

## 11.4 The Root Locus Method

Hitay Özbay

### Introduction

The root locus technique is a graphical tool used in feedback control system analysis and design. It has been formally introduced to the engineering community by W. R. Evans [3,4], who received the Richard E. Bellman Control Heritage Award from the American Automatic Control Council in 1988 for this major contribution.

In order to discuss the root locus method, we must first review the basic definition of bounded input bounded output (BIBO) stability of the standard linear time invariant feedback system shown in Figure 11.22, where the plant, and the controller, are represented by their transfer functions  $P(s)$  and  $C(s)$ , respectively<sup>1</sup> The plant,  $P(s)$ , includes the physical process to be controlled, as well as the actuator and the sensor dynamics.

The feedback system is said to be stable if none of the closed-loop transfer functions, from external inputs  $r$  and  $v$  to internal signals  $e$  and  $u$ , have any poles in the closed right half plane,  $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$ . A necessary condition for feedback system stability is that the closed right half plane zeros of  $P(s)$  (respectively  $C(s)$ ) are distinct from the poles of  $C(s)$  (respectively  $P(s)$ ). When this condition holds, we say that there is no unstable pole-zero cancellation in taking the product  $P(s)C(s) = : G(s)$ , and then checking feedback system stability becomes equivalent to checking whether all the roots of

$$1 + G(s) = 0 \tag{11.45}$$

are in the open left half plane,  $\mathbb{C}_- := \{s \in \mathbb{C} : \text{Re}(s) < 0\}$ . The roots of Equation (11.1) are the closed-loop system poles. We would like to understand how the closed-loop system pole locations vary as functions of a real parameter of  $G(s)$ . More precisely, assume that  $G(s)$  contains a parameter  $K$ , so that we use the notation  $G(s) = G_K(s)$  to emphasize the dependence on  $K$ . The *root locus* is the plot of the roots of Equation (11.45) on the complex plane, as the parameter  $K$  varies within a specified interval.

The most common example of the root locus problem deals with the uncertain (or adjustable) gain as the varying parameter: when  $P(s)$  and  $C(s)$  are fixed rational functions, except for a gain factor,  $G(s)$  can be written as  $G(s) = G_K(s) = KF(s)$ , where  $K$  is the uncertain/adjustable gain, and

$$F(s) = \frac{N(s)}{D(s)} \quad \text{where} \quad \begin{aligned} N(s) &= \prod_{j=1}^m (s - z_j) \\ D(s) &= \prod_{i=1}^n (s - p_i), \end{aligned} \quad n \geq m \tag{11.46}$$

with  $z_1, \dots, z_m$ , and  $p_1, \dots, p_n$  being the open-loop system zeros and poles. In this case, the closed-loop system

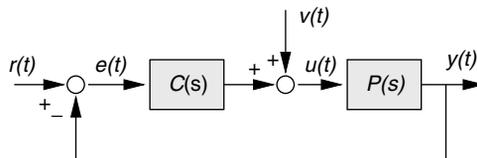


FIGURE 11.22 Standard unity feedback system.

<sup>1</sup>Here we consider the continuous time case; there is essentially no difference between the continuous time case and the discrete time case, as far as the root locus construction is concerned. In the discrete time case the desired closed-loop pole locations are defined relative to the unit circle, whereas in the continuous time case desired pole locations are defined relative to the imaginary axis.

poles are the roots of the characteristic equation

$$\chi(s) := D(s) + KN(s) = 0 \quad (11.47)$$

The *usual root locus* is obtained by plotting the roots  $r_1(K), \dots, r_n(K)$  of the characteristic polynomial  $\chi(s)$  on the complex plane, as  $K$  varies from 0 to  $+\infty$ . The same plot for the negative values of  $K$  gives the *complementary root locus*. With the help of the root locus plot the designer identifies the admissible values of the parameter  $K$  leading to a set of closed-loop system poles that are in the desired region of the complex plane. There are several factors to be considered in defining the “desired region” of the complex plane in which all the roots  $r_1(K), \dots, r_n(K)$  should lie. Those are discussed briefly in the next section. Section 11.3 contains the root locus construction procedure, and design examples are presented in section 11.4.

The root locus can also be drawn with respect to a system parameter other than the gain. For example, the characteristic equation for the system  $G(s) = G_\lambda(s)$ , defined by

$$G_\lambda(s) = P(s)C(s), \quad P(s) = \frac{(1 - \lambda s)}{s(1 + \lambda s)}, \quad C(s) = K_c \left( 1 + \frac{1}{T_I s} \right)$$

can also be transformed into the form given in Equation (11.47). Here  $K_c$  and  $T_I$  are given fixed PI (Proportional plus Integral) controller parameters, and  $\lambda > 0$  is an uncertain plant parameter. Note that the phase of the plant is

$$\angle P(j\omega) = -\frac{\pi}{2} - 2 \tan^{-1}(\lambda\omega)$$

so the parameter  $\lambda$  can be seen as the uncertain phase lag factor (for example, a small uncertain time delay in the plant can be modeled in this manner, see [9]). It is easy to see that the characteristic equation is

$$s^2(\lambda s + 1) + K_c(1 - \lambda s) \left( s + \frac{1}{T_I} \right) = 0$$

and by rearranging the terms multiplying  $\lambda$  this equation can be transformed to

$$1 + \frac{1(s^2 + K_c s + K_c/T_I)}{\lambda s(s^2 - K_c s - K_c/T_I)} = 0$$

By defining  $K = \lambda^{-1}$ ,  $N(s) = (s^2 + K_c s + K_c/T_I)$ , and  $D(s) = s(s^2 + K_c s - K_c/T_I)$ , we see that the characteristic equation can be put in the form of Equation (11.3). The root locus plot can now be obtained from the data  $N(s)$  and  $D(s)$  defined above; that shows how closed-loop system poles move as  $\lambda^{-1}$  varies from 0 to  $+\infty$ , for a given fixed set of controller parameters  $K_c$  and  $T_I$ . For the numerical example  $K_c = 1$  and  $T_I = 2.5$ , the root locus is illustrated in Figure 11.23.

The root locus construction procedure will be given in section 11.3. Most of the computations involved in each step of this procedure can be performed by hand calculations. Hence, an approximate graph representing the root locus can be drawn easily. There are also several software packages to generate the root locus automatically from the problem data  $z_1, \dots, z_m$ , and  $p_1, \dots, p_n$ .

If a numerical computation program is available for calculating the roots of a polynomial, we can also obtain the root locus with respect to a parameter which enters into the characteristic equation nonlinearly. To illustrate this point let us consider the following example:  $G(s) = G_{\omega_o}(s)$  where

$$G_{\omega_o}(s) = P(s)C(s), \quad P(s) = \frac{(s - 0.1)}{(s^2 + 1.2\omega_o s + \omega_o^2)(s + 0.1)}, \quad C(s) = \frac{(s - 0.2)}{(s + 2)}$$

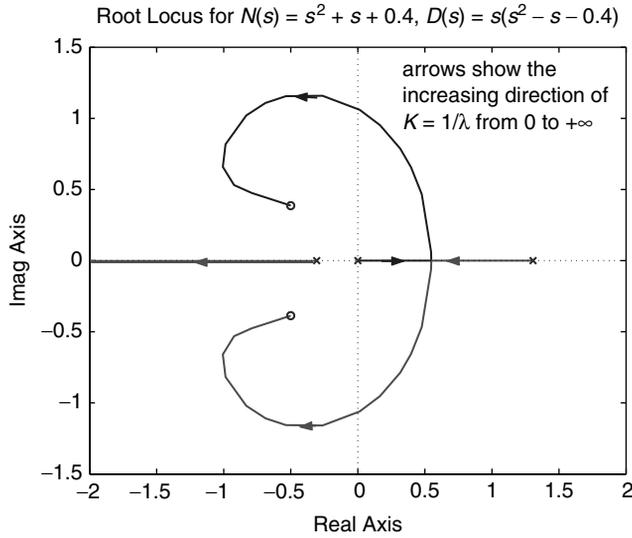


FIGURE 11.23 The root locus with respect to  $K = 1/\lambda$ .

Here  $\omega_o \geq 0$  is the uncertain plant parameter. Note that the characteristic equation

$$1 + \frac{\omega_o(1.2s + \omega_o)(s + 0.1)(s + 2)}{s^2(s + 0.1)(s + 2) + (s - 0.2)(s - 0.1)} = 0 \tag{11.48}$$

cannot be expressed in the form of  $D(s) + KN(s) = 0$  with a single parameter  $K$ . Nevertheless, for each  $\omega_o$  we can numerically calculate the roots of Equation (11.48) and plot them on the complex plane as  $\omega_o$  varies within a range of interest. Figure 11.24 illustrates all the four branches,  $r_1(K), \dots, r_4(K)$ , of the root locus for

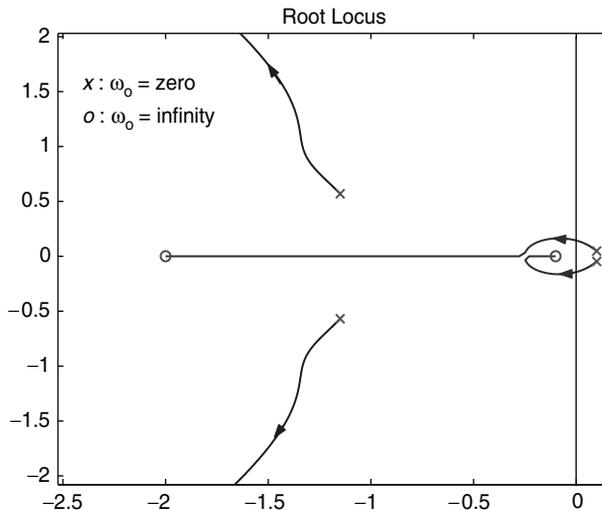


FIGURE 11.24 The root locus with respect to  $\omega_o$ .

this system as  $\omega_o$  increases from zero to infinity. The figure is obtained by computing the roots of Equation (11.48) for a set of values of  $\omega_o$  by using MATLAB.

