

# Chapter 4

## $p$ -Center Problems

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**Abstract** A  $p$ -center is a minimax solution that consists in a set of  $p$  points that minimizes the maximum distance between a demand point and a closest point belonging to that set. We present different variants of that problem. We review special polynomial cases, determine the complexity of the problems and present mixed integer linear programming formulations, exact algorithms and heuristics. Several extensions are also reviewed.

**Keywords**  $p$ -Center • Location in public sector • Minimax facility location

### 4.1 Introduction

Minimizing the total or average distance that potential users have to travel to reach a facility may not be the right decision criterion for placing a public facility. Total- or average distance minimization tends to favor clients who are clustered in population centers to the detriment of clients who are spatially dispersed. Discrimination of this kind with regard to accessibility may have a negative impact on remote clients in the case of an emergency service (ambulances, fire brigades, police stations, etc.). As a result, decision makers may want to consider a criterion focusing on clients who are the poorest served.

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The center problem, defined as finding a vertex whose distance to all the other vertices of a graph is minimum, has been known for a long time in graph theory (see, for instance, Berge 1967).

Hakimi (1964) introduced the absolute center problem to locate a police station or a hospital such that the maximum distance of the station to a set of communities connected by a highway system is minimized. Given a graph  $G = (V, E)$  with  $V = \{v_1, \dots, v_n\}$ , weight  $w_j$  for node  $v_j \in V$  and length  $\ell_{ij}$  for edge  $\{i, j\} \in E$  connecting nodes  $v_i$  and  $v_j$ , the aim of the *absolute center problem* is to find a point  $x$  on the nodes or edges such that  $\max_{j=1, \dots, n} w_j d(v_j, x)$  is minimized, where  $d(v_j, x)$  is the length of the shortest path between node  $v_j$  and point  $x$  (referred to as distance between  $v_j$  and  $x$ ). The optimal value is called the *absolute radius* of graph  $G$ . If  $x$  is limited to the nodes of  $G$ , then we obtain the *center* of graph  $G$  and the optimal value is the *radius* of  $G$ . The center of  $G$  is not necessarily an absolute center of  $G$ . In other words, the absolute radius can be smaller than the radius. To see this, consider a very simple example with two nodes of weight 1 and an edge connecting them with length 1. In this example, the absolute radius is 0.5 whereas the radius is 1.

Hakimi (1964) proposed a solution method to compute the absolute center of a graph and motivated further studies of this problem by casting it as a game. Two people, X and Y, are playing a game on a graph  $G$ . First player X chooses a point  $x$  in  $G$ , then player Y chooses a point  $y$  in  $G$  and X pays  $d(x, y)$  units to Y. When X chooses point  $x$ , Y chooses a point farthest from  $x$  to maximize his gain. Hence, player X computes the absolute radius of graph  $G$  to minimize his loss.

In the conclusion of his subsequent paper on median and covering problems, Hakimi (1965) mentions the generalization of the absolute center problem to the  $p$ -center problem. Given a set  $X_p = \{x_1, \dots, x_p\}$  of  $p$  points in  $G$ , the distance  $d(X_p, v_j)$  between  $X_p$  and node  $v_j$  is computed as  $\min_{i=1, \dots, p} d(x_i, v_j)$ . The  $p$ -center problem is to find a set  $X_p$  of  $p$  points in  $G$  such that  $\max_{j=1, \dots, n} w_j d(v_j, X_p)$  is minimized.

As defined above, the  $p$ -center problem is a network location problem. The literature contains several variants. In this chapter, we refer to the following variants:

- *vertex-restricted  $p$ -center problem*:  $X_p$  is restricted to be a subset of the node set;
- *unweighted  $p$ -center problem*: all node weights are equal;
- *discrete  $p$ -center problem*: the graph  $G = (J \cup I, E)$  is bipartite and complete with  $I$  denoting the set of possible facility locations and  $J$  denoting the set of demand points.

One can find a discussion of several theoretical results and exact methods for the  $p$ -center problem on general and tree networks in Tansel (2011). A large scale review of the exact and heuristic methods proposed for the  $p$ -center and capacitated  $p$ -center problems is provided by Calik (2013).

