A Survey of Results on Primes in Short Intervals

Cem Yalçın Yıldırım

§1. Introduction

Prime numbers have been a source of fascination for mathematicians since antiquity. The proof that there are infinitely many prime numbers is attributed to Euclid (fourth century B.C.). The basic method of determining all primes less than a given number N is the sieve of Eratosthenes (third century B.C.). Diophantus (third (?) century A.D.) was occupied with finding rational number solutions to equations, extending ancient knowledge from Babylon and India on Pythagorean triples. The books of Diophantus lay lost for ages. It took thousands of years before new aspects of primes were brought into light until chiefly Fermat and Mersenne (c.1640), influenced by Bachet's (1621) translation into Latin of the extant books of Diophantus, announced various criteria on divisibility by primes, assertions on primes possessing special forms, and solutions to Diophantine equations.

A major breakthrough was Euler's discovery (1737) of the identity

$$\prod_p (1 - \frac{1}{p^s})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

Here $s = \sigma + it \in \mathbb{C}$, $\sigma, t \in \mathbb{R}$, and (1.1) is meaningful for $\sigma > 1$ where both sides are absolutely convergent. This identity of a sum over the natural numbers and a product over the primes is an analytic way of expressing the property of unique factorization of natural numbers into primes. Euler considered (1.1) and similar identities with natural number values of s. It was Riemann (1859) who initiated the study of the quantity in (1.1) as an analytic function of a complex variable. The two expressions in (1.1) represent the *Riemann zeta-function* $\zeta(s)$ in the half-plane $\sigma > 1$. Riemann's aim was to prove the conjecture of Legendre and Gauss on the number $\pi(x)$ of primes $p \leq x$, that

$$\pi(x) \sim \frac{x}{\log x}, \quad (x \to \infty). \tag{1.2}$$

This goal was attained in 1896 independently by Hadamard and de la Vallée Poussin who were by then equipped with some essential knowledge on entire functions.

Riemann showed that $\zeta(s)$ can be continued analytically over the whole complex plane, being meromorphic with a simple pole at $s = 1$, and satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s). \tag{1.3}$$

The value of $\zeta(s)$ can be calculated at any s with $\sigma > 1$ to any desired accuracy from the expressions in (1.1). Then, using (1.3), $\zeta(s)$ can also be calculated for any s with $\sigma < 0$. In the rather mysterious strip $0 \leq \sigma \leq 1$, one may use

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (x)x^{-s-1}\,dx \quad (\sigma > 0), \tag{1.4}$$

where (x) is the fractional part of x. This is the analytic continuation of $\zeta(s)$ to $\sigma > 0$, obtained by applying partial summation to the series in (1.1).

At $s = -2, -4, -6 \ldots$, where $\Gamma(\frac{s}{2})$ has poles, $\zeta(s)$ vanishes - these are called the trivial zeros. Upon developing general results on entire functions, Hadamard (1893) deduced that $\zeta(s)$ has infinitely many *nontrivial zeros* in $0 \leq \sigma \leq 1$. The nontrivial zeros must be situated symmetrically with respect to the real axis, and by (1.3) also with respect to the line $\sigma = \frac{1}{2}$. Applying the argument principle, von Mangoldt (1895) gave the proof of Riemann's assertion that the number of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ is asymptotically $\frac{T}{2\pi}\log T$, as $T \to \infty$. It follows that if the zeros ρ are arranged in a sequence $\rho_n = \beta_n + i\gamma_n$ with $\gamma_{n+1} \geq \gamma_n$, then

$$\gamma_n \sim \frac{2\pi n}{\log n} \quad (n \to \infty). \tag{1.5}$$

Riemann's assertion that *all* of the nontrivial zeros lie on the critical line $\sigma = \frac{1}{2}$ is yet unproved. Known as the *Riemann Hypothesis* (RH), this has been one of the most profound problems of twentieth century mathematics. The Riemann Hypothesis settles the horizontal positioning of the zeros of $\zeta(s)$. In 1972 Montgomery came up with the *pair correlation conjecture* (MC), as to how the nontrivial zeros, assumed to be on the line $\sigma = \frac{1}{2}$, are distributed on this line.

In what follows we narrate the relation between $\zeta(s)$ and counting the number of primes (§2), some unproved strong assertions on the distribution of primes (§3), primes in arithmetic progressions (§4), the pair correlation conjecture (§5), some details of the connections between the distribution of primes and the zeta zeros (§6), and we give the proof of a theorem of Goldston and Yıldırım on primes in arithmetic progressions in short intervals (§7). Finally there are 'Further Notes' for each section.

§2. The explicit formula

The distribution of primes is closely linked with (the distribution of the nontrivial zeros of) the Riemann zeta-function. Such connections are already hinted at by (1.1). Taking the logarithmic derivative of the product in (1.1) gives

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \quad (\sigma > 1), \tag{2.1}$$

where $\Lambda(n)$ is von Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n \text{ is the power of a prime } p, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Defining

$$\psi(x) \;=\; \sum_{n\le x} \Lambda(n) \;=\; \sum_{p^m \le x} \log p \;, \tag{2.3}$$

and $\psi_0(x) = \psi(x) - \dfrac{\Lambda(x)}{2}$, one has

$$\psi_0(x) \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} [-\frac{\zeta'}{\zeta}(s)]\frac{x^s}{s}\,ds \quad (c>1). \tag{2.4}$$

Considering the integral from $c-iT$ to $c+iT$, and moving the line of integration all the way to the left in the complex plane one obtains, by the residue theorem, for any $x \geq 2$,

$$\begin{aligned}\psi_0(x) \;&=\; x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) \\ &\quad + O(\frac{x\log^2(xT)}{T}) + O(\log x \min(1, \frac{x}{T\langle x\rangle}))\end{aligned} \tag{2.5}$$

(here $\langle x\rangle$ denotes the distance from x to the nearest prime power - other than x itself if x is a prime power). Equation (2.5) is called the *explicit formula*; it provides an explicit link between a (weighted) count of the primes and a sum over the nontrivial zeros of $\zeta(s)$. (This form of (2.5) is more useful in applications than the form obtained by taking the limit $T \to \infty$ in (2.5)). The estimate for the sum over ρ in (2.5) depends upon our knowledge about the location of these zeros (this will be dwelt upon in §6). From de la Vallée Poussin's result (1899) that $\zeta(s) \neq 0$ for $\sigma > 1 - \dfrac{c_1}{\log t}$, (which could be derived from a relation between $\zeta(\sigma+it)$ and $\zeta(\sigma+2it)$ resting on the inequality $3 + 4\cos\theta + \cos 2\theta \geq 0$) it follows that

$$\psi(x) = x + O(x\exp[-c_2(\log x)^{\frac{1}{2}}]) \tag{2.6}$$

(here c_i are appropriate positive constants). This embodies the *prime number theorem* in the form (1.2). If one assumes RH, then all the ρ's have real part $\frac{1}{2}$, implying

$$\psi(x) \;=\; x + O(x^{\frac{1}{2}}\log^2 x), \tag{2.7}$$

with a much smaller error term than (2.6). The sharpest possible estimate for $\psi(x)$ was conjectured by Montgomery [38] by probabilistic arguments (upon assuming RH and that the imaginary parts $\gamma > 0$ of the nontrivial zeros are linearly independent) to be

$$\overline{\lim}\frac{\psi(x)-x}{\sqrt{x}(\log\log\log x)^2} = \pm\frac{1}{2\pi} \quad (x\to\infty). \tag{2.8}$$

In the opposite direction we note that Littlewood (1914) proved

$$\psi(x) - x = \Omega_{\pm}(x^{\frac{1}{2}}\log\log\log x) \quad (x\to\infty) \tag{2.9}$$

(for the proof see Ingham's tract [31]).

After the prime number theorem it is natural to ask for which functions $\Phi(x)$, as $x\to\infty$,

$$\pi(x+\Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x}\,? \tag{2.10}$$

Here one would try to find $\Phi(x)$ as slowly increasing as possible. Heath-Brown [28] proved that one can take $\Phi(x) = x^{\frac{7}{12}-\epsilon(x)}$ ($\epsilon(x)\to 0$, as $x\to\infty$), and assuming RH $\Phi(x) = x^{\frac{1}{2}+\epsilon}$ is allowed. Of course $\Phi(x)$ cannot be too small, and we know due to Rankin [46] that there exist intervals around x of length $> c\dfrac{\log x\log_2 x\log_4 x}{(\log_3 x)^2}$ ($\log_k$ is the k-fold iterated logarithm) which don't contain a prime. Moreover Maier [35] showed that (2.10) is false even for $\Phi(x)$ as large as $(\log x)^\lambda$ with any $\lambda > 1$, contrary to what was expected from the heuristic probabilistic arguments of Cramér [7]. On the other hand Selberg [51] showed assuming RH that, (2.10) holds for almost all x if $\dfrac{\Phi(x)}{(\log x)^2}\to\infty$ as $x\to\infty$. Here what is meant by 'almost all x' is that, while $X\to\infty$ the measure of the set of $x\in[0,X]$ for which (2.10) doesn't hold is $o(X)$. Without assuming RH, this almost-all result is known to hold with $\Phi(x) = x^{\frac{1}{6}+\epsilon}$ (Huxley [30]).

