# 49

# Decentralized Control




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## 49.1 Introduction

The complexity and high performance requirements of present-day industrial processes place increasing demands on control technology. The orthodox concept of driving a large system by a central computer has become unattractive for either economic or reliability reasons. New emerging notions are subsystems, interconnections, distributed computing, parallel processing, and information constraints, to mention a few. In complex systems, where databases are developed around the plants with distributed sources of data, a need for fast control action in response to local inputs and perturbations dictates the use of distributed (that is, decentralized) information and control structures.

The accumulated experience in controlling complex industrial processes suggests three basic reasons for using decentralized control structures:

1. dimensionality,
2. information structure constraints, and
3. uncertainty.

Because the amount of computation required to analyze and control a system of large dimension grows faster than its size, it is beneficial to decompose the system into subsystems, and design controls for each subsystem independently based on the local subsystem dynamics and its interconnections. In this way, special structural features of a system can be used to devise feasible and efficient decentralized strategies for solving large control problems previously impractical to solve by "one–shot" centralized methods.

A restriction on what and where the information is delivered in a system is a standard feature of interconnected systems. For example, the standard automatic generation control in power systems is decentralized because of the cost of excessive information requirements imposed by a centralized control strategy over distant geographic areas. The structural constraints on information make the centralized methods for control and estimation design difficult to apply, even to systems with small dimensions.

It is a common assumption that neither the internal nor the external nature of complex systems can be known precisely in deterministic or stochastic terms. The essential uncertainty resides in the interconnections between different parts of the system (subsystems). The local characteristics of each individual subsystem can be satisfactorily modeled in most practical situations. Decentralized control strategies are inherently robust with respect to a wide variety of structured and unstructured perturbations in the interconnections. The strategies can be made reliable to both interconnection and controller failures involving individual subsystems.

In decentralized control design, it is customary to use a wide variety of disparate methods and techniques that originated in system and control theory. Graph–theoretic methods have been devised to identify the special structural features of the system, which may help us cope with dimensionality problems and formulate a suitable decentralized control strategy. The concept of vector Liapunov functions, each component of which determines the stability of a part of the system where others do not, is a powerful method for the stability analysis of large interconnected systems. Stochastic modeling and decentralized control


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have been used in a broad range of situations, involving LQG design, Kalman filtering, Markov processes, and stability analysis and design. Robustness considerations of decentralized control have been carried out since the early stages of its evolution, often preceding a similar development in the centralized control theory. Especially popular have been the adaptive decentralized schemes because of their flexibility and ability to cope efficiently with perturbations in both the interactions and the subsystems of a large system.

The objective of this chapter is to introduce the concept and methods of decentralized control. Due to a large number of results and techniques available, only the basic theory and practice of decentralized control will be reviewed. At the end of the chapter is a discussion of the larger background listing the books and survey papers on the subject. References related to more sophisticated treatment of decentralized control and the relevant applications are also discussed.

## 49.2 The Decentralized Control Problem

To introduce the decentralized control problem, consider two inverted penduli coupled by a spring as shown in Figure 49.1. The control objective is to keep the penduli in the upright position by applying feedback control via the inputs $u_1$ and $u_2$. The linearized equations of motion in the vicinity of $\theta_1 = \theta_2 = 0$ are

$$\begin{aligned} m\ell^2\ddot{\theta}_1 &= mg\ell\theta_1 - ka^2(\theta_1 - \theta_2) + u_1, \\ m\ell^2\ddot{\theta}_2 &= mg\ell\theta_2 - ka^2(\theta_2 - \theta_1) + u_2. \end{aligned} \tag{49.1}$$

By choosing the state vector $x = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)^T$ and the input vector $u = (u_1, u_2)^T$, the state space representation of the system is

$$\begin{aligned} \mathbf{S}:\ \dot{x} &= \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ \frac{g}{\ell} - \frac{ka^2}{m\ell^2} & 0 & \frac{ka^2}{m\ell^2} & 0 \\ \hline 0 & 0 & 0 & 1 \\ \frac{ka^2}{m\ell^2} & 0 & \frac{g}{\ell} - \frac{ka^2}{m\ell^2} & 0 \end{array}\right] x \\ &+ \left[\begin{array}{c|c} 0 & 0 \\ \frac{1}{m\ell^2} & 0 \\ \hline 0 & 0 \\ 0 & \frac{1}{m\ell^2} \end{array}\right] u\,. \end{aligned} \tag{49.2}$$