This chapter is organized as follows. We review some polynomial cases, identify the complexity of the problems in general and present some approximation results in Sect. 4.2. Section 4.3 is devoted to the mixed integer linear programming models and algorithms for solving  $p$ -center problems. Heuristics are discussed in Sect. 4.4 and some extensions of the  $p$ -center problem are considered in Sect. 4.5. Section 4.6 concludes the chapter.

## 4.2 Polynomial Cases, Complexity and Approximation Results

An algorithm to compute an absolute center of a graph was proposed by Hakimi (1964). The idea is to compute, for each edge, an optimal point assuming that the center is restricted to be on that edge. Such an optimal point is called a local center of that edge. Then the algorithm finds the best local center. Hence, the overall complexity is equal to the number of edges multiplied by the complexity of computing a local center of an edge.

The computation of a local absolute center is based on the observation that the objective function is piecewise linear on each edge and that local minima correspond to *intersection points* and vertices (see Miniéka 1970). A point  $x$  on edge  $\{v_k, v_m\} \in E$  qualifies as an intersection point if there exist two distinct nodes  $v_i, v_j \in V$  such that  $x$  is the unique point on  $\{v_k, v_m\}$  for which  $d(v_i, x) = d(v_i, v_k) + d(v_k, x) = d(x, v_j) = d(x, v_m) + d(v_m, v_j)$ .

It follows from this definition that the number of intersection points on an edge is bounded by  $O(n^2)$ . Nevertheless, Kariv and Hakimi (1979) observed that at most  $n + 1$  such points can be local minima of the objective function. The resulting algorithm proposed by Kariv and Hakimi (1979) solves the absolute center problem in  $O(|E|n + n^2 \log n)$  time.

An algorithm for finding an absolute center in the weighted case can be derived along the same lines. First, a solution can also be found in the set of local centers, i.e., solutions to the problems in which the solution is restricted to be on an edge. Then, the objective function remains piecewise linear on each edge but the slopes of the linear pieces depend on the vertex weights  $w_j$ . Kariv and Hakimi (1979) showed that, on an edge, at most  $3n - 2$  intersection points can determine a local minima. A point  $x$  on an edge  $\{v_k, v_m\}$  is now an intersection point if there exist two distinct nodes  $v_i, v_j \in V$  such that  $x$  is the unique point on  $\{v_k, v_m\}$  for which  $w_i d(v_i, x) = w_i (d(v_i, v_k) + d(v_k, x)) = w_j d(x, v_j) = w_j (d(x, v_m) + d(v_m, v_j))$ . The complexity of the resulting algorithm proposed by Kariv and Hakimi (1979) is  $O(|E|n \log n)$ .

Goldman (1972) proposed an  $O(n^2)$  algorithm to find an absolute center of a tree in the unweighted case. The algorithm checks whether an edge contains an absolute center and if not, searches the two subtrees obtained by deleting this edge. Handler (1973) proposed an  $O(n)$  algorithm exploiting the fact that the midpoint of a longest

path of the tree is an absolute center and that the distance is a convex function along any path of the tree. Given any node  $v_i$ , the algorithm first determines the vertex  $v_j$  whose distance to  $v_i$  is maximum, then determines the node  $v_k$  whose distance to  $v_j$  is maximum. The path linking  $v_j$  and  $v_k$  is a longest one and the absolute center is its midpoint.

Kariv and Hakimi (1979) provided an  $O(n \log n)$  algorithm for the weighted center problem on a tree, which was improved to  $O(n)$  by Megiddo (1983).

For an arbitrary graph  $G$  and  $p \geq 2$ , Kariv and Hakimi (1979) proved that the  $p$ -center problem is NP-hard even on a planar graph where the maximum degree is 3 and all node weights and edge lengths are equal to 1. The result is also true for the vertex-restricted problem. The authors show that the problem with  $p \geq 2$  can be solved in  $O(n^2 \log n)$  time when  $G$  is a tree.

Hochbaum and Shmoys (1985) developed a two-approximation algorithm for the unweighted discrete problem with  $I = J$  and edge lengths satisfying the triangle inequality. The algorithm runs in  $O(|E| \log |E|)$  time. Hsu and Nemhauser (1979) proved that it is NP-hard to find an approximation with a better guarantee. Dyer and Frieze (1985) gave an  $O(np)$  algorithm with a guarantee of  $\min\{3, 1 + \alpha\}$ , where  $\alpha$  is the ratio of the largest weight and the minimum weight. In the unweighted case, this guarantee is 2.