§3. Some unproved conjectures on the distribution of primes

In this section we briefly relate some of the deepest conjectures on the distribution of primes. One of the oldest of all is the *Goldbach conjecture* (1742), that every even number > 2 is the sum of two prime numbers. The furthest that has been proved in this direction is the remarkable theorem of Chen [3], that every sufficiently large even number can be expressed as the sum of a prime and a number which has at most two prime factors - counted with multiplicity. It should also be noted that Vinogradov, using his method of estimating exponential sums, proved that every sufficiently large odd number can be expressed as a sum of three primes. The methods developed for attacking the Goldbach conjecture can also be used for other problems of an additive nature. But still we do not know whether or not there are infinitely many twin primes (e.g. p and $p+2$ both prime). The more general situation was asserted as the *prime r-tuple conjecture* by Hardy and Littlewood [25]. The r-tuple conjecture is an asymptotic formula for the number $\pi_{\mathbf{d}}(N)$ of positive integers $n \leq N$ for which $n+d_1, \ldots, n+d_r$ are all prime (here $d_1, \ldots, d_r$ are distinct integers and $\mathbf{d} = (d_1, \ldots, d_r)$). The formula is

$$\pi_{\mathbf{d}}(N) \sim \wp_{\mathbf{d}} \frac{N}{\log^r N} \quad (N \to \infty) \tag{3.1}$$

when $\wp_{\mathbf{d}} \neq 0$, where

$$\wp_{\mathbf{d}} = \prod_p \frac{p^{r-1}(p - \nu_{\mathbf{d}}(p))}{(p-1)^r},$$

and $\nu_{\mathbf{d}}(p)$ is the number of distinct residue classes modulo p occupied by $d_1, \ldots, d_r$. With $r = 1$, this reduces to the prime number theorem. For $r \geq 2$ the conjecture remains unproved for any $\mathbf{d}$. Assuming that for each r, (3.1) holds uniformly for $1 \leq d_1, \ldots, d_r \leq h$, Gallagher [14] showed that if $P_k(h, N)$ is the number of integers $n \leq N$ for which the interval $(n, n+h]$ contains exactly k primes, then $P_k(\lambda \log N, N) \sim N \frac{e^{-\lambda}\lambda^k}{k!}$ as $N \to \infty$, i.e. the distribution tends to the Poisson distribution with parameter λ.

A heuristic way, depending on the prime number theorem and the counting of appropriate residue classes to certain moduli, of deriving the r-tuple conjecture (in the special case $r = 2$, $d_1 = 0$, $d_2 = 2$) can be found in the book of Hardy and Wright [26, §22.20]). Hardy and Littlewood developed the *circle method* for attacking such additive arithmetical problems, which when written in the form of summations can be re-expressed as integrals over the circle $|z| = \varrho < 1$ with a power series of radius of convergence 1 in the integrand. The main contribution comes from those z's with arguments close to fractions with small denominators while $\varrho \to 1$. The arithmetical information is then extracted from the singularities of the power series on the unit circle.

For an upper bound on the difference between consecutive primes Cramér [7] conjectured on probabilistic grounds that

$$\limsup_n \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1, \tag{3.2}$$

where p_n is the n-th prime. The known estimates for this limit, even under the unproved assumptions RH and MC, fall dismally short of Cramér's guess:

$$\begin{array}{lll} p_{n+1} - p_n \ll & p_n^{0.535} & \text{(unconditional)} \quad [1] \\ p_{n+1} - p_n \ll & p_n^{\frac{1}{2}} \log p_n & \text{(on RH)} \quad [6] \\ p_{n+1} - p_n = & o((p_n \log p_n)^{\frac{1}{2}}) & \text{(on RH + MC)} \quad [19] \end{array} \tag{3.3}$$

(in (6.1) and (6.15) below, further conditional estimates for the difference between consecutive primes are given).

§4. Primes in arithmetic progressions

Dirichlet (1837) proved that if a and q are two coprime natural numbers, then there are infinitely many primes of the form $kq + a$. Davenport begins his book [8] with the remark that this work of Dirichlet may be regarded as the origin of analytic number theory.

The proof involved the so-called *Dirichlet's L-functions*, defined by

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \tag{4.1}$$

in $\sigma > 1$ where the series is absolutely convergent. Here χ is a *Dirichlet's character to the modulus q*, a function of an integer variable n which is multiplicative and periodic with period q. It follows that if $(n,q)=1$, then $\chi(n)$ is a root of unity. For $(n,q)>1$, it is convenient to define $\chi(n)=0$. The character χ_0 which assumes the value 1 at all n coprime to q is called the *principal character*. It could be that for values of n coprime to q, the least period of $\chi(n)$ is a proper divisor of q, in which case χ is called an *imprimitive* character, and otherwise *primitive*. There are $\phi(q)$ characters in all to the modulus q, which form an abelian group (defining $\chi_1\chi_2(n)=\chi_1(n)\chi_2(n)$) isomorphic to the group of relatively prime residue classes to the modulus q. The characters satisfy

$$\sum_{n\,(\mathrm{mod}\,q)}\chi(n)=\begin{cases}\phi(q), & \text{if } \chi=\chi_0,\\ 0, & \text{otherwise,}\end{cases}$$

or equivalently

$$\sum_{\chi\,(\mathrm{mod}\,q)}\chi(n)=\begin{cases}\phi(q), & \text{if } n\equiv 1\,(\mathrm{mod}\,q),\\ 0, & \text{otherwise.}\end{cases}$$

Thus by using Dirichlet's characters we can select from integers in a given set those that are in a particular residue class modulo q as in (4.3) below. It also follows that for nonprincipal χ the series in (4.1) is conditionally convergent in the strip $0<\sigma\le 1$. Dirichlet's proof hinges on the fact that $L(1,\chi)\neq 0$ for nonprincipal χ. The theory of Dirichlet's L-functions parallels that of $\zeta(s)$ for the most part, and the *Generalized Riemann Hypothesis* (GRH) states that all zeros of Dirichlet's L-functions lie on the line $\sigma=\frac{1}{2}$.

The main question is for which ranges of the relevant variables are the primes evenly distributed with respect to the permissible congruence classes modulo q. To what extent and in which sense this distribution is even has been an active area of research. Analogous to (2.3), for $(a,q)=1$ let

$$\psi(x;q,a)=\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\Lambda(n). \tag{4.2}$$

Writing $\psi(x,\chi)=\sum_{n\le x}\chi(n)\Lambda(n)$, we have

$$\psi(x;q,a)=\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\overline{\chi}(a)\psi(x,\chi). \tag{4.3}$$

Proceeding as in the proof of the prime number theorem one aims for a result of the type

$$\psi(x;q,a)=\frac{x}{\phi(q)}(1+o(1)). \tag{4.4}$$

Unconditionally the *Siegel-Walfisz theorem* says that (4.4) holds uniformly for $q<(\log x)^N$ with any fixed $N>0$, while assuming GRH yields

$$\psi(x;q,a)=\frac{x}{\phi(q)}+O(x^{\frac{1}{2}}\log^2 x)\quad (q\le x). \tag{4.5}$$

Just as for (2.6) and (2.7), these results depend on the knowledge of the zero-free region to the left of $\sigma=1$ for Dirichlet's L-functions. The Siegel-Walfisz restriction on the range of q is quite severe. On the other hand (4.5) implies (4.4) for q almost up to $x^{\frac{1}{2}}$. It is natural to wonder whether the error term in (4.5) need really be so large. With regard to this we make the following observation. Littlewood's result (2.9) was preceeded by the weaker $\psi(x)-x=\Omega_\pm(x^{\frac{1}{2}})$ due to E. Schmidt (1903). A proof of this is in Ingham's tract [31, Thm. 33], with the constants implied in $\Omega_\pm$ being $\pm\frac{1}{|\frac{1}{2}+i\gamma_1|}$ ($\frac{1}{2}+i\gamma_1$ is the zero of $\zeta(s)$ with the least positive γ_1; $\gamma_1\approx 14.13$ [52, §15.2]). If we adapt this proof for $\psi(x,\chi)$, using (4.3) we have

$$\psi(x;q,a)-\frac{x}{\phi(q)}=\Omega_\pm\Big(\frac{x^{\frac{1}{2}}}{\phi(q)}\Big), \tag{4.6}$$

assuming GRH and some extra hypotheses. In these Ω-results assuming RH or GRH is not a burden, because zeros off the critical line, if they ever exist, would cause greater oscillations of the error term in the prime number theorem. The extra hypotheses are $L(\frac{1}{2}, \chi) \neq 0$ for all $\chi \pmod q$, and the critical zero of $\prod_{\chi \pmod q} L(s, \chi)$ with the least positive ordinate is a zero of just one of the functions $L(s, \chi)$. If this zero $\frac{1}{2} + i\gamma_{1,q}$ has multiplicity m_1 then (4.6) holds with the constants $\pm \frac{m_1}{|\frac{1}{2}+i\gamma_{1,q}|}$ (the latter condition may be somewhat relaxed or modified, and the constants would accordingly be modified). The best that can be hoped for was conjectured by Friedlander and Granville [12] as

$$\psi(x; q, a) \ll_\epsilon \frac{x}{\phi(q)} \quad (q < \frac{x}{(\log x)^{2+\epsilon}}), \tag{4.7}$$

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O((\frac{x}{q})^{\frac{1}{2}} x^\epsilon) \quad (q \leq x). \tag{4.8}$$

When various averages over q and a are taken results that hold in greater ranges of the parameters can be obtained. The *Bombieri-Vinogradov theorem* says that given any constant $A > 0$, we have

$$\sum_{q \leq Q} \max_{y \leq x} \max_{\substack{a \\ (a,q)=1}} |\psi(y; q, a) - \frac{y}{\phi(q)}| \ll \frac{x}{(\log x)^A} \tag{4.9}$$

with $Q = \dfrac{x^{\frac{1}{2}}}{(\log x)^B}$ where $B = B(A)$, thereby saving an arbitrary power of $\log x$ from the trivial estimate (see e.g. [36, Chapter 15]). The meaning of the Bombieri-Vinogradov theorem is that the asymptotic formula for $\psi(x; q, a)$ usually holds for q roughly as large as $x^{\frac{1}{2}}$, the same extent that can be handled by GRH for an individual $\psi(y; q, a)$, compared with q restricted to powers of $\log x$ in the Siegel-Walfisz theorem. The Elliott-Halberstam conjecture is that (4.7) should hold beyond $x^{\frac{1}{2}}$ up to $Q = x^{1-\epsilon}$. It has been proved by Friedlander et al. [13] that it cannot hold up to $\dfrac{x}{(\log x)^C}$. The important ingredients in the proof of the Bombieri-Vinogradov theorem include a so-called *large-sieve* inequality

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_\chi{}^* |\sum_{M+1}^{M+N} a_n \chi(n)|^2 \ll (N + Q^2) \sum_{M+1}^{M+N} |a_n|^2 \tag{4.10}$$

(here $\sum_{\chi}^{*}$ denotes a sum over all primitive characters $\chi(\bmod q)$), and the *Pólya-Vinogradov inequality* for a nonprincipal character to the modulus q,

$$\sum_{M+1}^{M+N} \chi(n) \ll q^{\frac{1}{2}} \log q. \tag{4.11}$$

The *Barban-Davenport-Halberstam theorem*, the proof of which also depends upon the large-sieve inequality (4.10), reads in its asymptotic form (proved by Montgomery)

$$\sum_{q\le Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} |\psi(x;q,a) - \frac{x}{\phi(q)}|^2 \sim Qx\log x \quad (\frac{x}{(\log x)^A} \le Q \le x), \tag{4.12}$$

where $A > 0$ is any fixed number ([36, Thm. 17.2]). Upon GRH this holds for $x^{\frac{1}{2}}\log^2 x \le Q \le x$ ([11], [21]). Here the range of q is much longer but a mean-square over the residue classes is considered instead of the maximum in Bombieri-Vinogradov theorem.

