n_{k1} = n_{k2}$ and, by (3.5) at $t = t_i$, we obtain

$$\mathbf{c}_k^T \dot{\mathbf{x}}(t_i, \mathbf{b}) \;\; = (\lambda_1 - \lambda_2) e^{\lambda_1 t_i} n_{k1} = -(\lambda_1 - \lambda_2) e^{\lambda_2 t_i} n_{k2}. \tag{3.6}$$

Thus, if $\mathbf{b} \in int\mathbf{S}$, then just before the hit, the trajectory was moving towards $\mathbf{s}_i$ if and only if $\mathbf{c}_k^T \dot{\mathbf{x}}(t_i, \mathbf{b}) < 0$. It follows that a trajectory starting inside $\mathbf{S}$ hits B_i and moves out of $\mathbf{S}$ if and only if $n_{k1} < 0$, $n_{k2} > 0$. Similarly, if $\mathbf{b} \notin \mathbf{S}$, then it hits B_i and is moving in $\mathbf{S}$ if and only if $n_{k1} > 0$, $n_{k2} < 0$. This analysis yields Table 3.1.

Table 3.1: Conditions on $\mathbf{b} \in \mathbf{S}$ for exit from $\mathsf{B}_i, i = 1, 2$.

n_{k1}	n_{k2}	Hit B_i	Direction
-	-	no	-
-	+	yes	out
+	-	yes	in
+	+	no	-

Table 3.1 can now be used to classify each mode according to the behavioral characteristics of trajectories starting inside its defining cone.

Consider the positioning of the eigevectors in Figure 3.1a, where the eigenvector $\mathbf{v}_1$ associated with the larger eigenvalue is inside the cone and $\mathbf{v}_2$ associated with the smaller eigenvalue is exterior to the cone. It is easy to see that the geometry of the Figure 3.1a, we can conclude that $\mathbf{c}_1^T \mathbf{v}_1 > 0, \mathbf{c}_1^T \mathbf{v}_2 < 0, \mathbf{c}_2^T \mathbf{v}_1 > 0$, and $\mathbf{c}_2^T \mathbf{v}_2 > 0$. Moreover, it is clear from the Figure 3.1a that $\mathbf{v}_1$ is bifurcating the cone into two regions and we can calculate the signs of $\mathbf{w}_1^T \mathbf{b}$ and $\mathbf{w}_2^T \mathbf{b}$ for all initial conditions inside both regions separately. For the region 1, we have $\mathbf{w}_1^T \mathbf{b} > 0$ and $\mathbf{w}_2^T \mathbf{b} < 0$, and for region 2, we have $\mathbf{w}_1^T \mathbf{b} > 0$ and $\mathbf{w}_2^T \mathbf{b} > 0$. Based on these signs of CV and WS matrices, we determine the signs of n_{k1} and n_{k2} for both regions. It turns out that, $n_{11} > 0, n_{12} > 0, n_{21} > 0$ and $n_{22} < 0$ for region 1 and $n_{11} > 0, n_{12} < 0, n_{21} > 0$ and $n_{22} > 0$ for region 2, respectively. Employing the results from Table 3.1, we can characterize this given configuration as a sink.

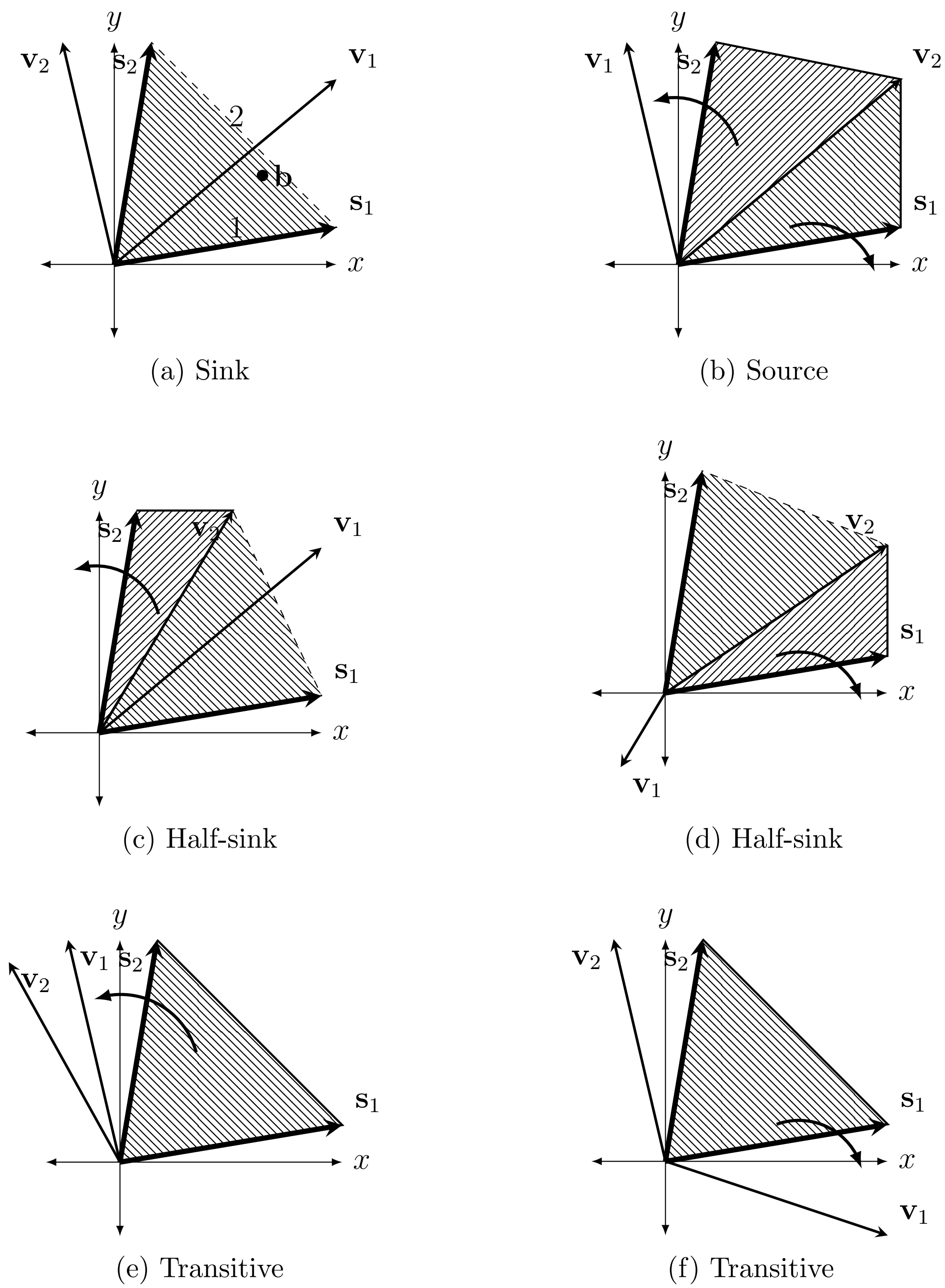
Remark 3.1.1 *To see that* $\mathbf{c}_1^T\mathbf{v}_2 < 0$*, note* $\mathbf{v}_2 = \alpha\mathbf{s}_2 + \beta(-\mathbf{s}_1)$ *for* $\alpha, \beta > 0$ *but* $\alpha = \mathbf{c}_2^T\mathbf{v}_2, -\beta = \mathbf{c}_1^T\mathbf{v}_2$ *so that* $\mathbf{c}_1^T\mathbf{v} < 0$ *(and* $\mathbf{c}_2^T\mathbf{v}_2 > 0$*).*

Based on the same analysis, we provide basins for distinct eigenvalues associated with positively oriented eigenvectors for a typical source in Figure 3.1b, half-sinks in Figure 3.1c, and Figure 3.1d and 1-transitive modes in Figure 3.1e, and Figure 3.1f, respectively.

Exploiting the geometry of eigenvectors with distinct eigenvalues, the characterization of a given mode in $\mathbf{R}^2$ can be summarized as follows:

- A mode is a sink if and only if $\mathbf{v}_1$ is inside and $\mathbf{v}_2$ is exterior to its cone.
- A mode is a source if and only if $\mathbf{v}_2$ is inside and $\mathbf{v}_1$ is exterior to its cone.
- A mode is transitive if and only if $\mathbf{v}_1$, and $\mathbf{v}_2$ are both exterior to its cone. The direction of the trajectory is towards $\mathbf{v}_1$ following the short route.
- A mode is a half-sink if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are both inside its cone. The direction of the trajectories starting inside the transitive half is towards $\mathbf{v}_1$ following the short route.

In order to determine the corresponding digraph of a given CLS, we can represent every planar mode defined by a cone with a node and two associated edges. A transitive cone can be represented as a node with one outgoing and one incoming edge. Similarly, a source can be represented as a node with two outgoing edges and a sink can be represented as a node with two incoming edges. A half-sink which can be represented as two disjoint nodes with one incoming edge to its sink sub-cone and one outgoing edge from its transitive part. Note that the direction of trajectories associated with the transitive and half-sink modes are lacking in these representations. The graph however of a given CLS will be well-defined whenever the CLS is well-posed. Figure 3.2 provides a general idea about the representations of individual modes.



(a) Sink (b) Source (c) Half-sink (d) Half-sink (e) Transitive (f) Transitive

Figure 3.1: Basins for distinct eigenvalues for 2D conewise linear systems.

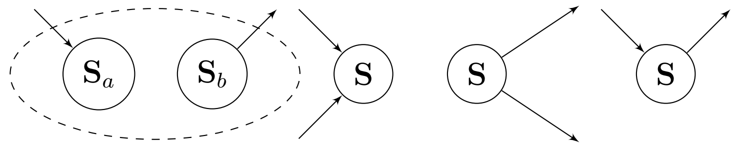


Figure 3.2: Node-edge representations of a half-sink, sink, source, and one-transitive mode.

We now give a stability result that applies to a CLS with no cyclic trajectories. The stated result is a very simple special case of the general stability results of [27] or [49]. Our purpose here is to provide a comparison with the analogous 3D result derived below in Section 3.6.

Theorem 3.1.1 *Suppose a CLS (3.1) with N modes consists only of sinks, sources, half-sinks and 1-transitive modes. If such a CLS is well-posed, then it corresponds to a 2-regular (at most two edges per node) digraph that is free of self-loops and loops of length two. If the CLS is well-posed and its digraph is acyclic, then it is globally asymptotically stable if and only if it is such that*

i. the largest eigenvalue of every sink, and half-sink mode is negative, and

ii. the smallest eigenvalue of every source mode is negative.

Proof. If a CLS is well-posed then, by the fact that smooth continuation is possible in only one of the modes, there can be no initial condition at a boundary that would result in trajectories outgoing or incoming from different modes. Since the modes are restricted to pure transitive modes, sinks, half-sinks, and sources (no mixed type boundaries), it follows that self-cycles are avoided.

To prove the second statement, first note that every acyclic digraph must include among its nodes at least one source and at least one sink or atleast one half-sink (see e.g., Property 19.12 in [63]). Since every acyclic digraph has a topological sort, there are no cycles and every trajectory starting at any mode should enter a sink or half-sink except the initial conditions starting at rays defined by the eigenvectors associated with the smallest eigenvalues of the sources.

The condition (i) is necessary and sufficient, for such trajectories to converge to the origin and condition (ii) is necessary and sufficient, for all other trajectories to converge to the origin after arriving at a sink or sink sector of half-sink. $\square$

Corollary 3.1.1 *Given a well-posed CLS (3.1) with N modes i.e., no self-loops and loops of length two. If the CLS consists of transitive and half-sink modes only, with at least one half-sink, then it is globally asymptotically stable if and only if largest eigenvalue of every half-sink mode is negative.*

Proof. The only possible loop in a well-posed planar (2D) CLS is of length N i.e., for a loop to be completed, the trajectory initiated in an arbitrary node must revisit it after going through all other nodes. The cyclicity of such directed cycle can be broken by introducing at least a pair of disjoint nodes (or equivalently, half-sink) in its directed path.

Now, as mentioned earlier, every acyclic digraph has a topological sort, there are no cycles and every trajectory starting at any mode should enter a sink sector of any of the half-sinks. For all trajectories to converge to the origin after arriving at the sink sector, it is necessary and sufficient to have negative eigenvalues. The proof is completed. $\square$

3.2 Third Order Conewise Linear Systems

We now consider third order conewise linear systems as special cases of (2.1) with $n = 3$ given by

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_1, \\ A_2\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_2, \\ \vdots & \vdots \quad \vdots \\ A_N\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_N, \end{cases} \tag{3.7}$$

where $A_i \in \mathbf{R}^{3\times 3}$ and, with $C_i \in \mathbf{R}^{3\times 3}$,

$$\mathbf{S}_i := \{\mathbf{x} \in \mathbf{R}^3 : C_i\mathbf{x} \geq 0\},$$

for $i = 1, 2, ..., N$. We assume that (3.7) is also memoryless and let

$$S_i = \begin{bmatrix} \mathbf{s}_{i1} & \mathbf{s}_{i2} & \mathbf{s}_{i3} \end{bmatrix} := C_i^{-1} = \begin{bmatrix} \mathbf{c}_{i1}^T \\ \mathbf{c}_{i2}^T \\ \mathbf{c}_{i3}^T \end{bmatrix}^{-1}$$

so that $\det S_i > 0$. It is easy to see that, if each $\mathbf{S}_i$, $i = 1, ..., N$ is strictly contained in a half-space, then $\mathbf{S}_i$ is a tri-hedral convex cone

$$\mathbf{S}_i = cone\{\mathbf{s}_{i1}, \mathbf{s}_{i2}, \mathbf{s}_{i3}\},$$

where $cone\{\mathbf{v}_1, ..., \mathbf{v}_l\} = \{\alpha_1 \mathbf{v}_1 + ... + \alpha_l \mathbf{v}_l : \alpha_j \geq 0, \, j = 1, ..., l\}$. The faces of each $\mathbf{S}_i$ are three planar cones

$$\mathsf{B}_{ik} = cone\{\mathbf{s}_{il}, \mathbf{s}_{im}\}, \; k \neq l \neq m \in \{1, 2, 3\}$$

The **extreme rays** of each $\mathbf{S}_i$ are the rays defined by its boundary vectors $\alpha \mathbf{s}_{i1}, \beta \mathbf{s}_{i2}, \gamma \mathbf{s}_{i3}, \; \alpha, \beta, \gamma > 0$. Note that because $\det S_i > 0$, the cross products $\mathbf{s}_{i1} \times \mathbf{s}_{i2}$, $\mathbf{s}_{i2} \times \mathbf{s}_{i3}$, $\mathbf{s}_{i3} \times \mathbf{s}_{i1}$ are positively oriented according to the right-hand rule.