## Desired Pole Locations

The performance of a feedback system depends heavily on the location of the closed-loop system poles  $r_i(K) = 1, \dots, n$ . First of all, for stability we want  $r_i(K) \in \mathbb{C}_-$  for all  $i = 1, \dots, n$ . Clearly, having a pole “close” to the imaginary axis poses a danger, i.e., “small” perturbations in the plant might lead to an unstable feedback system. So the desired pole locations must be such that stability is preserved under such perturbations (or in the presence of uncertainties) in the plant. For second-order systems, we can define certain stability robustness measures in terms of the pole locations, which can be tied to the characteristics of the step response. For higher order systems, similar guidelines can be used by considering the dominant poles only.

In the standard feedback control system shown in Figure 11.22, assume that the closed-loop transfer function from  $r(t)$  to  $y(t)$  is in the form

$$T(s) = \frac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}, \quad 0 < \zeta < 1, \quad \omega_o \in \mathbb{R}$$

and  $r(t)$  is the unit step function. Then, the output is

$$y(t) = 1 - \frac{e^{-\zeta\omega_o t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta), \quad t \geq 0$$

where  $\omega_d := \omega_o \sqrt{1-\zeta^2}$  and  $\theta := \cos^{-1}(\zeta)$ . For some typical values of  $\zeta$ , the step response  $y(t)$  is as shown in Figure 11.25. The maximum *percent overshoot* is defined to be the quantity

$$\text{PO} := \frac{y_p - y_{ss}}{y_{ss}} \times 100\%$$

where  $y_p$  is the peak value. By simple calculations it can be seen that the peak value of  $y(t)$  occurs at the time

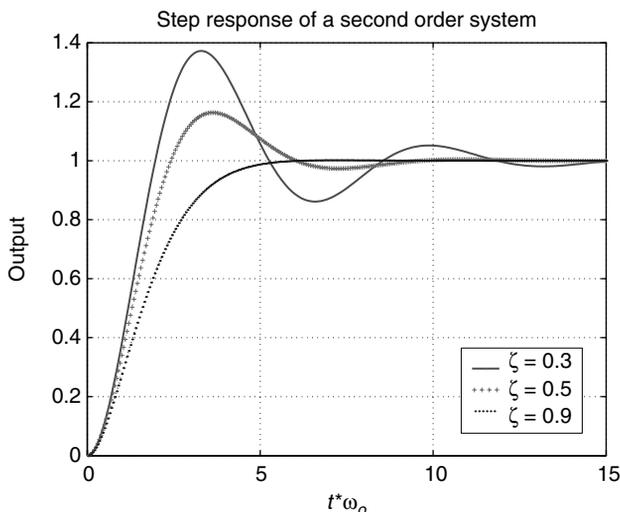
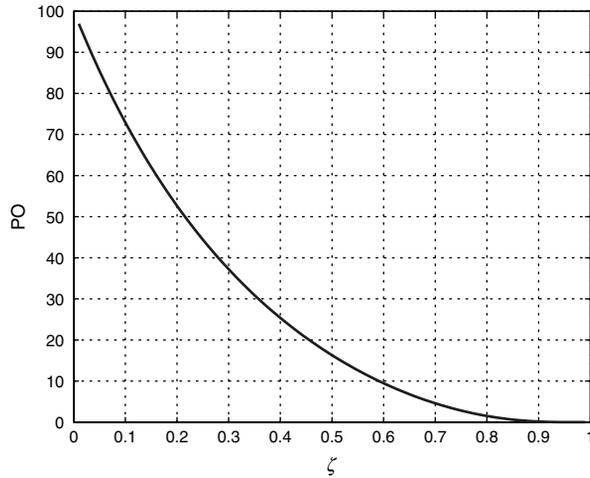


FIGURE 11.25 Step response of a second-order system.

FIGURE 11.26 PO versus  $\zeta$ .

instant  $t_p = \pi/\omega_d$ , and

$$PO = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

Figure 11.26 shows PO versus  $\zeta$ . The *settling time* is defined to be the smallest time instant  $t_s$ , after which the response  $y(t)$  remains within 2% of its final value, i.e.,

$$t_s := \min \{t' : |y(t) - y_{ss}| \leq 0.02y_{ss} \forall t \geq t'\}$$

Sometimes 1% or 5% is used in the definition of settling time instead of 2%; conceptually, there is no difference. For the second-order system response, we have

$$t_s \approx \frac{4}{\zeta\omega_o}$$

So, in order to have a fast settling response, the product  $\zeta\omega_o$  should be large.

The closed-loop system poles are

$$r_{1,2} = -\zeta\omega_o \pm j\omega_o\sqrt{1-\zeta^2}$$

Therefore, once the maximum allowable settling time and PO are specified, we can define the region of desired pole locations by determining the minimum allowable  $\zeta$  and  $\zeta\omega_o$ . For example, let the desired PO and  $t_s$  be bounded by

$$PO \leq 10\% \quad \text{and} \quad t_s \leq 8s$$

The PO requirement implies that  $\zeta \geq 0.6$ , equivalently  $\theta \leq 53^\circ$  (recall that  $\cos(\theta) = \zeta$ ). The settling time requirement is satisfied if and only if  $\text{Re}(r_{1,2}) \leq -0.5$ . Then, the region of desired closed-loop poles is the shaded area shown in Figure 11.27. The same figure also illustrates the region of desired closed-loop poles for similar design requirements in the discrete time case.

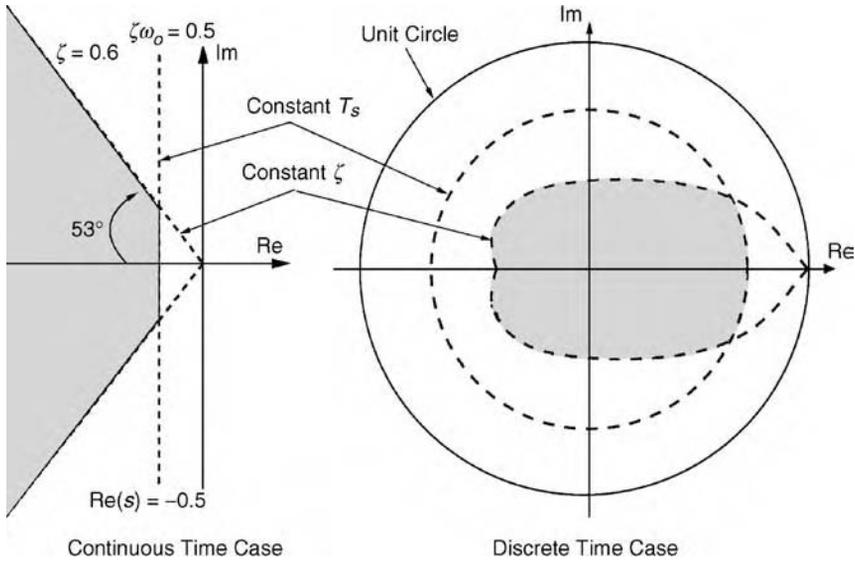


FIGURE 11.27 Region of the desired closed-loop poles.

If the order of the closed-loop transfer function  $T(s)$  is higher than two, then, depending on the location of its poles and zeros, it may be possible to approximate the closed-loop step response by the response of a second-order system. For example, consider the third-order system

$$T(s) = \frac{\omega_o^2}{(s^2 + 2\zeta\omega_o s + \omega_o^2)(1 + s/r)} \quad \text{where } r \gg \zeta\omega_o$$

The transient response contains a term  $e^{-rt}$ . Compared with the envelope  $e^{-\zeta\omega_o t}$  of the sinusoidal term,  $e^{-rt}$  decays very fast, and the overall response is similar to the response of a second-order system. Hence, the effect of the third pole  $r_3 = -r$  is negligible.

Consider another example,

$$T(s) = \frac{\omega_o^2[1 + s/(r + \epsilon)]}{(s^2 + 2\zeta\omega_o s + \omega_o^2)(1 + s/r)} \quad \text{where } 0 < \epsilon \ll r$$

In this case, although  $r$  does not need to be much larger than  $\zeta\omega_o$ , the zero at  $-(r + \epsilon)$  cancels the effect of the pole at  $-r$ . To see this, consider the partial fraction expansion of  $Y(s) = T(s)R(s)$  with  $R(s) = 1/s$ :

$$Y(s) = \frac{A_0}{s} + \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \frac{A_3}{s + r}$$

where  $A_0 = 1$  and

$$A_3 = \lim_{s \rightarrow -r} (s + r)Y(s) = \frac{\omega_o^2}{2\zeta\omega_o r - (\omega_o^2 + r^2)} \left( \frac{\epsilon}{r + \epsilon} \right)$$

Since  $|A_3| \rightarrow 0$  as  $\epsilon \rightarrow 0$  the term  $A_3 e^{-rt}$  is negligible in  $y(t)$ .