§5. Pair correlation and simple zeros

In 1972 Montgomery [37], manipulating the explicit formula, was led to define the function $F(\alpha, T)$ as

$$F(\alpha,T) = (\frac{T}{2\pi}\log T)^{-1} \sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma') \tag{5.1}$$

($\frac{1}{2}+i\gamma$ and $\frac{1}{2}+i\gamma'$ run through the zeros of $\zeta(s)$), where $w(u)$ is a suitable weight function; in [37] it was $w(u) = \frac{4}{4+u^2}$ but other weight functions can also be used (cf. (5.8) below and Hejhal's $w(u) = e^{-au^2}$ in [29]). By using the large sieve result (a quantitative form of Parseval's identity for Dirichlet series)

$$\int_0^T |\sum_n a_n n^{-it}|^2\, dt = \sum_n |a_n|^2 (T + O(n)), \tag{5.2}$$

Montgomery showed that upon RH

$$F(\alpha,T) = (1+o(1))T^{-2\alpha}\log T + \alpha + o(1) \quad (0 \le \alpha \le 1) \tag{5.3}$$

as $T \to \infty$. For larger α in bounded intervals Montgomery, drawing upon the prime r-tuple conjecture with $r = 2$, conjectured that

$$F(\alpha, T) = 1 + o(1) \quad (1 \le \alpha \le A). \tag{5.4}$$

(In (5.3) and (5.4) the estimates are uniform in the respective domains of α).

Convolving $F(\alpha, T)$ in (5.1) with a kernel $\hat{r}(\alpha)$ gives

$$\sum_{0<\gamma,\gamma'\le T} r((\gamma-\gamma')\frac{\log T}{2\pi})w(\gamma-\gamma') = (\frac{T}{2\pi}\log T)\int_{-\infty}^{\infty} F(\alpha,T)\hat{r}(\alpha)\,d\alpha\,, \tag{5.5}$$

where r and $\hat{r}$ are Fourier transforms of each other,

$$\hat{r}(\alpha) = \int_{-\infty}^{\infty} r(u)e^{-2\pi i\alpha u}\,du\,.$$

Since on RH, $F(\alpha, T)$ can be calculated for $|\alpha| \le 1$ as in (5.3), one can use (5.5) with $\hat{r}(\alpha)$ supported in $[-1, 1]$ to see the implications of RH. By taking $r(u) = (\frac{\sin \pi\alpha u}{\pi\alpha u})^2$ Montgomery derived that at least $\frac{2}{3}$ of the zeros of $\zeta(s)$ are simple. The pair correlation conjecture (5.4) implies that almost all zeros are simple. In this connection we mention that *Mertens hypothesis* in its weaker form

$$\int_1^X (\frac{M(x)}{x})^2\,dx = O(\log X), \tag{5.6}$$

where

$$M(x) = \sum_{n\le x}\mu(n) \tag{5.7}$$

($\mu(n)$ is the Möbius function) implies that all zeros of $\zeta(s)$ which are on the critical line are simple (see [52, §14.29]). Recall that the Riemann Hypothesis is equivalent to $|M(x)| = O(x^{\frac{1}{2}+\epsilon})$ ([52, §14.25]). The Mertens conjecture, in the form $|M(x)| < x^{\frac{1}{2}}$, was disproved by Odlyzko and te Riele [42].

Goldston [17] showed assuming RH, that the following asymptotic estimates as $T \to \infty$ are equivalent:

(i) $\int_a^{a+\delta} F(\alpha,T)\,d\alpha \sim \delta \quad (\text{fixed } \alpha \geq 1, \delta > 0)$

(ii) $\sum_{\substack{0<\gamma,\gamma' \leq T \\ 0<\gamma-\gamma' \leq \frac{2\pi\beta}{\log T}}} 1 \sim (\frac{T}{2\pi}\log T)\int_0^{\beta} 1-(\frac{\sin \pi u}{\pi u})^2\,du \quad (\text{fixed } \beta > 0)$

(iii) $\int_1^{T^\kappa} (\psi(u+\frac{u}{T})-\psi(u)-\frac{u}{T})^2 u^{-2}\,du \sim (\kappa-\frac{1}{2})\frac{\log^2 T}{T}$,
for fixed $\kappa \geq 1$ (if $0 < \kappa \leq 1$, then this integral is $\sim \frac{\kappa^2}{2}\frac{\log^2 T}{T}$ assuming RH).

Some of these results have been extended to Dirichlet's L-functions and primes in arithmetic progressions by Özlük [43] and Yıldırım [55]. By estimating a q-analogue of $F(\alpha,T)$,

$$F_Q(\alpha) = \sum_{q\leq Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma,\gamma'} Q^{i\alpha(\gamma-\gamma')}(\frac{\sin\gamma}{\gamma})^2(\frac{\sin\gamma'}{\gamma'})^2, \tag{5.8}$$

(in the innermost summation, assuming GRH, $\frac{1}{2}+i\gamma$ and $\frac{1}{2}+i\gamma'$ run through the zeros of $L(s,\chi)$) in $0 \leq \alpha \leq 2-\epsilon$, Özlük showed that at least $\frac{11}{12}$ of the zeros of all Dirichlet's L-functions are simple. It was because an ensemble of L-functions were considered together that the barrier $\alpha \leq 1$ could be overcome.

It was shown by Goldston and Montgomery [20] that upon RH, assuming

$$F(T,x) \sim \frac{1}{2\pi} T \log T$$

for $x^{B_1}(\log x)^{-3} \leq T \leq x^{B_2}(\log x)^3$, $(0 < B_1 \leq B_2 \leq 1)$ implies

$$\int_1^x \{\psi(y(1+\delta))-\psi(y)-y\delta\}^2\,dy \sim \frac{1}{2}\delta x^2 \log\frac{1}{\delta}, \tag{5.9}$$

uniformly for $x^{-B_2} \leq \delta \leq x^{-B_1}$. (There is also a converse implication. Another such equivalence was mentioned above). Upon RH, for $0 < \delta \leq 1$ this integral is $\ll \delta x^2(\log\frac{2}{\delta})^2$ (see [50]; also in Eq. (7.9) below we mention a lower bound of the correct order of magnitude). Yıldırım defined a function which correlates the zeros of all pairs of Dirichlet's L-functions to the same modulus. Under a conjecture for this function, analogous to MC, the author got the asymptotic result corresponding to (5.9) for the second moment for primes in an individual arithmetic progression.

The appearance in (ii) of $1-(\frac{\sin \pi u}{\pi u})^2$ as the pair correlation function of the zeros of $\zeta(s)$ has opened up new avenues of progress. The eigenvalues of a random complex Hermitian matrix of large order taken from the Gaussian Unitary Ensemble (GUE) have the same pair correlation function. So one might expect that there exists a linear operator whose eigenvalues characterize the zeros of $\zeta(s)$. Recently higher order correlations of zeros of $\zeta(s)$ have been under study. Hejhal [29] calculated the triple correlation function, similar to Montgomery's work. Rudnick and Sarnak [48] [49] defined the n-level correlation sums for the Riemann zeta-function and more general L-functions. They showed that the n-level correlations are in accordance with the predictions by the GUE model. Farmer [9] has given some consequences of the 'GUE Hypothesis' that the distribution of gaps between the zeros of $\zeta(s)$ is like the distribution of gaps between the eigenvalues of large random Hermitian matrices. The numerical results of Odlyzko [40], [41] constitute great evidence for the truth of RH and the GUE model. Also the heuristic and non-rigourous methods of Bogomolny and Keating (in a series of articles, the last being [2]) assuming the Hardy-Littlewood conjecture with $r = 2$, indicate that the n-level correlations of zeta zeros are in agreement with the results for GUE beyond the ranges that were possible in the works of Hejhal, and Rudnick – Sarnak who assumed merely RH. The equivalence of (iii) with the pair correlation conjecture (cf. also (5.9)) reflects that the second moment for primes is determined by the pair correlation of the zeros of $\zeta(s)$. The higher moments analogues of (5.9) (which can be expected to be calculable from the general r-tuple conjecture (3.1), see Gallagher [14]), and their connections with the distribution of zeta zeros seems not to have been worked out yet.