We note here that if, in a 3D piecewise linear system (3.1), there is a mode defined on a half-space or a cone larger than a half-space, then it can be split into two modes having the same dynamics (the same A-matrix) so that each is still defined on a convex cone and can be still modeled by (3.7).

Given a mode i, its eigenvalues will be denoted by $\lambda_{i1}, \lambda_{i2}, \lambda_{i3} \in \mathbf{C}$ and, in case of real eigenvalues, they will be indexed so that $\lambda_{i1} \geq \lambda_{i2} \geq \lambda_{i3}$.

3.2.1 Some More Definitions and Facts

This section introduces basic definitions and facts for third order conewise linear systems that are essential for the construction of our main results in the following chapters.

Definition 3.2.1 $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *is* positively oriented *if* $\mathbf{s}_1 \cdot \mathbf{s}_2 \times \mathbf{s}_3 = \det[\mathbf{s}_1 \; \mathbf{s}_2 \; \mathbf{s}_3] > 0$

Definition 3.2.2 *A vector* $\mathbf{v}$ *is* inside *cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *if* $\mathbf{v} = \alpha\mathbf{s}_1 + \beta\mathbf{s}_2 + \gamma\mathbf{s}_3$ *for some* $\alpha, \beta, \gamma > 0$. *A vector* $\mathbf{v}$ *is* interior *to cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *if* $\mathbf{v}$ *or* $-\mathbf{v}$ *is inside the cone. It is* exterior *to cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *if it is not interior to it and it is not on the boundary cone*$\{\mathbf{s}_i, \mathbf{s}_j\}$, $(i, j) \in \{1, 2, 3\}$.

Fact 3.2.1 *A vector* $\mathbf{v}$ *is inside cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *if and only if*

$$\mathbf{v} \cdot (\mathbf{s}_2 \times \mathbf{s}_3), \mathbf{v} \cdot (\mathbf{s}_3 \times \mathbf{s}_1),\ \mathbf{v} \cdot (\mathbf{s}_1 \times \mathbf{s}_2) \tag{3.8}$$

have the same sign as $\mathbf{s}_1 \cdot (\mathbf{s}_2 \times \mathbf{s}_3)$.

Fact 3.2.2 *Let* $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *and* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *be positively oriented. Let*

$$C := [\mathbf{s}_1,\ \mathbf{s}_2,\ \mathbf{s}_3]^{-1} = S^{-1}, W = [\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3]^{-1} = V^{-1}.$$

Let

$$C = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix},\ W = \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \end{bmatrix}.$$

Then,

$$\mathbf{c}_i = \mathbf{s}_j \times \mathbf{s}_k,\ \mathbf{w}_i = \mathbf{v}_j \times \mathbf{v}_k,\ \{i, j, k\} = \{1, 2, 3\}$$
$$\mathbf{c}_i^T\mathbf{v} = \mathbf{v} \cdot (\mathbf{s}_j \times \mathbf{s}_k),\ \mathbf{w}_i^T\mathbf{b} = \mathbf{b} \cdot (\mathbf{v}_j \times \mathbf{v}_k).$$

3.3 Characterization of a Single Mode in $\mathbf{R}^3$

We now focus on a single mode i (and temporarily drop index i that designates a mode) to consider

$$\dot{\mathbf{x}} = A\mathbf{x},\ \mathbf{x} \in \mathbf{S} = cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}, \tag{3.9}$$

and $S = [\mathbf{s}_1\ \ \mathbf{s}_2\ \ \mathbf{s}_3]$ with $\det S > 0$. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{R}^3$ be such that

$$AV = V\Lambda,\ V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix},$$

where Λ is a Jordan form of A. Also let

$$W = \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \end{bmatrix} := V^{-1}.$$

Depending on whether the eigenvalues are real or non-real and distinct or repeated, there are a total of seven possible Jordan forms and $\mathbf{v}_2$ and/or $\mathbf{v}_3$ may be either generalized eigenvectors or the real or imaginary parts of non-real eigenvectors. The word "eigenvector" will refer to a true eigenvector. Let $\mathbf{x}(t, \mathbf{b})$ denote the solution (trajectory) of (3.9) resulting from an initial condition $\mathbf{b}$ inside the cone $\mathbf{S}$. Consider the identity $CV = (WS)^{-1}$ in explicit form

$$\begin{bmatrix} \mathbf{c}_1^T\mathbf{v}_1 & \mathbf{c}_1^T\mathbf{v}_2 & \mathbf{c}_1^T\mathbf{v}_3 \\ \mathbf{c}_2^T\mathbf{v}_1 & \mathbf{c}_2^T\mathbf{v}_2 & \mathbf{c}_2^T\mathbf{v}_3 \\ \mathbf{c}_3^T\mathbf{v}_1 & \mathbf{c}_3^T\mathbf{v}_2 & \mathbf{c}_3^T\mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \mathbf{w}_1^T\mathbf{s}_1 & \mathbf{w}_1^T\mathbf{s}_2 & \mathbf{w}_1^T\mathbf{s}_3 \\ \mathbf{w}_2^T\mathbf{s}_1 & \mathbf{w}_2^T\mathbf{s}_2 & \mathbf{w}_2^T\mathbf{s}_3 \\ \mathbf{w}_3^T\mathbf{s}_1 & \mathbf{w}_3^T\mathbf{s}_2 & \mathbf{w}_3^T\mathbf{s}_3 \end{bmatrix} = I \tag{3.10}$$

We can note that, if (k, l, m) is a positive permutation of $(1, 2, 3)$, then $\mathbf{c}_k^T\mathbf{v}_i = 0$ if and only if $\mathbf{v}_i$ is in $\mathbf{s}_m\mathbf{s}_l$-plane and $\mathbf{w}_m^T\mathbf{s}_i = 0$ if and only if $\mathbf{s}_i$ is in $\mathbf{v}_m\mathbf{v}_l$-plane, for any $i \in \{1, 2, 3\}$.

Definition 3.3.1 *A nonzero vector* $\mathbf{v}$ *is* **exterior to S** *if for all* $\alpha \in \mathbf{R}$ *it holds that* $\alpha\mathbf{v} \notin \mathbf{S}$. *It is* **interior to S** *if for some* $\alpha \neq 0$, $\alpha\mathbf{v} \in \mathbf{S}$.

Fact 3.3.1 *A real eigenvector* $\mathbf{v}_k$ *of* A *is exterior to* $\mathbf{S}$ *if and only if all entries of the* k*-th column of the matrix* CV *are nonzero and do not have the same sign, i.e., iff the set* $\{\mathbf{c}_1^T\mathbf{v}_k, \mathbf{c}_2^T\mathbf{v}_k, \mathbf{c}_3^T\mathbf{v}_k\}$ *has a positive and a negative element.*

The following repeats Def. 2.3.1 for the special case of $\mathbf{R}^3$.

Definition 3.3.2 *i) A mode is a* **sink** *if for every* $\mathbf{b} \in \mathbf{S}$ *and for all* $t \geq 0$, $\mathbf{x}(t, \mathbf{b}) \in \mathbf{S}$. *ii) A mode is* k-**transitive** *for* $k \in \{1, 2, 3\}$ *if for every* $0 \neq \mathbf{b} \in \mathbf{S}$, *there exists a finite* $t_k^* > 0$ *such that* $\mathbf{x}(t_k^*, \mathbf{b}) \in \mathsf{B}_k$ *and* $\mathbf{x}(t, \mathbf{b}) \notin \mathsf{B}_l \cup \mathsf{B}_m$ *for any* $t \in (0, t_k^*)$. *iii) A mode is* lm-**transitive** *for* $l, m \in \{1, 2, 3\}$ *if for every*

$0 \neq \mathbf{b} \in \mathbf{S}$*, there exists a finite* $t_k^* > 0$ *such that* $\mathbf{x}(t_k^*, \mathbf{b}) \in \mathsf{B}_l \cup \mathsf{B}_m$ *and* $\mathbf{x}(t, \mathbf{b}) \notin \mathsf{B}_k$ *for any* $t \in (0, t_k^*)$*. iv) A mode is a* **source** *if, first, for* $k \in \{1, 2, 3\}$ *and for every* $0 \neq \mathbf{b} \in \mathbf{B}_k$*, there exists a finite* $\epsilon > 0$ *such that* $\mathbf{x}(t, \mathbf{b}) \notin \mathbf{S}$ *for all* $t \in (0, \epsilon]$ *and, second, for all* $\mathbf{b} \in \mathbf{S}$ *except those on a cone of dimension one or two, there exists* $t^* > 0$ *such that* $\mathbf{x}(t, \mathbf{b}) \in \mathsf{B}_k \cup \mathsf{B}_l \cup \mathsf{B}_m$ *for all* $t \in (0, t^*)$*.*

We now derive expressions for trajectories $\mathbf{x}(t, \mathbf{b})$ of (3.9) resulting from an initial condition $\mathbf{b} \in \mathbf{S}$, determine the conditions under which trajectories may intersect one of the boundaries $\mathsf{B}_k = \mathit{cone}\{\mathbf{s}_l, \mathbf{s}_m\}$, and derive the expressions for the values of such trajectories at the boundaries of $\mathbf{S}$. We will assume that the single mode is well-posed.

3.3.1 Real and Distinct Eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$

If all eigenvalues are real and distinct, then all eigenvectors are also real and Jordan form in this case is $\Lambda = \mathit{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. By well-posedness assumption, we have that none is at a boundary, i.e.,

$$\mathbf{c}_k^T \mathbf{v}_i \neq 0, \;\; k, i = 1, 2, 3, \tag{3.11}$$

The unique solution of (3.9) for the initial state at $\mathbf{b}$, the *trajectory*, is given by

$$\mathbf{x}(t, \mathbf{b}) = e^{\lambda_1 t} \mathbf{w}_1^T \mathbf{b}\, \mathbf{v}_1 + e^{\lambda_2 t}\, \mathbf{w}_2^T \mathbf{b}\, \mathbf{v}_2 + e^{\lambda_3 t} \mathbf{w}_3^T \mathbf{b}\, \mathbf{v}_3. \tag{3.12}$$

This hits the $\mathbf{s}_l\mathbf{s}_m$-plane at a finite time $t_k > 0$ if and only if $\mathbf{c}_k^T \mathbf{x}(t_k, \mathbf{b}) = 0$, where

$$\mathbf{c}_k^T \mathbf{x}(t, \mathbf{b}) = e^{\lambda_1 t}\, n_{k1} + e^{\lambda_2 t}\, n_{k2} + e^{\lambda_3 t} n_{k3},$$

with

$$n_{ki} := \mathbf{c}_k^T \mathbf{v}_i\, \mathbf{w}_i^T \mathbf{b}, \;\; i = 1, 2, 3.$$

If $\mathbf{b} \in \mathbf{R}^n$, then the trajectory $\mathbf{x}(t_k, \mathbf{b}) = 0$ hits the border B_k at time $t_k > 0$ and is moving towards $\mathbf{s}_k$ if and only if $\mathbf{c}_k^T \mathbf{x}(t_k, \mathbf{b}) = 0$ and $\mathbf{c}_k^T \dot{\mathbf{x}}(t_k, \mathbf{b}) > 0$. It moves towards $-\mathbf{s}_k$ if and only if $\mathbf{c}_k^T \dot{\mathbf{x}}(t_k, \mathbf{b}) < 0$. We can thus write

$$\begin{aligned} \mathbf{c}_k^T \dot{\mathbf{x}}(t_k, \mathbf{b}) \;\; &= (\lambda_1 - \lambda_3) e^{\lambda_2 t_k} [e^{(\lambda_1 - \lambda_2) t_k} n_{k1} + p n_{k2}] \\ &= (\lambda_1 - \lambda_3) e^{\lambda_3 t_k} [e^{(\lambda_1 - \lambda_3) t_k} (1 - p) n_{k1} - p n_{k3}] \\ &= (\lambda_1 - \lambda_3) e^{\lambda_3 t_k} [-e^{(\lambda_2 - \lambda_3) t_k} (1 - p) n_{k2} - n_{k3}]. \end{aligned} \tag{3.13}$$

and check the sign in order to determine the direction of a trajectory after hitting the border. This analysis yields the last column of Table 3.2.