In summary, if there is an approximate pole–zero cancellation in the left half plane, then this pole–zero pair can be taken out of the transfer function  $T(s)$  to determine PO and  $t_s$ . Also, the poles closest to the imaginary axis dominate the transient response of  $y(t)$ . To generalize this observation, let  $r_1, \dots, r_n$  be the poles of  $T(s)$ , such that  $\text{Re}(r_k) \ll \text{Re}(r_2) = \text{Re}(r_1) < 0$ , for all  $k \geq 3$ . Then, the pair of complex conjugate poles  $r_{1,2}$  are called the *dominant poles*. We have seen that the desired transient response properties, e.g., PO and  $t_s$ , can be translated into requirements on the location of the dominant poles.

## Root Locus Construction

As mentioned above, the root locus primarily deals with finding the roots of a characteristic polynomial that is an affine function of a single parameter,  $K$ ,

$$\chi(s) = D(s) + KN(s) \quad (11.49)$$

where  $D(s)$  and  $N(s)$  are fixed monic polynomials (i.e., coefficient of the highest power is normalized to 1). If  $N$  and/or  $D$  are not monic, the highest coefficient(s) can be absorbed into  $K$ .

### Root Locus Rules

Recall that the usual root locus shows the locations of the closed-loop system poles as  $K$  varies from 0 to  $+\infty$ . The roots of  $D(s)$ ,  $p_1, \dots, p_n$ , are the poles, and the roots of  $N(s)$ ,  $z_1, \dots, z_m$ , are the zeros, of the open-loop system,  $G(s) = KF(s)$ . Since  $P(s)$  and  $C(s)$  are proper,  $G(s)$  is proper, and hence  $n \geq m$ . So the degree of the polynomial  $\chi(s)$  is  $n$  and it has exactly  $n$  roots.

Let the closed-loop system poles, i.e., roots of  $\chi(s)$ , be denoted by  $r_1(K), \dots, r_n(K)$ . Note that these are functions of  $K$ ; whenever the dependence on  $K$  is clear, they are simply written as  $r_1, \dots, r_n$ . The points in  $\mathbb{C}$  that satisfy (Equation (11.49)) for some  $K > 0$  are on the root locus. Clearly, a point  $r \in \mathbb{C}$  is on the root locus if and only if

$$= -\frac{1}{F(r)} \quad (11.50)$$

The condition (Equation (11.50)) can be separated into two parts:

$$|K| = -\frac{1}{|F(r)|} \quad (11.51)$$

$$\angle K = 0^\circ = -(2\ell + 1) \times 180^\circ - \angle F(r), \quad \ell = 0, \pm 1, \pm 2, \dots \quad (11.52)$$

The phase rule (Equation (11.52)) determines the points in that are on the root locus. The magnitude rule (Equation (11.51)) determines the gain  $K > 0$  for which the root locus is at a given point  $r$ . By using the definition of  $F(s)$ , (Equation (11.52)) can be rewritten as

$$(2\ell + 1) \times 180^\circ = \sum_{i=1}^n \angle(r - p_i) - \sum_{j=1}^m \angle(r - z_j) \quad (11.53)$$

Similarly, (Equation (11.51)) is equivalent to

$$K = \frac{\prod_{i=1}^n |r - p_i|}{\prod_{j=1}^m |r - z_j|} \quad (11.54)$$

### Root Locus Construction

There are several software packages available for generating the root locus automatically for a given  $F = N/D$ . In particular, the related MATLAB commands are `rllocus` and `rllocfind`. In many cases, approximate root locus can be drawn by hand using the rules given below. These rules are determined from the basic definitions Equation (11.49), Equation (11.51), and Equation (11.52).

1. The root locus has  $n$  branches:  $r_1(K), \dots, r_n(K)$ .
2. Each branch starts ( $K \cong 0$ ) at a pole  $p_i$  and ends (as  $K \rightarrow \infty$ ) at a zero  $z_j$ , or converges to an asymptote,  $Me^{j\alpha_\ell}$  where  $M \rightarrow \infty$  and

$$\alpha_\ell = \frac{2\ell + 1}{n - m} \times 180^\circ, \quad \ell = 0, \dots, (n - m - 1)$$

3. There are  $(n - m)$  asymptotes with angles  $\alpha_\ell$ . The center of the asymptotes (i.e., their intersection point on the real axis) is

$$\sigma_a = \frac{\sum_{i=1}^n P_i - \sum_{j=1}^m Z_j}{n - m}$$

4. A point  $x \in \mathbb{R}$  is on the root locus if and only if the total number of poles  $p_i$ 's and zeros  $z_j$ 's to the right of  $x$  (i.e., total number of  $p_i$ 's with  $\text{Re}(p_i) > x$  plus total number of  $z_j$ 's with  $\text{Re}(z_j) > x$ ) is odd. Since  $F(s)$  is a rational function with real coefficients, poles and zeros appear in complex conjugates, so when counting the number of poles and zeros to the right of a point  $x \in \mathbb{R}$  we just need to consider the poles and zeros on the real axis.
5. The values of  $K$  for which the root locus crosses the imaginary axis can be determined from the Routh–Hurwitz stability test. Alternatively, we can set  $s = j\omega$  in Equation (11.49) and solve for real  $\omega$  and  $K$  satisfying

$$D(j\omega) + KN(j\omega) = 0$$

Note that there are two equations here, one for the real part and one for the imaginary part.

6. The break points (intersection of two branches on the real axis) are feasible solutions (satisfying rule 4) of

$$\frac{d}{ds} F(s) = 0 \tag{11.55}$$

7. Angles of departure ( $K \cong 0$ ) from a complex pole, or arrival  $K \rightarrow +\infty$  to a complex zero, can be determined from the phase rule. See example below.

Let us now follow the above rules step by step to construct the root locus for

$$F(s) = \frac{(s + 3)}{(s - 1)(s + 5)(s + 4 + j2)(s + 4 - j2)}$$

First, enumerate the poles and zeros as  $p_1 = -4 + j2, p_2 = -4 - j2, p_3 = -5, p_4 = 1, z_1 = -3$ . So,  $n = 4$  and  $m = 1$ .

1. The root locus has four branches.
2. Three branches converge to the asymptotes whose angles are  $60^\circ, 180^\circ,$  and  $-60^\circ$ , and one branch converges to  $z_1 = -3$ .
3. The center of the asymptotes is  $\sigma = (-12 + 3)/3 = -3$ .
4. The intervals  $(-\infty, -5]$  and  $[-3, 1]$  are on the root locus.

5. The imaginary axis crossings are the feasible roots of

$$(\omega^4 - j12\omega^3 - 47\omega^2 + j40\omega - 100) + K(j\omega + 3) = 0 \tag{11.56}$$

for real  $\omega$  and  $K$ . Real and imaginary parts of Equation (11.56) are

$$\omega^4 - 47\omega^2 - 100 + 3K = 0$$

$$j\omega(-12\omega^2 + 40 + K) = 0$$

They lead to two feasible pairs of solutions ( $K = 100/3, \omega = 0$ ) and ( $K = 215.83, \omega = \pm 4.62$ ).

6. Break points are the feasible solutions of

$$3s^4 + 36s^3 + 155s^2 + 282s + 220 = 0$$

Since the roots of this equation are  $-4.55 \pm j1.11$  and  $-1.45 \pm j1.11$ , there is no solution on the real axis, hence no break points.

7. To determine the angle of departure from the complex pole  $p_1 = -4 + j2$ , let  $\Delta$  represent a point on the root locus near the complex pole  $p_1$ , and define  $v_i, i = 1, \dots, 5$ , to be the vectors drawn from  $p_i$  for  $i = 1, \dots, 4$ , and from  $z_1$  for  $i = 5$ , as shown in Figure 11.28. Let  $\theta_1, \dots, \theta_5$  be the angles of  $v_1, \dots, v_5$ . The phase rule implies

$$(\theta_1 + \theta_2 + \theta_3 + \theta_4) - \theta_5 = \pm 180^\circ \tag{11.57}$$

As  $\Delta$  approaches  $p_1$ ,  $\theta_1$  becomes the angle of departure and the other  $\theta_i$ 's can be approximated by the angles of the vectors drawn from the other poles, and from the zero, to the pole  $p_1$ . Thus  $\theta_1$  can be solved from Equation (11.57), where  $\theta_2 \approx 90^\circ, \theta_3 \approx \tan^{-1}(2), \theta_4 \approx 180^\circ - \tan^{-1}(\frac{2}{5})$ , and  $\theta_5 \approx 90^\circ + \tan^{-1}(\frac{1}{2})$ . That yields  $\theta_1 \approx -15^\circ$ .

The exact root locus for this example is shown in Figure 11.29. From the results of item 5 above, and the shape of the root locus, it is concluded that the feedback system is stable if

$$33.33 < K < 215.83$$

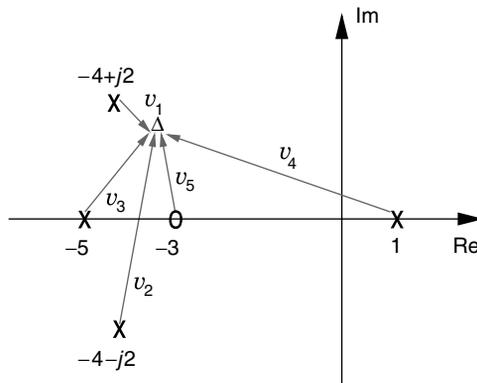


FIGURE 11.28 Angle of departure from  $-4 + j2$ .