§6. Sums over zeta zeros and the error term in the prime number theorem

The pair correlation conjecture (5.4), assumed in varying degrees of strength depending on the problem, implies (see Heath-Brown [27])

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0, \tag{6.1}$$

$$\psi(x) = x + o(x^{\frac{1}{2}} \log^2 x) \tag{6.2}$$

(cf. Eq.s (2.6)-(2.9) and (3.3)). Such implications are natural, as (5.4) (or weaker forms of it) imply nontrivial estimates on sums like $\sum_\rho \frac{x^\rho}{\rho}$, the quantity which appears in the explicit formula (2.5). To describe this briefly we take the pair correlation function in the form

$$F(T,x) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2}, \tag{6.3}$$

and let

$$\sum(T,v,u) = \sum_{0<\gamma\le T} e(\gamma(v+u)). \tag{6.4}$$

Then

$$\int_{-\infty}^{\infty} 2\pi e^{-4\pi|v|} |\sum(T,v,\log x)|^2\, dv = F(T,x), \tag{6.5}$$

and from here it follows that

$$\sum_{0<\gamma\le T} x^{i\gamma} \ll T^{\frac{1}{2}} \{\max_{t\le T} F(t,x)\}^{\frac{1}{2}}. \tag{6.6}$$

So, roughly speaking, the assumption that the size of $F(T,x)$ (for appropriate ranges of T and x) is $O(T\log T)$ will save a $\log^{\frac{1}{2}} T$ from the trivial estimate

$$\sum_{0<\gamma\le T} x^{i\gamma} \ll T\log T. \tag{6.7}$$

Since $\psi_0(x)$ is discontinuous at the prime powers, so is $\sum_\rho \frac{x^\rho}{\rho}$ $(= \lim_{T\to\infty} \sum_{|\gamma|<T} \frac{x^\rho}{\rho})$ by the explicit formula (2.5); the series is boundedly

convergent in fixed intervals $1 < a \le x \le b$. The sum $\sum_{0<\gamma\le T} x^\rho$ is also discontinuous at the prime powers. By considering $\int \frac{\zeta'(s)}{\zeta(s)} x^s\, ds$ taken around a rectangular contour Landau [33] proved that

$$\sum_{0<\gamma\le T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(\log T) \tag{6.8}$$

for every fixed $x > 1$, as $T \to \infty$. Gonek [23] proved a uniform (in both x and T) version of (6.8), that for $x, T > 1$,

$$\begin{aligned} \sum_{0<\gamma\le T} x^\rho &= -\frac{T}{2\pi}\Lambda(x) + O(x\log 2x \log\log 3x) \\ &\quad + O(\log x \min(T, \frac{x}{\langle x\rangle})) + O(\log 2T \min(T, \frac{1}{\log x})). \end{aligned} \tag{6.9}$$

(It is possible to calculate the sum also for $0 < x < 1$ from (6.9) by using the symmetry of the zeros of $\zeta(s)$ with respect to the critical line). In (6.9) if one assumes RH, then

$$\sum_{0<\gamma\le T} x^{i\gamma} \ll (Tx^{-\frac{1}{2}} + x^{\frac{1}{2}}) \log x \log\log x. \tag{6.10}$$

Comparison with (6.7) shows that (6.10) is nontrivial for $2 \le x \le T^{2-\epsilon}$. If one assumes further that $x^{i\gamma}$'s behave like independent random variables (cf. (2.8)), then one expects that for almost all $x > 1$

$$\sum_{0<\gamma\le T} x^{i\gamma} \ll T^{\frac{1}{2}+\epsilon}. \tag{6.11}$$

By Dirichlet's theorem on Diophantine approximation there exist arbitrarily large x with

$$\sum_{0<\gamma\le T} x^{i\gamma} \gg T\log T, \tag{6.12}$$

so (6.11) doesn't hold for all $x > 1$. The observation (6.11) along with the heuristics of using $\frac{n}{\log n}$ in place of γ_n (see (1.5)) in $\sum x^{i\gamma}$ led Gonek to conjecture that

$$\sum_{0<\gamma\le T} x^{i\gamma} \ll (Tx^{-\frac{1}{2}+\epsilon} + T^{\frac{1}{2}}x^\epsilon) \quad (x, T \ge 2). \tag{6.13}$$

This would imply that for $1 \leq h \leq x$

$$\psi(x+h) - \psi(x) = h + O(h^{\frac{1}{2}} x^{\epsilon}), \tag{6.14}$$

which in turn implies

$$p_{n+1} - p_n \ll p^{\epsilon} \tag{6.15}$$

(cf. (3.3) and (6.2)).

Averages of the error term in the prime number theorem have also been of interest not just for their own sake but because quantities involving them crop up in many problems (e.g. Eq.s (7.15) and (7.24) below). For brevity call

$$R(x) = \psi(x) - x\,. \tag{6.16}$$

The results on the order of magnitude of $R(x)$ or various averages of it are in correspondence with the estimates for sums over zeta zeros as clearly seen from the explicit formula. Upon RH one has (2.7), and one can only hope to have improvements in the logarithmic part (cf. (2.8), (2.9), (6.2)). By (2.9) it is known that, as $x \to \infty$, $R(x)$ changes sign infinitely many times. Cramér showed that on RH

$$\int_1^X \frac{(R(u))^2}{u}\, du = O(X), \tag{6.17}$$

$$\int_1^X \left(\frac{R(u)}{u}\right)^2 du \sim C \log X\,. \tag{6.18}$$

Gallagher's article [15] contains compact proofs of such results. By Cauchy-Schwarz inequality (6.17) implies

$$\int_1^X |R(u)|\, du = O(X^{\frac{3}{2}}).$$

Pintz [44] has shown that for all sufficiently large X

$$\frac{X^{\frac{3}{2}}}{400} \leq \int_1^X |R(u)|\, du \leq X^{\frac{3}{2}}\,, \tag{6.19}$$

where the lower bound is unconditional and the upper bound depends essentially on RH. Jurkat [32], by developing concepts on almost-periodic functions proved upon RH that, with $d(x) = \dfrac{1}{\log\log x}$

$$\frac{1}{d(x)} \int_x^{x + x\, d(x)} \frac{R(u)}{u^{\frac{1}{2}}} \frac{du}{u} = \Omega_{\pm}(\log\log\log x) \tag{6.20}$$

(with the implied constants $\pm\frac{1}{2}$), and that this cannot be improved upon much for he also showed that this quantity is $O((\log\log\log x)^2)$. Eq. (6.20) implies (2.9), Littlewood's result without averages. From (6.19) and (2.7) we see that as $x \to \infty$, $|R(x)|$ spends most of its time roughly around the value $x^{\frac{1}{2}}$ (instead of much smaller values), and (6.20) reveals the existence of quite long intervals throughout which $|R(x)|$ is almost as large as possible.

§7. Some recent results on the second moments for primes

In this section, as an example of recent work in our topic, we present a theorem of Goldston and Yıldırım. This is Thm. 3 of [22], where the details of the proof have not been included. The results given by Eq.s (7.2), (7.6) and (7.9) below are also proved in [22]. We define for

$$x \geq 2, \quad 1 \leq q \leq x, \quad 1 \leq h \leq x,$$

$$I(x,h,q) = \sum_{a(q)}{}^{*} \int_x^{2x} \left(\psi(y+h;q,a) - \psi(y;q,a) - \frac{h}{\phi(q)} \right)^2 dy, \tag{7.1}$$

where $\sum_{a(q)}^{*}$ is the sum over a reduced set of residues modulo q.

If $h \leq q$, the interval $(y, y+h]$ contains at most one integer which belongs to the congruence class $a \,(\mathrm{mod}\ q)$, so the situation is rather trivial and one has unconditionally

$$I(x,h,q) \sim hx\log x \quad (h \leq q). \tag{7.2}$$

So henceforth we will take $1 \leq q \leq h \leq x$.

It was shown by Prachar [45] that, assuming GRH

$$I(x,h,q) \ll hx\log^2 qx. \tag{7.3}$$

It is possible to evaluate the asymptotic value of $I(x,h,q)$, as $x \to \infty$, assuming RH and a strong form of the twin prime conjecture (the case $r = 2$ of the Hardy-Littlewood conjecture mentioned in §3 above). Let

$$N_1 = N_1(k) = \max(0, -k), \qquad N_2 = N_2(x,k) = \min(x, x-k),$$

$$E(x,k) = \sum_{N_1(k) < n \le N_2(x,k)} \Lambda(n)\Lambda(n+k) - \mathfrak{S}(k)(x - |k|), \tag{7.4}$$

where

$$\mathfrak{S}(k) = \begin{cases} 2C \prod_{\substack{p|k \\ p>2}} \left(\frac{p-1}{p-2}\right), & \text{if } k \text{ is even, } k \ne 0; \\ 0, & \text{if } k \text{ is odd,} \end{cases}$$

with

$$C = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Assuming the twin prime conjecture in the form that for $0 < |k| \le x$, and some given $\epsilon \in (0, \frac{1}{2})$

$$E(x,k) \ll x^{\frac{1}{2}+\epsilon}, \tag{7.5}$$

it follows for $1 \le \frac{h}{q} \le x^{\frac{1}{2}-\epsilon}$ and $h \le x$ that

$$I(x,h,q) \sim hx \log \frac{xq}{h}. \tag{7.6}$$

Upon GRH only, it can be shown that (7.6) holds for almost all q with $h^{\frac{3}{4}} \log^5 x \le q \le h$ (see [22]). Moreover by Goldston's method ([18]) of using the auxiliary arithmetic function

$$\lambda_R(n) = \sum_{r \le R} \frac{\mu^2(r)}{\phi(r)} \sum_{\substack{d|r \\ d|n}} d\mu(d), \tag{7.7}$$

which in relevant cases mimics the behaviour of $\Lambda(n)$, starting from the inequality

$$\sum_{a(q)} \int_x^{2x} \Big| \sum_{\substack{y < n \le y+h \\ n \equiv a(q)}} (\Lambda(n) - \lambda_R(n)) \Big|^2 dy \ge 0 \tag{7.8}$$

a lower-bound of the correct order of magnitude for $I(x,h,q)$ can be obtained. The last inequality enables one to replace the troublesome sums involving $\Lambda(n)\Lambda(n+k)$'s by sums of $\lambda_R(n)\Lambda(n+k)$'s and $\lambda_R(n)\lambda_R(n+k)$'s, which can be evaluated with no need for a conjecture like (7.5). The result is that for any $\epsilon > 0$, and $0 \le \alpha \le \frac{1}{3}$, where we write $\frac{h}{q} = x^\alpha$,

$$I(x,h,q) \ge (\frac{1}{2} - \frac{3}{2}\alpha - \epsilon) hx \log x. \tag{7.9}$$

Let us remember that Lavrik [34] showed that for $B > 0$ and $y \ge y_0(B)$,

$$\sum_{k=1}^{\frac{y}{2}} (E(y,k))^2 \ll y^2 (\log y)^{-B}. \tag{7.10}$$

This was used by Montgomery [36] in proving the asymptotic version of the Barban-Davenport-Halberstam theorem (4.10), in the light of which we expect that assuming only GRH one can get an asymptotic result for sums of $I(x,h,q)$ over certain ranges of q. This is indeed the case, and here we will prove the following theorem.

