Damping Mechanisms

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1 Linear Response and Damping

Introduction

The term *damping* is used to describe the means by which oscillation amplitudes are reduced through irreversible removal of vibratory energy in a mechanical system or a component. Dissipation, on the other hand, refers to the mechanism by which irreversible energy transfer, from vibratory to thermal, takes place. In this sense, damping is a macro-scale manifestation of atomic-scale dissipation.

High damping is desirable to attain low vibration and noise levels whereas low damping is desirable for increased sensitivity in sensors and certain precision instrumentation.

Damping is most obvious at resonance where the stiffness and inertia forces become equal. As a result, damping is a key factor in predicting vibration response of structures.

As we will see in the following sections, there are numerous paths to damping and in a complex structure several means of damping may take place simultaneously at different locations throughout the structure. Accordingly, in determining the response of a vibrating structure, the total effect of all types of damping that may be distributed throughout a structure must be taken into account.

Measurements of damping normally indicate the total damping a system experiences. It is difficult to isolate a component or a subsystem or a material within a system and measure its damping. In describing the various damping mechanisms, we will examine each through its effect on a single-degree-of-freedom (sdof) oscillator.

In this section, we will review the response of a simple oscillator and examine the role of damping on it and review the basic methods of measurement criteria for damping properties of structures. However, we will

P. Hagedorn, G. Spelsberg-Korspeter (Eds.), *Active and Passive Vibration Control of Structures*, CISM International Centre for Mechanical Sciences
DOI 10.1007/ 978-3-7091-1821-4_6

not consider here the role of damping in dynamic behaviors such as chaos, stability, etc.

Dissipation of vibratory energy takes place in both fluid and solid media, initiated by a number of possible macro activities. Accordingly, we will consider damping methods to reflect the media in which dissipation takes place when addressing damping methods in the next section. Models of fundamental dissipation mechanisms that describe energy transfer from ordered energy to disordered or thermalized energy are briefly summarized in the last section.

1.1 Simple Harmonic Oscillator

We employ the simple harmonic oscillator as the platform to describe damping models and measures through its linear response.

Initial Value Problem Equation of motion for free vibrations of an undamped sdof oscillator with mass M and stiffness K can be expressed as

$$M\ddot{\eta} + K\eta = 0$$

or in a simplified form

$$\ddot{\eta} + \omega_0^2 \eta = 0$$

where $\omega_0 = \sqrt{K/M}$ is the natural frequency of the oscillator. General solution for displacement η can be expressed as

$$\eta = A\cos(\omega_0 t - \phi).$$

Expressing the initial conditions at $t = 0$ as $\eta(0) = \eta_0$ and $\dot{\eta}(0) = \dot{\eta}_0$, we can write

$$\eta_0 = A\cos(\phi) \qquad \dot{\eta}_0 = \omega_0 A\sin(\phi)$$

and the vibration amplitude in terms of the initial conditions becomes

$$A = \sqrt{\eta_0^2 + (\dot{\eta}_0/\omega_0)^2}.$$

Free Damped Motion When damping is proportional to oscillator velocity, represented by a constant of proportionality C, the equation of motion becomes:

$$M\ddot{\eta} + C\dot{\eta} + K\eta = 0$$

with a corresponding solution for free vibrations:

$$\eta(t) = A\, e^{-\gamma t}\cos(\omega_d t - \phi)$$

where $\omega_d = \omega_0\sqrt{1-\zeta^2} = \sqrt{\omega_0^2 - \gamma^2}$ is the damped natural frequency and $\gamma = C/2M$ is the *decay constant* and is related to the *damping ratio* $\zeta = \gamma/\omega_0 = C/C_c$, which is the ratio of damping constant to its critical value $C_c = 2\sqrt{KM}$.

For *underdamped* cases, $\zeta < 1$, response to initial conditions can be written as

$$\eta(t) = e^{-\zeta\omega_0 t}\left[\frac{\dot{\eta}_0 + \zeta\omega_0\eta_0}{\omega_d}\sin\omega_d t + \eta_0\cos\omega_d t\right].$$

When $\gamma > \omega_0$, or $\zeta > 1$, ω_d becomes complex and oscillations are not possible and the system is referred as *overdamped*.

Between these two cases, when $\gamma = \omega_0$ or $\zeta = 1$, the oscillator is considered *critically damped*.

Forced Motion Response η of a sdof oscillator to a force $F(t)$ can be described with:

$$M\ddot{\eta} + C\dot{\eta} + K\eta = F(t). \tag{1}$$

Fourier transforming the motion equation (1) according to

$$\eta(\omega) = \int_{-\infty}^{\infty}\eta(t)e^{-j\omega t}dt$$

we obtain response equation in the frequency domain:

$$\left[-M\omega^2 - j\omega C + K\right]\eta(\omega) = F(\omega).$$

► **Harmonic Excitation** Response of a simple oscillator to harmonic excitation $F_0e^{j\omega t}$ can be expressed in terms of receptance (or compliance) frequency response function (FRF), $H(\omega)$, of the oscillator

$$\eta(\omega) = H(\omega)F_0e^{-j\omega t}$$

where

$$H(\omega) = \left\{K\left[1 - (\omega/\omega_0)^2 - j(2\zeta\omega/\omega_0)\right]\right\}^{-1}.$$

Frequency response function is the Fourier transform of the impulse response function and is generally a complex quantity:

$$H(\omega) = |H(\omega)|\, e^{j\Phi}$$

where

$$\tan\Phi = \frac{\omega C}{K - \omega^2 M} = \frac{2\zeta(\omega/\omega_0)}{1 - (\omega/\omega_0)^2}.$$

Instead of displacement FRF, if we use the velocity FRF, we can write

$$\dot{\eta} = -j\omega H(\omega) F_0 e^{-j\omega t} = Y(\omega) F_0 e^{-j\omega t}$$

where the mobility relates to the impedance expression as $Y(\omega) = 1/Z(\omega)$. The relationship between mobility and receptance is

$$Y(\omega) = -j\omega H(\omega) = \omega |H(\omega)| e^{j\theta}$$

where $\theta = \Phi - \pi/2$, since $\exp(-j\pi/2) = -j$.

Similarly acceleration and excitation force are related through acceleerance (or inertance) $\mathcal{A}(\omega) = (-j\omega)^2 \eta / F$ and

$$\mathcal{A}(\omega) = -j\omega Y(\omega) = \omega |Y(\omega)| e^{-j\alpha} = -\omega^2 H(\omega)$$

and $\alpha = \theta - \pi/2 = \Phi - \pi$.

►**Impulse or Step Excitation** Impulse response of an undamped simple oscillator can be considered as equivalent to response to an initial velocity and described as:

$$h(t) = \frac{1}{M\omega_0} \sin \omega_o t, \quad t > 0$$

and when damping is present:

$$h(t) = \frac{1}{M\omega_d} e^{-\zeta\omega_d t} \sin \omega_d t, \quad t > 0.$$

►**Arbitrary Excitation** Response $\eta(t)$ of a system to an arbitrary excitation can be obtained through a convolution integral of the input $F(t)$ and the impulse response $h(t)$ of the linear system:

$$\eta(t) = \int_{-\infty}^{\infty} F(\tau)\, h(t-\tau) d\tau = \int_{-\infty}^{\infty} F(t-\tau)\, h(\tau) d\tau. \tag{2}$$

However, for the system to be causal, its impulse response also must be causal:

$$h(t) = 0 \quad \text{for} \quad t < 0.$$

The causality condition states that response must follow the excitation and not anticipate or precede it. Invoking causality, the limits of the convolution expression given in (2) can be modified as:

$$\eta(t) = \int_{0}^{t} F(\tau)\, h(t-\tau) d\tau = \int_{0}^{t} F(t-\tau)\, h(\tau) d\tau.$$

1.2 Causality

The most significant consequence of causality emerges from the Fourier transform of a causal impulse response function. Causal functions exhibit a strong linkage between the real and imaginary parts of their Fourier transforms, expressed in terms of Hilbert transform pairs or Kramer-Krönig relations.

The spectrum of $h(t)$, namely the frequency response function $H(\omega)$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

must reflect in its spectrum the causal properties of $h(t)$. Accordingly, its inverse transform must have the following properties (Pierce, 2008):

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t} dt = \begin{cases} 0 & \text{if } t < 0 \\ h(t) & \text{if } t > 0 \end{cases}. \tag{3}$$

The frequency response function $H(\omega)$ that satisfies the causality condition in Eq. (3), can be obtained by Fourier transforming the $h(t)$ by expressing its causal property with a unit step, or Heaviside's, function $\mathcal{U}(t)$,

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\{h(t)\,\mathcal{U}(t)\}$$

which can be expressed as a convolution of the Fourier Transforms of h and $\mathcal{U}$

$$H(\omega) = \mathcal{F}[h(t)] * \mathcal{F}[\mathcal{U}(t)]$$

where the Fourier transform of the unit step function is

$$\mathcal{F}[\mathcal{U}(t)] = \int_{-\infty}^{\infty} \mathcal{U}(t) e^{-j\omega t} dt = \left[\pi\delta(\omega) - j\left\{\frac{1}{\omega}\right\}\right].$$

It is understood that the second term in the brackets is interpreted as a distribution and when combined with a function its Cauchy principle value (**p.v.**) is taken. Substituting in the convolution equation above

$$H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega') \left[\pi\delta(\omega - \omega') - \left\{\frac{j}{\omega - \omega'}\right\}\right] d\omega'$$

$$H(\omega) = \frac{1}{2} H(\omega) - \frac{j}{2\pi}\ \mathbf{p.v.} \int_{-\infty}^{\infty} \left\{\frac{H(\omega')}{\omega - \omega'}\right\}\ d\omega'$$

$$H(\omega) = -\frac{j}{\pi}\ \mathbf{p.v.} \int_{-\infty}^{\infty} \left\{ \frac{H(\omega')}{\omega - \omega'} \right\}\ d\omega'$$

switching ω and ω' produces Hilbert transform of $H(\omega)$:

$$H(\omega) = \frac{j}{\pi}\ \mathbf{p.v.} \int_{-\infty}^{\infty} \left\{ \frac{H(\omega')}{\omega' - \omega} \right\}\ d\omega'.$$

Separating the real and imaginary parts of $H(\omega)$ yields:

$$H_R(\omega) = -\frac{1}{\pi}\ \mathbf{p.v.} \int_{-\infty}^{\infty} \frac{H_I(\omega')}{\omega' - \omega} d\omega' \tag{4}$$

$$H_I(\omega) = \frac{1}{\pi}\ \mathbf{p.v.} \int_{-\infty}^{\infty} \frac{H_R(\omega')}{\omega' - \omega} \omega'. \tag{5}$$

These relations between the real and imaginary parts of the frequency response function represent and assure the causality of the impulse response. They also show that when the real part of a causal function is known, the imaginary part can be obtained and vice versa.