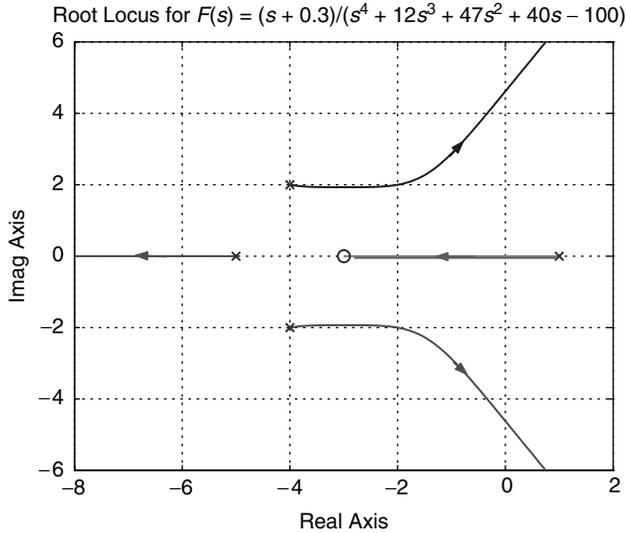


FIGURE 11.29 Root locus for  $F(s) = \frac{(s + 3)}{(s - 1)(s + 5)(s + 4 + j2)(s + 4 - j2)}$

i.e., by simply adjusting the gain of the controller, the system can be made stable. In some situations we need to use a dynamic controller to satisfy all the design requirements.

**Design Examples**

**Example 11.1**

Consider the standard feedback system with a plant

$$P(s) = \frac{1}{0.72(s + 1)(s + 2)}$$

and design a controller such that

- the feedback system is stable,
- $PO \leq 10\%$ ,  $t_s \leq 4$  s, and steady state error is zero when  $r(t)$  is unit step,
- steady state error is as small as possible when  $r(t)$  is unit ramp.

It is clear that the second design goal cannot be achieved by a simple proportional controller. To satisfy this condition, the controller must have a pole at  $s = 0$ , i.e., it must have integral action. If we try an integral control of the form  $C(s) = K_c/s$ , with  $K_c > 0$ , then the root locus has three branches, the interval  $[-1, 0]$  is on the root locus; three asymptotes have angles  $\{60^\circ, 180^\circ, -60^\circ\}$  with a center at  $\sigma_a = -1$ , and there is only one break point at  $-1 + \frac{1}{\sqrt{3}}$ , see Figure 11.30. From the location of the break point, center, and angles of the asymptotes, it can be deduced that two branches (one starting at  $p_1 = -1$ , and the other one starting at  $p_3 = 0$ ) always remain to the right of  $p_1$ . On the other hand, the settling time condition implies that the real parts of the dominant closed-loop system poles must be less than or equal to  $-1$ . So, a simple integral control does not do the job. Now try a PI controller of the form

$$C(s) = K_c \left( \frac{s - z_c}{s} \right), \quad K_c > 0$$

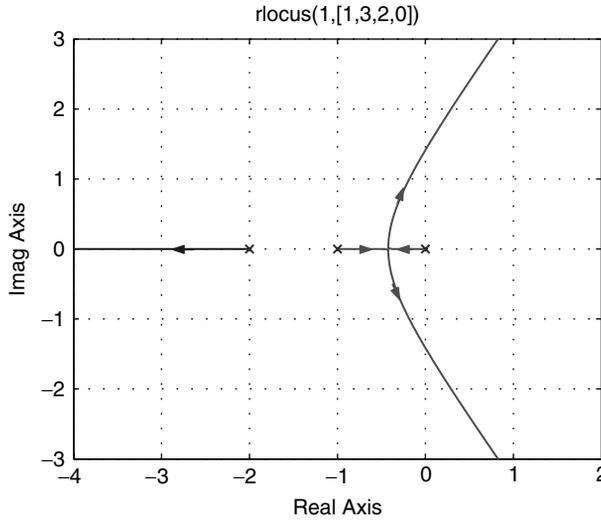


FIGURE 11.30 Root locus for Example 1.

In this case, we can select  $z_c = -1$  to cancel the pole at  $p_1 = -1$  and the system effectively becomes a second-order system. The root locus for  $F(s) = 1/s(s + 2)$  has two branches and two asymptotes, with center  $\sigma_a = -1$  and angles  $\{90^\circ, -90^\circ\}$ ; the break point is also at  $-1$ . The branches leave  $-2$  and  $0$ , and go toward each other, meet at  $-1$ , and tend to infinity along the line  $\text{Re}(s) = -1$ . Indeed, the closed-loop system poles are

$$r_{1,2} = -1 \pm \sqrt{1 - K}, \quad \text{where } K = K_c/0.72$$

The steady state error, when  $r(t)$  is unit ramp, is  $2/K$ . So  $K$  needs to be as large as possible to meet the third design condition. Clearly,  $\text{Re}(r_{1,2}) = -1$  for all  $K \geq 1$ , which satisfies the settling time requirement. The percent overshoot is less than 10% if  $\zeta$  of the roots  $r_{1,2}$  is greater than 0.6. A simple algebra shows that  $\zeta = 1/\sqrt{K}$ , hence the design conditions are met if  $K = 1/0.36$ , i.e.  $K_c = 2$ . Thus a PI controller that solves the design problem is

$$C(s) = 2 \left( \frac{s + 1}{s} \right)$$

The controller cancels a stable pole (at  $s = -1$ ) of the plant. If there is a slight uncertainty in this pole location, perfect cancellation will not occur and the system will be third-order with the third pole at  $r_3 \cong -1$ . Since the zero at  $z_o = -1$  will approximately cancel the effect of this pole, the response of this system will be close to the response of a second-order system. However, we must be careful if the pole-zero cancellations are near the imaginary axis because in this case small perturbations in the pole location might lead to large variations in the feedback system response, as illustrated with the next example.

**Example 11.2**

A flexible structure with lightly damped poles has transfer function in the form

$$P(s) = \frac{\omega_1^2}{s^2(s^2 + 2\zeta\omega_1s + \omega_1^2)}$$

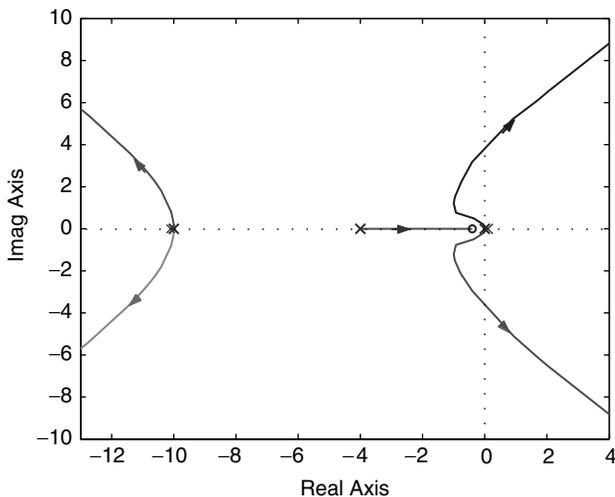


FIGURE 11.31 Root locus for Example 11.2(a).

By using the root locus, we can see that the controller

$$C(s) = K_c \frac{(s^2 + 2\zeta\omega_1 s + \omega_1^2)(s + 0.4)}{(s + r)^2(s + 4)}$$

stabilizes the feedback system for sufficiently large  $r$  and an appropriate choice of  $K_c$ . For example, let  $\omega_1 = 2$ ,  $\zeta = 0.1$ , and  $r = 10$ . Then the root locus of  $F(s) = P(s)C(s)/K$ , where  $K = K_c\omega_1^2$  is as shown in Figure 11.31. For  $K = 600$ , the closed-loop system poles are

$$\{-10.78 \pm j2.57, -0.94 \pm j1.61, -0.2 \pm j1.99, -0.56\}$$

Since the poles  $-0.2 \pm j1.99$  are canceled by a pair of zeros at the same point in the closed-loop system transfer function  $T = G(1 + G)^{-1}$ , the dominant poles are at  $-0.56$  and  $-0.94 \pm j1.61$  (they have relatively large negative real parts and the damping ratio is about 0.5).

Now, suppose that this controller is fixed and the complex poles of the plant are slightly modified by taking  $\zeta = 0.09$  and  $\omega_1 = 2.2$ . The root locus corresponding to this system is as shown in Figure 11.32. Since lightly damped complex poles are not perfectly canceled, there are two more branches near the imaginary axis. Moreover, for the same value of  $K = 600$ , the closed-loop system poles are

$$\{-10.78 \pm j2.57, -1.21 \pm j1.86, 0.05 \pm j1.93, -0.51\}$$

In this case, the feedback system is unstable.