### 4.3 Exact Methods for $p$ -Center Problems

We first observe that the different variants of the  $p$ -center problem can be transformed into a discrete  $p$ -center problem and solved as such.

In the case of the vertex-restricted  $p$ -center problem, the set  $I$  of possible locations and the set  $J$  of demand points are both equal to the set of vertices  $V$ .

The weighted and unweighted absolute  $p$ -center problems enjoy the same property as their single facility counterpart: an optimal solution can always be found in the set of vertices and intersection points. This follows from the fact that each point  $x_i$  of an optimal solution  $X_p$  must be a local minimizer of the function given by the maximum (possibly weighted) distance to the vertices that are allocated to  $x_i$ , i.e., which are closer to that  $x_i$  than to any other point of  $X_p$ . To transform an absolute  $p$ -center problem into a discrete  $p$ -center problem one thus simply sets  $I = V \cup P$ , where  $P$  denotes the set of intersection points, and  $J = V$ .

The remainder of this section is now devoted to models and algorithms for solving the discrete  $p$ -center problem.

Several methods based on solving finite series of an auxiliary problem called the set covering problem are developed. The set covering problem is a kind of covering problem (see Chap. 5), which is closely related to the  $p$ -center problem. Given a zero-one matrix  $A = [a_{ji}]$ , the set covering problem consists of finding a set of columns at minimum cost that covers the rows of the matrix  $A$ . In order to minimize the number of facilities required to serve all customers within a given radius value

$r$ , one can solve a set covering problem with unit column costs by constructing  $A$  as follows:

$$a_{ji} = \begin{cases} 1, & \text{if } d(j, i) \leq r, \\ 0, & \text{otherwise} \end{cases} \quad \forall j \in J, i \in I.$$

If the optimal value of the set covering problem is greater than  $p$ , then the optimal value of the  $p$ -center problem needs to be greater than  $r$ ; if it is less than or equal to  $p$ , then it means that the optimal value of the  $p$ -center problem is less than or equal to  $r$ .

The first set covering based approach was proposed by Miniéka (1970). Let  $r_1 < r_2 < \dots < r_K$  be an ordering of the distinct distance values in the distance matrix  $D = [d_{ji}] : d_{ji} = d(j, i), i \in I, j \in J$  and  $R = \{r_1, r_2, \dots, r_K\}$ . The method by Miniéka (1970) solves the set covering problem for a smaller value in  $R$  not yet considered at each step by updating the matrix  $A$ . The algorithm terminates when the optimal value of the set covering problem is greater than  $p$ . Since the number of different distance values in  $D$  is at most  $|I| \cdot |J|$ , the algorithm converges to an optimal solution in a finite number of steps.

Garfinkel et al. (1977) improved the set covering based approach by Miniéka (1970) by first finding a heuristic solution, then, reducing the search space of the radius values and eliminating some of the intersection points. They also reduce the size of the set covering matrix by using standard matrix reductions and heuristic techniques. For the selection of the radius values at the next step, they proposed using a bisection or binary search strategy instead of moving to the next smaller radius value.

The first mixed integer programming (MIP) formulation for the discrete  $p$ -center problem was proposed by Daskin (2013). The following decision variables are defined:  $y_i = 1$  if a facility is placed at node  $i \in I$  and 0 otherwise,  $x_{ij} = 1$  if  $j \in J$  is assigned to a facility placed at  $i \in I$  and 0 otherwise. The formulation by Daskin can be stated as follows:

$$\text{Minimize} \quad z \quad (4.1)$$

$$\text{subject to} \quad \sum_{i \in I} d_{ji} x_{ij} \leq z \quad \forall j \in J, \quad (4.2)$$

$$\sum_{i \in I} x_{ij} = 1 \quad \forall j \in J, \quad (4.3)$$

$$x_{ij} \leq y_i \quad \forall i \in I, j \in J, \quad (4.4)$$

$$\sum_{i \in I} y_i \leq p, \quad (4.5)$$

$$y_i \in \{0, 1\} \quad \forall i \in I, \quad (4.6)$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J. \quad (4.7)$$

The objective function (4.1) together with (4.2) ensure that the objective value is no less than the maximum of the distances between demand points and their facilities. Constraints (4.3) establish the assignment of each demand point to exactly one facility. Constraints (4.4) avoid assignment of demand points to locations with no facility. Constraint (4.5) restricts the number of facilities to  $p$ . Constraints (4.6) and (4.7) are the binary restrictions.