Since in vibrations we deal with positive frequencies, the integral in the Hilbert transform pair (4) and (5) can be re-expressed so long as H has the property $H(-\omega) = H^*(\omega)$ that allows us to make the substitutions $H_R(-\omega) = H_R(\omega)$ and $H_I(-\omega) = -H_I(\omega)$ in the following derivations:

$$H_R(\omega) = -\frac{1}{\pi}\ \mathbf{p.v.} \int_{-\infty}^{0} \frac{H_I(\omega')}{\omega' - \omega} d\omega' - \frac{1}{\pi}\ \mathbf{p.v.} \int_{0}^{\infty} \frac{H_I(\omega')}{\omega' - \omega} d\omega'$$

$$H_R(\omega) = \frac{1}{\pi}\ \mathbf{p.v.} \int_{0}^{\infty} \frac{H_I(-\omega')}{\omega' + \omega} d\omega' - \frac{1}{\pi}\ \mathbf{p.v.} \int_{0}^{\infty} \frac{H_I(\omega')}{\omega' - \omega} d\omega'$$

$$H_R(\omega) = -\frac{1}{\pi}\ \mathbf{p.v.} \int_{0}^{\infty} H_I(\omega') \left(\frac{1}{\omega' + \omega} + \frac{1}{\omega' - \omega} \right) d\omega'$$

$$H_R(\omega) = -\frac{2}{\pi}\ \mathbf{p.v.} \int_{0}^{\infty} H_I(\omega') \left(\frac{\omega'}{\omega'^2 - \omega^2} \right) d\omega'. \tag{6}$$

Similarly,

$$H_I(\omega) = \frac{2}{\pi}\ \mathbf{p.v.} \int_{0}^{\infty} H_R(\omega') \left(\frac{\omega}{\omega'^2 - \omega^2} \right) d\omega'. \tag{7}$$

The last two equations (6) and (7) are known as the Kramers-Krönig relations that are used to describe causal impulse response functions (viz., Waters et al., 2005).

1.3 Damping Measurement Criteria

Since damping can only be measured *indirectly* by observing the response of a system, we review below the different means by which damping can be characterized. This section also introduces the terms commonly used in association with damping in vibrating systems. For example, in elastic systems, a measure of damping during time-dependent or cyclic motion is defined as the dissipated part $\triangle W$ of total elastic energy stored, W, during one cycle. Their ratio is called the *specific damping ratio*

$$\Psi = \frac{\triangle W}{W}$$

and the corresponding *loss factor* is defined as

$$\chi = \frac{\Psi}{2\pi} = \frac{1}{2\pi}\frac{\triangle W}{W}.$$

As described later, loss factor is related to the Q-value as:

$$\chi = \frac{1}{Q}.$$

Logarithmic Decrement Logarithmic decrement method is used in conjunction with decaying free vibration response of an oscillator taking advantage of the exponential nature of the response envelope as described in the transient or complementary solution expressions above:

$$\eta(t) = Ae^{-\gamma t}\cos(\omega_d t - \alpha).$$

Again, the quantity $\gamma = C/2M$ is the *decay constant* and its inverse is the *decay time*, τ. Hence, the amplitude variation depends on time exponentially, $\exp(-t/\tau)$.

The relative change of the amplitude in one period is called the *logarithmic decrement*, δ, which describes the rate of decay

$$\delta = \gamma T_d = \gamma 2\pi/\omega_d = (\pi C/\omega_0 M)(\omega_0/\omega_d).$$

Logarithmic decrement is also related to the *damping ratio* or *damping factor* ζ:

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

and for very small values of $\zeta << 1$

$$\delta \approx 2\pi\zeta.$$

Logarithmic decrement is determined by measuring the response at two maxima on its envelope that are apart by one or more periods:

$$\delta = \frac{1}{n} \ln \frac{\eta^{(0)}}{\eta^{(n)}}$$

where n is the number of periods between the measurement positions with amplitudes $\eta^{(0)}, \eta^{(n)}$ measured at times $t = t_0, t_n$, respectively.

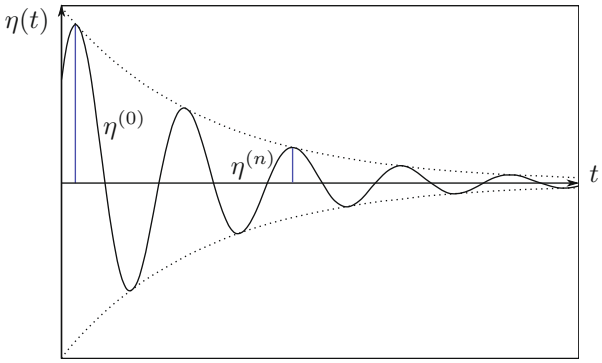

Figure 1. Logarithmic decay of amplitude of damped vibrations of a sdof oscillator.

Quality Factor Damping values can be directly obtained from the frequency response functions. Normalizing the displacement amplitude of a harmonically forced oscillator with the static displacement η_{st} gives:

$$\left| \frac{\eta(\omega)}{\eta_{st}} \right| = \frac{1}{\sqrt{[1 - (\omega/\omega_0)^2]^2 + 4\zeta^2(\omega/\omega_0)^2}}$$

where $\eta_{st} = F_0/K$.

From the response measurements, the maximum amplitude is measured at approximately $\omega \approx \omega_0$

$$\left[\frac{|\eta(\omega)|}{\eta_{st}} \right]_{max} \approx \frac{1}{2\zeta} = Q$$

resulting in the *Q-value*. *Quality Factor* of the system, which is inversely related to the damping factor ζ, can now be directly measured from the FRF.

Using Q-value to determine amplitude is useful when damping is low and the resonant amplitudes are high. In frequency response plots, bandwidth of the resonance at half-power points provide another measurement method.

Frequency Response & Half-Power At half power points in an FRF, we can write

$$\frac{Q}{\sqrt{2}} = \frac{1}{\{[1-(\omega/\omega_0)^2]^2 + 4\zeta^2(\omega/\omega_0)^2\}^{1/2}} = \frac{1}{2\sqrt{2}\zeta}.$$

Expanding the denominator

$$(\omega/\omega_0)^4 - (\omega/\omega_0)^2(2-4\zeta^2) + (1-8\zeta^2) = 0$$

produces the roots:

$$(\omega_1/\omega_0)^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1+\zeta^2}$$

$$(\omega_2/\omega_0)^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1+\zeta^2}.$$

For small values of damping ratios, such that $\zeta < 0.05$

$$(\omega_1/\omega_0)^2 \approx 1 - 2\zeta$$

$$(\omega_2/\omega_0)^2 \approx 1 + 2\zeta.$$

Subtracting these equations

$$\omega_2^2 - \omega_1^2 = 4\zeta\omega_0^2$$

$$(\omega_2 - \omega_1)2\omega_0 = 4\zeta\omega_0^2.$$

Then the damping ratio follows as

$$\zeta = \frac{\triangle\omega}{2\omega_0}$$

where $\triangle\omega = \omega_2 - \omega_1$ is the bandwidth at half-power points. Effect of damping on displacement amplitude is illustrated in Fig. (2) with transfer functions for different damping values.

There are numerous other approaches to measure and quantify damping in materials and mechanical systems such as using Nyquist plots and Bode plots that can be found in the literature (viz., Mead, 1998; Nashif et al., 1985).

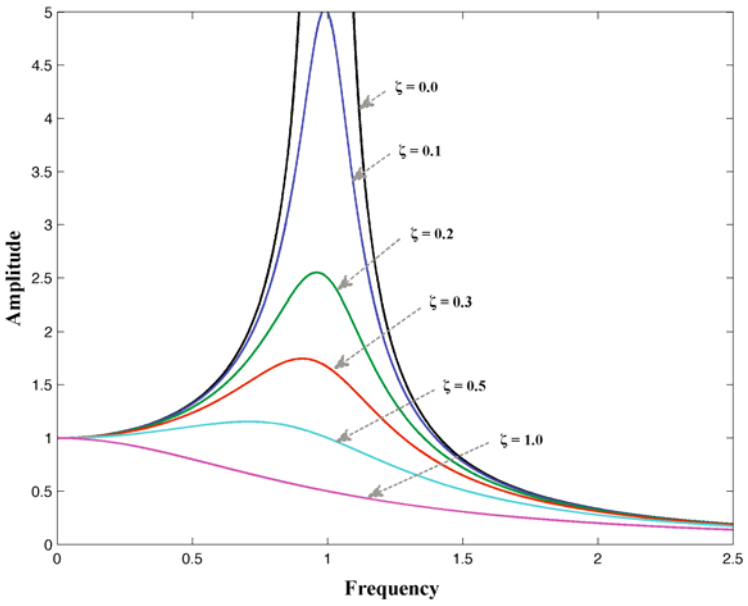

Figure 2. Frequency response function of a SDOF oscillator for different damping values.