If on the other hand, $\mathbf{b} \in \mathsf{B}_k$, then $\mathbf{c}_k^T\mathbf{b} = 0$ and $n_{k1} + n_{k2} + n_{k3} = 0$ so that

$$n_{k1} + pn_{k2} = (1-p)n_{k1} - pn_{k3} = -(1-p)n_{k2} - n_{k3}. \tag{3.14}$$

A conewise linear system is well-posed if and only if smooth continuation is possible from M_j to M_k. This will be the case whenever each mode M_j is such that (C_j, A_j) is observable and at $C_j\mathbf{x} = 0$, it holds that $sign(n_{jl}) = sign(n_{kl})$.

Fact 3.3.2 *A single mode is not well-posed if and only if for some* $k \in \{1, 2, 3\}$, *there exists* $\mathbf{b} \in \mathsf{B}_k$ *for which (3.14) is zero.*

Proof. If $\mathbf{b} \in \mathsf{B}_k$, then $\mathbf{c}_k^T\mathbf{b} = 0$ so that equality (3.14) holds. The trajectory starting at such a $\mathbf{b}$ remains in B_k if and only if $\mathbf{c}_k^T\dot{\mathbf{x}}(0, \mathbf{b}) = 0$, which holds if and only if (3.14) is zero. □

We caution here that the mode being well-posed discards the possibility of a trajectory being tangent to B_k at any $t \geq 0$. This is, of course a major assumption and eliminates the necessity of dealing with the second derivative of $\mathbf{c}_k^T\mathbf{x}(t, \mathbf{b})$. Also note that if (3.14) is positive (resp., negative), then the trajectory starting at B_k moves in the direction of $\mathbf{s}_k$ (resp., $-\mathbf{s}_k$), and conversely.

Note that $n_{ki} = 0$ only when $\mathbf{w}_i^T\mathbf{b} = 0$, by (3.11), or equivalently, when $\mathbf{b}$ is on the $\mathbf{v}_h\mathbf{v}_j$-plane, with (h, i, j) being a permutation of $(1, 2, 3)$. In such a case, the trajectory that starts with $\mathbf{b}$ remains in the $\mathbf{v}_h\mathbf{v}_j$-plane for all $t \geq 0$. Also since, by $\mathbf{v}_1\mathbf{w}_1^T + \mathbf{v}_2\mathbf{w}_2^T + \mathbf{v}_3\mathbf{w}_3^T = I$, we have $n_{k1} + n_{k2} + n_{k3} = \mathbf{c}_k^T\mathbf{b}$, the equality $\mathbf{c}_k^T\mathbf{x}(t, \mathbf{b}) = 0$ holds if and only if

$$\mathcal{L}: \ \chi_1\, n_{k1} + \chi_2\, n_{k2} + \mathbf{c}_k^T\mathbf{b} = 0, \tag{3.15}$$

where

$$\chi_1 := e^{(\lambda_1-\lambda_3)t} - 1 \geq 0, \chi_2 := e^{(\lambda_2-\lambda_3)t} - 1 \geq 0, \ \forall\, t \geq 0.$$

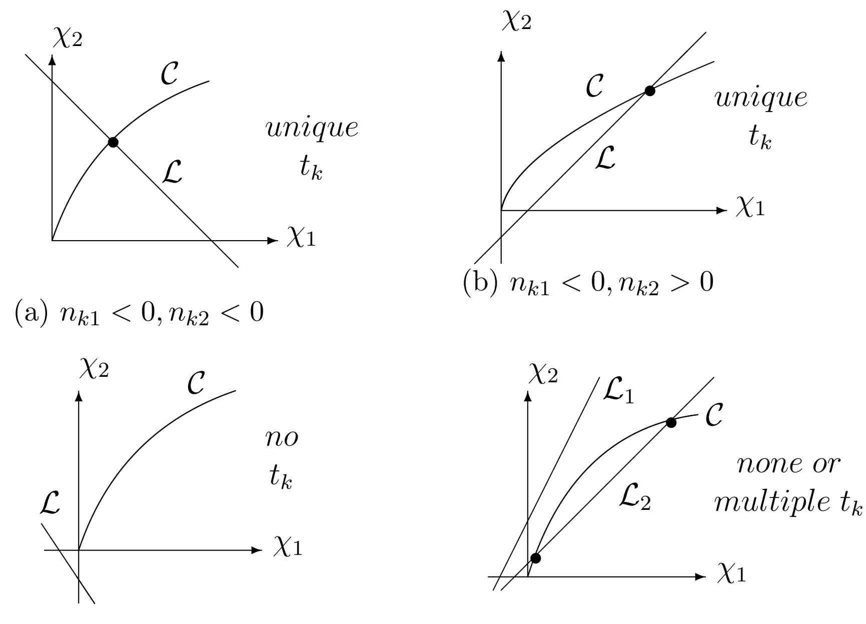
These time-dependent variables are related by

$$\mathcal{C}: \; \chi_2 = (\chi_1 + 1)^p - 1, \; p := \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}, \tag{3.16}$$

which describes a parametric curve that monotonically increases with curvature downwards in $\chi_1\chi_2$-plane as shown in Figure 3.3. Thus, an intersection with $\mathbf{s}_l\mathbf{s}_m$-plane, i.e., the border B_k, at a finite time occurs if and only if the line $\mathcal{L}$ of (3.15) intersects the curve $\mathcal{C}$ of (3.16) in the first quadrant of the $\chi_1\chi_2$-plane. The value of the parameter $t > 0$ at such an intersection is set equal to t_k. If, in addition, the two inequalities

$$\begin{aligned} \mathbf{c}_l^T \mathbf{x}(t_k, \mathbf{b}) &= \chi_1 \, n_{l1} + \chi_2 \, n_{l2} + \mathbf{c}_l^T \mathbf{b} > 0, \\ \mathbf{c}_m^T \mathbf{x}(t_k, \mathbf{b}) &= \chi_1 \, n_{m1} + \chi_2 \, n_{m2} + \mathbf{c}_m^T \mathbf{b} > 0, \end{aligned} \tag{3.17}$$

also hold for all $t \in (0, t_k)$, then t_k is the first time instant at which the trajectory intersects a border (among the three) so that $t_k = t_k^*$ in the context of Definition 3.3.2(iii).



(a) $n_{k1} < 0, n_{k2} < 0$

(b) $n_{k1} < 0, n_{k2} > 0$

(c) $n_{k1} > 0, n_{k2} > 0$

(d) $n_{k1} > 0, n_{k2} < 0$

Figure 3.3: Existence of t_k when $\mathbf{c}_k^T \mathbf{b} > 0$.

Figure 3.3 examines whether $\mathcal{L}$ and $\mathcal{C}$ intersect under four possible sign patterns for n_{k1} and n_{k2}. If $n_{k1} < 0$, then they intersect for both positive and negative n_{k2}, as illustrated in Figures 3.3(a) and 3.3(b). If $n_{k1} > 0$, then no intersection exists when $n_{k2} > 0$ and the intersection conditionally exists when $n_{k2} < 0$. These cases are shown in Figures 3.3(c) and (d).

Table 3.2: Conditions on $\mathbf{b} \in int\mathbf{S}$ for exit from B_k.

Type	n_{k1}	n_{k2}	n_{k3}	Exists	# t_k	Moves
1	-	-	+	yes	1	out
2	-	+	+	yes	1	out
3	-	+	-	yes	1	C(b)
4	+	-	+	C(a)	2	C(c)
5	+	-	-	no	0	—
6	+	+	-	no	0	—
7	+	+	+	no	0	—
8	-	-	-	no	0	—

Table 3.2 is summarized and enhanced, by including singular cases of $\mathbf{w}_1^T\mathbf{b} = 0$, $\mathbf{w}_2^T\mathbf{b} = 0$, or $\mathbf{w}_3^T\mathbf{b} = 0$ (initial condition being on one of the $\mathbf{v}_k\mathbf{v}_l$-planes) in the following.

Lemma 3.3.1 *Let (k, l, m) be a positive permutation of $(1, 2, 3)$ and let $\mathbf{S} =$ cone$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ be positively oriented. A trajectory (3.12), starting with $\mathbf{b} \in int\mathbf{S}$, intersects $\mathbf{s}_l\mathbf{s}_m$-plane at some $t = t_k > 0$ and goes out of $\mathbf{S}$ if and only at least two among $\{n_{k1}, n_{k2}, n_{k3}\}$ are nonzero and one of the following holds:*

i) $n_{k1} < 0$, $n_{k3} > 0$,

ii) $n_{k1} < 0$, $n_{k3} < 0$, $n_{k2} > 0$, *and at the time of intersection C(b) holds,*

iii) $n_{k1} > 0, n_{k3} > 0, n_{k2} < 0$, *C(a) holds, and at a time of intersection C(c) holds,*

$$\mathrm{C(a)} : yes \Leftrightarrow \left(\frac{-pn_{k2}}{n_{k1}}\right)^{\frac{p}{1-p}} \geq \frac{-n_{k3}}{(1-p)n_{k2}}, \ \frac{-pn_{k2}}{n_{k1}} > 1$$

$$\mathrm{C(b)} : out \Leftrightarrow \frac{-pn_{k2}}{n_{k1}} < e^{(\lambda_1-\lambda_2)t_k}$$

$$\mathrm{C(c)} : out \Leftrightarrow \frac{-pn_{k2}}{n_{k1}} > e^{(\lambda_1-\lambda_2)t_k}$$

iv) $n_{k1} = 0,\ n_{k3} > -n_{k2} > 0,$ *or*

$n_{k2} = 0,\ n_{k3} > -n_{k1} > 0,$ *or*

$n_{k3} = 0,\ n_{k2} > -n_{k1} > 0.$

Definition 3.3.3 *A cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ **excludes** *cone*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *if* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *are exterior to it and neither of the three planes cone*$\{\mathbf{v}_1, \mathbf{v}_2, \}$, *cone*$\{\mathbf{v}_2, \mathbf{v}_3\}$, *cone*$\{\mathbf{v}_3, \mathbf{v}_1\}$ *has an intersection with it. Two cones are* **mutually exclusive** *if both exclude one another.*

It is straightforward to see that *cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ excludes *cone*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if each row of the matrix WS has constant sign and each column of CV has mixed signs. The notation $rs\{CV\} = (-; -; +)$, for instance, will mean that the first and second rows of CV contain negative, and the third, positive entries. They are mutually exclusive if and only if both WS and CV have constant row signs with mixed signs in every column. In Figure 2.2, for instance, sign patterns for CV and WS corresponding to two possible mutually exclusive cases are shown.

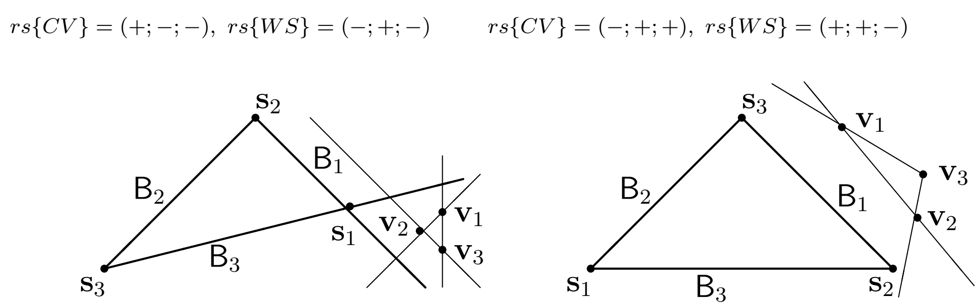


Figure 3.4: Two mutually exclusive $\mathbf{S}$ and *cone*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cases with sign patterns of CV, WS matrices.

A consequence of WS having constant row signs is that signs of the entries of the column vector $W\mathbf{b} = [\mathbf{w}_1^T\mathbf{b}\ \mathbf{w}_2^T\mathbf{b}\ \mathbf{w}_3^T\mathbf{b}]^T$ are constant over $\mathbf{b} \in \mathbf{S}$. If, in addition, CV has also constant row sign, then it follows that signs of the triplets (n_{i1}, n_{i2}, n_{i3}) are also constant over $\mathbf{b} \in \mathbf{S}$. Having mixed column signs in WS and

CV implies that every triplet has mixed signs and that one triplet for $i = k, l, m$ has the negative sign pattern of the other two. In Figure 3.4, for instance, the sign patterns are, respectively,

$$\begin{array}{c|ccc} & n_{i1} & n_{i2} & n_{i3} \\ \hline i=1 & - & + & - \\ i=2 & + & - & + \\ i=3 & + & - & + \end{array} \qquad \begin{array}{c|ccc} & n_{i1} & n_{i2} & n_{i3} \\ \hline i=1 & - & - & + \\ i=2 & + & + & - \\ i=3 & + & + & - \end{array}$$

Theorem 3.3.1 *Let* (k, l, m) *be a positive permutation of* $(1, 2, 3)$ *and let* $\mathbf{S} =$ *cone*$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ *be positively oriented. Let a mode* $\mathbf{S}$ *with real eigenvalues* $\lambda_1 > \lambda_2 > \lambda_3$ *be mutually exclusive with its cone of eigenvalues cone*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

a. The cone $\mathbf{S}$ *is* k*-transitive if and only if it holds that*

$$\begin{array}{lll} i) & (1-p)n_{k1} - p\,n_{k3} < 0 & \forall\, \mathbf{b} \in \mathsf{B}_k, \\ ii) & (1-p)n_{l1} - p\,n_{l3} > 0 & \forall\, \mathbf{b} \in \mathsf{B}_l, \\ iii) & (1-p)n_{m1} - p\,n_{m3} > 0 & \forall\, \mathbf{b} \in \mathsf{B}_m. \end{array}$$

b. The cone $\mathbf{S}$ *is* lm*-transitive if and only if it holds that*

$$\begin{array}{lll} iv) & (1-p)n_{k1} - p\,n_{k3} > 0 & \forall\, \mathbf{b} \in \mathsf{B}_k, \\ v) & (1-p)n_{l1} - p\,n_{l3} < 0 & \forall\, \mathbf{b} \in \mathsf{B}_l, \\ vi) & (1-p)n_{m1} - p\,n_{m3} < 0 & \forall\, \mathbf{b} \in \mathsf{B}_m. \end{array}$$

Proof. We first note that any trajectory that starts in the cone must eventually go out of the cone since all eigenvectors are exterior. It must thus intersect one of the border planes at a nonzero point (as the origin is a global equilibrium point). By the fact that $\mathbf{S}$ and *cone*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are mutually exclusive, both WS and CV have constant row signs with mixed signs in every column. This implies that each n_{ij}, $i, j = 1, 2, 3$ has constant sign over $\mathbf{b} \in \mathbf{S}$ and that every triplet (n_{i1}, n_{i2}, n_{i3}) has mixed signs with one triplet for $i = k, l, m$ having negative sign pattern of the other two. This, in particular, eliminates the possibilities of Type-7 and Type 8 of Table 3.2 in our mutually exclusive case.