Daskin (2013) also proposed a set covering based algorithm, in which the radius value of the set covering problem is selected from an interval of real numbers between pre-determined lower and upper bounds. At each step of the algorithm, the interval is halved and one of the segments is removed depending on whether the objective value of the set covering problem is greater than  $p$  or less than or equal to  $p$ .

Ihan and Pinar (2001) proposed a two-phase extension of the algorithm developed by Daskin (2013). In the first phase, they solve the linear programming (LP) relaxation of the feasibility problem defined by (4.5), (4.6), and

$$\sum_{i \in I} a_{ji} y_i \geq 1, \quad \forall j \in J, \quad (4.8)$$

iteratively for fixed  $r$  values to obtain a relatively tight lower bound for the  $p$ -center problem. In the second phase, they restrict the interval of the radius values with the lower bound obtained in the first phase and solve the integer programming (IP) version of the same feasibility problem iteratively to obtain the optimal value of the  $p$ -center problem.

Elloumi et al. (2004) proposed a new IP formulation for the  $p$ -center problem. This formulation utilizes the fact that the optimal value of the  $p$ -center problem is restricted to a finite set of distance values. They introduced additional binary variables  $z^k$ ,  $k = 2, \dots, K$ , with  $z^k = 0$  if all demand points can be covered by  $p$  facilities within a radius value of  $r_{k-1}$  and  $z^k = 1$  otherwise. The formulation is given below:

$$\text{Minimize} \quad r_1 + \sum_{k=2}^K (r_k - r_{k-1}) z^k \quad (4.9)$$

$$\text{subject to} \quad (4.5), (4.6),$$

$$\sum_{i \in I} y_i \geq 1, \quad (4.10)$$

$$z^k + \sum_{i: d_{ji} < r_k} y_i \geq 1 \quad \forall j \in J, k = 2, \dots, K, \quad (4.11)$$

$$z^k \in \{0, 1\} \quad k = 2, \dots, K. \quad (4.12)$$

Constraint (4.10) eliminates the solutions with no open facility. Constraints (4.11) and the objective function (4.9) ensure that all demand points are served by a facility within the smallest possible distance.

A semi-relaxation of this formulation, which is obtained by removing the binary restriction on the  $y$  variables, provides the best known lower bound for the  $p$ -center problem. This lower bound can be obtained by solving a finite series of LP problems, which are the LP relaxations of the set covering problems. Elloumi et al. (2004) also provided an exact algorithm that combines the important properties of the algorithms of Miniéka (1970) and İlhan and Pınar (2001). Their algorithm uses the two-phase idea and a binary search strategy similar to the algorithm by İlhan and Pınar (2001), but restricts the set of radius values to solve the set covering problems with the finite radius set  $R$  as in Miniéka (1970).

Calik and Tansel (2013) developed new IP formulations and a new exact algorithm based on the decomposition of their models for solving the  $p$ -center problem. They associated a binary variable  $u_k$  with  $r_k$ , for each  $k \in \{1, \dots, K\}$ . In particular,  $u_k$  is equal to 1 if  $r_k$  is selected as the optimal value and 0 otherwise. Initially, they proposed the following formulation:

$$\text{Minimize} \quad \sum_{k=1}^K r_k u_k \quad (4.13)$$

subject to (4.5), (4.6),

$$\sum_{i: d_{ji} \leq r_k} y_i \geq u_k \quad \forall j \in J, k = 1, \dots, K, \quad (4.14)$$

$$\sum_{k=1}^K u_k = 1, \quad (4.15)$$

$$u_k \in \{0, 1\} \quad k = 1, \dots, K. \quad (4.16)$$

Constraint (4.15) sets exactly one of the variables  $u_k$  to 1 and the corresponding  $r_k$  value is selected as the optimal value according to the objective function (4.13). Constraints (4.14) ensure that each customer is served within the selected radius by at least one facility. Constraints (4.16) are binary restrictions. The authors proposed a tightened formulation by using a relationship between their formulation and the formulation proposed by Elloumi et al. (2004). In this formulation, constraints (4.14) are replaced with constraints (4.17) given below:

$$\sum_{i: d(i,j) \leq r_k} y_i \geq \sum_{q=1}^k u_q, \quad \forall j \in J, k = 1, \dots, K. \quad (4.17)$$