2 Damping in Structures

Vibratory energy is dissipated in structures through numerous damping mechanisms. Several such damping mechanisms may take place simultaneously in a complex a mechanical system, or even in a simple component. The total damping that characterizes a system is a combination of energy dissipation effected by different mechanisms throughout the structure.

Damping mechanisms addressed here can be considered largely in three categories according to their fundamental mode of energy dissipation:

- dissipation within a solid,
- dissipation within or to a fluid medium, and
- dissipation at the interfaces between solids or between a solid and a fluid.

Some of the damping mechanisms described below are common to almost all systems and yet others are more specific to operating conditions and even length scales involved, particularly in cases of design and manufacture of MEMS and nano-scale devices.

2.1 Dissipation within Solids: Material or Internal Damping

Internal or material damping refers to inherent energy dissipation during cyclic motion or deformation of a material. The kinetic energy is irreversibly converted to thermal energy through one or more mechanisms. These mechanisms are associated with the internal structure of the material and have different length scales, such as those associated with dislocations, grain boundaries, or atomic motion. Internal damping also refers to those dissipations that arise from thermal, electronic and magnetic fields in the materials. As such, effectiveness of internal damping mechanisms range over different temperatures and frequencies.

Internal damping properties of materials can be enhanced by changing the molecular structure of the materials, or by, using alloys and viscoelastic materials. For example, carbon in cast iron is known to increase its damping properties. In the case of composites, however, macroscopic modification of the material structure provides for increased damping properties. Various fiber enhancements and foam-type structures are other examples.

In engineering, internal damping is generally characterized by a single value and, where appropriate, with frequency and temperature dependence. Internal damping is, in fact, an aggregate of energy dissipation due to numerous microscopic sources and mechanisms in a material.

This section attempts to delineate and explain some of these fundamental mechanisms of energy dissipation in types of solids that are of interest

in structural vibrations, namely anelastic and viscoelastic materials. We exclude here nonlinear elasticity and plasticity and consider linear elasticity to explain the anelastic properties.

Ideal (linear) elasticity assumes an instantaneous relationship between stress and strain with a unique equilibrium value and a perfectly reversible deformation, i.e., with a complete recovery. On the other hand, *anelasticity* exhibits the same properties of recoverability and linearity but without the instantaneous response. In an anelastic material, a unique equilibrium value of strain corresponds to every stress (and vice versa), but the equilibrium is reached after a finite time rather than instantaneously. In *viscoelasticity*, in addition to time dependence, the initial equilibrium is not completely recoverable.

Anelastic solids, also described as thermodynamic solids, reach a thermodynamic equilibrium in response to a change in applied external forces. Through self-adjustment the solid reaches the new equilibrium through a process called as *anelastic relaxation*, which takes place a over period of *relaxation time*. For instance, when a constant stress is applied, anelastic relaxation manifests itself as a time-dependent (or frequency-dependent) equilibriation of strain and vice versa. This external manifestation of anelasticity reflects the thermodynamic equilibriation of internal variables in the solid. In this manner, for each stress level, a strain relaxation develops in conjunction with a new internal equilibrium of the solid. In anelastic solids the stress and strain relationship has different moduli corresponding to the initial and new equilibria and such a change in the modulus requires a transport process, for example, of atomic migration, dislocation displacements, grain boundary sliding, and phase transformations as well as thermal relaxation all of which lead to anelastic behavior.

Thermodynamic damping is the most fundamental internal damping mechanism as it can develop without the presence of material inhomogeneities. In the presence of material inhomogeneities, other peaks with respective Lorentz distributions appear. Among these, Zener peaks refer to dislocation relaxation, Bordoni peaks describe grain boundary relaxation, and Snoek peaks refer to defect pair reorientation. Presence of multiple such relaxation times may lead to multiple or broadened Debye peaks (Lifshitz and Roukes, 2000).

A three-parameter solid is used commonly to describe thermoelastic damping or thermoelastic relaxation of materials, which is referred as Zener model or standard linear solid. The Zener model consists of either a Maxwell model (spring and damping elements in series) parallel to another spring or a Voigt element (spring and damping elements in series) in series with another spring. Such models are found to effectively represent anelastic

material behavior.

In what follows, we summarize the derivation of relaxation relations with Zener's anelastic solid model based on the thermodynamic model mentioned above. The resulting expressions describe dissipation due to inhomogeneities in a material, representing *thermoelastic damping*.

The thermoelastic damping expression has the form of a Lorentz distribution, with a maximum sometimes called the Debye peak, magnitude of which is the *relaxation strength* and the peak frequency corresponds to the inverse of relaxation. Different anelastic relaxations have different frequencies and temperature dependence. When multiple peaks develop, their effects are superposed.

2.2 Zener's Anelasticity Model

A homogeneous material subject to homogeneous stress undergoes thermal relaxation through heat exchange with its environment. However, if the stress or strain field is inhomogeneous, the resulting temperature gradients can lead to thermal relaxation through internal flow or ''thermal currents'' to reach new equilibrium from one part of the material to another (Nowick and Berry, 1972). Such coupling between stress fields and thermal fields in a solid gives rise to thermoelastic damping. *Thermoelastic coupling* is quantified by *thermal expansion coefficient*, α, as the coupling constant. Thermal expansion, change in strain due to change in externally applied temperature and the converse, and thermoelastic effect that describes small changes in temperature due to isentropic changes in dilatational stress are examples of thermoelastic coupling.

Thermoelastic effect can be considered by combining strain induced by temperature change with that obtained under pure elastic conditions, by considering strain to depend only on stress, σ, and temperature, T,

$$\epsilon = \frac{\sigma}{E_R} + \alpha \triangle T \tag{8}$$

where E_R is the *relaxed* or *isothermal modulus*, α represents the linear thermal expansion coefficient, and $\triangle T$ is the deviation from standard temperature (Zener, 1948).

The variation in temperature, $\triangle T$, is caused by either diffusion or change in strain. Temperature change caused by diffusion (or equalization or relaxation) of thermal fluctuations, can be approximated by

$$\left(\frac{d}{dt}\triangle T\right)_{diffusion} = -\frac{1}{\tau}\triangle T \tag{9}$$

where τ is the *relaxation* time. Relaxation time may have different values depending on the stress and strain restrictions imposed. For example, τ_ϵ represents the relaxation time for stress relaxation and temperature relaxation under constant strain. Analogously, a relaxation time τ_σ is defined for strain relaxation and temperature relaxation under constant stress. Just as an increase in temperature leads to an increase in length, an *adiabatic* increase in length leads to a decrease the temperature:

$$\left(\frac{d}{dt}\triangle T\right)_{adiabatic} = -\gamma\dot{\epsilon} \tag{10}$$

with $\gamma = (\partial T/\partial\epsilon)_{adiabatic}$. Combining the two mechanisms of temperature change in (9) and (10), we have:

$$\frac{d}{dt}\triangle T = -\frac{1}{\tau_\epsilon}\triangle T - \gamma\dot{\epsilon}. \tag{11}$$

Isolating $\triangle T$ from the coupled equation (8) and substituting it and its derivative in equation (11) above eliminates $\triangle T$ and leaves us with the stress-strain relationship:

$$E_R\,\epsilon + E_U\,\tau_\epsilon\,\dot{\epsilon} = \sigma + \tau_\epsilon\,\dot{\sigma} \tag{12}$$

where the unrelaxed modulus is

$$E_U = (1 + \alpha\,\gamma)E_R.$$

Rewriting Eq. (12) leads to an expression:

$$\sigma + \tau_\epsilon\dot{\sigma} = E_R(\epsilon + \tau_\sigma\dot{\epsilon}) \tag{13}$$

which describes the deviation from an elastic (Hook) solid into an anelastic, or standard, solid described by Zener (1948).

The relationship between τ_σ and τ_ϵ can be obtained by integrating both sides of (13) over a very small time δt, which yields

$$\int_0^{\delta t}(\sigma + \tau_\epsilon\dot{\sigma})dt = E_R\int_0^{\delta t}(\epsilon + \tau_\sigma\dot{\epsilon})dt. \tag{14}$$

The first term on each side of (14) vanishes as $\delta t \to 0$ leaving

$$\tau_\epsilon\,\triangle\sigma = E_R\,\tau_\sigma\,\triangle\epsilon.$$

In this case, the relation between the changes in stress and strain take place over such a short time that there is no time for relaxation to take effect and

the relationship between them, an adiabatic process, is through an *unrelaxed elastic modulus*, E_U as

$$\triangle\sigma = E_U \triangle\epsilon$$

and

$$\frac{\tau_\sigma}{\tau_\epsilon} = \frac{E_U}{E_R}. \tag{15}$$

Deviation of the ratio in (15) from unity indicates relaxation of stress or strain.