### Example 11.3

One of the most important examples of mechatronic systems is the DC motor. An approximate transfer function of a DC motor [8, pp. 141–143] is in the form

$$P_m(s) = \frac{K_m}{s(s + 1/\tau_m)}, \quad \tau_m > 0$$

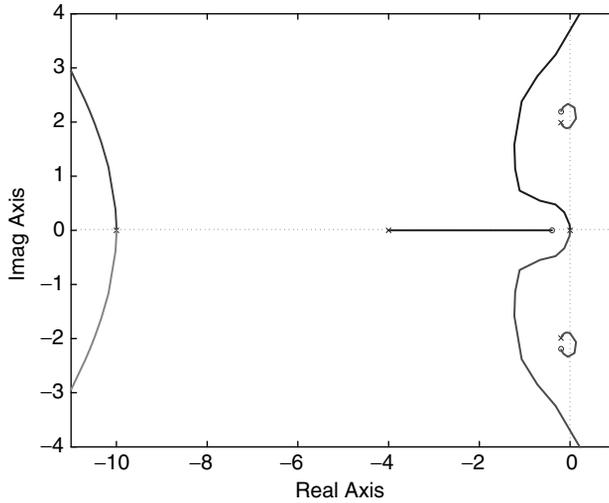


FIGURE 11.32 Root locus for Example 11.2(b).

Also note that if  $\tau_m$  is large, then  $P_m(s) \approx P_b(s)$ , where  $P_b(s) = K_b/s^2$ , is the transfer function of a rigid beam. In this example, the general class of plants  $P_m(s)$  will be considered. Assuming that  $p_m = -1/\tau_m$  and  $K_m$  are given, a first-order controller

$$C(s) = K_c \left( \frac{s - z_c}{s - p_c} \right) \tag{11.58}$$

will be designed. The aim is to place the closed-loop system poles far from the Im-axis. Since the order of  $F(s) = P_m(s)C(s)/K_mK_c$  is three, the root locus has three branches. Suppose the desired closed-loop poles are given as  $p_1, p_2$ , and  $p_3$ . Then, the pole placement problem amounts to finding  $\{K_c, z_c, p_c\}$  such that the characteristic equation is

$$\chi(s) = (s - p_1)(s - p_2)(s - p_3) = s^3 - (p_1 + p_2 + p_3)s^2 + (p_1p_2 + p_1p_3 + p_2p_3)s - p_1p_2p_3$$

But the actual characteristic equation, in terms of the unknown controller parameters, is

$$\chi(s) = s(s - p_m)(s - p_c) + k(s - z_c) = s^3 - (p_m + p_c)s^2 + (p_m p_c + K)s - Kz_c$$

where  $K := K_m K_c$ . Equating the coefficients of the desired  $\chi(s)$  to the coefficients of the actual  $\chi(s)$ , three equations in three unknowns are obtained:

$$\begin{aligned} p_m + p_c &= p_1 + p_2 + p_3 \\ p_m p_c + K &= p_1 p_2 + p_1 p_3 + p_2 p_3 \\ K z_c &= p_1 p_2 p_3 \end{aligned}$$

From the first equation  $p_c$  is determined, then  $K$  is obtained from the second equation, and finally  $z_c$  is computed from the third equation.

For different numerical values of  $p_m, p_1, p_2$ , and  $p_3$  the shape of the root locus is different. Below are some examples, with the corresponding root loci shown in Figure 11.33 to Figure 11.35.

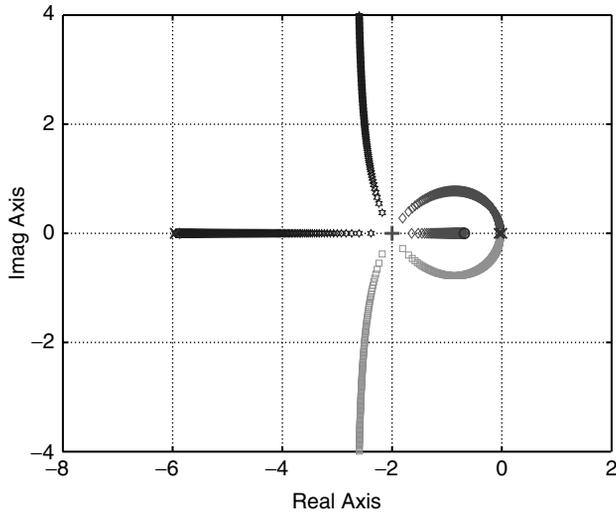


FIGURE 11.33 Root locus for Example 11.3(a).

(a)  $p_m = -0.05, p_1 = p_2 = p_3 = -2 \Rightarrow$

$$K = 11.70, \quad p_c = -5.95, \quad z_c = -0.68$$

(b)  $p_m = -0.5, p_1 = -1, p_2 = -2, p_3 = -3 \Rightarrow$

$$K = 8.25, \quad p_c = -5.50, \quad z_c = -0.73$$

(c)  $p_m = -5, p_1 = -11, p_2 = -4 + j_1, p_3 = -4 - j_1 \Rightarrow$

$$K = 35, \quad p_c = -14, \quad z_c = -5.343$$

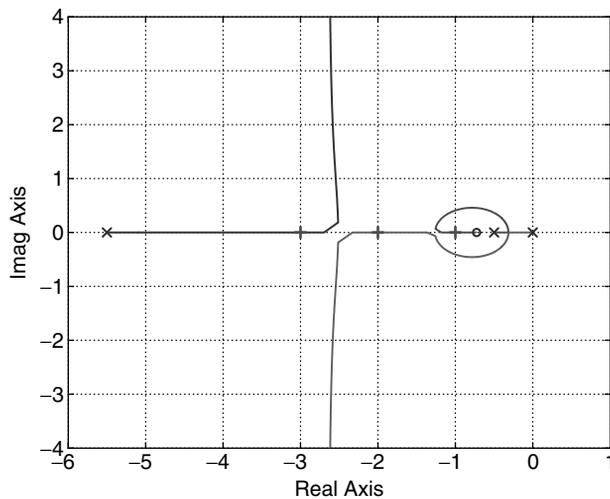


FIGURE 11.34 Root locus for Example 11.3(b).

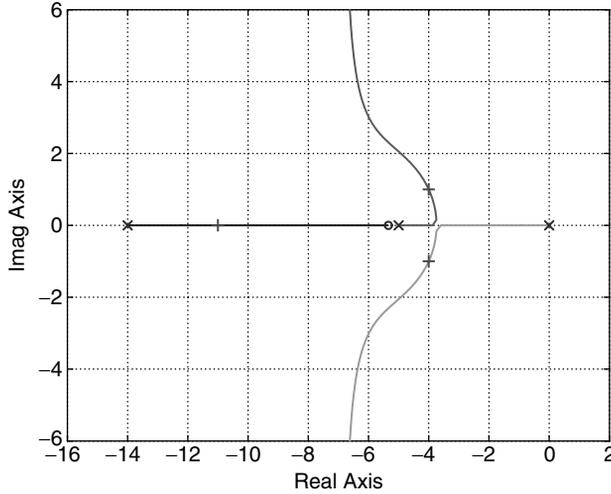


FIGURE 11.35 Root locus for Example 11.3(c).

**Example 11.4**

Consider the open-loop transfer function

$$P(s)C(s) = K_c \frac{(s^2 - 3s + 3)(s - z_c)}{s(s^2 + 3s + 3)(s - p_c)}$$

where  $K_c$  is the controller gain to be adjusted, and  $z_c$  and  $p_c$  are the controller zero and pole, respectively. Observe that the root locus has four branches except for the non-generic case  $z_c = p_c$ . Let the desired dominant closed-loop poles be  $r_{1,2} = -0.4$ . The steady state error for unit ramp reference input is

$$e_{ss} = \frac{p_c}{K_c z_c}$$

Accordingly, we want to make the ratio  $K_c z_c / p_c$  as large as possible.

The characteristic equation is

$$\chi(s) = s(s^2 + 3s + 3)(s - p_c) + K_c(s^2 - 3s + 3)(s - z_c)$$

and it is desired to be in the form

$$\chi(s) = (s + 0.4)^2(s - r_3)(s - r_4)$$

for some  $r_{3,4}$  with  $\text{Re}(r_{3,4}) < 0$ , which implies that

$$\chi(s)|_{s=-0.4} = 0, \quad \frac{d}{ds} \chi(s)|_{s=-0.4} = 0 \quad (11.59)$$

Conditions (Equation (11.59)) give two equations:

$$0.784(0.4 + p_c) - 4.36K_c(0.4 + z_c) = 0$$

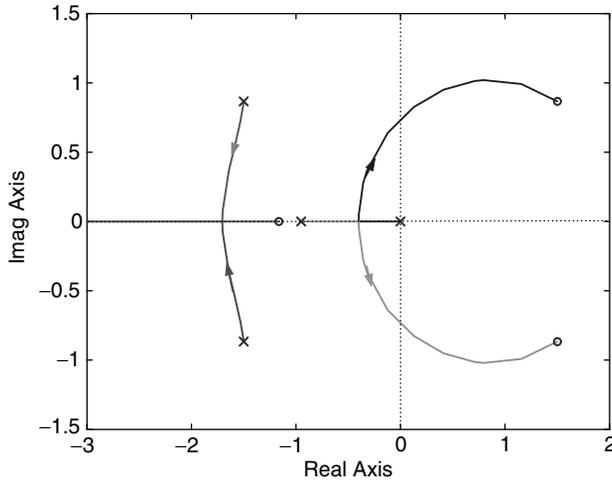


FIGURE 11.36 Root locus for Example 11.4.

$$4.36K_c - 0.784 - 1.08(0.4 + p_c) + 3.8K_c(0.4 + z_c) = 0$$

from which  $z_c$  and  $p_c$  can be solved in terms of  $K_c$ . Then, by simple substitutions, the ratio to be maximized,  $K_c z_c / p_c$ , can be reduced to

$$\frac{K_c z_c}{p_c} = \frac{3.4776K_c - 0.784}{24.2469K_c - 3.4776}$$

The maximizing value of  $K_c$  is 0.1297; it leads to  $p_c = -0.9508$  and  $z_c = -1.1637$ . For this controller, the feedback system poles are

$$\{-1.64 + j0.37, -1.64 - j0.37, -0.40, -0.40\}$$

The root locus is shown in Figure 11.36.