The fundamental restriction in choosing the feedback laws to control the system S is that each input $u_1$ and $u_2$ can depend only on the local states $x_1 = (\theta_1, \dot{\theta}_1)^T$ and $x_2 = (\theta_2, \dot{\theta}_2)^T$ of the corresponding penduli, that is, $u_1 = u_1(x_1)$ and $u_2 = u_2(x_2)$. This restriction is called the *decentralized information structure constraint.*

Since the system S is linear, a natural choice is the linear control laws

$$u_1 = k_1^T x_1\,,\quad u_2 = k_2^T x_2 \tag{49.3}$$

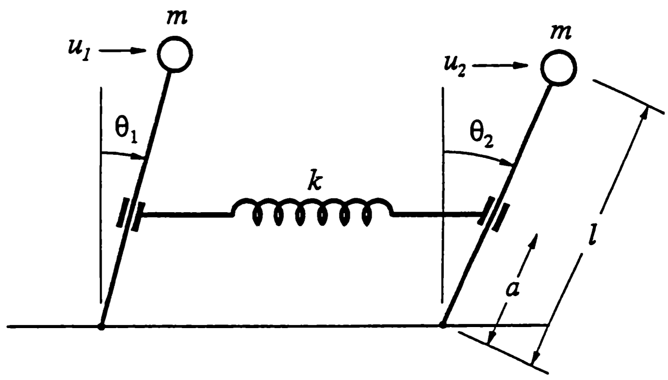


**Figure 49.1** Inverted penduli.

where the feedback gain vectors $k_1 = (k_{11}, k_{12})^T$ and $k_2 = (k_{21}, k_{22})^T$ should be selected to *stabilize* the system S, that is, hold the penduli in the upright position.

In control design, it is fruitful to recognize the structure of the system S as an interconnection

$$\begin{aligned} \mathbf{S}:\ \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_1 \\ &\quad + e\begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_1 + e\begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_2, \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_1 \\ &\quad + e\begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_1 + e\begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_2, \end{aligned} \tag{49.4}$$

of two subsystems

$$\begin{aligned} \mathbf{S}_1:\ \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_1, \\ \mathbf{S}_2:\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u_2, \end{aligned} \tag{49.5}$$

where $\alpha = g/\ell$, $\beta = 1/m\ell^2$, $\gamma = \bar{a}^2k/m\ell^2$, and $e = (a/\bar{a})^2$. One reason is that, in designing control for interconnected systems, the designer has to account for essential *uncertainty* in the interconnections among the subsystems. Though models of the subsystems are commonly available with sufficient accuracy, the shape and size of the interconnections cannot be predicted satisfactorily either for modeling or operational reasons. In the example, the interconnection parameter $e = a/\bar{a}$ is the uncertain height of the spring which is normalized by its nominal value $\bar{a}$.

An equally important reason for decomposition is present when controlling large dynamic systems. In complex systems with many variables, most of the variables are *weakly coupled,* if coupled at all, and the behavior of the overall system is dominated by strongly connected variables. Considerable conceptual and numerical simplification can be gained by controlling the strongly coupled variables with decentralized control.

## 49.3 Plant and Feedback Structures

Consider a linear constant system

$$\mathbf{S}\colon\ \begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{49.6}$$

as an interconnected system

$$\mathbf{S}\colon\ \begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + \sum_{j \in \mathcal{N}} (A_{ij} x_j + B_{ij} u_j), \\ y_i &= C_i x_i + \sum_{j \in \mathcal{N}} C_{ij} x_j, \quad i \in \mathcal{N}, \end{aligned} \tag{49.7}$$

which is composed of $N$ subsystems

$$\mathbf{S}_i\colon\ \begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i, \\ y_i &= C_i x_i, \quad i \in \mathcal{N}, \end{aligned} \tag{49.8}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $y_i(t) \in \mathbb{R}^{\ell_i}$ are the state, input, and output of the subsystem $\mathbf{S}_i$ at a fixed time $t \in \mathbb{R}$. All matrices have proper dimensions, and $\mathcal{N} = \{1, 2, \ldots, N\}$. At present we are interested in *disjoint* decompositions, that is,

$$\begin{aligned} x &= (x_1^T, x_2^T, \ldots, x_N^T)^T, \\ u &= (u_1^T, u_2^T, \ldots, u_N^T)^T, \\ y &= (y_1^T, y_2^T, \ldots, y_N^T)^T, \end{aligned} \tag{49.9}$$

and where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^\ell$ are the state, input, and output of the overall system $\mathbf{S}$, so that

$$\begin{aligned} \mathbb{R}^n &= \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_N}, \\ \mathbb{R}^m &= \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \ldots \times \mathbb{R}^{m_N}, \\ \mathbb{R}^\ell &= \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \times \ldots \times \mathbb{R}^{\ell_N}. \end{aligned} \tag{49.10}$$