The semi relaxations of these formulations, in which the binary restriction of the  $y$ -variables are removed, provide the tight lower bound obtained by Elloumi

et al. (2004). The algorithm developed by Calik and Tansel (2013) solves their formulations for restricted sets of radius values iteratively to converge to an optimal solution. They proposed several selection strategies for a two-element specialization of their algorithm. They also utilize the matrix reduction rules known for the set covering problem in their restricted formulations when solving large problems.

In the recent studies, instances from the OR-Library (Beasley 1990) and TSPLIB (Reinelt 1991) have been used for making computational experiments. The data for the uncapacitated  $p$ -median problem found in the OR-Library consists of 40 instances with  $n = 100 - 900$  and  $p = 5 - (n/3)$ . This data was used in the experiments conducted by Ilhan and Pınar (2001), Elloumi et al. (2004), and Calik and Tansel (2013). In addition to these instances, Elloumi et al. (2004) used the instances u1060, r11323 and u1817 ( $n = 1060, 1323, \text{ and } 1817$ , respectively) and Calik and Tansel (2013) used the instances u1817, d15112, and pcb3038 ( $n = 1817, 2500, \text{ and } 3038$ , respectively) from the TSPLIB.

#### 4.4 Heuristics

Mladenović et al. (2003) introduced the first meta-heuristic approaches for finding approximate solutions to the  $p$ -center problem. They proposed a multistart local search algorithm (M-I), a chain substitution Tabu Search (TS) algorithm, and a variable neighborhood search (VNS) algorithm and conducted large scale experiments on 40  $p$ -median instances from the OR-Library and instances with up to 3,038 nodes from TSPLIB. These experiments reveal that their algorithms outperform the algorithm proposed by Hochbaum and Shmoys (1985). Among the three heuristics proposed, TS and VNS algorithms outperform M-I algorithm, VNS performs the best on the average in terms of both the solution quality and solution time; however, TS provides slightly better results for the instances with smaller  $p$  values.

Pullan (2008) proposed a memetic genetic algorithm (PBS) for the vertex-restricted  $p$ -center problem, which combines a population based meta-heuristic with a local search algorithm. By using the phenotype crossover and directed mutation tools of the genetic algorithm, a wide range of elite starting solutions are generated and then, these solutions are improved to local optimality by using a local search algorithm. From the computational experiments using the instances previously tackled by Mladenović et al. (2003), an improvement in the CPU times and in the objective value of some problems is observed when PBS is compared with the VNS algorithm. The PBS algorithm can be executed also in a parallel processing mode. The experiments conducted by increasing the number of parallel processors utilized in the algorithm provide better CPU times.

Salhi and Al-Khedhairi (2010) obtained tight lower and upper bounds by using a three-level meta-heuristic and integrated these bounds with the algorithm by Daskin (2013) to solve the vertex-restricted  $p$ -center problem. In the first and second levels of the algorithm, a variable neighborhood strategy is utilized with distinct neighborhood structures. In the third level, a perturbation mechanism is introduced



to avoid sticking at local optima. The computational experiments conducted on the 40  $p$ -median instances of the OR-Library revealed that the utilization of these bounds decreases the solution times of Daskin's algorithm.

Other than the meta-heuristic algorithms, Martinich (1988) proposed a vertex closing approach for the vertex-restricted  $p$ -center problem on complete networks with distance values that satisfy the triangle inequality. Initially, the algorithm places a facility on each node and considers the problem of finding  $n - p$  facilities to close so that the maximum of the distances between the nodes and their facilities is minimized. In this study, the optimal solutions were characterized with the embedded sub-graphs of the original graph. From this analysis, initial lower and upper bounds were obtained, two polynomial time algorithms were proposed and procedures to verify the optimality of the solutions for several special cases were developed. In terms of the number of instances solved to optimality, they outperform the algorithm by Hochbaum and Shmoys (1985).

Bozkaya and Tansel (1998) showed that there exists a spanning tree of any connected network such that the optimal absolute  $p$ -center of this tree is optimal also for the network under consideration. They conducted experiments on two classes of spanning trees to observe how often these trees provide the optimal solution. They concluded that these two classes of spanning trees do not always include the optimizing tree, but they do in most of the instances.