Assuming $\epsilon = \dot{\epsilon} = 0$, equation (13) becomes:

$$\sigma + \tau_\epsilon \dot{\sigma} = 0$$

with the solution

$$\sigma(t) = \sigma_0 e^{-t/\tau_\epsilon}$$

where $\sigma_0 = \sigma(0)$. When an initial strain ϵ_0 is suddenly applied at $t = 0$, the relaxation of stress follows

$$\sigma(t) = E_R\epsilon_0 + (\sigma_0 - E_R\epsilon_0)e^{-t/\tau_\epsilon}.$$

After the relaxation is completed, the relationship is simply $\sigma(t) = E_R\epsilon_0$ and, hence, the modulus is the relaxed elastic modulus. Analogously, for a suddenly applied stress σ_0, the corresponding strain time history becomes (Beltzer, 1988)

$$\epsilon(t) = \frac{1}{E}_R \sigma_0 + \left(\epsilon_0 - \frac{1}{E}_R \sigma_0\right) e^{-t/\tau_\sigma}.$$

Substituting harmonic excitation and response expressions

$$\sigma(t) = \sigma_0 e^{-j\omega t} \text{ and } \epsilon(t) = \epsilon_0 e^{-j(\omega t - \phi)}$$

in the anelastic solid expression (13) yields:

$$\sigma_0 = \frac{1 - j\omega\tau_\sigma}{1 - j\omega\tau_\epsilon} E_R\, \epsilon_0 = \mathcal{E}\epsilon_0. \tag{16}$$

The angle by which strain lags behind stress is a measure of internal friction:

$$\tan\phi = Q^{-1} = \frac{\Im\{\mathcal{E}\}}{\Re\{\mathcal{E}\}} = \frac{\omega(\tau_\epsilon - \tau_\sigma)}{1 + (\omega\tau_\epsilon)(\omega\tau_\sigma)}$$

which can be simplified as a Lorentzian distribution:

$$\tan\phi = \frac{E_R - E_U}{\bar{E}} \frac{\omega\bar{\tau}}{1 + (\omega\bar{\tau})^2} \tag{17}$$

with the geometric means: $\bar{\tau} = \sqrt{\tau_\sigma \tau_\epsilon}$ and $\bar{E} = \sqrt{E_R E_U}$. The first part of the internal loss expression (17) represents the relative difference in relaxed and unrelaxed elastic moduli and the second part represents its frequency dependence. The relative difference of the moduli is defined as the *relaxation strength*

$$\triangle_E = \frac{E_R - E_U}{\bar{E}}.$$

The loss in Eq.(17) has a maximum value at $\omega\bar{\tau} = 1$:

$$(\tan\phi)_{max} = Q^{-1}_{max} = \frac{1}{2}\ \triangle_E.$$

The magnitude of the complex modulus $\mathcal{E}$ in (16), the ratio of stress to strain, is expressed as:

$$|\mathcal{E}| = \frac{1+\omega^2\tau_\sigma^2}{1+\omega^2\tau_\sigma\tau_\epsilon} E_R = E_U - \frac{E_U - E_R}{1+\omega^2\bar{\tau}^2}$$

for which approximate values are expressed for low and high frequencies as (Beltzer, 1988):

$$\left.\begin{aligned} E_U - |\mathcal{E}| &= \frac{E_U - E_R}{(\omega\bar{\tau})^2} \\ \tan\phi &= \frac{E_U - E_R}{\bar{E}\,(\omega\bar{\tau})} \end{aligned}\right\} \qquad \omega\bar{\tau} \gg 1$$

$$\left.\begin{aligned} |\mathcal{E}| - E_R &= (E_U - E_R)\ (\omega\bar{\tau})^2 \\ \tan\phi &= \frac{E_U - E_R}{\bar{E}}\ (\omega\bar{\tau}) \end{aligned}\right\} \qquad \omega\bar{\tau} \ll 1$$

which can be further simplified as:

$$|\mathcal{E}| = \begin{cases} E_U & \omega\bar{\tau} \gg 1 \\ E_R & \omega\bar{\tau} \ll 1 \end{cases}.$$

2.3 Thermoelastic Damping

It is known that a homogeneous material under homogeneous stress can undergo thermal relaxation only by heat exchange with its surroundings since there is no other heat flux path. However, if the stress field changes periodically, as it does during vibration, it gives rise to periodic changes in temperature, even if the material is homogeneous, resulting in temperature gradients. Heat flux due to temperature gradients lead to increase in entropy

indicating an increase of internal energy by reducing mechanical energy (Zener, 1940).

Thermoelastic damping that develops during vibration of a homogeneous system has been calculated for transverse and longitudinal waves by computing the values for relaxation strength and relaxation time. An example of thermoelastic damping is given for a beam of thickness h that vibrates in flexure at a frequency f (Zener, 1937, 1938)

$$Q^{-1}(f,T) = \triangle_T \frac{f \cdot f_0}{f^2 + f_0^2}$$

which has the same functional form as a Lorentz distribution and a Debye peak as a function of frequency given by:

$$f_0 = \frac{\pi k_T}{2h^2 \rho\, C_p}$$

where ρ is the density and C_p is the specific heat capacity under constant pressure or stress and the relaxation strength is defined as

$$\triangle_T = \alpha^2 \frac{E_U T}{\rho\, C_p}.$$

The relaxation time, in terms of the Debye peak frequency, f_0, is

$$\tau_T = 1/2\pi f_0 = h^2/\pi^2 D_{th}$$

with the thermal diffusivity $D_{th} = k_T/\rho C_p$ and k_T is the thermal conductivity. These relations are used in various forms to determine thermoelastic damping in mechanical systems, particularly in micro- and nano-mechanical systems where it can have a significant influence.

2.4 Viscoelastic Damping

The basic relations between stress and strain no longer hold through a simple proportionality of a modulus for linear viscoelastic materials under a time-dependent stress and strain. Compared with a completely recoverable behavior of an *anelastic* solid, *viscous* and *viscoplastic* properties describe nonrecoverable behavior and *viscoelasticity* falls in between and may have both recoverable and nonrecoverable parts. Such a partially recoverable behavior can be modeled using a four-parameter model that consists of a Voigt model in series with a spring and a damping element, which is commonly used to describe viscoelastic behavior. Origin of such models

may be explained through a partial differential equation of arbitrary order (Nowick and Berry, 1972).

$$a_0\sigma + a_1\frac{\partial\sigma}{\partial t} + a_2\frac{\partial^2\sigma}{\partial t^2} + \cdots + a_n\frac{\partial^n\sigma}{\partial t^n}\cdots = b_0\epsilon + b_1\frac{\partial\epsilon}{\partial t} + b_2\frac{\partial^2\epsilon}{\partial t^2} + \cdots + b_n\frac{\partial^n\epsilon}{\partial t^n}\cdots$$

This differential stress-strain equation can be used to express most of the complex relationships for a viscoelastic material.

For example, by keeping only a_0 and b_0, it describes an elastic solid:

$$\sigma = E\epsilon$$

or, keeping a_0 and b_1 and setting other coefficients to zero produces Newton's law of viscosity with the coefficient of viscosity η_0:

$$\sigma = \eta_0\left(\frac{\partial}{\partial t}\right)\epsilon.$$

The Voigt model that is commonly used to describe viscoelastic solids has a spring and a dashpot with a_0, b_0, b_1

$$\sigma = \left[E + \eta_0\left(\frac{\partial}{\partial t}\right)\right]\epsilon = E\epsilon + \eta_0\,\dot{\epsilon}.$$

The anelastic solid described earlier has an additional spring term that makes it a Voigt model in series with a spring:

$$\sigma + \tau_\epsilon\,\dot{\sigma} = E_R\,\epsilon + E_U\,\tau_\epsilon\,\dot{\epsilon}.$$

where the relaxation time τ_ϵ and the relaxed and unrelaxed elastic moduli E_R and E_U are those described earlier.

2.5 Friction Damping

Miscroslip vs. Sliding Friction or contact damping refers to conversion of the kinetic energy associated with the relative motion of two surfaces in contact to thermal energy. Contact stresses generate inhomogeneous stress-strain fields on and near the surfaces leading to temperature gradients and, thus, transport of thermal energy from the contact areas.

Contacts that generate friction damping can be characterized by the relative motion between the surfaces: microslip and sliding. When contact is between nominally conforming surfaces that do not have a relative rigid-body motion, contact behavior is sometimes described as micromotion or microslip, and may not reach slip or sliding conditions. Friction remains

more or less in the "static" range and is associated with the tangential stiffness of the contact zone. Examples of microslip can be found in bolted or riveted joints, braided wire ropes, and inserted gas turbine blades.

When the contacting surfaces have a relative whole-body motion as in the case of brakes, damper rings in gears, and Lanchester dampers, full slip can develop between the surfaces. In such cases, friction damping has a preferred range of normal contact force within which it becomes most effective. Below such an optimum normal force, excess relative motion at the interface develops without significant energy dissipation. Above it, excess pressure inhibits the relative sliding motion for friction to act as an effective damper (Akay, 2002).

The simplest case to illustrate the effects of friction on vibration is an oscillator with a friction damper as represented in the following equation of motion (Den Hartog, 2013):

$$m\ddot{x} + kx = F_0 \cos \omega t - \mu N \mathbf{sgn}(\dot{x}).$$

Action of a friction damper on a simple oscillator is analogous to the fluid damper where the drag force acts against the direction of motion of the oscillator. There is rich literature on the dynamic response of systems in the presence of sliding friction including on the damping effects of friction (viz., Dowell and Schwartz, 1983b,a; Dowell, 1986, 1983).

Focusing on microslip or quasi-static contact damping, knowledge of friction characteristics is necessary to predict its damping effects, much like any other dynamic problem that involves friction. However, this knowledge is normally obtained through measurements, since we do not yet have an acceptable model of friction that is based on first principles.

Friction damping that develops during microslip, by necessity, is associated with the tangential stiffness between the contacting surfaces. The tangential forces that develop not only resist relative motion but effectively change the boundary conditions and the resonant frequencies in the structure and thus lead to nonlinearities in the response.

Considering a simple oscillator with friction force resisting its motion, its motion can be described by:

$$m\ddot{x} + c\dot{x} + kx = f_0 \cos \omega t - f_\mu(t)$$

where the nonlinear friction force is approximated by a spring and a viscous damper acting in the direction of motion

$$f_\mu(t) \approx k_e x + c_e \dot{x}$$

where k_e and c_e represent the effective values of stiffness and damping, respectively, and are found from measurements (Filippi et al., 2004).

Granular Damping Granular damping, or particle damping, in vibrations refers to absorption of waves or oscillations through the use of a collection of particles or granular materials. Granular materials are known to provide effective means of dissipation of vibratory energy largely through inelastic collisions and friction among the granules.