Theorem. *Assume the Generalized Riemann Hypothesis. Then we have, for $h^{\frac{1}{2}} \log^6 x \le Q \le h \le x$, as $x \to \infty$*

$$\sum_{q \le Q} I(x,h,q) \sim Qhx \log(\frac{xQ}{h}).$$

In fact we shall obtain the more detailed formula (7.48). Note that the range of validity of this theorem is even greater than that of (7.6) when h is very close to x. In the proof we will use methods and results of Friedlander and Goldston [11].

Proof. Expanding the integrand of (7.1) we have

$$\begin{aligned} I(x,h,q) &= \sum_{a(q)}^{*} \int_x^{2x} (\psi(y+h;q,a)-\psi(y;q,a))^2 \, dy \\ &\quad -\frac{2h}{\phi(q)} \sum_{a(q)}^{*} \int_x^{2x} (\psi(y+h;q,a)-\psi(y;q,a)) \, dy + \frac{h^2 x}{\phi(q)} \\ &= S_1 - \frac{2h}{\phi(q)} S_2 + \frac{h^2 x}{\phi(q)}, \text{ say.} \end{aligned} \tag{7.11}$$

Here

$$S_2 = \sum_{(n,q)=1} \Lambda(n) f(n,x,h), \tag{7.12}$$

where

$$f(n,x,h) = \int_{[x,2x]\cap[n-h,n)} 1\, dy = \begin{cases} n-x, & \text{if } x \le n < x+h\ , \\ h, & \text{if } x+h \le n \le 2x, \\ 2x-n+h, & \text{if } 2x < n \le 2x+h, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\sum_{\substack{x\le n\le 2x+h \\ (n,q)>1}} \Lambda(n) = \sum_{p|q} \sum_{\substack{\nu \\ x \le p^\nu \le 2x+h}} \log p \ll \sum_{p|q} \log p \ll \log q, \tag{7.13}$$

we can lift the condition $(n,q)=1$, so that

$$S_2 = \sum_{x<n\le 2x+h} \Lambda(n) f(n,x,h) + O(h \log q).$$

The sums involving $f(n,x,h)$ will be evaluated by the following partial summation formula. Let $C(x) = \sum_{n\le x} c_n$. Then

$$\sum_{x<n\le 2x+h} c_n f(n,x,h) = \int_{2x}^{2x+h} C(u)\, du - \int_x^{x+h} C(u)\, du + h(c_{x+h} - c_{2x}) \tag{7.14}$$

($c_v = 0$ if v is not an integer). Taking $C(x) = \psi(x)$ in (7.14) and recalling (6.16) we obtain

$$S_2 = hx + \{\int_{2x}^{2x+h} - \int_x^{x+h} R(u)\, du\} + O(h \log x). \tag{7.15}$$

From (7.11) we have

$$\begin{aligned} S_1 &= \sum_{\substack{x<n\le 2x+h\\(n,q)=1}} \Lambda^2(n) f(n,x,h)\\ &\quad +2\sum_{\substack{0<k\le h\\k\equiv 0(q)}} \sum_{\substack{x<n\le 2n+h-k\\(n(n+k),q)=1}} \Lambda(n)\Lambda(n+k) f(n,x,h-k). \end{aligned} \tag{7.16}$$

A calculation similar to (7.13) shows that we may drop the conditions $(n,q)=1$ and $(n(n+k),q)=1$ in the above sums with an error $\ll \frac{h^2}{q}\log^2 x$. Thus we have

$$S_1 = S_3 + 2S_4 + O(\frac{h^2}{q}\log^2 x), \tag{7.17}$$

where

$$S_3 = \sum_{x<n\le 2x+h} \Lambda^2(n) f(n,x,h), \tag{7.18}$$

and

$$S_4 = \sum_{0<j\le h/q} \sum_{x<n\le 2x+h-jq} \Lambda(n)\Lambda(n+jq) f(n,x,h-jq). \qquad 7.19)$$

To evaluate S_3 call

$$P(x) = \sum_{n\le x} \Lambda^2(n) - x\log x + x, \tag{7.20}$$

and apply (7.14) with $c_n = \Lambda^2(n)$ to get

$$\begin{aligned} S_3 &= \frac{(2x+h)^2}{2}\log(2x+h) - \frac{(2x)^2}{2}\log 2x\\ &\quad -\frac{(x+h)^2}{2}\log(x+h) + \frac{x^2}{2}\log x - \frac{3xh}{2}\\ &\quad +\{\int_{2x}^{2x+h} - \int_x^{x+h} P(u)\,du\} + O(h\log^2 x). \end{aligned} \tag{7.21}$$

Taking $C(u) = \sum_{0<n\le u} \Lambda(n)\Lambda(n+jq)$ in (7.14) we have

$$\begin{aligned} S_4 &= \int_{2x}^{2x+h} \sum_{0<j\le \frac{u-2x}{q}} \sum_{n\le u-jq} \Lambda(n)\Lambda(n+jq)\,du\\ &\quad -\int_x^{x+h} \sum_{0<j\le \frac{u-x}{q}} \sum_{n\le u-jq} \Lambda(n)\Lambda(n+jq)\,du + O(\frac{h^2}{q}\log^2 x). \end{aligned} \tag{7.22}$$

Note that by (7.4) the difference of the two integrals in (7.22) is

$$x\sum_{0<j\le h/q}(h-jq)\mathfrak{S}(jq)+\{\int_{2x}^{2x+h}\sum_{0<j\le\frac{u-2x}{q}}-\int_{x}^{x+h}\sum_{0<j\le\frac{u-x}{q}}E(u,jq)\,du\} \tag{7.23}$$

On combining (7.11), (7.15), (7.17), (7.21), and effecting the cancellations that occur in (7.21), we obtain

$$\begin{aligned} I(x,h,q) &= hx\log x + x^2\log\Big(\frac{(1+\frac{h}{2x})^2}{(1+\frac{h}{x})^{\frac{1}{2}}}\Big) + hx\Big(\log\Big(\frac{4(1+\frac{h}{2x})^2}{(1+\frac{h}{x})}\Big)-\frac{3}{2}\Big) \\ &\quad + \frac{h^2}{2}\log\Big(\frac{2x+h}{x+h}\Big) - \frac{h^2x}{\phi(q)} + 2S_4 + O\Big(\frac{h^2\log^2 x}{q}\Big) \\ &\quad + \{\int_{2x}^{2x+h} - \int_{x}^{x+h}\Big(P(u)-\frac{2h}{\phi(q)}R(u)\Big)\,du\} \end{aligned} \tag{7.24}$$

If we use (7.22), (7.23) and (7.5) in (7.24), we obtain (7.6). Also (7.2) follows from (7.24) on using the prime number theorem (2.6) to estimate the integrals in (7.24) (when $\frac{h}{q}\le 1$, S_4 is void, and instead of the O-term of (7.24) there is $O(h\log^2 x)$. Note also that $\frac{q}{\phi(q)} \ll \log\log q$ [26, §18.4] to deal with the term $\frac{h^2x}{\phi(q)}$ in (7.24)).

Now note that by (7.3)

$$\sum_{q\le Q} I(x,h,q) = \sum_{Q_0\le q\le Q} I(x,h,q) + O(Q_0 hx\log^2 x), \tag{7.25}$$

so we must calculate $\sum_{Q_0<q\le Q} S_4$. Letting

$$S_u(\alpha) = \sum_{n\le u}\Lambda(n)e(n\alpha), \tag{7.26}$$

$$W_{v,u}(\alpha) = \sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}} e(-jq\alpha), \tag{7.27}$$

we can express

$$\sum_{n\le u-jq}\Lambda(n)\Lambda(n+jq) = \int_0^1 |S_u(\alpha)|^2 e(-jq\alpha)\,d\alpha, \tag{7.28}$$

and upon taking $v = \min(Q, u)$,

$$\sum_{Q_0 < q \le Q} S_4 = \int_{Q_0}^{h} \int_0^1 (|S_{u+2x}(\alpha)|^2 - |S_{u+x}(\alpha)|^2) W_{v,u}(\alpha)\, d\alpha\, du + O(h^2 \log^3 x). \tag{7.29}$$

We shall need an upper bound on the size of $W_{v,u}(\alpha)$. Changing the order of summation in (7.27) gives

$$W_{v,u}(\alpha) = \sum_{0<j\le \frac{u}{Q_0}} \sum_{Q_0<q\le \frac{u}{j}} e(-jq\alpha) - \sum_{0<j\le \frac{u}{v}} \sum_{v<q\le \frac{u}{j}} e(-jq\alpha). \tag{7.30}$$