The necessity of the conditions (i)-(iii) for k-transitivity and, of (iv)-(vi) for lm-transitivity, is clear since the required transitivity properties need to be satisfied by initial conditions starting at a border as well.

If (i) holds, then $n_{k1} > 0, n_{k3} < 0$ is not possible on B_k and hence anywhere in $\mathbf{S}$, which implies that (n_{k1}, n_{k2}, n_{k3}) is of Type-1 to Type-4 of Table 3.2. The conditions (ii) and (iii) eliminates the possibility of Type 1 and Type 2 for both borders B_l and B_m. This means that if (n_{k1}, n_{k2}, n_{k3}) is of Type-1 or Type-2, then (n_{l1}, n_{l2}, n_{l3}) and (n_{m1}, n_{m2}, n_{m3}) are of Type-5 or Type-6 so that B_k is the only possible border of exit and the mode is k-transitive. If (n_{k1}, n_{k2}, n_{k3}) is of Type-3 or Type-4, then all three (n_{i1}, n_{i2}, n_{i3}) are of Type-3 or Type-4 of Table 3.2. The conditions on direction of trajectories imposed on the border planes by (i)-(iii) ensure that trajectories are all exiting via B_k. This establishes that (i)-(iii) are sufficient for k-transitivity.

If (vi) holds, then (n_{k1}, n_{k2}, n_{k3}) is not of Type-1 or Type-2 of Table 3.2, whereas, by (iv) and (v), (n_{l1}, n_{l2}, n_{l3}) and (n_{m1}, n_{m2}, n_{m3}) are not of Type-5 or Type-6. It follows that (n_{k1}, n_{k2}, n_{k3}) is of Type-5 or Type-6, in which case the mode is lm-transitive, or it is of Type-3 or Type 4, in which case all borders are of Type-3 or Type 4. If (b.ii) holds, then all three (n_{i1}, n_{i2}, n_{i3}) are of Type-3 or Type-4 of Table 3.2. The conditions on direction of trajectories imposed on the border planes ensure that trajectories are all exiting via B_l or B_m. This establishes that (iv)-(vi) are sufficient for lm-transitivity. $\square$

Remark 3.3.1 *Since conditions (i)-(vi) of Theorem 3.3.1 need be satisfied at the respective borders, they can be replaced by similar conditions of (3.14) by replacing the pair (n_{i1}, n_{i3}) by (n_{i1}, n_{i2}) or by (n_{i2}, n_{i3}).*

The characterization of transitive cones obtained in Theorem 3.3.1 easily extends to A-matrices with repeated eigenvalues but having a diagonal Jordan form. It is also possible to derive similar results for A-matrices having arbitrary Jordan forms and A-matrices with a conjugate pair of eigenvalues. This thesis focuses only on the case of real distinct eigenvalues since it is presently difficult to bind all these sporadic results together in a concise treatment.

Example 3.3.1 *As an illustration of the result of Theorem 3.3.1, consider*

$$\mathbf{S}_1 = cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\},\ \mathbf{S}_2 = cone\{\mathbf{s}_2, \mathbf{s}_4, \mathbf{s}_3\},$$

where the boundary vectors are

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{s}_4 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}. \tag{3.18}$$

The state matrices for the CLS are given as:

$$A_1 = \begin{bmatrix} -0.7027 & -1.0054 & -0.9730 \\ -0.0622 & -1.0243 & -0.4784 \\ 0.0973 & 0.7946 & 0.2270 \end{bmatrix}, A_2 = \begin{bmatrix} 0.90 & -0.94 & -0.38 \\ 3.20 & -2.64 & -0.88 \\ -2.60 & 1.72 & 0.24 \end{bmatrix}.$$

The eigenvalues for Mode 1 and Mode 2 are: Mode 1 ($\lambda_{11} = -0.4, \lambda_{12} = -0.5, \lambda_{13} = -0.6$)*, and Mode 2* ($\lambda_{21} = -0.4, \lambda_{22} = -0.5, \lambda_{23} = -0.6$)*, respectively. The eigenvector matrices V_1 and V_2 can be chosen as*

$$V_1 = \begin{bmatrix} 3.0 & 2.0 & 9.0 \\ 2.0 & 2.5 & 1.5 \\ -3.0 & -3.0 & -2.5 \end{bmatrix}, V_2 = \begin{bmatrix} -1 & -7 & -1 \\ -3 & -8 & -2 \\ 4 & -6 & 1 \end{bmatrix}.$$

Figure 3.5, and Figure 3.6 shows the relative positions of the eigenvectors with respect to the cones $\mathbf{S}_1, \mathbf{S}_2$. By Theorem 3.3.1, it follows that the modes 1 and 2 are, respectively, two and one transitive modes.

3.4 Structural Analysis

Let us call a property of a mode **structural** if it only depends on the eigenvalues and the eigenvectors of that mode. It will be noticed that the conditions in Theorem 3.3.1 may sometimes be determined solely with the signs of $\{n_{k1}, n_{k2}, n_{k3}\}$, which are in turn constant over the whole cone in the mutually exclusive case. This observation gives the following structural properties.

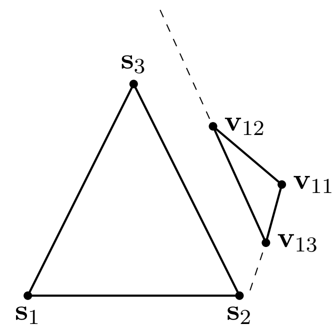


Figure 3.5: Positions of the $cone\{\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}\}$ relative to $cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$. The mode is 2-transitive.

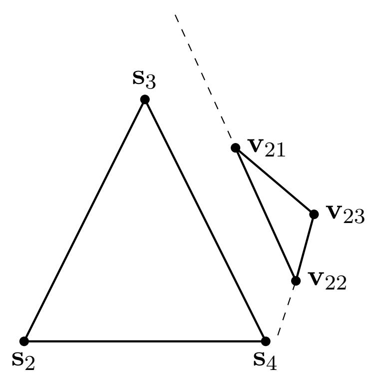


Figure 3.6: Positions of the $cone\{\mathbf{v}_{21}, \mathbf{v}_{22}, \mathbf{v}_{23}\}$ relative to $cone\{\mathbf{s}_2, \mathbf{s}_4, \mathbf{s}_3\}$. The mode is 1-transitive.

Theorem 3.4.1 *[60] If a mode* $\mathbf{S}$ *with real eigenvalues* $\lambda_1 > \lambda_2 > \lambda_3$ *excludes its cone of eigenvalues* $cone\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*, then it is transitive from one or two of its borders. Moreover, the following hold.*

a. The cone $\mathbf{S}$ *is structurally k-transitive if and only if it holds that*

$$\begin{array}{lll} i) & n_{k1} < 0, n_{k3} > 0 & for\ \mathbf{b} = \mathbf{s}_l\ and\ \mathbf{b} = \mathbf{s}_m, \\ ii) & n_{l1} > 0, n_{l3} < 0 & for\ \mathbf{b} = \mathbf{s}_m\ and\ \mathbf{b} = \mathbf{s}_k, \\ iii) & n_{m1} > 0, n_{m3} < 0 & for\ \mathbf{b} = \mathbf{s}_k\ and\ \mathbf{b} = \mathbf{s}_m. \end{array}$$

b. The cone $\mathbf{S}$ *is structurally lm-transitive if and only if it holds that*

$$\begin{array}{lll} iv) & n_{k1} > 0, n_{k3} < 0 & for\ \mathbf{b} = \mathbf{s}_l\ and\ \mathbf{b} = \mathbf{s}_m, \\ v) & n_{l1} < 0, n_{l3} > 0 & for\ \mathbf{b} = \mathbf{s}_m\ and\ \mathbf{b} = \mathbf{s}_k, \\ vi) & n_{m1} < 0, n_{m3} > 0 & for\ \mathbf{b} = \mathbf{s}_k\ and\ \mathbf{b} = \mathbf{s}_m. \end{array}$$

The result just stated depends only on the relative position of the triple of vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. For instance, under the assumption that these mutually exclusive triplets are both positively oriented in space according to the right hand rule. Then all possible geometric configurations are illustrated in Figures 3.7 and 3.8, where $(k, l, m) = (1, 2, 3)$ for simplicity. The cases a, b, e, f are structurally transitive cases, whereas c and d are not. Depending on the sign of the "speed" (3.14) at a border, which determine the direction of a trajectory that starts at that border, they can be classified as one or two-transitive cases.

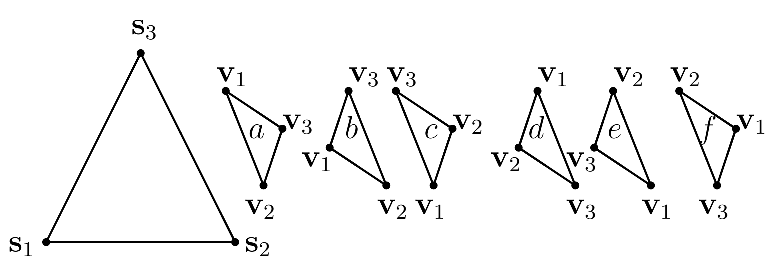


Figure 3.7: Mutually exclusive orientations of the $cone\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ relative to $cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$. The configurations a, b, e, f are structurally transitive.

Using the cross product properties of the vectors, the sign patterns of the $W\mathbf{S}$ matrix for each of the permutations are tabulated as

Perm. of eigenvectors in Figure 3.7	Sign pattern of corresponding $W\mathbf{S}$
a	$rs\{W\mathbf{S}\} = (+;+;-)$
b	$rs\{W\mathbf{S}\} = (+;-;-)$
c	$rs\{W\mathbf{S}\} = (+;-;+)$
d	$rs\{W\mathbf{S}\} = (-;+;-)$
e	$rs\{W\mathbf{S}\} = (-;-;+)$
f	$rs\{W\mathbf{S}\} = (-;+;+)$

Table 3.3: Sign patterns for different permutations of eigenvectors.

Since all the permutations(eigenvector $cone\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$) are on the same side of $cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, the sign pattern of the CV matrix will remain same for all the six cones $a - to - f$ i.e.,

$$rs\{CV\} = (-;+;+). \tag{3.19}$$

Similarly, Figure 3.8 demonstrates the eigenvector configurations when the choice of the eigenvectors in Figure 3.7 are replaced by their negatives. Although these configurations appear to be different apparently, nevertheless they possess the same structural and conditional transitivity properties as of Figure 3.7 as the sign patterns in corresponding WS remain the same.

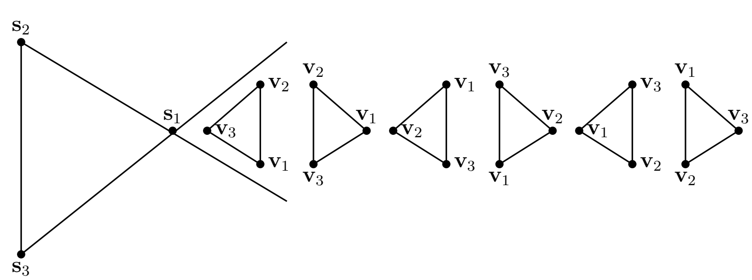


Figure 3.8: Further mutually exclusive orientations of the $cone\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ relative to $cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$. Third and fourth configurations are not structural but the others are structurally one or two-transitive.

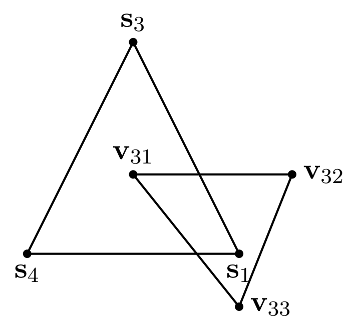


Figure 3.9: Position of the $cone\{\mathbf{v}_{31}, \mathbf{v}_{32}, \mathbf{v}_{33}\}$ relative to $cone\{\mathbf{s}_4, \mathbf{s}_1, \mathbf{s}_3\}$ for the sink.

3.5 Characterization of Sinks and Sources

This section discusses the characterization of a given mode as a source or a sink by exploiting the geometry of associated eigenvectors relative to the cone. For the results, we investigate the orientation of the eigenvectors and as well as the sign of associated eigenvalues. We provide two theorems that characterize a given mode as a sink and a source, respectively.