Physical properties of individual granular materials very much differ from their ensemble properties in terms of elasticity and dissipation. The collective behavior is governed not only by the physical properties of individual grains but also through their interactions with each other and the manner by which the ensemble is contained. These parameters include friction between the particles, filling factor or packing force, shape of the device in which they are contained and frequency and amplitude of the vibrations to which they are subjected. Based on these factors, the collective behavior of granular particles may be a plug-like solid, a fluid, or a gas, each phase having different dissipation characteristics. Elastic particles, such as ball bearings, absorb vibration energy effectively only when exposed to a vibration field collectively, similar to the so-called beanbag absorbers. Although each ball bearing may rebound upon impact on an elastic solid, when collected in a flexible container such as a bag, they behave inelastically, due to friction among the particles and due to diffusion of their energy (Jaeger et al., 1996).

Packing force of granular materials in a rigid container determines their density and, thus, directly affects their collective behavior. Very high packing forces can severely limit relative motion among the grains and thus reduce damping effectiveness.

2.6 Damping in Fluid Media

Radiation Damping Radiation damping describes a broad range of damping effects such as those associated with gravitational fields, quantum mechanics and optics, but we focus here on radiation damping effects on engineering structures. Radiation damping is a term also used in connection with civil engineering piles and footings to describe soil-structure interaction.

Radiation damping of a structure refers to energy lost from a vibrating surface through sound radiation. As a simple demonstration, we consider a sdof oscillator in a waveguide with the usual properties K, M excited by a harmonic force $F_0 e^{i\omega t}$ such that $2\pi c/\omega > D$, where c is the speed of sound and D is the diameter of the duct, so that radiation can be assumed to be a plane wave. Since the pressure acts on the mass against its surface, the forces arising from radiation have opposite signs and add together to form

the total acoustic radiation force and can be expressed as

$$F_a = 2\pi(D^2/4)\rho c\,\dot{\eta}$$

where ρ is the density of the fluid medium in the duct, and η represents the displacement of the oscillator. The equation of motion for the oscillator then becomes

$$M\frac{d^2\eta}{dt^2} + (\pi D^2/2)\rho c\frac{d\eta}{dt} + K\eta = F_0 e^{i\omega t}.$$

The harmonic solution yields a compliance expression:

$$\left|\frac{\eta}{F}\right| = \frac{1}{\sqrt{(K - M\omega^2)^2 + (\pi D^2 \rho c\omega/2)^2}}.$$

We note here that radiation damping is frequency dependent, which becomes significant in cases of radiation damping that involves higher-order systems, such as plates and beams that have their own modal and critical frequencies.

The rather straightforward approach and the expression described above takes a more complicated form in the case of more complex structures. Since radiation damping results from loss of energy radiated into the adjacent medium, its prediction requires modeling vibrations of the structure coupled with the medium. In media such as air, radiation damping from heavy machinery may be negligible whereas in water or soil, it is more significant. The fluid-structure coupling determines both the frequencies and magnitude of damping due to radiation. Although approximate expressions are available, more complete expressions for damping require solution of the coupled equations.

By defining a damping factor for radiation damping as the ratio of acoustic energy radiated to the maximum kinetic energy stored per cycle of vibration in such a structure, an approximate value can be given for the first mode of a rectangular plate with fixed edges as (Mangiarotty, 2005; Mead, 1998):

$$\delta_{ac} = 1.155X10^{-5}\frac{\rho_0}{\rho_m}\sqrt{\frac{E}{\rho_m}}\left(N + \frac{1}{N}\right)$$

where $N = a/b$ is the ratio of its length to width and ρ_m and E are the material elastic properties and ρ_0 is the density of the fluid medium. For a simply-supported panel a similar result is given as:

$$\delta_{ac} = 1.155X10^{-5}\frac{\rho_0}{\rho_m}\sqrt{\frac{E}{\rho_m}}\left(N^2 + \frac{2}{3} + \frac{1}{N^2}\right)^{1/2}.$$

We note that acoustic damping factor of a given material in these expressions depends on the density of the medium and the ratio of its length to width of the plate.

Fluid Damping - Drag Different from radiation damping, fluid damping refers to energy lost through drag forces on a solid body moving in a fluid. The fundamental mechanisms may involve vortex shedding, but not always.

Drag force F_D is proportional to the square of the relative velocity and, thus, is dominant at high relative velocities and may be negligible at low velocities. Drag force is also proportional to the density of the fluid medium and to the cross-sectional area of the body facing the flow:

$$F_D = -c_D S \left(\frac{1}{2} \rho v^2 \right).$$

The drag coefficient c_D depends on the shape of the body and usually found empirically. Because drag force acts against the relative motion, it is usually accompanied by a sign switching function that depends on the direction of velocity v. For a unidirectional motion, drag force is:

$$F_D = -c_D S \left(\frac{1}{2} \rho v^2 \right) \mathbf{sgn}(v).$$

The corresponding equation of motion for a simple oscillator becomes:

$$m\ddot{x} + \frac{1}{2} c_D \rho S \, \dot{x}^2 \, \mathbf{sgn}(\dot{x}) + kx = 0.$$

Analogous to the dry friction problems, the above equation can be expressed to obtain the phase plane for $(x, \dot{x})$ by a first-order differential equation for $\dot{x}^2$:

$$\frac{d\dot{x}^2}{dx} + \frac{1}{2} \frac{c_D \rho S}{m} \, \dot{x}^2 \, \mathbf{sgn}(\dot{x}) = -2 \frac{k}{m} x.$$

Examples of its solution can be found elsewhere (Kneubühl, 1997).

Squeeze-Film or Fluid-Film Damping

►**Air-Film Dampers** It is known that when two plates are brought in close contact with each other (for example, through spot-welding), the vibrational damping rate and the attenuation of radiated sound from the plates are markedly enhanced. This method of vibration damping and noise reduction can be used effectively in several applications where standard viscoelastic layer damping cannot be used, such as in hostile chemical environments and/or at high temperatures. For example, vibrations of combustion chambers can be reduced by applying patches of metal plates at strategic locations on the chambers. Other applications include damping of vibrations in centrifugal separators and circular saw blades. In the latter case, damping can be obtained by means of the thin air layer between the blade and a rigid block placed in close proximity of the blade (Meins, 1963; Allen, 1977). They are particularly useful to reduce vibration of moving devices, such rotating shafts.

The principle of operation is based on the motion of fluid in a narrow gap between a moving vibrating surface and a fixed one. As the moving surfaces oscillate toward and away from the fixed surface, pressure in the fluid layer periodically increases and decreases, squeezing the fluid out and pulling it back in, respectively. When the layer thickness is small, fluid motion is largely normal to the oscillation direction and parallel to the surfaces. The energy required to pump the fluid either through its edges or from a region of compression to a region of rarefaction under a vibrating plate is supplied by the motion of the plate and results in the damping of its motion.

The flow impedance per unit length of a thin film of fluid with density ρ_l can be expressed as (Morse, 1986; Ingard and Akay, 1987)

$$Z = R + jX = -j\omega\rho_l/(1 - F)$$

with

$$F(x) = x(1 + j)\tanh[(1 - j)/2x]$$

where $x = d_\nu/D$ is the ratio of viscous boundary layer and fluid layer thickness. Approximate expressions of F are given as:

$$F(x) \approx \begin{cases} (1 + j)x & x << 1 \\ 1 + (j/6x^2) - (1/30x^4) & x >> 1 \end{cases}$$

with these values the flow impedance becomes:

$$Z = \begin{cases} \rho_l\omega\left[(d_\nu/D) - j\right] & d_\nu/D << 1 \\ \rho_l\omega\left[12\mu/D^2 - j(1.2\omega)\right] & d_\nu/D >> 1 \end{cases}$$

where μ is the viscosity of the fluid.

At very low frequencies, where traditional damping techniques are not as effective, flow resistance per unit length becomes $R = 12\mu/D^2$ and the corresponding damping factor for bending vibrations of a plate takes the form:

$$\delta \sim (\lambda/D)(\lambda\mu/D^2)(1/\omega_0 m)$$

where λ represents the bending wavelength of the plate with mass m per unit length vibrating at frequency ω_0.

Q-value of the plate due to fluid layer damping is obtained as

$$Q = \omega_0 m D^3/\mu\lambda^2.$$

Further damping can develop from acoustic streaming that develops in a squeeze film between two flat surfaces. Beyond a certain oscillation frequency and amplitude combination, in addition to the periodic flow in and out of the layer edges, continuous streams develop changing the flow pattern in the layer (Akay and Xu, 1998).

2.7 Other Fluid Damping Mechanisms

It is worth mentioning other fluid damping mechanisms details of which can be found in literature.

►**Couette Flow Damping** Similar to squeeze film damping but the surfaces move parallel to each other developing a unidirectional flow. Its effects on MEMS devices continue to be of interest.

►**Damping in Porous Materials: Biot Damping** Named after M. A. Biot who developed the theory of mechanics for porous media, Biot damping describes dissipative effects of a material with fluid-filled pores (Biot and Tolstoy, 1992). Damping in porous materials results from a combination of the damping in the solid porous structure, the fluid that saturates it, and the relative motion at the interface of the two (Göransson, 2006). The skeleton of the porous solid carries the stress waves as the acoustic pressure waves propagate through the fluid medium. The configuration and properties of the porous material and the fluid pressure influence the energy balance between the two media and thus the resulting dissipation. Biot damping is particularly significant in sound transmission and soil consolidation problems in geophysics.

►**Aerodynamic & Hydrodynamic Damping** Aerodynamic damping is commonly used in connection with wind effects on structures such as buildings, airplanes, and cables. The forces generated by wind may attenuate vibrations but more commonly induce vibrations and instability. When the wind forces cause instability, they are described as negative damping forces. In aerodynamics, terms pitch-, yaw-, and roll-damping refer to moments due to differential forces rather than dissipation of energy.