A compact description of the interconnected system $\mathbf{S}$ is

$$\mathbf{S}\colon\ \begin{aligned} \dot{x} &= A_D x + B_D u + A_C x + B_C u \\ y &= C_D x + C_C x, \end{aligned} \tag{49.11}$$

where

$$\begin{aligned} A_D &= \text{diag}\{A_1, A_2, \ldots, A_N\}, \\ B_D &= \text{diag}\{B_1, B_2, \ldots, B_N\}, \\ C_D &= \text{diag}\{C_1, C_2, \ldots, C_N\}, \end{aligned} \tag{49.12}$$

and the coupling block matrices are

$$A_C = (A_{ij}), \quad B_C = (B_{ij}), \quad C_C = (C_{ij}). \tag{49.13}$$

The collection of $N$ decoupled subsystems is described by

$$\mathbf{S}_D\colon\ \begin{aligned} \dot{x} &= A_D x + B_D u \\ y &= C_D x, \end{aligned} \tag{49.14}$$

obtained from (49.11) by setting the coupling matrices to zero.

Important special classes of interconnected systems are input ($B_C = 0$) and output ($C_C = 0$) decentralized systems, where inputs and outputs are not shared among the subsystems. Input–output decentralized systems are described as

$$\mathbf{S}\colon\ \begin{aligned} \dot{x} &= A_D x + B_D u + A_C x \\ y &= C_D x, \end{aligned} \tag{49.15}$$

where both $B_C$ and $C_C$ are zero. This structural feature helps to a great extent when decentralized controllers and estimators are designed for large plants.

A *static decentralized state feedback*,

$$u = -K_D x, \tag{49.16}$$

is characterized by a block–diagonal gain matrix,

$$K_D = \text{diag}\{K_1, K_2, \ldots, K_N\}, \tag{49.17}$$

which implies that each subsystem $\mathbf{S}_i$ has its individual control law,

$$u_i = -K_i x_i, \quad i \in \mathcal{N}, \tag{49.18}$$

with a constant gain matrix $K_i$. The control law $u$ of (49.16), which is equivalent to the totality of subsystem control laws (49.18), obeys the decentralized information structure constraint requiring that each subsystem $\mathbf{S}_i$ is controlled on the basis of its locally available state $x_i$. The closed–loop system is described as

$$\hat{\mathbf{S}}\colon\ \dot{x} = (A_D - B_D K_D C_D)x + A_C x. \tag{49.19}$$

When *dynamic output feedback* is used under decentralized constraints, then controllers of the following type are considered:

$$\mathbf{C}_i\colon\ \begin{aligned} \dot{z}_i &= F_i z_i + G_i y_i, \\ u_i &= -H_i z_i - K_i y_i, \quad i \in \mathcal{N}, \end{aligned} \tag{49.20}$$

which can be written in a compact form as a single decentralized controller defined as

$$\mathbf{C}_D\colon\ \begin{aligned} \dot{z} &= F_D z + G_D y, \\ u &= -H_D z - K_D y, \end{aligned} \tag{49.21}$$

where

$$\begin{aligned} z &= (z_1^T, z_2^T, \ldots, z_N^T)^T, \quad y = (y_1^T, y_2^T, \ldots, y_N^T)^T, \\ u &= (u_1^T, u_2^T, \ldots, u_N^T)^T, \end{aligned} \tag{49.22}$$

are the state $z \in \mathbb{R}^r$, input $y \in \mathbb{R}^\ell$, and output $u \in \mathbb{R}^m$ of the controller $\mathbf{C}_D$. By combining the system $\mathbf{S}$ and the decentralized dynamic controller $\mathbf{C}_D$, we get the composite closed–loop system as

$$\mathbf{S}\&\mathbf{C}_D\colon\ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \tag{49.23}$$
$$\begin{bmatrix} A_D - B_D K_D C_D + A_C & -B_D H_D \\ G_D C_D & F_D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

## 49.4 Decentralized Stabilization

The fundamental problem in decentralized control theory and practice is choosing individual subsystem inputs to stabilize the overall interconnected system. In the previous section, the plant structures have been described, where the plant, inputs and outputs are all decomposed with each local controller responsible for the corresponding subsystem. While this is the most common situation in practice, it is by no means all inclusive. It is often advantageous, and sometime necessary, to decentralize the inputs and outputs without decomposing the plant. This is the situation that we consider first.