## 11.4 Complementary Root Locus

In the previous section, the root locus parameter  $K$  was assumed to be positive and the phase and magnitude rules were established based on this assumption. There are some situations in which controller gain can be negative as well. Therefore, the complete picture is obtained by drawing the usual root locus (for  $K > 0$ ) and the complementary root locus (for  $K < 0$ ). The complementary root locus rules are

$$\ell \times 360^\circ = \sum_{i=1}^n \angle(r - p_i) - \sum_{j=1}^n \angle(r - z_j), \quad \ell = 0, \pm 1, \pm 2, \dots \quad (11.60)$$

$$|K| = \frac{\prod_{i=1}^n |r - p_i|}{\prod_{j=1}^m |r - z_j|} \quad (11.61)$$

Since the phase rule (Equation (11.60)) is the  $180^\circ$  shifted version of (Equation (11.53)), the complementary root locus is obtained by simple modifications in the root locus construction rules. In particular, the number

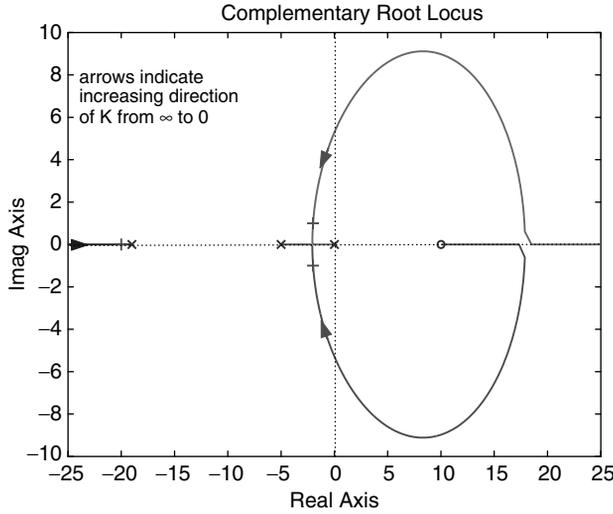


FIGURE 11.37 Complementary root locus for Example 11.3.

of asymptotes and their center are the same, but their angles  $\alpha_\ell$ 's are given by

$$\alpha_\ell = \frac{2\ell}{(n - m)} \times 180^\circ, \quad \ell = 0, \dots, (n - m - 1)$$

Also, an interval on the real axis is on the complementary root locus if and only if it is not on the usual root locus.

**Example 11.3 (revisited)**

In the Example 11.3 given above, if the problem data is modified to  $p_m = -5$ ,  $p_1 = -20$ , and  $p_{2,3} = -2 \pm j$ , then the controller parameters become

$$K = -10, \quad p_c = -19, \quad z_c = 10$$

Note that the gain is negative. The roots of the characteristic equation as  $K$  varies between 0 and  $-\infty$  form the complementary root locus; see Figure 11.37.

**Example 11.4 (revisited)**

In this example, if  $K$  increases from  $-\infty$  to  $+\infty$ , the closed-loop system poles move along the complementary root locus, and then the usual root locus, as illustrated in Figure 11.38.

**11.5 Root Locus for Systems with Time Delays**

The standard feedback control system considered in this section is shown in Figure 11.39, where the controller  $C$  and plant  $P$  are in the form

$$C(s) = \frac{N_c(s)}{D_c(s)}$$

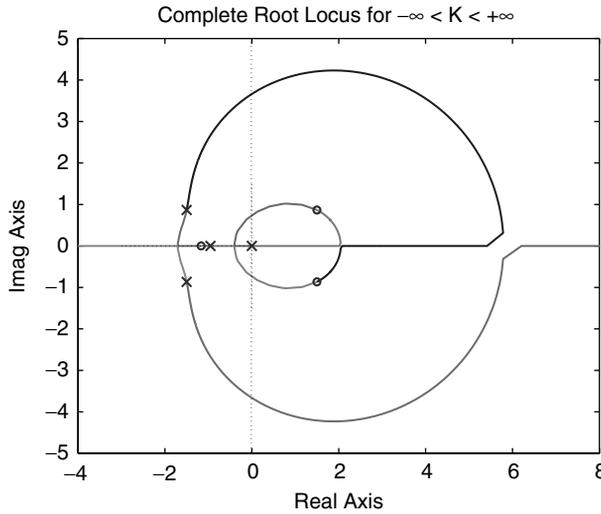


FIGURE 11.38 Complementary and usual root loci for Example 11.4.

and

$$P(s) = e^{-hs} P_0(s) \quad \text{where } P_0(s) = \frac{N_p(s)}{D_p(s)}$$

with  $(N_c, D_c)$  and  $(N_p, D_p)$  being coprime pairs of polynomials with real coefficients.<sup>1</sup> The term  $e^{-hs}$  is the transfer function of a pure delay element (in Figure 11.39 the plant input is delayed by  $h$  seconds). In general, time delays enter into the plant model when there is

- a sensor (or actuator) processing delay, and/or
- a software delay in the controller, and/or
- a transport delay in the process.

In this case the open-loop transfer function is

$$G(s) = G_h(s) = e^{-hs} G_0(s)$$

where  $G_0(s) = P_0(s)C(s)$  corresponds to the no delay case,  $h = 0$ .

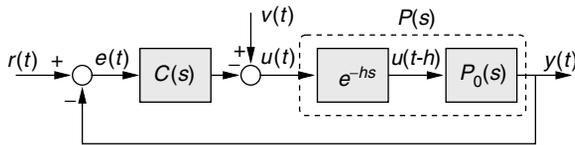


FIGURE 11.39 Feedback system a with time delay.

<sup>1</sup>A pair of polynomials is said to be coprime pair if they do not have common roots

Note that magnitude and phase of  $G(j\omega)$  are determined from the identities

$$|G(j\omega)| = |G_0(j\omega)| \quad (11.62)$$

$$\angle G(j\omega) = -h\omega + \angle G_0(j\omega) \quad (11.63)$$

### Stability of Delay Systems

Stability of the feedback system shown in Figure 11.39 is equivalent to having all the roots of

$$\chi(s) = D(s) + e^{-hs}N(s) \quad (11.64)$$

in the open left half plane,  $\mathbb{C}_-$ , where  $D(s) = D_c(s)D_p(s)$  and  $N(s) = N_c(s)N_p(s)$ . We assume that there is no unstable pole-zero cancellation in taking the product  $P_0(s)C(s)$ , and that  $\deg(D) > \deg(N)$  (here  $N$  and  $D$  need not be monic polynomials). Strictly speaking,  $\chi(s)$  is not a polynomial because it is a transcendental function of  $s$ . The functions of the form (Equation (11.64)) belong to a special class of functions called *quasi-polynomials*. The closed-loop system poles are the roots of (Equation (11.64)).

Following are known facts (see [1,10]):

- (i) If  $r_k$  is a root of Equation (20), then so is  $\bar{r}_k$ . (i.e., roots appear in complex conjugate pairs as usual)
- (ii) There are infinitely many poles  $r_k \in \mathbb{C}$ ,  $k = 1, 2, \dots$ , satisfying  $\chi(r_k) = 0$ .
- (iii) And  $r_k$ 's can be enumerated in such a way that  $\Re(r_k + 1) \leq \Re(r_k)$  moreover,  $\Re(r_k) \rightarrow -\infty$  as  $k \rightarrow \infty$ .

### Example 11.5

If  $G_h(s) = e^{-hs}/s$ , then the closed-loop system poles  $r_k$ , for  $k = 1, 2, \dots$ , are the roots of

$$1 + \frac{e^{-h\sigma_k} e^{-jh\omega_k}}{\sigma_k + j\omega_k} e^{\pm j2k\pi} = 0 \quad (11.65)$$

where  $r_k = \sigma_k + j\omega_k$  for some  $\sigma_k, \omega_k \in \mathbb{R}$ . Note that  $e^{\pm j2k\pi} = 1$  for all  $k = 1, 2, \dots$ . Equation (11.45) is equivalent to the following set of equations:

$$e^{-h\sigma_k} = |\sigma_k + j\omega_k| \quad (11.66)$$

$$\pm(2k-1)\pi = h\omega_k + \angle(\sigma_k + j\omega_k), \quad k = 1, 2, \dots \quad (11.67)$$

It is quite interesting that for  $h = 0$  there is only one root  $r = -1$ , but even for infinitesimally small  $h > 0$  there are infinitely many roots. From the magnitude condition (Equation (11.66)), it can be shown that

$$\sigma_k \geq 0 \Rightarrow |\omega_k| \leq 1 \quad (11.68)$$

Also, for  $\sigma_k \geq 0$  the phase  $\angle(\sigma_k + j\omega_k)$  is between  $-\pi/2$  and  $+\pi/2$ , therefore Equation (11.67) leads to

$$\sigma_k \geq 0 \Rightarrow h|\omega_k| \geq \frac{\pi}{2} \quad (11.69)$$

By combining Equation (11.68) and Equation (11.69), it can be proven that the feedback system has no roots in the closed right half plane when  $h < \pi/2$ . Furthermore, the system is unstable if  $h \geq \pi/2$ . In particular, for  $h = \pi/2$  there are two roots on the imaginary axis, at  $\pm j1$ . It is also easy to show that, for any  $h > 0$  as  $k \rightarrow \infty$ , the roots converge to

$$r_k \rightarrow \frac{1}{h} \left[ -\ln\left(\frac{2k\pi}{h}\right) \pm j2k\pi \right]$$

As  $h \rightarrow 0$ , the magnitude of the roots converge to  $\infty$ .