Similarly, hydrodynamic damping used in describing motion of marine vessels, typically consist of any combination of radiation damping into the fluid medium, viscous damping and its contributions to turbulent boundary layer, the so-called wave drift damping that describes added resistance due to waves in sea, and damping that results from vortex shedding (Kareem and Gurley, 1996).

►**Damping with Magnetorheological & Ferro Fluids** Effective viscous and stiffness properties of a fluid, such as oil, can be adjusted as desired when mixed with ferrous particles and subjected to magnetic field. Magnetorheological (MR) fluids refer to liquids with micron-size particles unlike ferro fluids (FF) which contain nano-sized particles. The ability to control their physical characteristics allows their use as lubricants, seals and dampers, for example.

Damping Materials and Devices Choice of damping treatment is governed by the amount of vibration energy to be dissipated and by the operating conditions under which a damping treatment is used. Environmental conditions with hazardous chemicals or extreme temperatures may prohibit use of some of the treatments.

The basic mechanisms of dissipation are utilized in many different forms as mentioned above. In addition, use of tuned dampers, impact dampers, pendulum dampers also dissipate energy largely through momentum exchange between moving parts and the dissipation mechanisms within them. One can consider sloshing fluids in the same category.

3 Dissipation Mechanisms

Dissipation refers to conversion of kinetic energy associated with motion, oscillations or waves to heat, which is described as the thermalized oscillations of molecules of the medium in which the propagation takes place. Whether in a solid or a fluid, the conversion process leads to an increase in the kinetic energy of the molecules in the medium. In this sense, dissipation describes the conversion of ordered kinetic energy at macroscopic scale to disordered kinetic energy at the molecular scale. As described in the previous section, damping models contain empirical constants that describe dissipation through, for example, viscosity, coefficient of friction, or material losses in solids.

Modeling dissipation based on first principles, requires knowledge of the heat capacity of the medium, which describes the increase in its internal energy per unit temperature. The various damping mechanisms described earlier involve some form of external excitation that leads to an increase in internal energy of the medium. Since internal energy is proportional to the average kinetic energy of its molecules, understanding the dynamic response of molecules and accurate prediction of their behavior constitutes a key to modeling dissipation.

Dissipation is fundamentally a nonlinear process during which energy transfer takes place irreversibly (viz., Çelik and Akay, 2000). The first-principles based quantitative models that can accurately predict dissipation await further advances in molecular dynamics simulations. However, qualitative relationships exist for thermal energy at the molecular scale in terms of average kinetic energy of molecules. Simulation studies that consider solids as a set of oscillators in a lattice have been continuing since the introduction of the FPU problem (Fermi, Pasta, and Ulam, 1955; Ford, 1992). Simulations invariably assume that the vibrations of molecules are in thermal equilibrium, or "thermalized," such that all states of the molecules have equal probability of having equal energy. As a result, investigations of thermalization process look for conditions leading to energy equipartitioning. Chaotic response of, say, molecules in a lattice describing a solid, is considered by some as the indicator of thermalization. Later the KAM theorem explained the role of nonlinearities in such models indicating that not all nonlinearities lead to thermalization of their oscillations (Kolmogorov, 1979; Arnol'd, 1963; Moser, 1962; Salamon, 1986).

Notwithstanding the nonlinearity of a dissipation process at molecular level, linear dissipative media have long been modeled with a set of independent linear oscillators. In most of these approaches, a dissipative property emerges from the collective behavior of an ensemble of independent linear

oscillators. Such models have been developed to describe, for example, Landau damping, dissipation in Langevin equation, and virtual damping due to phase diffusion.

A final point to note about these models is that the linear oscillators describing a dissipative environment must have a continuous distribution and, by implication, the number of oscillators is very large, reaching infinity. Models based on such requirements fall short of describing conditions where the oscillator numbers are not necessarily very large. Special cases when a finite number of linear oscillators can be used to describe dissipation of a medium with near irreversibility is also described later in this section.

3.1 Linear Modeling of Dissipative Systems

In its most fundamental form, classical or quantum dissipation can be considered as an interaction of a particle with its environment. An approach, first introduced by Feynman and Vernon (1963), is to model the environment as a continuous set of non-interacting, linear independent oscillators into which energy flows from the particle of interest. Presumption of infinite number of linear oscillators in the environment permits irreversible energy flow into it, which then acts as an energy sink as briefly summarized below.

Motion for a harmonically bound particle of mass M and spring constant K, with N oscillators attached to it bilinearly, can be described by a set of coupled equations:

$$M\ddot{x}(t) + Kx(t) = \sum_n m\omega_n^2(x - q_n) + F(t) \tag{18}$$

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = \omega_n^2 x(t) \qquad n = 1.2.3 \ldots N \tag{19}$$

where $F(t)$ is an external force and the summation terms represent the force by the oscillators, each with displacement $q_n(t)$, describing the environment. The force by the oscillators consists of a dissipative, or systematic, part and a fluctuating, or random, part described below.

To determine the force exerted by the oscillators on the bound particle, response of each oscillator is expressed as:

$$q_n(t) = -\int_0^t \cos\omega_n(t-\tau)\ \dot{x}(\tau)\ d\tau + q_n(t_0)\cos\omega_n(t'-t_0) + \frac{\dot{q}_n(t_0)}{\omega_n}\sin\omega_n(t'-t_0)$$

where $q_n(t_0)$ and $\dot{q}_n(t_0)$ are values of $q_n(t)$ and $\dot{q}_n(t)$ at $t' = t_0$. Substituting for $q_n(t)$ from above in equation (18), with $t = t' - t_0$:

$$M\ddot{x}(t)+\left(M\Omega_0^2+\sum_{n=1}^{N}m\omega_n^2\right)x(t)+\int_0^t\sum_{n=1}^{N}m\omega_n^2\cos\omega_n(t-\tau)\dot{x}(\tau)d\tau = \sum_{n=1}^{N}m\omega_n^2\left[q_n(0)\cos\omega_n t+\frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t\right]+F(t). \tag{20}$$

We now set

$$\Gamma'(t)=\sum_{n=1}^{N}m\omega_n^2\cos(\omega_n\tau),\ \Pi'(t)=\sum_{n=1}^{N}m\omega_n^2\left[q_n(0)\cos\omega_n t+\frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t\right] \tag{21}$$

and the system frequency

$$\Omega^2=\Omega_0^2+\sum_{n=1}^{N}\frac{m}{M}\omega_n^2$$

where the prime indicates the discrete summations for Γ and Π. With these notations the equation of motion (22) takes the form:

$$M\ddot{x}(t)+M\Omega^2(t)+\int_0^t\Gamma'(\tau)\dot{x}(t-\tau)d\tau=\Pi'(t)+F(t). \tag{22}$$

Coupled equations of motion for an unforced ($F=0$) particle of mass M with attached oscillators that are initially quiescent has the form:

$$M\ddot{x}(t)+M\Omega^2x(t)+\int_0^t\Gamma'(\tau)\dot{x}(t-\tau)d\tau=0$$

$$\ddot{q}_n(t)+\omega_n^2q_n(t)=\omega_n^2x(t)\qquad n=1.2.3\ldots N.$$

The solution of these equations show recurrence: energy is exchanged between the particle and the attached oscillators. However, when the attached oscillators are distributed continuously with a particular frequency distribution, energy is irreversibly absorbed by the attached particles even in the absence of a dissipation source in the classical sense.

Feynman and Vernon (1963) showed how a continuous distribution of oscillators describe loss using a perfectly linear, undamped set of oscillators. A simple demonstration of their approach is made using the distribution introduced by Caldeira and Leggett (1983) for the oscillators as $G(\omega)=2M\gamma_0/\pi m\omega^2$ in making the summation Γ' an integral Γ:

$$\sum_n \to \int_0^\infty G(\omega)\, d\omega.$$

Expression $\Gamma(\tau)$ can be written and evaluated as

$$\Gamma(\tau) = \int_0^\infty G(\omega) m\omega^2 \cos(\omega\tau) d\omega = 2M\gamma_0\, \delta(\tau)$$

where the constant γ_0 is analogous to a velocity-dependent damping constant that introduces the familiar viscous damping into the equation of motion:

$$\ddot{x}(t) + \gamma_0 \dot{x}(t) + \Omega^2 x(t) = 0.$$

As shown later in this section, in cases where the summation can not be substituted by an integral, special frequency distributions are shown to closely imitate the dissipative behavior of a continuously distributed, i.e., infinite number of oscillators (Carcaterra and Akay, 2004, 2007, 2011; Carcaterra et al., 2006; Koç et al., 2005; Akay et al., 2005).

►**Dynamics of an Ensemble of Oscillators** It is instructive to examine the behavior of the attached oscillators independently of the particle to which they are attached.

Consider an ensemble of simple oscillators each with a slightly different frequency, ω. If the entire set of oscillators start with the same initial conditions, their motions become out of phase with time and the average displacement of the ensemble decays, displaying a process known as *decoherence* (Ng, 2006, 2010; Carcaterra and Akay, 2004).

Motion of an oscillator in the ensemble with a displacement y and angular frequency ω is described by

$$\ddot{y} + \omega^2 y = 0$$

with the general solution

$$y(t) = y_0 \cos\omega t + \dot{y}_0\, \frac{\sin\omega t}{\omega}$$

where y_0 and $\dot{y}_0$ represent the initial values of y and $\dot{y}$, respectively.