### 49.4.1 Decentralized Inputs and Outputs

Suppose that only the inputs and outputs, but not states, of system S in (49.6) are partitioned as in (49.9), and S is described as

$$\begin{aligned} \mathbf{S}\colon \dot{x} &= Ax + \sum_{i\in\mathcal{N}} \tilde{B}_i u_i, \\ y_i &= \tilde{C}_i x\,,\ i \in \mathcal{N}. \end{aligned} \tag{49.24}$$

Then, the controllers $\mathbf{C}_i$ of (49.20) still operate on local measurements $y_i$ to generate local controls $u_i$, but now they are collectively responsible for the whole system. In this case,

$$\mathbf{S\&C}_D\colon \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - BK_DC & -BH_D \\ G_DC & F_D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{49.25}$$

It is well–known that without the decentralization constraint on the controller, the closed–loop system of (49.25) can be stabilized if, and only if, the uncontrollable or unobservable modes of the open–loop system S are stable; or equivalently, the set of (centralized) fixed modes of S, which is defined as

$$\Lambda_C = \bigcap_K \sigma(A - BKC) \tag{49.26}$$

is included in the open left half plane, where $\sigma(.)$ denotes the set of eigenvalues of the indicated matrix. This basic result has been extended in [34] to decentralized control of S, where it was shown that the closed–loop system (49.25) can be made stable with suitable choice of the decentralized controllers $\mathbf{C}_i$ if, and only if, the set of decentralized fixed modes

$$\begin{aligned} \Lambda_D &= \bigcap_{K_D} \sigma(A - BK_DC) \\ &= \bigcap_{K_1,\ldots,K_N} \sigma(A - \sum_{i\in\mathcal{N}} \tilde{B}_i K_i \tilde{C}_i) \end{aligned} \tag{49.27}$$

is included in the open left half plane.

The result of [34] has been followed by extensive research on the following topics:

- state–space and frequency domain characterization of decentralized fixed modes,
- development of various techniques for designing decentralized controllers (*e.g.*, using static output feedback in all but one channel, distributing the control effort among channels, sequential stabilization, etc.)
- generalization of the concept of decentralized fixed modes to arbitrary feedback structure constraints,
- formulation of the concept of structurally fixed modes, and their algebraic and graph–theoretical characterization.

A useful and simple characterization of decentralized fixed modes was provided in [1]. For any subset $\mathcal{I} = \{i_1, \ldots, i_P\}$ of the index set $\mathcal{N}$, let $\mathcal{I}^C = \{j_1, \ldots, j_{N-P}\}$ denote the complement of $\mathcal{I}$ in $\mathcal{N}$, and define

$$\begin{aligned} \tilde{B}_{\mathcal{I}} &= [\tilde{B}_{i_1}, \tilde{B}_{i_2}, \ldots, \tilde{B}_{i_P}]\,, \\ \tilde{C}_{\mathcal{I}^C} &= \begin{bmatrix} \tilde{C}_{j_1} \\ \tilde{C}_{j_2} \\ \vdots \\ \tilde{C}_{j_{N-P}} \end{bmatrix}. \end{aligned} \tag{49.28}$$

Then a complex number $\lambda \in \mathbb{C}$ is a decentralized fixed mode of S if, and only if,

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & \tilde{B}_{\mathcal{I}} \\ \tilde{C}_{\mathcal{I}^C} & 0 \end{bmatrix} < n \tag{49.29}$$

for some $\mathcal{I} \subset \mathcal{N}$. This result relates decentralized fixed modes to transmission zeros of the systems $(A, \tilde{B}_{\mathcal{I}}, \tilde{C}_{\mathcal{I}^C})$, called the complementary subsystems. Thus, appearance of a fixed mode corresponds to a special pole–zero cancellation, which can not be removed by constant decentralized feedback. However, under mild conditions, such fixed modes can be eliminated by time–varying decentralized feedback.