As illustrated by the above example, property (iii) implies that for any given real number  $\sigma$  there are only finitely many  $r_k$ 's in the region of the complex plane

$$\mathbb{C}_\sigma := \{s \in \mathbb{C} : \text{Re}(s) \geq \sigma\}$$

In particular, with  $\sigma = 0$ , this means that the quasi-polynomial  $\chi(s)$  can have only finitely many roots in the right half plane. Since the effect of the closed-loop system poles that have very large negative real parts is negligible (as far as closed-loop systems' input-output behavior is concerned), only finitely many "dominant" roots  $r_k$ , for  $k = 1, \dots, m$ , should be computed for all practical purposes.

### Dominant Roots of a Quasi-Polynomial

Now we discuss the following problem: given  $N(s)$ ,  $D(s)$ , and  $h \geq 0$ , find the dominant roots of the quasi-polynomial

$$\chi(s) = D(s) + e^{-hs}N(s)$$

For each fixed  $h > 0$ , it can be shown that there exists  $\sigma_{\max}$  such that  $\chi(s)$  has no roots in the region,  $\mathbb{C}_{\sigma_{\max}}$ , see [11] for a simple algorithm to estimate  $\sigma_{\max}$ , based on Nyquist criterion. Given  $h > 0$  and a region of the complex plane defined by  $\sigma_{\min} \leq \text{Re}(s) \leq \sigma_{\max}$ , the problem is to find the roots of  $\chi(s)$  in this region.

Clearly, a point  $r = \sigma + j\omega$  in  $\mathbb{C}$  is a root of  $\chi(s)$  if and only if

$$D(\sigma + j\omega) = -e^{-h\sigma} e^{-jh\omega} N(\sigma + j\omega)$$

Taking the magnitude square of both sides of the above equation,  $\chi(r) = 0$  implies

$$A_\sigma(x) := D(\sigma + x)D(\sigma - x) - e^{-2h\sigma} N(\sigma + x)N(\sigma - x) = 0$$

where  $x = j\omega$ . The term  $D(\sigma + x)$  stands for the function  $D(s)$  evaluated at  $\sigma + x$ . The other terms of  $A_\sigma(x)$  are calculated similarly. For each fixed  $\sigma$ , the function  $A_\sigma(x)$  is a polynomial in the variable  $x$ . By symmetry, if  $x$  is a zero of  $A_\sigma(\cdot)$  then  $(-x)$  is also a zero.

If  $A_\sigma(x)$  has a root  $x_\ell$  whose real part is zero, set  $r_\ell = \sigma + x_\ell$ . Next, evaluate the magnitude of  $\chi(r_\ell)$ ; if it is zero, then  $r_\ell$  is a root of  $\chi(s)$ . Conversely, if  $A_\sigma(x)$  has no root on the imaginary axis, then  $\chi(s)$  cannot have a root whose real part is the fixed value of  $\sigma$  from which  $A_\sigma(\cdot)$  is constructed.

### Algorithm

Given  $N(s)$ ,  $D(s)$ ,  $h$ ,  $\sigma_{\min}$ , and  $\sigma_{\max}$ :

*Step 1.* Pick  $\sigma$  values  $\sigma_1, \dots, \sigma_M$  between  $\sigma_{\min}$  and  $\sigma_{\max}$  such that  $\sigma_{\min} = \sigma_1$ ,  $\sigma_i < \sigma_{i+1}$ , and  $\sigma_M = \sigma_{\max}$ . For each  $\sigma_i$  perform the following.

*Step 2.* Construct the polynomial  $A_i(x)$  according to

$$A_i(x) := D(\sigma_i + x)D(\sigma_i - x) - e^{-2h\sigma_i} N(\sigma_i + x)N(\sigma_i - x)$$

*Step 3.* For each imaginary axis roots  $x_\ell$  of  $A_i$ , perform the following test:

Check if  $|\chi(\sigma_i + x_\ell)| = 0$ ; if yes, then  $r = \sigma_i + x_\ell$  is a root of  $\chi(s)$ ; if not, discard  $x_\ell$ .

*Step 4.* If  $i = M$ , stop; else increase  $i$  by 1 and go to Step 2.

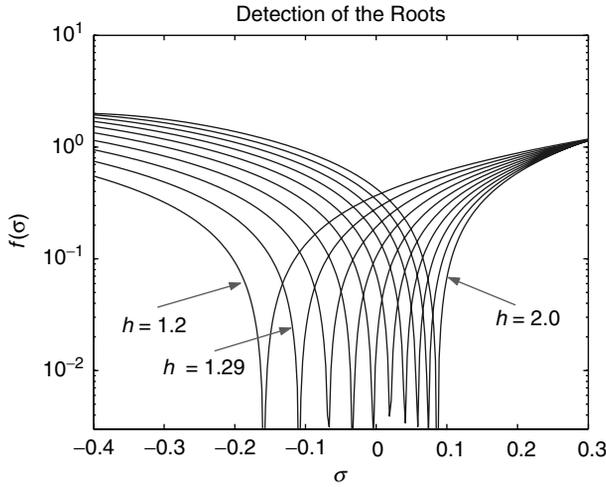


FIGURE 11.40 Detection of the dominant roots.

**Example 11.6**

We will find the dominant roots of

$$1 + \frac{e^{-hs}}{s} = 0 \tag{11.70}$$

for a set of critical values of  $h$ . Recall that Equation (11.70) has a pair of roots  $\pm j1$  when  $h = \pi/2 = 1.57$ . Moreover, dominant roots of Equation (11.70) are in the right half plane if  $h > 1.57$ , and they are in the left half plane if  $h < 1.57$ . So, it is expected that for  $h \in (1.2, 2.0)$  the dominant roots are near the imaginary axis. Take  $\sigma_{\min} = -0.5$  and  $\sigma_{\max} = 0.5$ , with  $M = 400$  linearly spaced  $\sigma_i$ 's between them. In this case

$$A_i(x) = \sigma_i^2 - e^{-2h\sigma_i} - x^2$$

Whenever  $e^{-2h\sigma_i} \geq \sigma_i^2$ ,  $A_i(x)$  has two roots:

$$x_\ell = \pm j\sqrt{e^{-2h\sigma_i} - \sigma_i^2}, \quad \ell = 1, 2$$

For each fixed  $\sigma_i$  satisfying this condition, let  $r_\ell = \sigma_i + x_\ell$  (note that  $x_\ell$  is a function of  $\sigma_i$ , so  $r_\ell$  is a function of  $\sigma_i$ ) and evaluate

$$f(\sigma_i) := \left| 1 + \frac{e^{-hr_\ell}}{r_\ell} \right|$$

If  $f(\sigma_i) = 0$ , then  $x_\ell$  is a root of 11.70. For 10 different values of  $h \in (1.2, 2.0)$ , the function  $f(\sigma)$  is plotted in Figure 11.40. This figure shows the feasible values of  $\sigma_i$  for which  $x_\ell$  (defined from  $\sigma_i$ ) is a root of Equation (11.70). The dominant roots of Equation (11.70), as  $h$  varies from 1.2 to 2.0, are shown in Figure 11.41. For  $h < 1.57$ , all the roots are in  $\mathbb{C}_-$ . For  $h > 1.57$ , the dominant roots are in  $\mathbb{C}_+$ , and for  $h = 1.57$ , they are at  $\pm j1$ .

**Root Locus Using Padé Approximations**

In this section we assume that  $h > 0$  is fixed and we try to obtain the root locus, with respect to uncertain/adjustable gain  $K$ , corresponding to the dominant poles. The problem can be solved by numerically calculating

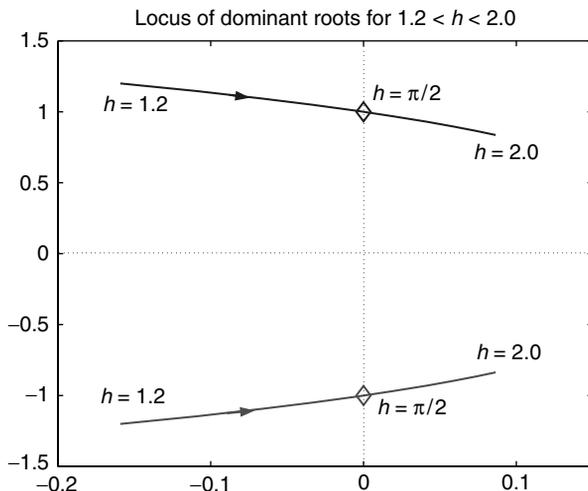


FIGURE 11.41 Dominant roots as  $h$  varies from 1.2 to 2.0.

the dominant roots of the quasi-polynomial

$$\chi(s) = D(s) + KN(s)e^{-hs} \quad (11.71)$$

for varying  $K$ , by using the methods presented in the previous section. In this section an alternative method is given that uses Padé approximation of the time delay term  $e^{-hs}$ . More precisely, the idea is to find polynomials  $N_h(s)$  and  $D_h(s)$  satisfying

$$e^{-hs} \approx \frac{N_h(s)}{D_h(s)} \quad (11.72)$$

so that the dominant roots

$$D(s)D_h(s) + KN(s)N_h(s) = 0 \quad (11.73)$$

closely match the dominant roots of  $\chi(s)$ , (11.71). How should we do the approximation (Equation (11.72)) for this match?