If all the oscillators start with the same initial conditions, the average displacement of the ensemble in time becomes

$$< y(t) >= y_0 \mathcal{H}(t) \int \rho(\omega) \cos\omega t\, d\omega, \qquad \dot{y}_0 = 0$$

alternatively, the average velocity is

$$< \dot{y}(t) >= \dot{y}_0 \mathcal{H}(t) \int \rho(\omega) \cos \omega t \; d\omega, \qquad y_0 = 0$$

where $\mathcal{H}$ represents the Heaviside step function, $\rho(\omega)$ is the distribution of the oscillation frequency in the ensemble, with the constraint

$$\int_{-\infty}^{\infty} \rho(\omega) \; d\omega = 1.$$

The impulse response of the ensemble is always real and vanishes for $t < 0$.

$$G(t) = \mathcal{H}(t) \int \rho(\omega) \cos \omega t \; d\omega, \qquad \dot{y}_0 = 0.$$

For ensemble frequencies having a Lorentz distribution with half-width-at-half-maximum $\triangle\omega$, we have

$$\rho(\omega) = \frac{1}{\pi} \; \frac{\triangle\omega}{(\omega - \bar{\omega})^2 + (\triangle\omega)^2}$$

where $\bar{\omega}$ is the mean angular frequency of the ensemble. For this distribution, the impulse response is

$$G(t) = \mathcal{H}(t) \Re e^{j(\bar{\omega} + j\triangle\omega)t} = \mathcal{H}(t) e^{-(\triangle\omega)t} \cos \bar{\omega} t$$

which shows that the average displacement of the oscillators decay with a relaxation time $\tau^{-1} \approx \triangle\omega$. The process described above is also called kinematic decoherence (Ng, 2006, 2010).

Phase Damping Instead of frequency distribution, if all oscillators are assumed to have the same frequency but each with a time-dependent phase, their collective response again exhibits dissipation. Since the response decay does not involve any dissipation in a real sense, sometimes phase damping is also called virtual damping when used in the context of phase noise in electronics (Ham and Hajimiri, 2003). Describing, as before, the response of a single oscillator as:

$$y(t) = y_0 \cos[\omega_0 t + \phi(t)]$$

where y_0, ω_0 and $\phi(t)$ are initial displacement, oscillation frequency and the phase variation, respectively.

Assuming that the phase distribution $\phi(t)$ can be characterized as a Wiener process, that is a zero-mean Gaussian random process, it can be shown that: $< \phi^2(t) >= 2Dt$ where D is the diffusion constant.

For a Gaussian distribution of $\phi(t)$ for all oscillators at any given time, t:

$$< \cos\phi >= e^{-<\phi^2>/2} \qquad < \sin\phi >= 0$$

and, therefore, the average response

$$< y(t) >= y_0\, e^{-\phi^2(t)/2}\ \cos\omega_0 t$$

and the autocorrelation

$$< y(t)y(t+\tau) >= \frac{1}{2}\, y_0^2\ e^{-Dt} \cos\omega_0\tau$$

show the decaying behavior of the collective response.

The corresponding power spectral density of the oscillators is a Lorentzian distribution.

$$S_{y,y}(f) = y_0^2\ \frac{D}{(\omega-\omega_0)^2 + D^2}$$

where the diffusion constant D represents the damping rate.

While the concept of virtual damping is used in connection with phase noise in electronics, as seen above it has a direct bearing on dissipation models in vibrations.

Forced Oscillations - Landau Damping When the object, to which the oscillators are attached, is subjected to a harmonic force, response of each oscillator is described by:

$$\ddot{y} + \omega^2 y = F_0 \cos\Omega t$$

with the general solution

$$y(t) = y_0 \cos\omega t + \dot{y}_0 \frac{\sin\omega t}{\omega} + \frac{F_0}{\omega^2 - \Omega^2}[\cos\Omega t - \cos\omega t].$$

Considering only the forced response, displacement of the center-of-mass of an ensemble with a distribution $\rho(\omega)$ can be expressed as:

$$< y(t) >= F_0 \int_{-\infty}^{\infty} \frac{\cos\Omega t - \cos\omega t}{\omega^2 - \Omega^2}\ \rho(\omega)\ d\omega. \tag{23}$$

For distribution $\rho(\omega)$ that is narrow and centered at $\bar{\omega}$ and without other peaks, when the ensemble is driven at a frequency close to the mean frequency, $\Omega \approx \bar{\omega}$, using the expansion $\omega = \Omega + (\omega - \Omega)$, $< y(t) >$ can be expressed as (Ng, 2006):

$$< y(t) >= \frac{F_0}{2\bar{\omega}} \left[\cos \Omega t \int_{-\infty}^{\infty} \frac{1 - \cos(\omega - \Omega)t}{\omega - \Omega} \rho(\omega) d\omega + \sin \Omega t \int_{-\infty}^{\infty} \frac{\sin(\omega - \Omega)t}{\omega - \Omega} \rho(\omega) d\omega \right].$$

The ensemble response has a fast oscillating component at frequency Ω and a slow-oscillating component $\omega - \Omega$ that acts like an envelope.

Noting that $\triangle\omega$ is a measure of the width of the frequency distribution $\rho(\omega)$, all transients vanish in the asymptotic behavior of the ensemble average displacement for $t \gg 1/\triangle\omega$ leaving:

$$< y(t) >= \frac{F_0}{2\bar{\omega}} \left[\cos \Omega t \ \ \mathbf{p.v.} \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega - \Omega} d\omega + \pi \rho(\Omega) \sin \Omega t \right].$$

For a narrow frequency distribution $\rho(\omega)$ centered around $\bar{\omega}$, the assembly average given in Eq. (23) can be approximated as (Ng, 2006, 2010)

$$< y(t) >= \frac{F_0 \sin \bar{\omega} t}{\bar{\omega}} \int_{-\infty}^{\infty} \frac{\sin \frac{1}{2}(\omega - \Omega)t}{\omega - \Omega}.$$

Considering the amplitude, A, of an oscillator with frequency ω as

$$\mathrm{A}(\omega) = \frac{F_0}{\bar{\omega}} \frac{\sin \frac{1}{2}(\omega - \Omega)t}{\omega - \Omega}$$

all oscillators with frequency ω are excited at $t = 0$ and reach a maximum at $t \approx \pi/(\omega - \Omega)$ and vanish again at $t \approx 2\pi/(\omega - \Omega)$. Such absorption and return of energy to the ensemble is a familiar display of recurrence. For frequencies ω closer to Ω, amplitude is larger and the return time is longer. For oscillators with $\omega = \Omega$, amplitude grows linearly with time and the absorption continues indefinitely. This process of vanishing amplitude growth except for a few oscillators near Ω is called the *Landau Damping* (Ng, 2006).

3.2 Energy Sinks

Energy sink is a concept based on the linear models of dissipation described above but adapted for use with a finite number of oscillators. As described above, linear oscillators when attached to a primary structure can irreversibly absorb its vibration energy provided their frequencies follow certain frequency distributions continuously. Since it is not practical in engineering applications to attach oscillators with a continuous frequency

distribution or, equivalently, to use an infinite number of them, the models are adapted for a finite set of oscillators (Carcaterra and Akay, 2004, 2007, 2011; Carcaterra et al., 2006; Koç et al., 2005; Akay et al., 2005).

The energy sinks described here rely on the use of a set of undamped, linear oscillators with a particular frequency distribution that enables it to rapidly absorb vibration energy from a structure and retain nearly all of it.

3.3 Dissipation in Finite Systems

Returning to the equation of motion (20) of an harmonically excited particle with a set of linear oscillators attached to it:

$$M\ddot{x}(t) + M\Omega^2 x(t) + \int_0^t \Gamma(\tau)\dot{x}(t-\tau)d\tau = F(t)$$

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = \omega_n^2 x(t) \qquad n = 1.2.3\ldots$$

where we have assumed quiescent initial conditions, $q_n(t_0) = 0$, $\dot{q}_n(t_0) = 0$, for the attached oscillators. The kernel of the dissipative term is

$$\Gamma(\tau) = \sum_n m_n \omega_n^2 \cos(\omega_n \tau).$$

In general, harmonic series such as $\Gamma(\tau)$ have properties similar to those of almost-periodic-functions and exhibit recurrence (Bohr, 1947), precluding the use of independent linear oscillators to model irreversible absorption of vibratory energy, except for early times of observation (Carcaterra and Akay, 2004).

However, it has been reported that when the attached oscillators follow particular frequency distributions, they very closely mimic the dissipation characteristics of an infinite number of oscillators, reducing and nearly eliminating the effects of recurrence (Carcaterra and Akay, 2004, 2007, 2011; Carcaterra et al., 2006; Koç et al., 2005; Akay et al., 2005).

►**Example** The prototypical system under consideration consists of a rigid primary structure with a substructure comprised of a set of linear oscillators of equal mass m_n attached to it, as described in Fig. 3. The system does not possess any mechanism to dissipate energy in the classical sense, thus stiffness alone characterizes the connections between the substructure and the primary structure. The total mass, m ($m = \sum m_n$), of the attachments is assumed to be fraction of the primary mass, M, say, $m/M \leq 0.1$. Under these conditions, the significance of the stiffness or frequency distribution is demonstrated for two different cases.

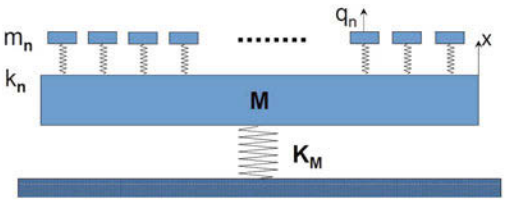

Figure 3. Schematic description of a SDOF structure with a set of linear oscillators attached to it.

For example, Figure 4 shows a typical impulse response of the primary structure with the attached set having a linear frequency distribution as shown in Fig. 4 (a). The oscillators have a constant frequency difference between the neighboring frequencies. As expected of linear oscillators with a linear frequency distribution, the response shows a recurrence (b); and as shown in Fig. 4(c), energy periodically returns to the primary structure when the number of oscillators is finite, in this case $N = 100$. The return time corresponds to the constant frequency difference $t^* = 2\pi/\Delta\omega$ (Koç et al., 2005).