We now employ the estimate of Vinogradov and Vaughan (see [8, Chapter 25])

$$\sum_{0<j\le \frac{u}{Y}} \sum_{Y<q\le \frac{u}{j}} e(-jq\alpha) \ll (\frac{u}{r} + \frac{u}{Y} + r) \log(\frac{2ru}{Y}), \tag{7.31}$$

which rests upon the assumption

$$\alpha = \frac{b}{r} + \beta, \quad |\beta| \le \frac{1}{r^2}, \quad (b, r) = 1, \tag{7.32}$$

to obtain for $v \ge Q_0$

$$W_{v,u}(\alpha) \ll (\frac{u}{r} + \frac{u}{Q_0} + r) \log(\frac{2ru}{Q_0}). \tag{7.33}$$

Letting

$$J_u(\beta) = \frac{\mu(r)}{\phi(r)} I_u(\beta), \quad I_u(\beta) = \sum_{n\le u} e(n\beta), \tag{7.34}$$

we will consider separately each term on the right-hand side of

$$\int_0^1 |S(\alpha)|^2 W(\alpha)\, d\alpha = \int_0^1 |J|^2 W(\alpha)\, d\alpha + \int_0^1 |S-J|^2 W(\alpha)\, d\alpha + 2\int_0^1 \mathfrak{Re}\{(\overline{S-J})J\} W(\alpha)\, d\alpha \tag{7.35}$$

where the subscripts (which are either $u + 2x$ or $u + x$ for S, and v, u for W) have been suppressed. The range of integration is decomposed into Farey arcs of order R; $M_R(r, b) = (\frac{b+b'}{r+r'}, \frac{b+b''}{r+r''}]$, where

$1 \le r \le R$, $(b,r) = 1$, and $\frac{b'}{r'} < \frac{b}{r} < \frac{b''}{r''}$ are consecutive fractions in the Farey sequence. We call $\theta_R(r,b)$ the translated interval $(\frac{-1}{r(r+r')}, \frac{1}{r(r+r'')}]$. Note that $(\frac{-1}{2rR}, \frac{1}{2rR}] \subset \theta_R(r,b) \subset (\frac{-1}{rR}, \frac{1}{rR}]$, and

$$\int_0^1 = \sum_{r\le R}\sum_{b(r)}^{*} \int_{\theta_R(r,b)}$$

(we will abbreviate $\theta_R(r,b)$ as θ). To estimate $\int_0^1 |S-J|^2 W(\alpha)\,d\alpha$, recall Lemma 7.1 of [11] which says upon GRH

$$\sum_{b(r)}^{*} \int_{-\delta}^{\delta} |S-J|^2\,d\beta \ll \delta r x(\log rx)^4, \quad (\delta \ge \frac{1}{x}). \tag{7.36}$$

Taking $\delta = \frac{1}{rR}$ (so that we must have $R \le x^{\frac{1}{2}}$), (7.33) and (7.36) give

$$\int_0^1 |S-J|^2 W(\alpha)\,d\alpha \ll \frac{xu}{R}\log^6 x + \frac{xu}{Q_0}\log^5 x + xR\log^5 x. \tag{7.37}$$

Next we have

$$\begin{aligned}&\int_0^1 \mathfrak{Re}\{(\overline{S-J})J\}W(\alpha)\,d\alpha \\ &\quad \ll \sum_{r\le R}(\frac{u}{r} + \frac{u}{Q_0} + r)\log(\frac{2ru}{Q_0})\frac{\mu^2(r)}{\phi(r)}\sum_{b(r)}^{*}\int_\theta |I||S-J|.\end{aligned} \tag{7.38}$$

Using $I \ll \min(x, ||\beta||^{-1})$, and the Cauchy-Schwarz inequality we get from (7.36)

$$\sum_{b(r)}^{*} \int_{-\frac{1}{x}}^{\frac{1}{x}} |I||S-J| \ll \sum_{b(r)}^{*} x^{\frac{1}{2}}(\int_{-\frac{1}{x}}^{\frac{1}{x}} |S-J|^2)^{\frac{1}{2}} \ll rx^{\frac{1}{2}}\log^2 x.$$

On the rest of $\theta_R(r,b)$, calling $u_j = (2^j rR)^{-1}$, we similarly see that

$$\sum_{b(r)}^{*} \int_{\frac{u_j}{2}}^{u_j} |I||S-J| \ll u_j^{-1}\sum_{b(r)}^{*}\int_{-u_j}^{u_j} |S-J| \ll rx^{\frac{1}{2}}\log^2 x.$$

There are $O(\log x)$ such $u_j's$. So (7.38) gives

$$\int_0^1 \mathfrak{Re}\{(\overline{S-J})J\}W(\alpha)\,d\alpha \ll ux^{\frac{1}{2}}\log^5 x + \frac{ux^{\frac{1}{2}}R}{Q_0}\log^4 x + R^2 x^{\frac{1}{2}}\log^4 x. \tag{7.39}$$

Next

$$\int_0^1 |J_{u+x}|^2 W_{v,u}(\alpha)\,d\alpha =$$

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi^2(r)}\sum_{b(r)}{}^{*} e(-\frac{jqb}{r})\int_{\theta}\sum_{n,m\le u+x}\ \sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}} e(n-m-jq\beta)\,d\beta.$$

Thinking of $\int_{\theta}$ as $\int_0^1-\int_{[0,1]\backslash\theta}$, we will first consider upon changing the order of summations

$$\sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}}\ \sum_{r\le R}\frac{\mu^2(r)}{\phi^2(r)}\sum_{b(r)}{}^{*} e(-\frac{jqb}{r})\int_0^1\sum_{n,m\le u+x} e((n-m-jq)\beta)\,d\beta. \tag{7.40}$$

With

$$c_r(k)=\sum_{b(r)}{}^{*} e(\frac{kb}{r})=\sum_{d|(r,k)} d\mu(\frac{r}{d})=\frac{\mu(\frac{r}{(r,k)})\phi(r)}{\phi(\frac{r}{(r,k)})},$$

for Ramanujan's sum (see [8, Chapter 20]), one has

$$\sum_{r=1}^{\infty}\frac{\mu^2(r)}{\phi^2(r)}c_r(k)=\mathfrak{S}(k).$$

We define

$$\mathfrak{S}_R(k)=\sum_{r\le R}\frac{\mu^2(r)}{\phi^2(r)}c_r(k),\quad \tilde{\mathfrak{S}}_R=\mathfrak{S}-\mathfrak{S}_R,$$

and (7.40) becomes

$$\begin{aligned}
&\sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}}\ \sum_{r\le R}\frac{\mu^2(r)}{\phi^2(r)}c_r(jq)\sum_{\substack{n,m\le u+x\\ n-m=jq}}1\\
&=\sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}}\mathfrak{S}_R(jq)\sum_{m\le u+x-jq}1\\
&=\sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}}\{(\mathfrak{S}(jq)-\tilde{\mathfrak{S}}_R(jq))(u+x-jq)+O(|\mathfrak{S}_R(jq)|)\}.
\end{aligned}$$

Recall from (7.29) that we need $\int_0^1(|J_{u+2x}|^2-|J_{u+x}|^2)W_{v,u}(\alpha)\,d\alpha$, so we should calculate

$$x\sum_{Q_0<q\le v}\ \sum_{0<j\le\frac{u}{q}}\{(\mathfrak{S}(jq)-\tilde{\mathfrak{S}}_R(jq))+O(u\log^2 x), \tag{7.41}$$

where the last error term is deduced from $\mathfrak{S}_R(jq) \ll \log R$. From the $\tilde{\mathfrak{S}}_R$ term we have

$$\begin{aligned} x \sum_{Q_0<q\leq v} \sum_{0<j\leq \frac{u}{q}} \sum_{r>R} \frac{\mu^2(r)}{\phi^2(r)} c_r(jq) &\ll x \sum_q \sum_j \sum_{r>R} \frac{\mu^2(r)}{\phi^2(r)} \sum_{d|(r,jq)} d \\ \ll x \sum_q \sum_j \sum_{d|jq} \frac{d\mu^2(d)}{\phi^2(d)} \sum_{r_1>\frac{R}{d}} \frac{\mu^2(r_1)}{\phi^2(r_1)} &\ll \frac{x}{R} \sum_q \sum_j \sum_{d|jq} \frac{d^2\mu^2(d)}{\phi^2(d)} \\ \ll \frac{x}{R} \sum_{d\leq u} \frac{d^2\mu^2(d)}{\phi^2(d)} \sum_{\substack{m\leq u \\ d|m}} \tau(m) &\ll \frac{x}{R} \sum_{d\leq u} \frac{d^2\mu^2(d)\tau(d)}{\phi^2(d)} \sum_{n\leq \frac{u}{d}} \tau(n) \\ \ll \frac{xu}{R} \log x \sum_{d\leq u} \frac{d^2\mu^2(d)\tau(d)}{\phi^2(d)} &\ll \frac{xu}{R} \log^3 x. \end{aligned} \tag{7.42}$$

Now we estimate

$$\begin{aligned} &\sum_{r\leq R} \sum_{b(r)}{}^{*} \int_{[0,1]\backslash\theta} |J_{u+x}|^2 W(\alpha)\, d\alpha \\ &\leq \sum_{r\leq R} \sum_{b(r)}{}^{*} \frac{\mu^2(r)}{\phi^2(r)} \int_{[0,1]\backslash\theta} |I(\beta)|^2 |W(\frac{b}{r}+\beta)|\, d\beta \end{aligned} \tag{7.43}$$

by taking a finer Farey decomposition of order T with $T > R$. Then $\theta_T(r,b) \subset \theta_R(r,b)$, so that

$$\begin{aligned} \int_{[0,1]\backslash\theta_R(r,b)} |I(\beta)|^2 \; |W(\frac{b}{r}+\beta)|\, d\beta \\ &= \int_{[0,1]\backslash M_R(r,b)} |I(\alpha-\frac{b}{r})|^2 |W(\alpha)|\, d\alpha \\ &\leq \int_{[0,1]\backslash M_T(r,b)} |I(\alpha-\frac{b}{r})|^2 |W(\alpha)|\, d\alpha \\ &= \sum_{t\leq T} \sum_{\substack{c(t) \\ \frac{c}{t}\neq\frac{b}{r}}}{}^{*} \int_{\theta_T(t,c)} |I(\frac{c}{t}-\frac{b}{r}+\beta)|^2 |W(\frac{c}{t}+\beta)|\, d\beta \end{aligned}$$