Theorem 3.5.1 *[60] A mode* $\mathbf{S}$ *with all eigenvalues real and distinct is a sink, i.e., for every* $\mathbf{b} \in \mathbf{S}$ *and for all* $t \geq 0$, $\mathbf{x}(t, \mathbf{b}) \in \mathbf{S}$, *if and only if*

i) $\mathbf{v}_1$ *is interior to* $\mathbf{S}$,

ii) *the plane cone*$\{\mathbf{v}_2, \mathbf{v}_3\}$ *does not intersect* $\mathbf{S}$, *and*

iii) *at all* $\mathbf{b} \in \mathsf{B}_k$ *at which* (n_{k1}, n_{k2}, n_{k3}) *'s of Type 4, the direction (3.14) is positive.*

Theorem 3.5.2 *[60] A mode* $\mathbf{S}$ *with all eigenvalues real and distinct is a source if and only if*

i) $\mathbf{v}_3$ *is interior to* $\mathbf{S}$,

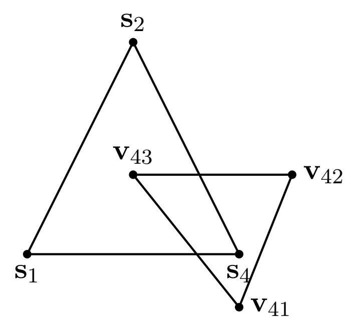


Figure 3.10: Position of the $cone\{\mathbf{v}_{41}, \mathbf{v}_{42}, \mathbf{v}_{43}\}$ relative to $cone\{\mathbf{s}_1, \mathbf{s}_4, \mathbf{s}_2\}$ for the source.

ii) *the plane* $cone\{\mathbf{v}_1, \mathbf{v}_2\}$ *does not intersect* $\mathbf{S}$*, and*

iii) *at all* $\mathbf{b} \in \mathsf{B}_k$ *at which* (n_{k1}, n_{k2}, n_{k3}) *is of Type 4, the direction (3.14) is negative.*

Example 3.5.1 *Let*

$$\mathbf{S}_3 = cone\{\mathbf{s}_4, \mathbf{s}_1, \mathbf{s}_3\}, \ \mathbf{S}_4 = cone\{\mathbf{s}_1, \mathbf{s}_4, \mathbf{s}_2\},$$

where the boundary vectors are as in (3.18). Consider

$$A_3 = \begin{bmatrix} -14.8583 & -5.0677 & 4.8103 \\ -6.8294 & -5.2371 & -0.7727 \\ -4.0378 & -5.3255 & -3.6047 \end{bmatrix}, A_4 = \begin{bmatrix} -11.8796 & 5.7854 & 6.3506 \\ 3.0231 & 3.0186 & -11.9621 \\ 3.0449 & 5.9158 & -17.8390 \end{bmatrix}$$

defining the dynamics of Mode 3 and 4. The eigenvalues are $(\lambda_{31} = -0.3, \lambda_{32} = -8.1, \lambda_{33} = -15.3)$ *and* $(\lambda_{41} = -0.3, \lambda_{42} = -8.1, \lambda_{43} = -15.3)$*, respectively. Suitable corresponding eigenvector matrices are*

$$V_3 = \begin{bmatrix} -0.3400 & -0.6205 & 18.9605 \\ 0.5420 & -2.3970 & 13.8550 \\ -0.4580 & -3.3970 & 12.8550 \end{bmatrix}, V_4 = \begin{bmatrix} -1.4240 & 4.1735 & -8.7495 \\ -1.8820 & 0.7765 & 4.1055 \\ -0.8820 & 1.7765 & 5.1055 \end{bmatrix}.$$

The orientation of $cone\{\mathbf{v}_{31}, \mathbf{v}_{32}, \mathbf{v}_{33}\}$ *relative to* $cone\{\mathbf{s}_4, \mathbf{s}_1, \mathbf{s}_3\}$ *and* $cone\{\mathbf{v}_{41}, \mathbf{v}_{42}, \mathbf{v}_{43}\}$ *relative to* $cone\{\mathbf{s}_1, \mathbf{s}_4, \mathbf{s}_2\}$ *are illustrated in Figure 3.9 and Figure 3.10, respectively. By Theorems 3.5.1 and 3.5.2, Mode 3 must be a sink and Mode 4, a source.*

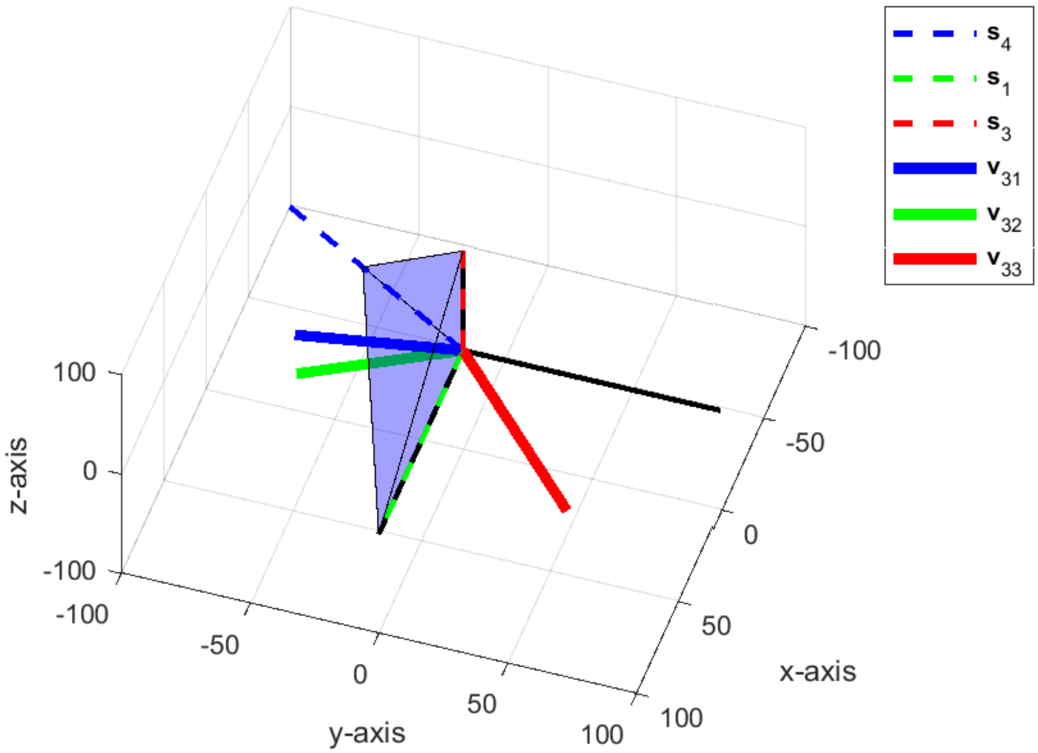


Figure 3.11: Position of the $cone\{\mathbf{v}_{31}, \mathbf{v}_{32}, \mathbf{v}_{33}\}$ relative to $cone\{\mathbf{s}_4, \mathbf{s}_1, \mathbf{s}_3\}$ for the sink in 3D.

The three dimensional figures in Figure 3.11 and Figure 3.12, show the positions of the eigenvectors for the two modes. Simulation results (not shown here) of trajectories for a grid of initial state values inside the respective cones further verify the behaviors as a sink and a source. This example will be continued in Example 3.6.1 below by adding two more modes, Mode 1 and 2, to cover the whole space and illustrate the stability result.

3.6 Stability of Conewise linear Systems

We now present the main result on stability of CLS (3.7). In doing this, we use the correspondence between a CLS that consists of either one or two-transitive

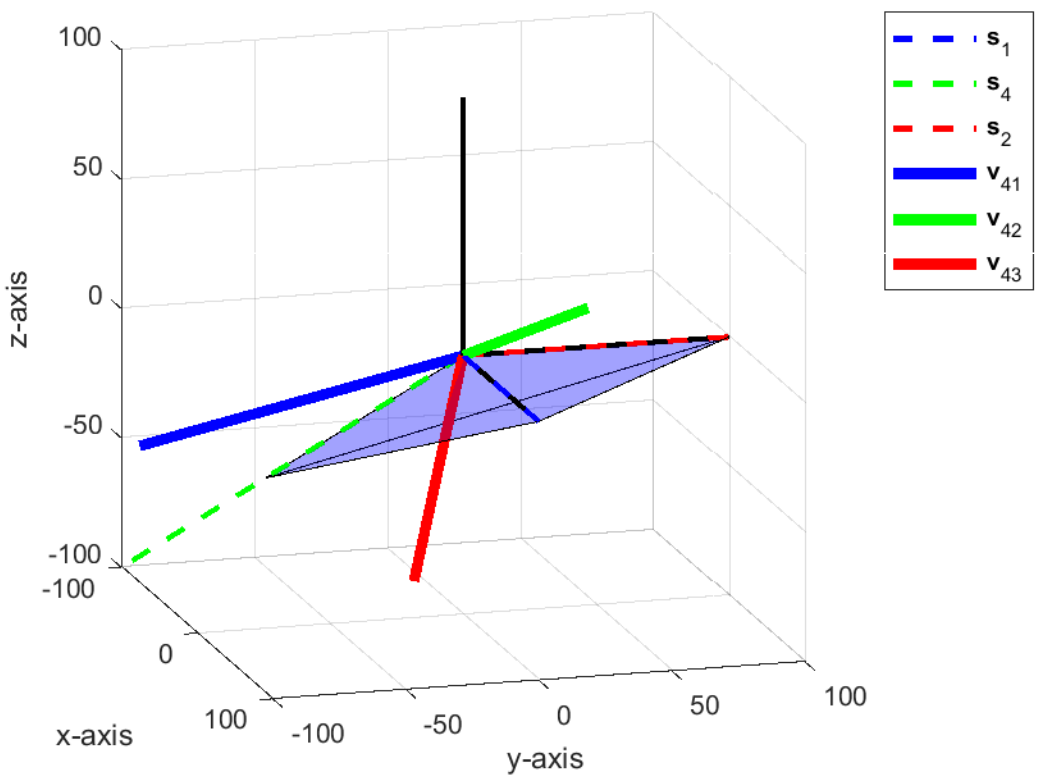


Figure 3.12: Position of the $cone\{\mathbf{v}_{41}, \mathbf{v}_{42}, \mathbf{v}_{43}\}$ relative to $cone\{\mathbf{s}_1, \mathbf{s}_4, \mathbf{s}_2\}$ for the source in 3D.

cones of the type described above, sinks, and sources and a digraph.

A cone that is one-transitive can be represented as a vertex with one outgoing and two incoming edges. Similarly, a cone that is two-transitive can be represented as a vertex with two outgoing and one incoming edges. A source will then be a vertex with three outgoing edges and a sink will have three incoming edges. A well-posed CLS that only contains modes of these four types is thus in one-to-one correspondence with a cubic digraph.

The following result gives a relation among the numbers of sinks, sources, and transitive nodes in a cubic digraph. This result is stated only because it is interesting on its own; as the relations given in the lemma will not be needed in our main result.

Lemma 3.6.1 *Given an arbitrary CLS with $N = 2M$ modes, where $M \geq 2$, suppose there are p sources, s sinks, r 1-transitive, and q 2-transitive modes. If the CLS is well-posed, then the numbers p, s of sources and sinks stand in the relation*

$$\begin{aligned} p &= M - \tfrac{1}{3}(2q + r), \\ s &= M - \tfrac{1}{3}(q + 2r) \end{aligned} \tag{3.20}$$

to the numbers q, r of transitive modes.

Proof. Recall that the number of tri-hedral cones that fill up the space must always be even, by Remark 2.1.3. If the cones are restricted to be a sink, source, one-transitive or two-transitive, then the resulting conewise system corresponds to a cubic digraph. By Lemma 2.4.2, the numbers q, p, r, and s must stand in the relation (3.20). □

Consider a multi-modal ($N > 2$) conewise linear system such that 3D-state space is partitioned into a finite number of non-overlapping convex cones on each of which the dynamics of the system is described by a linear vector differential equation.

Theorem 3.6.1 *Suppose a CLS (3.7) with N modes consists only of sinks, sources, 1-transitive, and 2-transitive modes. If such a CLS is well-posed, then its corresponding cubic digraph is free of self-loops and loops of length two. If the CLS is well-posed and its digraph is acyclic, then it is globally asymptotically stable if and only if it is such that*

i. the largest eigenvalue of every sink mode is negative, and

ii. the smallest eigenvalue of every source mode is negative.

Proof. If a CLS is well-posed then, by the fact that smooth continuation ıs possible in only one of the modes, there can be no initial condition at a boundary that would result in trajectories outgoing or incoming from different modes. Since the modes are restricted to pure transitive modes, sinks, and sources (no half-sinks and no mixed type boundaries), it follows that self-cycles are avoided.