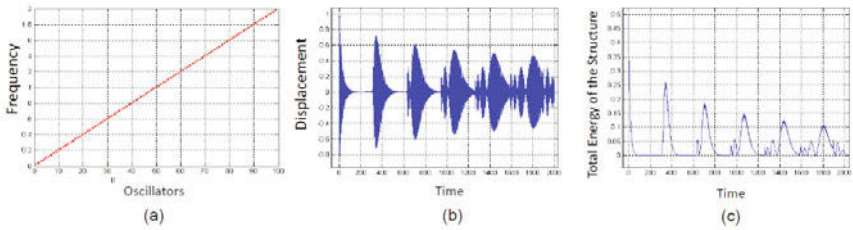

Figure 4. Response of a SDOF structure with attached linear oscillators as shown in Fig.(3): (a) Attached oscillators have a linear frequency distribution, (b) displacement response of the structure with periodic increases, (c) energy of the structure displaying recurrence.

It has been shown that there exist optimum frequency distributions for such finite sets that increase the period of recurrence, effectively absorbing vibratory energy in a nearly irreversible manner (Koç et al., 2005; Carcaterra and Akay, 2004).

In the example shown in Figure 5, for the particular frequency distribution shown in (a), recurrence nearly disappears in the response. A comparison of the energy of the primary oscillator in Figure 5 (c) with that in Figure 4 (c) also shows the reduction in the energy of the structure as a result of the frequency distribution of the attached oscillators. Experimental verifications of such systems have also been reported (Akay et al., 2005).

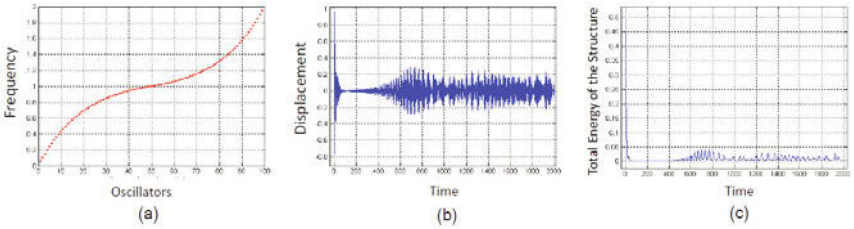

Figure 5. Response of the SDOF structure as in Fig.(3): (a) Frequency distribution among the attached oscillators, (b) displacement response of the structure is no longer periodic, (c) energy of the structure has been absorbed by the oscillators.

Compared with the conventional methods of vibration reduction, the concept of energy sinks presents a unique and viable alternative for cases where the classical vibration absorption or damping techniques have limited applicability, particularly at low frequencies and under transient conditions. Energy sinks described here consist of a set of oscillators that collectively absorb and retain energy when attached to a vibrating structure. An energy sink admits flow of energy from a transiently excited structure and retains it in the collective phase space of the oscillators.

3.4 Damping and Dissipation in an Ideal Gas

Damped motion of a piston in a cylinder presents another example of dissipation at the molecular level where the piston energy is irreversibly converted to thermal energy in a gas. Piston motion is similar to that of a particle motion in a thermal environment discussed earlier where its dissipation is modeled with independent linear oscillators. In this case, however, dissipation modeling is nonlinear and arises from individual impacts of particles on the piston.

Considering a prototypical system illustrated in Fig. 6 in which a piston

P with a cross-sectional surface area S moves freely without wall friction. It moves along the cylinder axis x at speed V_P, separating the two reservoirs of the piston, each maintained at temperatures such that $T_H > T_C$.

Gas is modeled as a set of freely moving particles with a velocity distribution, rather than as attached oscillators. Pressures in volumes H and C that act on the piston in opposite directions are calculated by summing the impulses from particles impinging on the two surfaces of the piston.

Considering first the pressure in the H reservoir, a single gas particle has speed components v_{Hx} and v'_{Hx} along x, before and after its impact on the piston surface, respectively, and are related through the piston velocity:

$$v'_{Hx} = -v_{Hx} + 2V_P.$$

Note that this relationship is valid for cases where the particle mass m is very small compared with the mass M of the piston. The momentum variation of a particle associated with the impact is $2m(V_P - v_{Hx})$. The force exerted on the piston surface H is then calculated as a sum of the x-components of the pulses:

$$F_H(V_P, t) = 2m \sum_i (v_{Hx_i} - V_P)\, \delta(t - t_{iH}) \tag{24}$$

where F_H depends both on time t and the piston speed V_P. This force evolves through a sequence of random impulses $\delta(t - t_{iH})$ of random amplitude and random time delay. Analogous to the Fluctuation-Dissipation

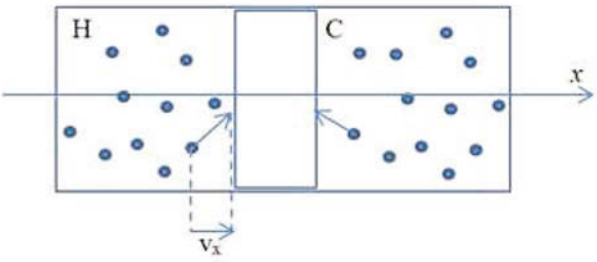

Figure 6. Piston in a cylinder.

Theorem, we separate the mean and fluctuating parts of F_H in Eq. (24). Designating by $n(v_{Hx})\, dv_{Hx}$ the number of particles per unit volume of gas that have the speed component v_{Hx}, the number dN of particles that hit the piston surface within the time interval dt can be expressed as:

$$dN = \frac{1}{2}(v_{Hx} - V_P)\; n(v_{Hx})\; S\, dv_{Hx}\, dt$$

the factor 1/2 comes from the consideration of particles traveling in only one direction along x. Then, the force produced on the piston by dN number of impinging particles in the reservoir H becomes:

$$dF_H(v_{Hx}) = mS\,(v_{Hx} - V_P)^2\,n(v_{Hx})\,dv_{Hx}.$$

Introduction of a velocity distribution $f_{Hx}(v_{Hx}) = n(v_{Hx})/N$ results in the corresponding average total force expression:

$$\bar{F}_H = mNS\int_0^\infty (V_P - v_{Hx})^2\,f_{Hx}\,dv_{Hx} \tag{25}$$

where N is the number of particles per unit volume. For piston velocities that are small compared with the particle velocity in the gas, we can assume the standard Maxwell distribution for f_{Hx}. The influence of piston motion can be evaluated by modifying the Maxwell distribution to include a drift velocity $V_{GH} = -V_P/2$ equivalent to the velocity of the center of mass of the gas in reservoir H, with respect to the piston reference frame:

$$f_{Hx} = \sqrt{\frac{m}{2\pi k T_H}}\;e^{-\frac{m}{2kT_H}(v_{Hx}-V_{GH})^2}. \tag{26}$$

The distribution in (26) is a valid approximation for small drift velocities compared to the gas particle velocity, $V_{GH} \ll \sqrt{\overline{v_{Hx}^2}}$, where the upper bar represents the average value calculated with the standard Maxwell distribution, in the absence of any drift effects ($V_P = 0$). Integrating equation (25) using (26) produces an average force on the piston facing reservoir H:

$$\bar{F}_H(V_P) = mNS\left[\bar{v}_{Hx}^2 - 3V_P\bar{v}_{Hx} + \frac{9}{8}V_P^2\right]$$

and similarly for its opposite side facing C:

$$\bar{F}_C(V_P) = mNS\left[\bar{v}_{Cx}^2 + 3V_P\bar{v}_{Cx} + \frac{9}{8}V_P^2\right].$$

These expressions differ only in the sign of the linear piston velocity terms. Independent of its direction piston motion induces damping. When both reservoirs are at the same temperature, equivalently when the piston is under pressure equilibrium, the average net force on the piston reduces to:

$$\bar{F}(V_P) = \bar{F}_H(V_P) - \bar{F}_C(V_P) = -6mNSV_P\;\bar{v}_x \tag{27}$$

which reveals the viscous nature of the average damping force. The instantaneous net force on the piston follows from equation (24):

$$F(V_P,t) = 2m\left[\sum_i (v_{xi}+V_P)\,\delta(t-t_{iH}) - \sum_i (v_{xi}-V_P)\,\delta(t-t_{iC})\right]. \tag{28}$$

The force (28) on the piston can be expressed as a combination of its average value and a fluctuating part:

$$F(V_P,t) = \bar{F}(V_P) + f(t) \tag{29}$$

where the mean value of force $\bar{F}(V_P)$ from Eq. (27) represents the damping force and $f(t)$ represents the fluctuating force that evolves due to impulses from the particles in the gas.

An explicit expression for the fluctuation can be obtained simply by considering the case for a stationary piston, and setting $V_P = 0$, in Eq. (29):

$$F(0,t) = \bar{F}(0) + f(t)$$

resulting in the expression for fluctuation:

$$f(t) = 2m\sum_i v_{xi}\delta(t-t_i).$$

Finally the general expression for the interaction force between the piston and the gas can be expressed as:

$$F(V_P,t) \approx -6mNSV_P\bar{v}_x + 2m\sum_i v_{xi}\delta(t-t_i).$$

This shows how the interaction with the particles in the reservoirs generates a viscous damping, with damping coefficients $C_D = 6mNS\bar{v}_x$, and a random force on the right-hand side. Moreover, it appears the two terms are not independent, since they appear to be both driven by the speed v_x of the particles in the gas. This fact again manifests the fluctuation-dissipation duality. Finally, note how the damping coefficient depends on the average speed $\bar{v}_x$ of the particles contained in the reservoirs, that it is proportional to the root of the temperature of the gas since $T \propto \overline{v_x^2} \propto \bar{v}_x^2$ and therefore $C_D \propto \sqrt{T}$.

Acknowledgement

Adnan Akay would like to express his gratitude to the Alexander von Humboldt Foundation for supporting this work through a Research Award.

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