The characterization of decentralized fixed modes above prompts a generalization of the concept to arbitrary feedback structures. Let $\bar{K} = (\bar{k}_{ij})$ be an $m \times l$ binary matrix such that $\bar{k}_{ij} = 1$ if, and only if, a feedback link from output $y_i$ to input $u_i$ is allowed. Thus $\bar{K}$ specifies a constraint on the feedback structure, a special case of which is decentralized feedback. In this case, permissible controllers have the structure

$$\begin{aligned} \mathbf{C}_{\bar{K}}\colon \dot{z}_i &= F_i z_i + \sum_{j\in\mathcal{J}_i} g_{ij} y_j \\ u_i &= -h_i^T z_i - \sum_{j\in\mathcal{J}_i} k_{ij} y_i \end{aligned} \tag{49.30}$$

where $\mathcal{J}_i = \{j\colon \bar{k}_{ij} = 1\}$.

Let $K$ denote any feedback matrix conforming to the structure of $\bar{K}$, that is, one with $k_{ij} = 0$ whenever $\bar{k}_{ij} = 0$. Then, the set

$$\Lambda_{\bar{K}} = \bigcap_K \sigma(A - BKC) \tag{49.31}$$

can conveniently be defined as the set of fixed modes with respect to the decentralized feedback structure constraint specified by $\bar{K}$. Then the closed–loop system consisting of S and the constrained controller $\mathbf{C}_{\bar{K}}$ can be stabilized if, and only if, $\Lambda_{\bar{K}}$ is included in the open left half-plane. Finally, it remains to characterize $\Lambda_{\bar{K}}$ as in (49.29). This, however, is quite automatic; consider the index sets $\mathcal{I} \subset \mathcal{M} = \{1, 2, \ldots, M\}$ and replace $\mathcal{I}^C$ by $\mathcal{J} = \cup_{i\in\mathcal{I}^C} \mathcal{J}_i$, where now $\mathcal{I}^C$ refers to the complement of $\mathcal{I}$ in $\mathcal{M}$.

### 49.4.2 Structural Analysis

Structural analysis of large scale systems via graph–theoretic concepts and methods offers an appealing alternative to quantitative analysis which often faces difficulties due to high dimensionality and lack of exact knowledge of system parameters. Equipped with the powerful tools of graph theory, structural analysis provides valuable information concerning certain qualitative properties of the system under study by practical tests and algorithms [30].

One of the earliest problems of structural analysis is the graph–theoretic formulation of controllability [20]. Consider an uncontrollable pair $(A, B)$. Loss of controllability is either due to a perfect matching of system parameters or due to an insufficient number of nonzero parameters, indicating a lack of sufficient linkage among system variables. In the latter case, the pair $(A, B)$ is structurally uncontrollable in the sense that all pairs having the same structure as $(A, B)$ are uncontrollable. Since the structure of $(A, B)$ can be described by a directed graph (as explained below for a more general case), structural controllability can be checked by graph–theoretic means. Indeed, $(A, B)$ is structurally controllable if, and only if, the system graph is input reachable (that is, each state variable is affected directly or indirectly by at least one input variable), and contains no dilations (that is, no subset of state variables exists whose number exceeds the total number of all state and input variables directly affecting these variables). These two conditions are equivalent to the spanning of the system graph by a minimal subgraph, called a cactus, which has a special structure.

The idea of treating controllability in a structural framework has led to formulation and graph–theoretic characterization of *structurally fixed modes* under constrained feedback [26]. Let $\mathbf{D} = (\mathcal{V}, \mathcal{E})$ be a directed graph associated with the system $\mathbf{S}$ of (49.6), where $\mathcal{V} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}$ is a set of vertices corresponding to inputs, states, and outputs of $\mathbf{S}$, and $\mathcal{E}$ is a set of directed edges corresponding to nonzero parameters of the system matrices $A$, $B$, and $C$. To every nonzero $a_{ij}$, there corresponds an edge from vertex $x_j$ to vertex $x_i$, to every nonzero $b_{ij}$, an edge from $u_j$ to $x_i$, and to every nonzero $c_{ij}$, one from $x_j$ to $y_i$. Given a feedback pattern $\bar{K}$ and adding to $\mathbf{D}$ a feedback edge from $y_j$ to $u_i$ for every $\bar{k}_{ij} = 1$, one gets a digraph $\mathbf{D}_{\bar{K}} = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_{\bar{K}})$ completely describing the structure of both the system $\mathbf{S}$ and the feedback constraint specified by $\bar{K}$.