Mihelič and Robič (2005) solved the vertex-restricted  $p$ -center problem by introducing a heuristic algorithm based on solving a finite series of minimum dominating set problems. Given a graph  $G = (V, E)$ , the minimum dominating set problem aims to find a node subset  $S \subset V$  of minimum cardinality so that any node in  $V \setminus S$  is adjacent to some node in  $S$ . They assumed that the underlying network is complete and the distance values satisfy the triangle inequality. The computational experiments performed on 40 standard test instances indicate that their algorithm performs much better than the other polynomial time heuristics found in the literature and competes with the best known non-polynomial time algorithms.

## 4.5 Variants

In this section, we briefly discuss some extensions of the  $p$ -center problem.

### 4.5.1 The Capacitated $p$ -Center Problem

One first variant concerns problems with capacitated facilities. There are few studies on this variant. Bar-Ilan et al. (1993) introduced a ten-approximation algorithm for the special case of unit demands. The guarantee was improved to 6 by Khuller and

Sussmann (2000). If multiple centers can be located at the same location, then the guarantee is further improved to 5.

Jaeger and Goldberg (1994) proposed a polynomial time algorithm for the capacitated  $p$ -center problem when the graph is a tree, capacities are equal, and multiple facilities can be located at the same location. In this work, the demand of a node can be split among different facilities.

Ozsoy and Pinar (2006) proposed an exact algorithm to solve the capacitated  $p$ -center problem. The idea is to see if the all nodes can be assigned within a given distance and update lower and upper bounds on the radius using this information. In the subproblem solved to see whether it is possible to assign all nodes within a given distance, the objective is to minimize the number of facilities required.

In addition to the subproblem solved by Ozsoy and Pinar (2006) to obtain bounds on the optimal radius, Albareda-Sambola et al. (2010) proposed a second approach where they solved the problem of maximizing the demand covered within a given distance using at most  $p$  facilities. They used bounds from the Lagrangian relaxation of the two subproblems to eliminate some radius values and concluded that the first approach for finding the minimum number of required facilities is a better approach. Based on this conclusion, they proposed an exact algorithm using binary search over possible values of the optimal radius.

A very large-scale neighborhood heuristic was developed by Scapparra et al. (2004). Two types of exchanges were considered. In a cyclic exchange, one takes a sequence of nodes that are served by different facilities and replaces the facility of each node with the facility of the next node in the sequence (the facility of the last node in the sequence becomes the facility of the first node). In a path exchange, we again take a sequence of nodes served by different facilities and replace the facility of each node with the facility of the next node. The facility of the last node is replaced by a facility different from the facilities of the nodes in the sequence. A relocation step that moves the facilities to better locations with respect to the set of nodes they are serving is also added to the algorithm.

Three data sets were used in the last three papers mentioned. The first data set contains 20 instances of the capacitated  $p$ -median problem from the OR-Library (Beasley 1990), with 50 and 100 nodes. The second data set is from Lorena and Senne (2004) and is also for the capacitated  $p$ -median problem. Here there are six instances with the number of nodes ranging from 100 to 402. Finally, Scapparra et al. (2004) provided a data set with 8 instances containing 100 and 150 nodes. Additional instances of the  $p$ -median problem were used by Albareda-Sambola et al. (2010). These authors also compared their approach with the one of Ozsoy and Pinar (2006).

#### ***4.5.2 The Conditional $p$ -Center Problem***

The second variant is the conditional  $p$ -center problem. In this variant, there are  $q$  existing facilities and additional  $p$  facilities are to be located so that the

maximum distance between a node and its facility (among  $p + q$  facilities) is minimized. Miniéka (1980) introduced the conditional 1-center problem. Drezner (1989) showed that the conditional  $p$ -center problem can be solved by solving  $O(\log n)$   $p$ -center problems. Suppose that the nodes are ranked in non-increasing order of their distances to their facilities (using the existing  $q$  facilities). Then there exists a node  $s$  such that the optimal value of the conditional  $p$ -center problem is equal to the maximum of the optimal value of the  $p$ -center problem solved for the first  $s$  nodes and the distance of the  $s + 1$ st node to its facility using the existing  $q$  facilities. The algorithm tries to find the best  $s$  using bisection.