To prove the second statement,first note that every acyclic digraph must include among its nodes at least one source and at least one sink (see e.g., Property 19.12 in [63]). Since every acyclic digraph has a topological sort, there are no cycles and every trajectory starting at any mode should enter a sink except the initial conditions starting at rays defined by the eigenvectors associated with the smallest eigenvalues of the sources. The condition (i) is necessary and sufficient, in view of Theorem 3.5.2, for such trajectories to converge to the origin and condition (ii) is necessary and sufficient, in view of Theorem 3.5.1, for all other trajectories to converge to the origin after arriving at a sink. □

Example 3.6.1 *As an illustration of the result of Theorem 3.6.1 and combining the four modes of Examples 3.3.1 and 3.5.1, we consider a CLS with four modes*

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_1, \\ A_2\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_2, \\ A_3\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_3, \\ A_4\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_4, \end{cases} \tag{3.21}$$

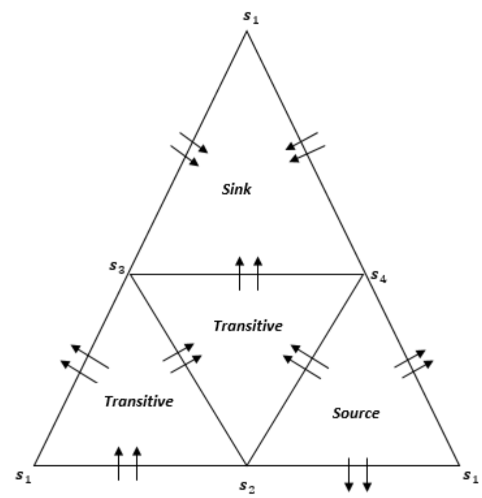


Figure 3.13: A GAS four-mode CLS.

for which the A-matrices and the cones are defined as in Examples 3.3.1 and 3.5.1 above. Recall from Example 3.3.1 that Mode 1 is 2-transitive from its faces $\{\mathbf{s}_2, \mathbf{s}_3\}$, *and* $\{\mathbf{s}_1, \mathbf{s}_3\}$ *and Mode 2 is 1-transitive from its face* $\{\mathbf{s}_3, \mathbf{s}_4\}$. *Also by Example 3.5.1, the modes 3 and 4 are source and sink, respectively. Figure 3.16 gives the corresponding digraph of this CLS. Its adjacency matrix is easily computed as*

$$P = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $tr(\exp(P) - I) = 0$ *the corresponding digraph is acyclic and hence, according to Theorem 3.6.1 CLS is GAS. Figure 3.15 shows the evolution of trajectories for an arbitrary initial condition interior to source (Mode 4), multiple colors have been used corresponding to evolution of trajectory in different modes, it is observed that sequence of the trajectory is Mode 4* $\to$ *Mode 1* $\to$ *Mode 2* $\to$ *Mode 3 and the states are converging to origin while residing in the sink (Mode 3).*

Remark 3.6.1 *The result of Theorem 3.6.1 can be strengthened to cover a CLS with a digraph containing cycles only if a suitable characterization of a cyclic trajectory (a trajectory that returns to the cone it starts at) converging to the*

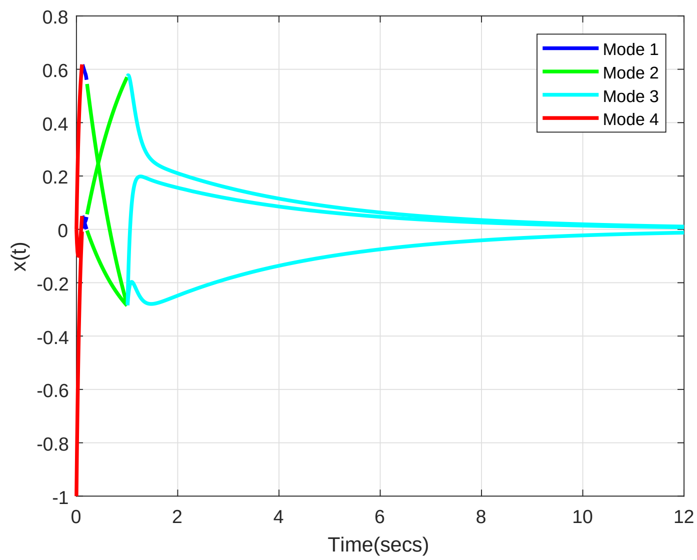


Figure 3.14: State trajectories for GAS CLS.

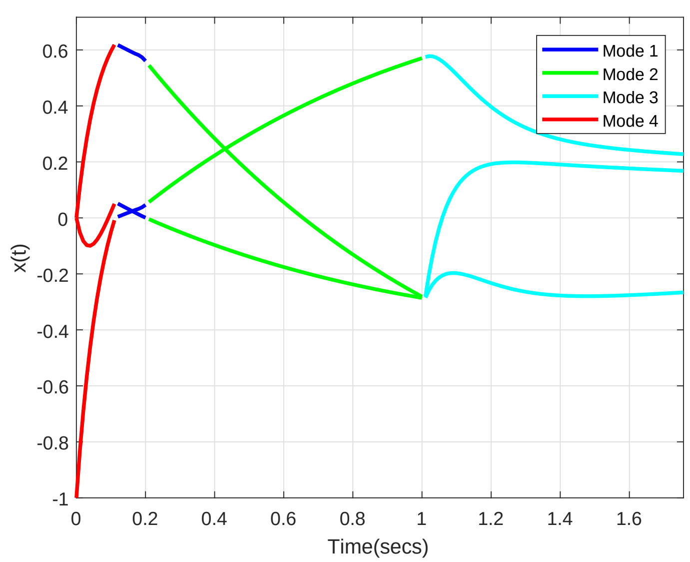


Figure 3.15: State trajectories for GAS CLS (zoomed).

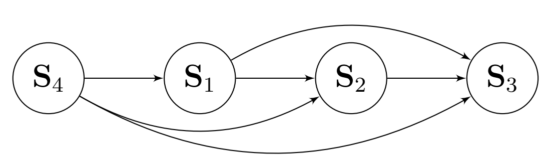


Figure 3.16: Topological sort of CLS in Figure 3.13.

origin is obtained. In 2D, the "factor of expansion" is a measure that makes this possible, [27], a counterpart of which is currently lacking in 3D.

Chapter 4

A CLS with Half-Sinks

The result presented in Theorem 3.6.1 can be strengthened by allowing transitive modes in which trajectories starting at a face of the cone do not have uniformly the same direction, i.e., a mode that is partially a sink or, in short, a half-sink. We see below that some such half-sinks can be symbolized with two nodes rather than just one, and such a CLS will still correspond to a suitable digraph.

This chapter presents a stability analysis of such a CLS that a practical non-linear RC-circuit containing ideal diodes can be modeled as .

4.1 A Piecewise Linear Network

The elements such as diodes, transistors or thyristors in a circuit leads to non-linearity and therefore results in hybrid networks. A thorough study of literature suggests many modeling techniques for hybrid systems, such as the complementarity approach [64] and piecewise linear descriptions [65]. An example of dynamic analysis of piecewise linear networks can be found in [16].

Consider the RC circuit in Fig. 4.1 coupled to two ideal diodes. We model the network as a discontinuous vector differential equation. The discontinuity

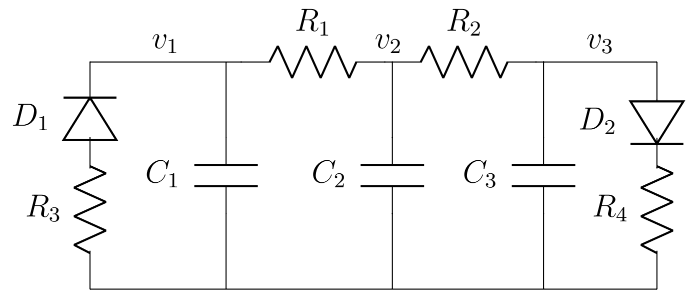


Figure 4.1: RC circuit with ideal diodes.

appears from the ideal behavior of the diodes. Depending on whether diodes are blocking or conducting, the network has $2^2 = 4$ different circuit topologies.

- *Circuit 1*: Both diodes are OFF : $v_1 > 0$ & $v_3 < 0$, both R_3 and R_4 are excluded from circuit.
- *Circuit 2*: D_1 is ON, D_2 id OFF : $v_1 < 0$ & $v_3 < 0$, R_3 is part of circuit whereas R_4 isn't.
- *Circuit 3*: D_1 is OFF, D_2 id ON : $v_1 > 0$ & $v_3 > 0$, R_4 is part of circuit whereas R_3 isn't.
- *Circuit 4*: Both diodes are ON : $v_1 < 0$ & $v_3 > 0$, both R_3 and R_4 are included.

Since these topologies do not satisfy the definition of a polyhedral convex cone (strictly contained ina half-space), we perform a reasonable partition to have a CLS and paying attention to avoid sliding modes at the boundaries. Considering the sign of v_2, we split each circuit topology into two convex cones. After splitting, we have 8 modes each corresponding to one octant in $\mathbf{R}^3$. We thus model the circuit as the following CLS: Let the state vector be $\mathbf{x} = [v_1\ v_2\ v_3]^T$, the vector of capacitor voltages. and let

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_1, \mathbf{S}_2 \\ A_2\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_3, \mathbf{S}_4 \\ A_3\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_5, \mathbf{S}_6 \\ A_4\mathbf{x} & if \quad \mathbf{x} \in \mathbf{S}_7, \mathbf{S}_8 \end{cases} \tag{4.1}$$

where the cones are defined as

$$\mathbf{S}_1 = cone\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\},\ \mathbf{S}_2 = cone\{\mathbf{s}_1, -\mathbf{s}_2, \mathbf{s}_3\},$$
$$\mathbf{S}_3 = cone\{-\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\},\ \mathbf{S}_4 = cone\{-\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\},$$
$$\mathbf{S}_5 = -\mathbf{S}_4,\ \mathbf{S}_6 = -\mathbf{S}_3,\ \mathbf{S}_7 = -\mathbf{S}_2,\ \mathbf{S}_8 = -\mathbf{S}_1,$$

and the boundary vectors are

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{s}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},\ \mathbf{s}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Using reasonable values of components in Fig. 4.1, one possible realization of state-matrices in (4.1) is

$$A_1 = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -5 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -7 \end{bmatrix},\ A_4 = \begin{bmatrix} -5 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -7 \end{bmatrix}.$$

All four state-matrices have real distinct eigenvalues so that the results of Theorems 3.3.1, 3.5.1, and 3.5.2 can be used.

Employing eigenvalue decomposition on state-matrices, we compute V_i and determine the sign patterns of $C_iV_i$and W_iS_i to define the orientation of the $cone\{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}\}$ relative to the $\mathbf{S}_i$ for each mode. Fig. 4.5 displays the resulting geometric figures of all modes and helps us to classify each mode as transitive, source, or sink. For instance, consider the first mode with $cone\{\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}\}$ and $\mathbf{S}_1$. The eigenvector planes divide the cone $\mathbf{S}_1$ into four sectors i.e., $1, 2, 3, 4$ as seen in Fig. 4.5. In each sector, the entries of $W_1\mathbf{b}$ have uniformly the same sign for all $\mathbf{b}$ in that sector. The sign pattern of C_1V_1 and W_1S_1 matrices are

$$C_1V_1 := \begin{bmatrix} + & - & + \\ - & + & + \\ - & - & - \end{bmatrix}, \; W_1S_1 := \begin{bmatrix} + & - & - \\ - & + & - \\ + & + & - \end{bmatrix}.$$

For the $\mathbf{b} \in 1, 2, 3, 4$, we can write the sign of $W_1\mathbf{b}$ as

	1	2	3	4
$\mathbf{w}_1^T\mathbf{b}$	$-$	$-$	$-$	$+$
$\mathbf{w}_2^T\mathbf{b}$	$-$	$-$	$+$	$-$
$\mathbf{w}_3^T\mathbf{b}$	$-$	$+$	$+$	$+$

Using the sign pattern of $C_1V_1$and $W_1\mathbf{b}$ for each sector, we determine the signs of $\{n_{k1}, n_{k2}, n_{k3}\}$ and employ the Table 3.2 to determine the transitivity of a given border. Table 4.1 summarizes the direction of trajectories on the bo faces of $\mathbf{S}_1$, where "OUT" means trajectories are going out, "IN" denotes incoming trajectories and whereas "MIXED" means trajectories going IN and OUT without violating well-posedness in that border. We note that the cone $\mathbf{S}_1$ is also transitive but it is neither one nor two-transitive because one of its faces may have both incoming and outgoing trajectories starting at an initial-state $\mathbf{b}$, depending on the sign of the speed of the trajectory at that $\mathbf{b}$. Since smooth continuation at such a border is preserved, the mode is still well-posed.

We can not represent such a cone using one node, as we were able to do in the case of one or two transitive modes, because one node would have had a self-loop. However, by using two nodes $\mathbf{S}_{1a}$ and $\mathbf{S}_{1b}$, we have the following digraph that corresponds to the cone $\mathbf{S}_1$ in a one-to-one manner. Note that a transitive cone, in one face of which we have mixed directions, is represented by two nodes of out-degree one $\mathbf{S}_{1a}$ and of out degree two $\mathbf{S}_{1b}$. Figure 4.2 shows the two-node representation of $\mathbf{S}_1$ and $\mathbf{S}_2$.

Table 4.2 provides the similar analysis for all cones. For the sake of brevity, we exclude the details of the analysis.

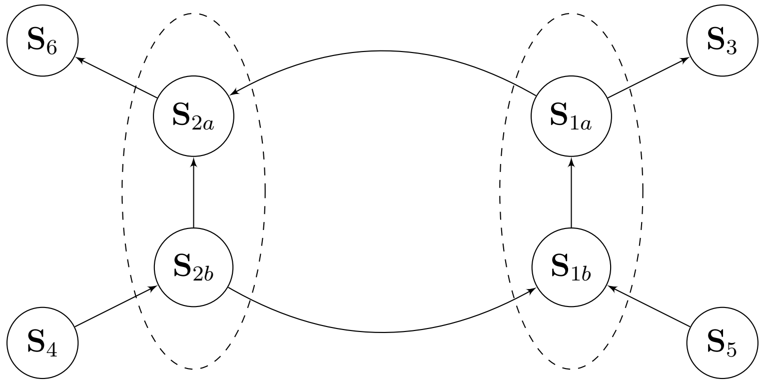


Figure 4.2: Two-node representation of cones $\mathbf{S}_1$ and $\mathbf{S}_2$.