On $\theta_T(t,c)$, we have $|\beta| \leq \frac{1}{tT} \leq \frac{1}{2rt}$ if $T \geq 2R$, so that

$$|I(\frac{c}{t}-\frac{b}{r}+\beta)| \ll ||\frac{c}{t}-\frac{b}{r}+\beta||^{-1} \ll 2||\frac{cr-bt}{rt}||^{-1}.$$

Hence by (7.33)

$$\sum_{t\le T}\sideset{}{^*}\sum_{\substack{c(t)\\ \frac{c}{t}\neq\frac{b}{r}}}\int_{\theta_T(t,c)}|I(\frac{c}{t}-\frac{b}{r}+\beta)|^2|W(\frac{c}{t}+\beta)|\,d\beta$$
$$\ll\sum_{t\le T}(\frac{u}{t}+\frac{u}{Q_0}+t)\log x\sideset{}{^*}\sum_{\substack{c(t)\\ \frac{c}{t}\neq\frac{b}{r}}}\frac{1}{tT}\frac{(rt)^2}{|cr-bt|^2},$$

where from each pair of residue classes $b\,(\mathrm{mod}\,r)$ and $c\,(\mathrm{mod}\,t)$ the b and c which minimize $|cr-bt|$ is chosen. For given r,t the number of representations of m as $m = cr-bt$ is (r,t) if $(r,t)|m$, and 0 otherwise. So

$$\sideset{}{^*}\sum_{b(r)}\sideset{}{^*}\sum_{\substack{c(t)\\ \frac{c}{t}\neq\frac{b}{r}}}\frac{1}{|cr-bt|^2}\le\frac{2\zeta(2)}{(r,t)}\ll 1.$$

Using these in (7.43) we get

$$\sum_{r\le R}\sideset{}{^*}\sum_{b(r)}\int_{[0,1]\backslash\theta}|J_{u+x}|^2W(\alpha)\,d\alpha\ll\frac{\log x}{T}\sum_{r\le R}\frac{r^2\mu^2(r)}{\phi^2(r)}\sum_{t\le T}u+\frac{uT}{Q_0}+t^2$$
$$\ll Ru\log x+\frac{uR^2}{Q_0}\log x+R^3\log x, \tag{7.44}$$

if we take $T = 2R$. Combining Eq.s (7.35), (7.37), (7.39-44) in (7.29), and recalling that $R\le x^{\frac{1}{2}}$ we have

$$\sum_{Q_0<q\le Q}S_4 = x\int_{Q_0}^{h}\sum_{Q_0<q\le v}\sum_{0<j\le\frac{u}{q}}\mathfrak{S}(jq)\,du$$
$$+O(\frac{xh^2}{R}\log^6 x)+O(xhR\log^5 x)+O(\frac{xh^2}{Q_0}\log^5 x). \tag{7.45}$$

The integral in (7.45) is evaluated by pulling the summations outside the integral sign as

$$\sum_{Q_0<q\le Q}\sum_{0<j\le\frac{h}{q}}(h-jq)\mathfrak{S}(jq)$$

which in turn is, by Proposition 3 and Lemma 6.1 of [11],

$$\frac{x}{2}\sum_{Q_0<q\le Q}\{\frac{h^2}{\phi(q)}-h\log(\frac{h}{q})-h(\gamma+\log 2\pi-1+\sum_{p|q}\frac{\log p}{p-1})\}$$
$$+O(\min(Q^{\frac{3}{2}}h^{\frac{1}{2}}x\log^{\frac{3}{2}}Q, Qhx)) \tag{7.46}$$

Hence by (7.24), (7.25), (7.45) and (7.46) we obtain

$$\begin{aligned}\sum_{q\le Q} I(x,h,q) \;=\; & Q\{\frac{(2x+h)^2}{2}\log(2x+h)-\frac{(2x)^2}{2}\log 2x \\ & -\frac{(x+h)^2}{2}\log(x+h)+\frac{x^2}{2}\log x-\frac{3xh}{2}\} \\ & +Qxh\log\frac{Q}{h}-Qxh(\gamma+\log 2\pi+\sum_p\frac{\log p}{p(p-1)}) \\ & +O(\min(Q^{\frac{3}{2}}h^{\frac{1}{2}}x\log^{\frac{3}{2}}Q, Qhx)) \\ & +O(\frac{xh^2}{R}\log^6 x)+O(xhR\log^5 x)+O(\frac{xh^2}{Q_0}\log^5 x) \\ & +O(Qhx^{\frac{1}{2}}\log^3 x)+O(Q_0hx\log^2 x).\end{aligned} \tag{7.47}$$

In writing (7.47), we have used the RH estimate (2.7) for $R(u)$ and $P(u)$ of (7.24). Choosing $Q_0=R=\frac{h^{\frac{1}{2}}}{2}$, the O-terms in (7.47) can be gathered in $O(\min(Q^{\frac{3}{2}}h^{\frac{1}{2}}x\log^{\frac{3}{2}}Q, Qhx))+O(xh^{\frac{3}{2}}\log^6 x)$, and (7.47) may be recast as

$$\begin{aligned}&\sum_{q\le Q} I(x,h,q) = Qhx\log(\frac{xQ}{h})+Qx^2\log(\frac{(1+\frac{h}{2x})^2}{(1+\frac{h}{x})^{\frac{1}{2}}})+ \\ & Qhx(\log(\tfrac{2}{\pi}\tfrac{(1+\frac{h}{2x})^2}{1+\frac{h}{x}})-\tfrac{3}{2}-\gamma-\sum_p\frac{\log p}{p(p-1)})+ \\ & \frac{Qh^2}{2}\log(\frac{2x+h}{x+h})+O(\min(Q^{\frac{3}{2}}h^{\frac{1}{2}}x\log^{\frac{3}{2}}Q, Qhx))+O(xh^{\frac{3}{2}}\log^6 x).\end{aligned} \tag{7.48}$$

This completes the proof of the theorem. The result of this theorem is expected to be related to the distribution and simplicity of zeros of Dirichlet's L-functions as was recounted on p. 319.

Further notes

Section 1: The book of Hardy and Wright [26] contains almost all of the classic results of number theory. For an exposition of the theory of the Riemann zeta-function, Dirichlet's L-functions and distribution of primes we refer the reader to the books of Davenport [8] and Ingham [31]. Titchmarsh's book (revised by Heath-Brown) [52] is an extensive treatise on the Riemann zeta-function.

From the work of Conrey [4] at least $\frac{2}{5}$ of the zeta zeros are known to lie on $\sigma = \frac{1}{2}$.

Section 2: We note that with more work involving estimates on exponential sums a greater region than de la Vallée Poussin's has been shown to be free of zeta zeros by Vinogradov, so (2.6) can be written with any number less than $\frac{3}{5}$ replacing $\frac{1}{2}$ (see [52, Chapter 6]).

There are many problems about the prime counting functions which have not been included in this survey, a few of which will be mentioned briefly here. By (2.9) it is clear that $\psi(x) - x$ changes sign infinitely often as $x \to \infty$. This is less involved than the problem that gave rise to it, the sign changes of $\pi(x) - \mathrm{li}(x)$, because $\psi(x)$ is more directly related to $\zeta(s)$ than is $\pi(x)$. Riemann had asserted that $\pi(x) < \mathrm{li}(x)$ for $x > 2$, which was proved to be false by Littlewood's result

$$\pi(x) - \mathrm{li}(x) = \Omega_{\pm}(\frac{x^{\frac{1}{2}}}{\log x} \log\log\log x) \quad (x \to \infty).$$

Skewes succeeded in giving an upper bound (a huge number) as to where the first change of sign occurs. The frequency of the sign changes was considered by Pólya, Ingham, Turán, Knapowski, Levinson, Pintz and others. There are also estimates concerning $\psi(x; q, a_1) - \psi(x; q, a_2)$ (or with π instead of ψ) originated by Landau, Ingham, Pólya and continued by Turán, Knapowski, Staś, Wiertelak. In these estimates generally q is taken to be either fixed or very small compared to x. For all these we refer the reader to Ingham's tract [31], Turán's Collected Works Vol. 3 [53] and Pintz's review articles therein.

Section 3: The prime r-tuple conjecture was put forth by Hardy and Littlewood [25]. Gallagher's paper [14] contains results related to this conjecture. The methods of attacking problems of an additive nature, and the conjectures on the distribution of primes mentioned in this section are treated in the books by Halberstam and Richert [24], Richert [47] and Vaughan [54].

Section 4: The latest studies related to the Barban-Davenport-Halberstam theorem were conducted by Friedlander and Goldston [11], and Goldston and Vaughan [21]. For in-depth comments on the error terms in the prime number theorems we refer the reader to Friedlander's survey article [10].

Section 5: With a better choice of $r(u)$ Montgomery showed (upon RH) that at least 0.6725.. of the zeros of $\zeta(s)$ are simple. Conrey, Ghosh and Gonek [5], assuming RH and an upper bound for averages of sixth moments of Dirichlet's L-functions, improved this to $\frac{19}{27}$. The sixth moment estimate is implied by the *Generalized Lindelöf Hypothesis* that for any $\epsilon > 0$

$$L(s,\chi) \ll_\epsilon (q(1+|t|))^\epsilon \qquad (\sigma \geq \frac{1}{2}).$$

The Generalized Riemann Hypothesis implies the Generalized Lindelöf Hypothesis. The entirely different approach in [5] rests upon using appropriate Dirichlet polynomials instead of some of the Dirichlet series involved, thus being able to handle certain higher-moment calculations.

For the theory of correlation functions and relations of zeta zeros to eigenvalues of a Hermitian operator the reader may consult the book by Mehta [39]. The links between the Gaussian unitary ensemble of random matrix theory and quantum chaology are recounted by Bogomolny and Keating [2].