Two systems are said to be structurally equivalent if they have the same system graphs. A system $\mathbf{S}$ is said to have structurally fixed modes with respect to a given $\bar{K}$ if every system structurally equivalent to $\mathbf{S}$ has fixed modes with respect to $\bar{K}$. Having structurally fixed modes is a common property of a class of systems described by the same system graph; if a system has no structurally fixed modes, then either it has no fixed modes, or if it does, arbitrarily small perturbations of system parameters can eliminate the fixed modes. As a result, if a system has no structurally fixed modes with respect to $\bar{K}$, then generically it can be stabilized by a constrained controller of the form defined in (49.30).

It was shown in [26] that a system $\mathbf{S}$ has no structurally fixed modes with respect to a feedback pattern $\bar{K}$ if, and only if

1. all state vertices of $\mathbf{D}_{\bar{K}}$ are covered by vertex disjoint cycles, and
2. no strong component of $\mathbf{D}_{\bar{K}}$ contains only state vertices, where a strong component is a maximal subgraph whose vertices are reachable from each other.

This simple graph–theoretic criterion has been used in an algorithmic way in problems such as choosing a minimum number of feedback links (or, if each feedback link is associated with a cost, choosing the cheapest feedback pattern) that avoid structurally fixed modes. As an example, consider a system with a system graph as in Figure 49.2. Let the costs of setting up feedback links (dotted lines) from each output to each input be given by a matrix

$$\begin{bmatrix} 6 & 2 \\ 3 & 7 \end{bmatrix}.$$

It can easily be verified that any feedback pattern of the form

$$\begin{bmatrix} 1 & * \\ * & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & 1 \\ 1 & * \end{bmatrix},$$

where $*$ stands for either a 0 or a 1, avoids structurally fixed modes. Clearly, the feedback patterns which contain the least number of links and which cost the least are, respectively,

$$\bar{K}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \bar{K}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

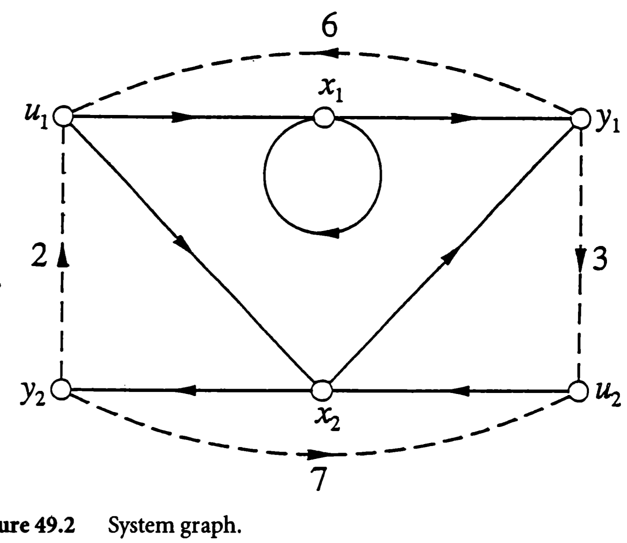


**Figure 49.2** System graph.

### 49.4.3 Decentrally Stabilizable Structures

Consider an interconnected system

$$\begin{aligned} \mathbf{S}\colon \dot{x}_i &= A_i x_i + B_i(u_i + \sum_{j \in \mathcal{N}} D_{ij} x_j) \\ y_i &= x_i \quad i \in \mathcal{N} \end{aligned} \tag{49.32}$$

which is a special case of the system $\mathbf{S}$ in (49.7) in that $A_{ij} = B_i D_{ij}$, $B_{ij} = 0$, $C_i = I$, and $C_{ij} = 0$. Assuming that the decoupled subsystems described by the pairs $(A_i, B_i)$ are controllable, it is easy to verify that S has no decentralized fixed modes. Thus S can be stabilized using a decentralized dynamic feedback controller of the form (49.21). However, because the subsystem outputs are the states, there should be no need to use dynamic controllers.