Table 4.1: Direction of Trajectories on the Boundary cones for $\mathbf{S}_1$.

sector	B_{1k}	B_{1m}	B_{1l}
1	OUT	IN	-
2	OUT	MIXED	IN
3	OUT	-	IN
4	-	OUT	IN

After characterization of all modes, we bring together all single cones such that $\bigcup_{j=1}^{8} \mathbf{S}_j = \mathbf{R}^3$. Figure 4.4 shows the distribution and structure of all cones in CLS. In Figure 4.4, the cubic digraph obtained is illustrated with arrows denoting the direction of the trajectories at the faces.

Figure 4.3 depicts the constructed digraph from Figure 4.4 and the corresponding adjacency matrix is

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Since P is nil-potent, the cubic digraph is acyclic. Moreover, all the sinks and the sources have negative eigenvalues; hence the system in (4.1) is GAS according to

Table 4.2: Characterization of convex cones for CLS in (4.1).

Cone	B_k	B_m	B_l	Type
$\mathbf{S}_1$	OUT	MIXED	IN	Half-sink
$\mathbf{S}_2$	OUT	MIXED	IN	Half-sink
$\mathbf{S}_3$	IN	IN	IN	Sink
$\mathbf{S}_4$	OUT	OUT	OUT	Source
$\mathbf{S}_5$	OUT	OUT	OUT	Source
$\mathbf{S}_6$	IN	IN	IN	Sink
$\mathbf{S}_7$	IN	MIXED	OUT	Half-sink
$\mathbf{S}_8$	OUT	MIXED	IN	Half-sink

Theorem 3.6.1. Figure 4.6, Figure 4.7, and Figure 4.8 demonstrate the evolution of trajectories initiated interior to disparate modes.

Although the boundary surfaces $cone\{\mathbf{s}_1, \mathbf{s}_3\}$, $cone\{-\mathbf{s}_1, -\mathbf{s}_3\}$ have trajectories of mixed directions, because they share the same A matrices, there is no loop of length two that can violate acyclicity.

Remark We have carried out a detailed simulation of all global trajectories obtained for a grid of initial state values in each one of the eight cones. In two modes (Mode 4 and Mode 5), trajectories that start on the extreme (boundary) rays behave erratically. This indicates that a problem with well-posedness exists at the common faces of these two modes.

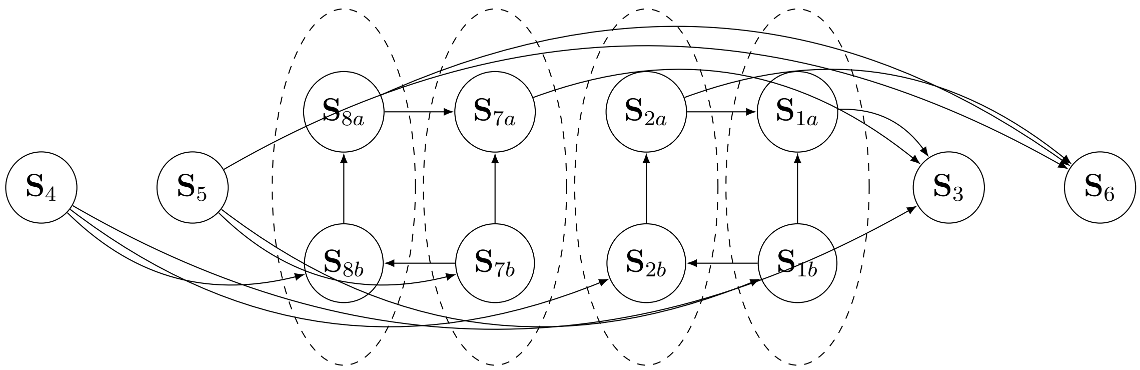


Figure 4.3: Topological sort of CLS in Fig. 4.4. Th four half-sinks can be modeled as two-node dashed ellipses, which behave as cubic nodes.

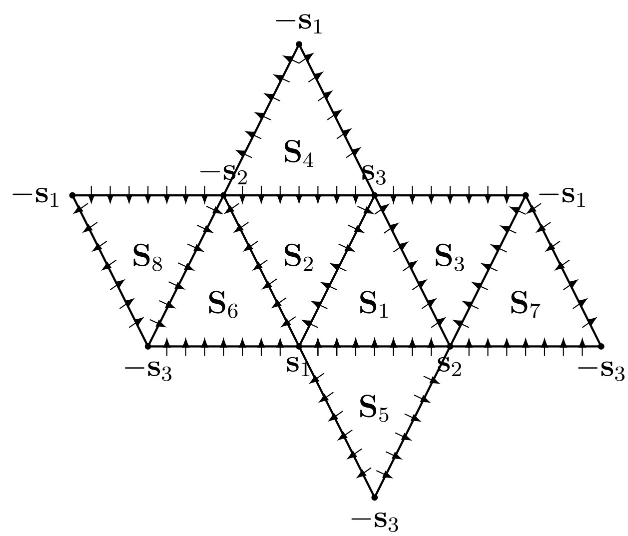


Figure 4.4: A 8-mode CLS with GAS.

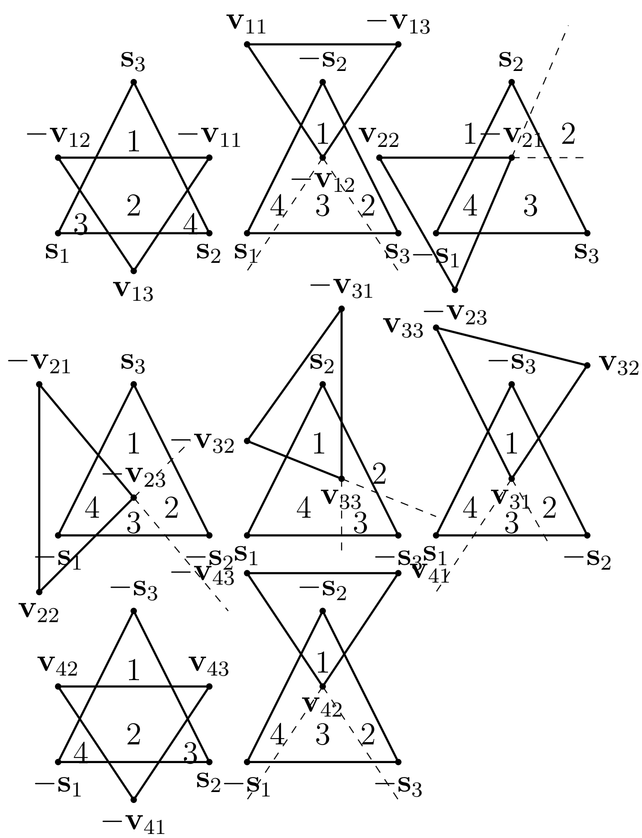


Figure 4.5: Positions of the $cone\{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}\}$ relative to $cone\{\mathbf{s}_{i1}, \mathbf{s}_{i2}, \mathbf{s}_{i3}\}$.

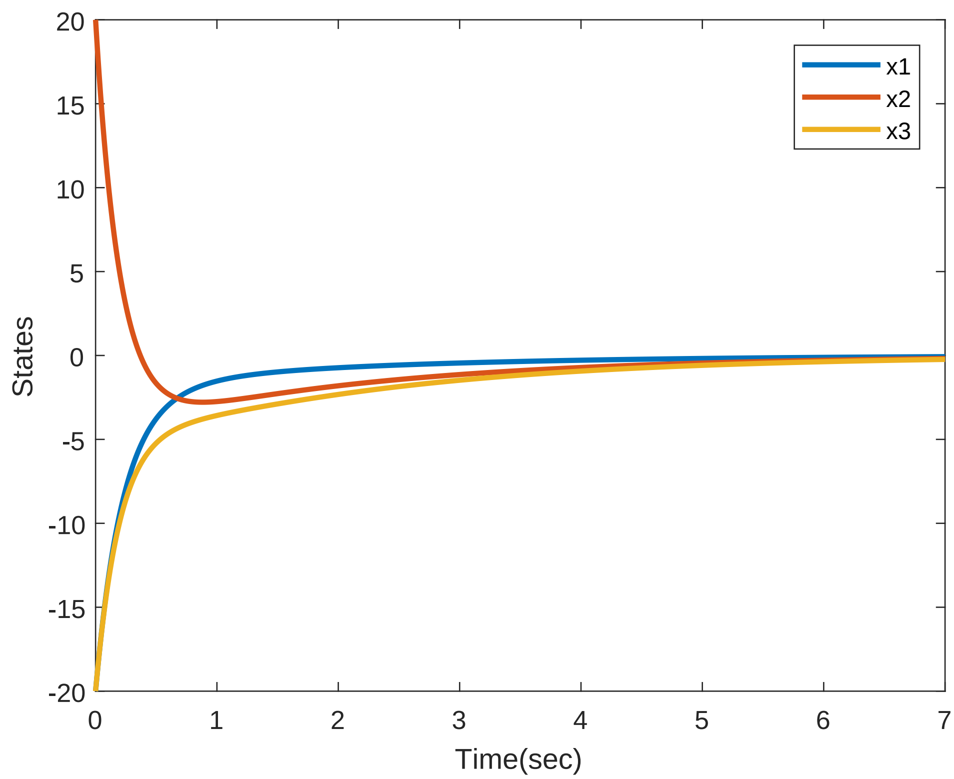


Figure 4.6: Evolution of state trajectories with initial conditions interior to $\mathbf{S}_4$ (source).

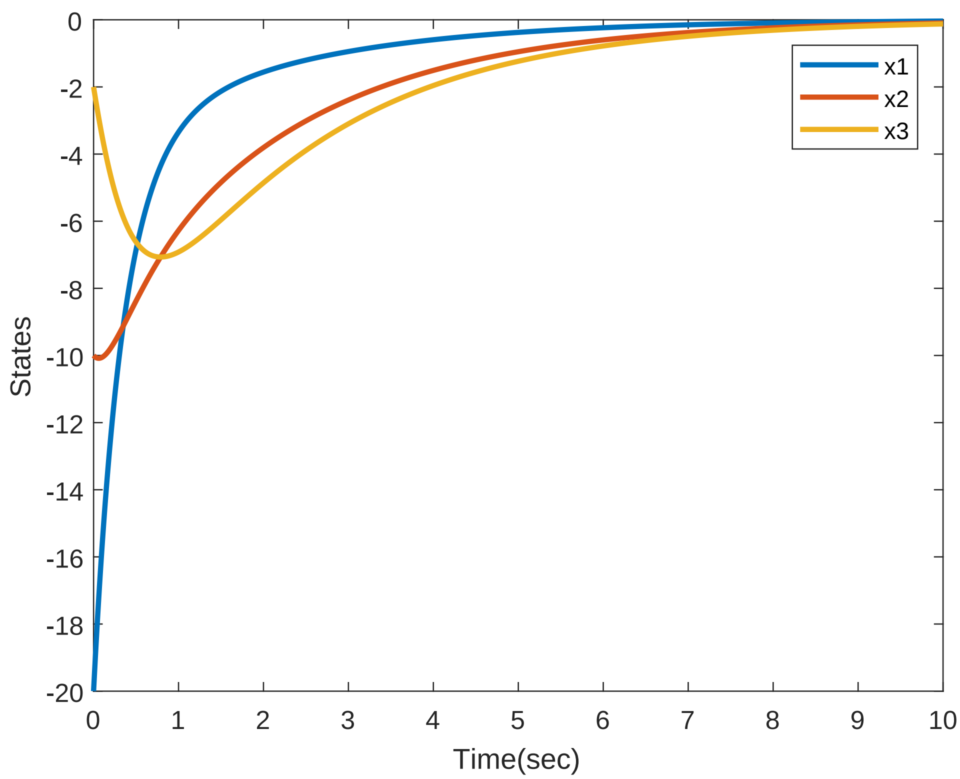


Figure 4.7: Evolution of state trajectories with initial conditions interior to $\mathbf{S}_3$ (sink).

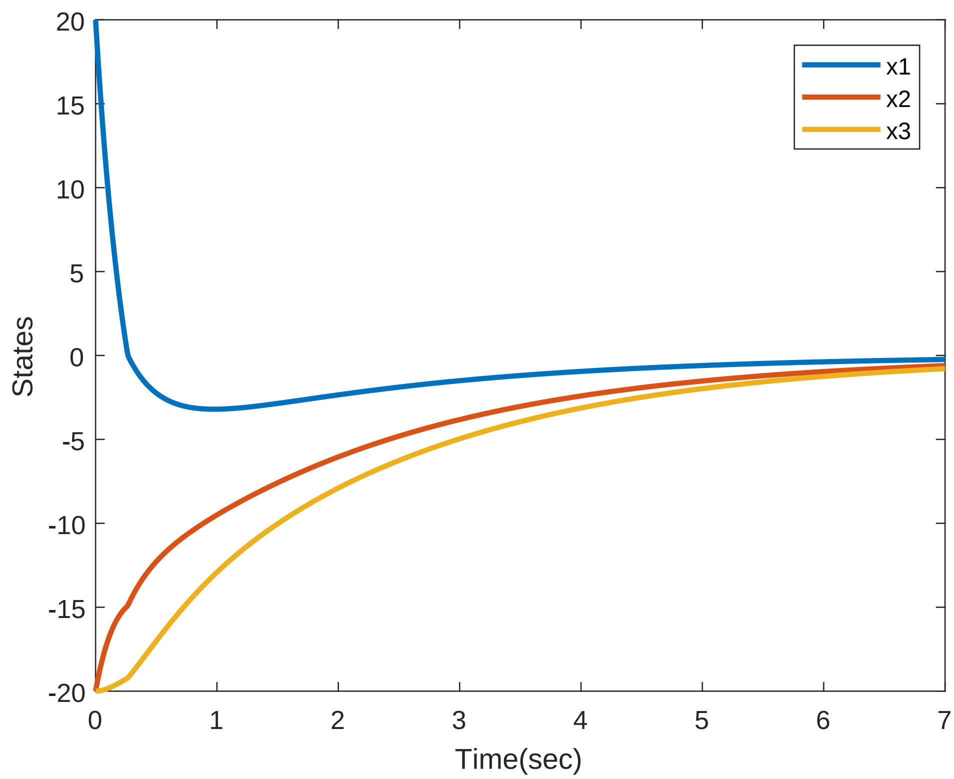


Figure 4.8: Evolution of state trajectories with initial conditions interior to $\mathbf{S}_1$ (transitive).