Section 6: For the convergence properties of $\sum \frac{x^\rho}{\rho}$ we refer the reader to Ingham [31, Chapters 4, 5]. It was remarked after Eq. (6.9) that one may consider $x \in (0,1)$ as well. In this case the explicit formula is

$$\sum_{n\le\frac{1}{x}}{}' \frac{\Lambda(n)}{n} = \log\frac{1}{x} - \gamma + \sum_{\rho}\frac{x^\rho}{\rho} - x + \frac{1}{2}\log\frac{1+x}{1-x}$$

($\sum'$ means that when $\frac{1}{x}\in\mathbb{Z}$, the term corresponding to $n=\frac{1}{x}$ is $\frac{1}{2}\frac{\Lambda(n)}{n}$; γ is Euler's constant). The series $\sum_{\gamma>0}(\frac{x^\rho}{\rho}+\frac{x^{\bar\rho}}{\bar\rho})$ exhibits a Gibbs phenomenon in the neighbourhood of the points $p^{\pm m}$. At $x=1$ this series is absolutely convergent, but no explicit formula is valid. This series cannot be boundedly convergent on either side of $x=1$ because of the logarithmic terms in (2.5) and the last formula. It is this infinite jump at $x=1$ that Jurkat [32] exploited remarkably.

Section 7: It was Selberg [51] who first obtained an upper bound (assuming RH) for the second moment on primes. The integral Selberg considered was similar to (5.9), but its integrand was damped by an extra factor of y^{-2}. The result expressed in (7.48) and other related results will be in a forthcoming paper by Goldston and Yıldırım [22].

References

[1] R.C. Baker and G. Harman - *The difference between consecutive primes*, Proc. London Math. Soc. (3) **72** (1996), 261-280.

[2] E. Bogomolny and J.P. Keating - *Random matrix theory and the Riemann zeros II: n-point correlations*, Nonlinearity **9** (1996), 911-935.

[3] J. Chen - *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica **16** (1973), 157-176.

[4] J.B. Conrey - *More than two fifths of the zeros of the Riemann zeta function are on the critical line*, J. reine angew. Math. **399** (1989), 1-26.

[5] J.B. Conrey, A. Ghosh, S.M. Gonek - *Simple zeros of the Riemann zeta-function*, Proc. London Math. Soc. (3) **76** (1998), 497-522.

[6] H. Cramér - *Some theorems concerning prime numbers*, Ark. Mat. Astronom. Fys. **15** (1920), 1-32.

[7] H. Cramér - *On the order of magnitude of the difference between consecutive prime numbers*, Acta Arithmetica **2** (1937), 23-46.

[8] H. Davenport - Multiplicative number theory (2-nd ed.), Springer-Verlag, 1980.

[9] D.W. Farmer - *Mean values of $\frac{\zeta'}{\zeta}$ and the GUE hypothesis*, Berkeley MSRI Preprint No. 017-95, (1995).

[10] J.B. Friedlander - *Irregularities in the distribution of primes*, in Advances in number theory (ed.s Gouvêa and Yui), Oxford 1993, 17-30 .

[11] J.B. Friedlander and D.A. Goldston - *Variance of distribution of primes in residue classes*, Quart. J. Math. Oxford (2) **47** (1995), 313-336.

[12] J.B. Friedlander and A. Granville - *Limitations to the equi-distribution of primes I*, Ann. Math. **129** (1989), 363-382.

[13] J.B. Friedlander, A. Granville, A. Hildebrand, H. Maier - *Oscillation theorems for primes in arithmetic progressions and for sifting functions*, J. Amer. Math. Soc. **4** (1991), 25-86.

[14] P.X. Gallagher - *On the distribution of primes in short intervals*, Mathematika **23** (1976), 4-9.

[15] P.X. Gallagher - *Some consequences of the Riemann hypothesis*, Acta Arithmetica **37** (1980), 339-343.

[16] D.A. Goldston - *The second moment for prime numbers*, Quart. J. Math. Oxford (2) **35** (1984), 153-163.

[17] D.A. Goldston - *On the pair correlation conjecture for zeros of the Riemann zeta-function*, J. reine angew. Math. **385** (1988), 24-40.

[18] D.A. Goldston - *A lower bound for the second moment of primes in short intervals*, Expo. Math. **13** (1995), 366-376.

[19] D.A. Goldston and D.R. Heath-Brown - *A note on the differences between consecutive primes*, Math. Annalen **266** (1984), 317-320.

[20] D.A. Goldston and H.L. Montgomery - *Pair correlation of zeros and primes in short intervals*, in Analytic Number Theory and Diophantine Problems; Proc. of a conference at Oklahoma State Univ. 1984, (ed.s Adolphson, Conrey, Ghosh and Yager) Birkhäuser, 183-203.

[21] D.A. Goldston and R.C. Vaughan - *On the Montgomery-Hooley asymptotic formula*, preprint (1996).

[22] D.A. Goldston and C.Y. Yıldırım - *Primes in short segments of arithmetic progressions*, preprint (1997) to appear in Canad. Math. J.

[23] S.M. Gonek - *An explicit formula of Landau and its applications to the theory of the zeta-function*, in A tribute to Emil Grosswald: Number theory and related analysis (ed.s Knopp and Sheingorn) Contemporary Math. Vol. 143, AMS, Providence 1993.

[24] H. Halberstam and H.-E. Richert - Sieve Methods, Academic Press, 1974.

[25] G.H. Hardy and J.E. Littlewood - *Some problems of 'Partitio Numerorum': III. On the expression of a number as a sum of primes*, Acta Mathematica **44** (1922), 1-70.

[26] G.H. Hardy and E.M. Wright - An introduction to the theory of numbers (5-th ed.), Oxford, 1985.

[27] D.R. Heath-Brown - *Gaps between primes, and the pair correlation of zeros of the zeta-function*, Acta Arithmetica **XLI** (1982), 85-99.

[28] D.R. Heath-Brown - *The number of primes in a short interval*, J. Reine Angew. Math. **397** (1989), 162-193.

[29] D.A. Hejhal - *On the triple correlation of zeros of the zeta function*, IMRN (1994) no.7.

[30] M.N. Huxley - *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164-170.

[31] A.E. Ingham - The distribution of prime numbers, Cambridge 1932.

[32] W.B. Jurkat - *The Mertens conjecture and related general Ω-theorems*, Proc. Symp. Pure Math. **24**, AMS, Providence (1973), 147-158.

[33] E. Landau - *Über die Nullstellen der Zetafunktion* Math. Annalen **71** (1911), 548-564.

[34] A.F. Lavrik - *On the twin prime hypothesis of the theory of primes by the method of I.M. Vinogradov*, Soviet Math. Dokl. **1** (1960), 700-702.

[35] H. Maier - *Primes in short intervals*, Michigan Math. J. **32** (1985), 221-225.

[36] H.L. Montgomery - Topics in multiplicative number theory, Lecture Notes in Math. **227**, Springer, Berlin, 1971.

[37] H.L. Montgomery - *The pair correlation of zeros of the zeta function*, Proc. Symp. Pure Math. **24**, AMS, Providence (1973), 181-193.

[38] H.L. Montgomery - *Sums of independent random variables*, Proc. Queen's Number Theory Conference, Queen's University, 1979.

[39] M.L. Mehta - Random Matrices (2nd ed.), Academic Press, 1991.

[40] A.M. Odlyzko - *On the distribution of spacings between zeros of zeta functions*, Math. Comp. **48**, (1987), 273-308.

[41] A.M. Odlyzko - *The* 10^{20} *zero of the Riemann zeta function and 70 million of its neighbors*, preprint, A.T. & T., (1989).

[42] A.M. Odlyzko and H.J.J. te Riele - *Disproof of the Mertens conjecture*, J. Reine Angew. Math. **357**, (1985), 138-160.

[43] A.E. Özlük - *Pair correlation of zeros of Dirichlet L-functions*, Thesis, University of Michigan, 1982.

[44] J. Pintz - *On the remainder term of the prime number formula and the zeros of Riemann's zeta-function*, Lecture Notes in Math. **1068** (ed. H. Jager), 186-197.

[45] K. Prachar - *Generalisation of a theorem of A. Selberg on primes in short intervals*, Topics in Number Theory, Colloq. Math. Soc. János Bolyai, Vol. 13, 267-280, Debrecen, 1974.

[46] R.A. Rankin - *The difference between consecutive primes*, J. London Math. Soc. **13** (1938), 242-247.

[47] H.-E. Richert - Lectures on Sieve Methods, Tata Institute, 1976.

[48] Z. Rudnick and P. Sarnak - *The n-level correlations of zeros of the zeta function*, C.R. Acad. Sci. Paris, **319**, Série I, 1994, 1027-1032.

[49] Z. Rudnick and P. Sarnak - *Zeros of Principal L-functions and random matrix theory*, Duke Math. J. **81** no.2, (1996), 269-322.

[50] B. Saffari and R.C. Vaughan - *On the fractional parts of* $\frac{x}{n}$ *and related sequences II*, Ann. Inst. Fourier (Grenoble) **27** no.2, (1977), 1-30.

[51] A. Selberg - *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** no.6, (1943), 87-105.

[52] E.C. Titchmarsh - The theory of the Riemann Zeta-function (2-nd. ed.) Oxford, 1986.

[53] P. Turán - Collected papers of Paul Turán Vol. 3, (ed. P. Erdös), Akademiai Kiado, Budapest, 1990.

[54] R.C. Vaughan - The Hardy-Littlewood method, Cambridge, 1981.

[55] C.Y. Yıldırım - *The pair correlation of zeros of Dirichlet L-functions and primes in arithmetic progressions*, Manuscripta Mathematica **72** (1991), 325-334.

Cem Yalçın Yıldırım
Mathematics Department
Bilkent University
Ankara 06533
Turkey
e-mail: `yalcin@fen.bilkent.edu.tr`