By using the stability robustness measures determined from the Nyquist stability criterion, we can show that for our purpose we may consider the following cost function in order to define a meaningful measure for the approximation error:

$$\Delta_h =: \sup_{\omega} \left| \frac{k_{\max} N(j\omega)}{D(j\omega)} \right| \left| e^{-jh\omega} - \frac{N_h(j\omega)}{D_h(j\omega)} \right|$$

where  $K_{\max}$  is the maximum value of interest for the uncertain/adjustable parameter  $K$ .

The  $\ell$  th order Padé approximation is defined as follows:

$$N_h(s) = \sum_{k=0}^{\ell} (-1)^k c_k h^k s^k$$

$$D_h(s) = \sum_{k=0}^{\ell} c_k h^k s^k$$

where coefficients  $c_k$ 's are computed from

$$c_k = \frac{(2\ell - k)! \ell!}{2\ell! k! (\ell - k)!}, \quad k = 0, 1, \dots, \ell$$

First- and second-order approximations are in the form

$$\frac{N_h(s)}{D_h(s)} = \begin{cases} \frac{1 - hs/2}{1 + hs/2}, & \ell = 1 \\ \frac{1 - hs/2 + (hs)^2/12}{1 + hs/2 + (hs)^2/12}, & \ell = 2 \end{cases}$$

Given the problem data  $\{h, K_{\max}, N(s), D(s)\}$ , how do we find the smallest degree,  $\ell$ , of the Padé approximation, so that  $\Delta_h \leq \delta$  (or  $\Delta_h/K_{\max} \leq \delta'$ ) for a specified error  $\delta$ , or a specified relative error  $\delta'$ ? The answer lies in the following result [7]: for a given degree of approximation  $\ell$  we have

$$\left| e^{-jh\omega} - \frac{N_h(j\omega)}{D_h(j\omega)} \right| \leq \begin{cases} 2 \left( \frac{eh\omega}{4\ell} \right)^{2\ell+1}, & \omega \leq \frac{4\ell}{eh} \\ 2, & \omega \geq \frac{4\ell}{eh} \end{cases}$$

In light of this result, we can solve the approximation order selection problem by using the following procedure:

1. Determine the frequency  $\omega_x$  such that

$$\left| \frac{K_{\max} N(j\omega)}{D(j\omega)} \right| \leq \frac{\delta}{2}, \quad \text{for all } \omega \geq \omega_x$$

and initialize  $\ell = 1$ .

2. For each  $\ell \geq 1$  define

$$\omega_\ell = \max \left\{ \omega_x, \frac{4\ell}{eh} \right\}.$$

and plot the function

$$\Phi_\ell(\omega) := \begin{cases} 2 \left| \frac{K_{\max} N(j\omega)}{D(j\omega)} \right| \left( \frac{eh\omega}{4\ell} \right)^{2\ell+1}, & \text{for } \omega \leq \frac{4\ell}{eh} \\ 2 \left| \frac{K_{\max} N(j\omega)}{D(j\omega)} \right|, & \text{for } \omega_\ell \geq \omega \geq \frac{4\ell}{eh} \end{cases}$$

3. Check If

$$\max_{\omega \in [0, \omega_x]} \Phi_\ell(\omega) \leq \delta \tag{11.74}$$

If yes, stop, this value of  $\ell$  satisfies the desired error bound:  $\Delta_h \leq \delta$ . Otherwise, increase  $\ell$  by 1, and go to Step 2. Note that the left-hand side of the inequality (Equation (11.74)) is an upper bound of  $\Delta_h$ .

Since we assumed  $\deg(D) > \deg(N)$ , the algorithm will pass Step 3 eventually for some finite  $\ell \geq 1$ . At each iteration, we have to draw the error function  $\Phi_\ell(\omega)$  and check whether its peak value is less than  $\delta$ . Typically, as  $\delta$  decreases,  $\omega_x$  increases, and that forces  $\ell$  to increase. On the other hand, for very large values of  $\ell$ , the relative magnitude  $c_0/c_\ell$  of the coefficients becomes very large, in which case numerical difficulties arise in analysis and simulations. Also, as time delay  $h$  increases,  $\ell$  should be increased to keep the level of the approximation error  $\delta$  fixed. This is a fundamental difficulty associated with time delay systems.

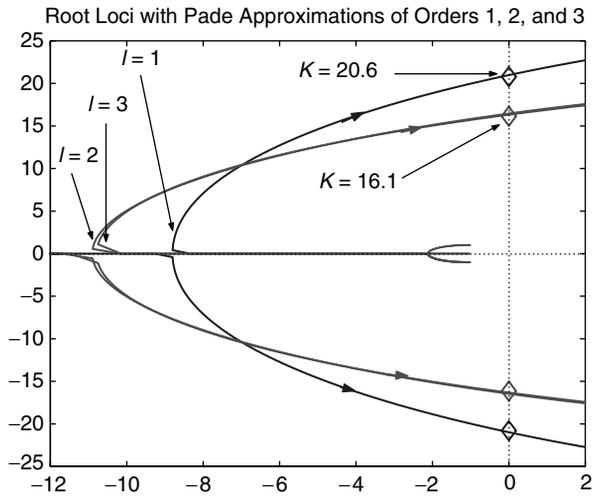


FIGURE 11.42 Dominant root for  $\ell = 1$

### Example 11.7

Let  $N(s) = s + 1$ ,  $D(s) = s^2 + 2s + 2$  and  $h = 0.1$ , and  $K_{\max} = 20$ . Then, for  $\delta' = 0.05$  applying the above procedure we calculate  $\ell = 2$  as the smallest approximation degree satisfying  $\Delta_{h_i}/K_{\max} < \delta'$ . Therefore, a second-order approximation of the time delay should be sufficient for predicting the dominant poles for  $K \in [0, 20]$ . Figure 11.42 shows the approximate root loci obtained from Padé approximations of degrees  $\ell = 1, 2, 3$ . There is a significant difference between the root loci for  $\ell = 1$  and  $\ell = 2$ . In the region  $\text{Re}(s) \geq -12$ , the predicted dominant roots are approximately the same for  $\ell = 2, 3$ , for  $K \in [0, 20]$ . So, we can safely say that using higher order approximations will not make any significant difference as far as predicting the behavior of the dominant poles for the given range of  $K$ .

### Notes and References

This section in the chapter is an edited version of related parts of the author's book [9]. More detailed discussions of the root locus method can be found in all the classical control books, such as [2, 5, 6, 8]. As mentioned earlier, extension of this method to discrete time systems is rather trivial: the method to find the roots of a polynomial as a function of a varying real parameter is independent of the variable  $s$  (in the continuous time case) or  $z$  (in the discrete time case). The only difference between these two cases is the definition of the desired region of the complex plane: for the continuous time systems, this is defined relative to the imaginary axis, whereas for the discrete time systems the region is defined with respect to the unit circle, as illustrated in Figure 11.27.

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## 11.5 Compensation

*Charles L. Phillips and Royce D. Harbor*

**Compensation** is the process of modifying a closed-loop control system (usually by adding a *compensator* or *controller*) in such a way that the compensated system satisfies a given set of design specifications. This section presents the fundamentals of compensator design; actual techniques are available in the references.

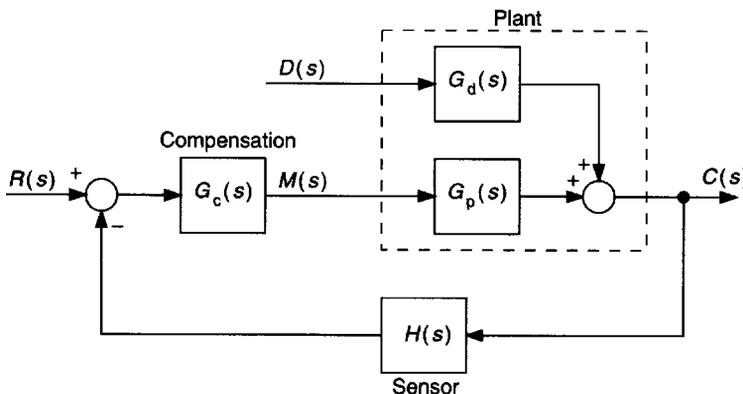
A single-loop control system is shown in Figure 11.43. This system has the transfer function from input  $R(s)$  to output  $C(s)$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} \quad (11.75)$$

and the characteristic equation is

$$1 + G_c(s)G_p(s)H(s) = 0 \quad (11.76)$$

where  $G_c(s)$  is the *compensator* transfer function,  $G_p(s)$  is the *plant* transfer function, and  $H(s)$  is the *sensor* transfer function. The transfer function from the disturbance input  $D(s)$  to the output is  $G_d(s)/[1 + G_c(s)G_p(s)H(s)]$ . The function  $G_c(s)G_p(s)H(s)$  is called the *open-loop function*.



**FIGURE 11.43** A closed-loop control system. (Source: C.L. Phillips and R.D. Harbor, *Feedback Control Systems*, 2nd ed., Englewood Cliffs, N.J.: Prentice-Hall, 1991, p. 161. With permission.)