Chapter 5

Conclusions and Future Directions

We have developed a useful framework to analyze the global asymptotic stability of three dimensional conewise linear systems having distinct real eigenvalues using a direct approach. The approach requires a behavioral classification of the trajectories of a given single mode by examining how trajectories evolve inside the cone associated with the mode. A mode can be a source, sink, transitive from one or two of its faces but it can also be of a mixed type that we called half-sink. We have illustrated that, to a great extent, the relative positions of its eigenvectors to the extreme rays that define its cone determine the behavioral characteristics of a mode. But there are cases in which the values of the entries in the A-matrix of the mode makes a difference. That is, there are situations in which two modes defined by the same cones and the same positioning of the eigenvectors relative to that cone will behave differently due to a small difference in their A-matrices. Luckily, these cases can be identified and we have used the concepts of "mutually exclusive" and "structurally transitive" to distinguish the behaviorally robust modes from those that are not.

If the single modes consisting of sources, sinks, and transitive modes from one or two faces are brought together to form an N-mode CLS and if this CLS is well-posed, then we can use tools from graph theory to examine stability. The main result we have been able to obtain is that if there are no cycles in the corresponding digraph of a CLS, then its stability is determined by the smallest eigenvalue of its sources and the largest eigenvalues of its sinks.

This result appears amenable to extension to include half-sinks of certain types. We have illustrated such an extension by modeling a non-linear piecewise linear electrical network as a conewise linear system and by performing its stability analysis again using a graphical approach.

It is worth mentioning that our main stability result is generalizable to n-dimensional conewise linear systems whenever the system contains sources, sinks, and transitive modes from their $n-1$-dimensional faces. The correspondence of such a CLS with a digraph would be straightforward. The problem to be solved there is the behavioral characterization of a single n-dimensional mode in a manner similar to what we have done in $n = 3$.

In three dimension, we will next direct our efforts towards a behavioral characterization of modes with non-diagonalizable and/or complex eigenvalued A-matrices. A more ambitious goal is to devise a measure that generalizes the "factor of expansion" of the planar $n = 2$ case in order to extend the results on asymptotic stability to three dimensional CLS that contain global trajectories that are circular, that is trajectories that meet a particular mode infinitely many times.

Bibliography

[1] M. Wicks, P. Peleties, and R. DeCarlo, "Switched controller synthesis for the quadratic stabilisation of a pair of unstable linear systems," *European Journal of Control*, vol. 4, no. 2, pp. 140–147, 1998.

[2] A. J. Van Der Schaft and J. M. Schumacher, *An Introduction to Hybrid Dynamical Systems*, vol. 251. Springer London, 2000.

[3] W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 751–760, 1999.

[4] Z. Sun and S. S. Ge, *Stability Theory of Switched Dynamical Systems*. Springer Science & Business Media, 2011.

[5] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE control systems magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[6] W. M. G. van Bokhoven, *Piecewise-linear modelling and analysis*. Kluwer Technische Boeken Deventer, The Netherlands, 1981.

[7] M. K. Camlibel, J.-S. Pang, and J. Shen, "Conewise linear systems: non-zenoness and observability," *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1769–1800, 2006.

[8] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 555–559, April 1998.

[9] R. Iervolino, F. Vasca, and L. Iannelli, "A cone-copositive approach for the stability of piecewise linear differential inclusions," in *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pp. 1062–1067, IEEE, 2011.

[10] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic control*, vol. 54, no. 2, pp. 308–322, 2009.

[11] J. Mohammadpour and C. W. Scherer, *Control of linear parameter varying systems with applications*. Springer Science & Business Media, 2012.

[12] S. Ahmed, F. Mazenc, and H. Özbay, "Dynamic output feedback stabilization of switched linear systems with delay via a trajectory based approach," *Automatica*, vol. 93, pp. 92–97, 2018.

[13] M. di Bernardo, C. I. H. Velasco, U. Montanaro, and S. Santini, "Experimental implementation and validation of a novel minimal control synthesis adaptive controller for continuous bimodal piecewise affine systems," *Control Engineering Practice*, vol. 20, no. 3, pp. 269–281, 2012.

[14] H. Molla-Ahmadian, F. Tahami, A. Karimpour, and N. Pariz, "Hybrid control of DC-DC series resonant converters: The direct piecewise affine approach," *IEEE Transactions on Power Electronics*, vol. 30, no. 3, pp. 1714–1723, 2015.

[15] I. Uyanık, M. M. Ankaralı, N. J. Cowan, U. Saranlı, and Ö. Morgül, "Identification of a vertical hopping robot model via harmonic transfer functions," *Transactions of the Institute of Measurement and Control*, vol. 38, no. 5, pp. 501–511, 2016.

[16] W. Heemels, M. K. Camlibel, and J. M. Schumacher, "On the dynamic analysis of piecewise-linear networks," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 3, pp. 315–327, 2002.

[17] I. Dobson, "Stability of ideal thyristor and diode switching circuits," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, no. 9, pp. 517–529, 1995.

[18] N. Inaba and S. Mori, "Chaos via torus breakdown in a piecewise-linear forced van der Pol oscillator with a diode," *IEEE Transactions on Circuits and Systems*, vol. 38, no. 4, pp. 398–409, 1991.

[19] S. Duan, J. Ni, and A. G. Ulsoy, "Modeling and control of an automotive all-wheel drive clutch as a piecewise affine system," *Journal of Dynamic Systems, Measurement, and Control*, vol. 136, no. 1, p. 011008, 2014.

[20] M. Rewienski and J. White, "A trajectory piecewise-linear approach to model order reduction and fast simulation of nonlinear circuits and micromachined devices," *IEEE Transactions on computer-aided design of integrated circuits and systems*, vol. 22, no. 2, pp. 155–170, 2003.

[21] A. B. Ozgüler, "Conflictual peacetime international politics," *Bilkent University*, 2013.

[22] M. E. Sezer and A. B. Ozguler, "A dynamic allocation scheme for a multi-agent nash equilibrium," *WSEAS Transactions on Systems and Control*, vol. 1, no. 2, pp. 262–266, 2006.

[23] J. M. Gonçalves, A. Megretski, and M. A. Dahleh, "Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2089–2106, 2003.

[24] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," in *1997 European Control Conference (ECC)*, pp. 2005–2010, IEEE, 1997.

[25] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *Proceedings of the 1998 American Control Conference. ACC (IEEE Cat. No. 98CH36207)*, vol. 6, pp. 3659–3664, IEEE, 1998.

[26] S. Pettersson, *Modelling, Control and Stability Analysis of Hybrid Systems*. Chalmers University of Technology, 1996.

[27] Y. Iwatani and S. Hara, "Stability tests and stabilization for piecewise linear systems based on poles and zeros of subsystems," *Automatica*, vol. 42, no. 10, pp. 1685–1695, 2006.

[28] A. Arapostathis and M. E. Broucke, "Stability and controllability of planar, conewise linear systems," *Systems & Control Letters*, vol. 56, no. 2, pp. 150–158, 2007.

[29] M. Camlibel, W. Heemels, and J. Schumacher, "Stability and controllability of planar bimodal linear complementarity systems," in *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, vol. 2, pp. 1651–1656, IEEE, 2003.

[30] Z. Xi, G. Feng, and T. Hesketh, "Piecewise sliding-mode control for T-S fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 4, pp. 707–716, 2011.

[31] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[32] A. J. van der Schaft and J. Schumacher, "Complementarity modeling of hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 483–490, 1998.

[33] K. H. Johansson, A. Rantzer, and K. J. Aström, "Fast switches in relay feedback systems," *Automatica*, vol. 35, no. 4, pp. 539–552, 1999.

[34] C. Pittet, S. Tarbouriech, and C. Burgat, "Stability regions for linear systems with saturating controls via circle and popov criteria," in *Proceedings of the 36th IEEE Conference on Decision and Control*, vol. 5, pp. 4518–4523, IEEE, 1997.

[35] Y. Mori, T. Mori, and Y. Kuroe, "A solution to the common Lyapunov function problem for continuous-time systems," in *Proceedings of the 36th*

IEEE Conference on Decision and Control, vol. 4, pp. 3530–3531, IEEE, 1997.

[36] R. N. Shorten and K. S. Narendra, "A sufficient condition for the existence of a common Lyapunov function for two second order linear systems," in *Proceedings of the 36th IEEE Conference on Decision and Control*, vol. 4, pp. 3521–3522, IEEE, 1997.

[37] Z. Sun, "Stability of piecewise linear systems revisited," *Annual Reviews in Control*, vol. 34, no. 2, pp. 221–231, 2010.

[38] R. N. Shorten, O. Mason, F. O'Cairbre, and P. Curran, "A unifying framework for the SISO circle criterion and other quadratic stability criteria," *International Journal of Control*, vol. 77, no. 1, pp. 1–8, 2004.

[39] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on automatic control*, vol. 43, no. 4, pp. 475–482, 1998.

[40] A. N. Michel and B. Hu, "Towards a stability theory of general hybrid dynamical systems," *Automatica*, vol. 35, no. 3, pp. 371–384, 1999.

[41] R. A. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[42] D. Liberzon, *Switching in Systems and Control*. Springer Science & Business Media, 2003.

[43] Z. Sun, *Switched Linear Systems: Control and Design*. Springer Science & Business Media, 2006.

[44] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM review*, vol. 49, no. 4, pp. 545–592, 2007.

[45] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *IEEE transactions on automatic control*, vol. 45, no. 10, pp. 1864–1876, 2000.

[46] M. K. Camlibel, W. Heemels, and J. Schumacher, "Algebraic necessary and sufficient conditions for the controllability of conewise linear systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 762–774, 2008.

[47] J. Shen, "Observability analysis of conewise linear systems via directional derivative and positive invariance techniques," *Automatica*, vol. 46, no. 5, pp. 843–851, 2010.

[48] K. Liu, Y. Yao, B. Yang, V. Balakrishnan, and Y. Guo, "Exponential stability analysis of planar piecewise-linear systems: An integral function approach," *International Journal of Control, Automation and Systems*, vol. 10, no. 2, pp. 203–212, 2012.

[49] A. Abdullahi, "Stability of planar piecewise linear systems: A geometric approach," Master's thesis, Bilkent University, 2015.

[50] V. Eldem and I. Öner, "A note on the stability of bimodal systems in with discontinuous vector fields," *International Journal of Control*, vol. 88, no. 4, pp. 729–744, 2015.

[51] V. Eldem and G. Şahan, "On the stability of bimodal systems in $\mathcal{R}^3$," in *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, pp. 3220–3225, IEEE, 2009.

[52] R. Iervolino, F. Vasca, and L. Iannelli, "Cone-copositive piecewise quadratic Lyapunov functions for conewise linear systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 11, pp. 3077–3082, 2015.

[53] G. Feng, "Controller design and analysis of uncertain piecewise-linear systems," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 2, pp. 224–232, 2002.

[54] D. Da-Wei and Y. Guang-Hong, "Static output feedback control for discrete-time piecewise linear systems: an LMI approach," *Acta Automatica Sinica*, vol. 35, no. 4, pp. 337–344, 2009.

[55] A. Lj. Juloski, W. Heemels, and S. Weiland, "Observer design for a class of piecewise linear systems," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 17, no. 15, pp. 1387–1404, 2007.

[56] R. Diestel, *Graph Theory*. Springer Publishing Company, Incorporated, 2018.

[57] C. Davis, "Theory of positive linear dependence," *American Journal of Mathematics*, vol. 76, no. 4, pp. 733–746, 1954.

[58] R. G. Regis, "On the properties of positive spanning sets and positive bases," *Optimization and Engineering*, vol. 17, no. 1, pp. 229–262, 2016.

[59] J.-i. Imura and A. Van Der Schaft, "Characterization of well-posedness of piecewise-linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 9, pp. 1600–1619, 2000.

[60] B. Özgüler and M. Zakwan, "Stability of conewise linear systems in 3D:a geometric approach," *Bilkent University*, 2019.

[61] K. J. Guo, *Simple eigenvalues of graphs and digraphs*. PhD thesis, Science:, 2015.

[62] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*. MIT press, 2009.

[63] R. Sedgewick, *Algorithms in Java, Part 5: Graph Algorithms: Graph Algorithms*. Addison-Wesley Professional, 2003.

[64] W. Heemels and B. Brogliato, "The complementarity class of hybrid dynamical systems," *European Journal of Control*, vol. 9, no. 2-3, pp. 322–360, 2003.

[65] D. Leenaerts and W. M. Van Bokhoven, *Piecewise Linear Modeling and Analysis*. Springer Science & Business Media, 2013.