Berman and Simchi-Levi (1990) solved the conditional  $p$ -center problem by solving a  $p + 1$  center problem. They add a dummy demand node and a dummy possible location. The distance from a demand node to the dummy location is the distance of that node to its facility considering the existing facilities. The distance of the dummy demand node to the dummy location is zero and its distance to the other possible locations is a very large number. As a result, an optimal solution to the  $p + 1$ -center problem includes the dummy facility location and opens  $p$  other facilities. Berman and Drezner (2008) improved this approach and showed that the conditional  $p$ -center problem can be solved by solving a  $p$ -center problem where the distance between a node and a potential facility is set to the minimum of this distance and the distance between this node and the closest existing facility.

### 4.5.3 *The Continuous $p$ -Center Problem*

The next variant is the continuous  $p$ -center problem. When demand points are continuously distributed over the whole graph, a set  $X_p$  of  $p$  points of the graph minimizing the largest distance from a demand point to a closest point of  $X_p$  is called a continuous  $p$ -center.

In the single facility case, i.e., when  $p = 1$ , the problem can still be solved by choosing a best solution among all the local continuous centers, i.e., solutions to continuous center problem in which the location is restricted to an edge. On an edge, the objective function is again piecewise linear with  $O(|E|)$  breakpoints. Based on these facts,  $O(|E|^2 \log(|E|))$  algorithms were proposed by Hansen et al. (1991) and Tamir (1988).

On a tree, the absolute center coincides with the unweighted absolute center.

For the continuous  $p$ -center problem, Tamir (1987) identified a finite set of rational numbers containing the optimal solution value. Hence, a continuous  $p$ -center can be found by solving a finite number of continuous set covering problems, i.e., problems in which one looks for the smallest set of facilities needed to cover all points of the graph (vertices and interior points to edges) within a given maximum distance.

#### 4.5.4 The $p$ -Center Problem with Uncertain Parameters

Finally, we consider the variants with uncertain parameters. Averbakh and Berman (1997) studied the minmax regret version of the problem where the node weights are uncertain within given intervals. They showed that the robust version of the problem can be reduced to the resolution of  $n + 1$  deterministic problems. Averbakh (1997) showed that the robust 1-center problem is strongly NP-hard on general networks when there is uncertainty in edge lengths. Averbakh and Berman (2000) developed polynomial time algorithms for the robust weighted 1-center problem with uncertainty in both node weights and edge lengths on a tree network.

## 4.6 Conclusions

We conclude this chapter with some future research directions. The majority of the solution methods proposed for the  $p$ -center problem are based on either the set covering or the dominating set problems. Well known optimization methods such as the cutting plane, branch-and-cut, Benders decomposition, or dynamic programming are rarely used. Recently, Calik (2013) provided a Benders decomposition method to solve the vertex restricted  $p$ -center problem and developed a branch-and-cut method for the capacitated  $p$ -center problem with multiple allocation. The experimental study conducted by Calik (2013) revealed that the utilization of a branch-and-cut method enables obtaining optimal solutions of large instances in small CPU time. The multiple allocation variant, which was previously studied by Jaeger and Goldberg (1994) on trees, is also an open research area for the capacitated  $p$ -center problem.

Although there are many studies for the  $p$ -center problem on trees, the capacitated version is not extensively investigated. The only study on this problem considers facilities with identical capacities and allows multi centers and multiple allocation. Hence investigating the capacitated  $p$ -center problem on tree networks with non-identical capacities, without multi centers and/or with single allocation might be a worthwhile undertaking.

Another variant of the  $p$ -center problem that has recently attracted the attention of the researchers is the fault tolerant  $p$ -center problem. This is a generalization of the  $p$ -center problem in which each customer is assigned to  $\alpha$  different facilities. The idea is to make back-up services available in case of a failure of some facilities. The fault tolerance can also be taken into account for the capacitated  $p$ -center problem. Among the existing studies for the fault tolerant  $p$ -center and capacitated  $p$ -center problems, Krumke (1995), Khuller et al. (2000), and Chechik and Peleg (2012) focus on approximation algorithms and a recent study by Chen and Chen (2013) presents two exact algorithms. Therefore, developing different exact approaches and meta-heuristic algorithms for this problem might appeal to the researchers.

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