Choose the decentralized constant state feedbacks in (49.18) to place the subsystem poles at $-\mu_{il}\rho, i \in \mathcal{N}, l = 1, 2, \ldots, n_i$, where $-\mu_{il}$ are distinct negative real numbers, and $\rho$ is a parameter. Then a suitable change of coordinate frame transforms the closed-loop system of (49.19) into the form

$$\hat{\mathbf{S}}\colon \dot{x} = (-\rho M + \hat{A}_C)x\ , \tag{49.33}$$

where $M = \text{diag}\{M_1, M_2, \ldots, M_N\}$, with $M_i = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_{in_i}\}$, and $\hat{A}_C$ is independent of the parameter $\rho$. Clearly, $\hat{\mathbf{S}}$ is stable for a sufficiently large $\rho$.

The success of this high–gain decentralized stabilization technique results from the special structure of the interconnections among the subsystems. The interconnections from other subsystems affect a particular subsystem in the same way its local input does. This makes it possible to neutralize potentially destabilizing effects of the interconnections by a local state feedback and provide a high degree of stability to the decoupled subsystems. This special interconnection structure is termed the "matching conditions" [18].

Decentralized stabilizability of interconnected systems satisfying the matching conditions has motivated research in characterizing other decentrally stabilizable interconnection structures. Below, another such interconnection structure is described, where single–input subsystems are considered for convenience.

Let the interconnected system be described as

$$\mathbf{S}\colon \dot{x}_i = A_i x_i + b_i u_i + \sum_{j\in\mathcal{N}} A_{ij}x_j\ ,\ \ i \in \mathcal{N} \tag{49.34}$$

where, without loss of generality, the subsystem pairs $(A_i, b_i)$ are assumed to be in controllable canonical form. For each interconnection matrix $A_{ij}$, define an integer $m_{ij}$ as

$$m_{ij} = \begin{cases} \max\{q - p : a_{pq}^{ij} \neq 0\}, & A_{ij} \neq 0, \\ -n, & A_{ij} = 0, \end{cases} \tag{49.35}$$

Thus, $m_{ij}$ is the distance between the main diagonal and a line parallel to the main diagonal which borders all nonzero elements of $A_{ij}$.

For an index set $\mathcal{I} \subset \mathcal{N}$, let $\mathcal{I}_P$ denote any permutation of $\mathcal{I}$. Then, the system S in (49.34) is stabilizable by decentralized constant state feedback if

$$\sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{I}_P}} (m_{ij} - 1) < 0 \tag{49.36}$$

for all $\mathcal{I}$ and all permutations $\mathcal{I}_P$ [14], [30]. In the case of matching interconnections, $m_{ij} = n_j - n_i$, so that (49.36) guarantees decentralized stabilizability even when the elements of the interconnection matrices $A_{ij}$ are bounded nonlinear, time–varying functions of the state variables. Therefore, the condition (49.36) and, thus, the matching conditions, are indeed structural conditions.

### 49.4.4 Vector Liapunov Functions

A general way to establish the stability of nonlinear interconnected systems is to apply the Matrosov–Bellman concept of vector Liapunov functions [17]. The concept has been developed to provide an efficient method of checking the stability of linear interconnected systems controlled by decentralized feedback [30]. First, each subsystem is stabilized using local state or output feedback. Then, for each stable closed–loop (but decoupled) subsystem, a Liapunov function is chosen using standard methods. These functions are stacked to form a vector of functions, which can then be used to form a single scalar Liapunov function for the overall system. The function establishes stability if we show positivity of the leading principal minors of a constant aggregate matrix whose dimension equals the number of subsystems.

Consider the linear interconnected system of (49.7),

$$\begin{aligned} \mathbf{S}\colon \dot{x}_i \ &= \ A_i x_i + B_i u_i \\ &\quad + \sum_{j\in\mathcal{N}} e_{ij}A_{ij}x_j\ ,\quad i \in \mathcal{N}, \end{aligned} \tag{49.37}$$

where the output $y_i$ is not included and $B_{ij} = 0$. We inserted the elements of $e_{ij} \in [0, 1]$ of the $N \times N$ interconnection matrix $E = (e_{ij})$ to capture the presence of uncertainty in coupling between the subsystems

$$\mathbf{S}_i\colon \dot{x}_i = A_i x_i + B_i u_i\ , \tag{49.38}$$

as illustrated by the example of the two penduli above.

We assume that each pair $(A_i, B_i)$ is controllable and assign the eigenvalues $-\sigma_1^i \pm j\omega_1^i, \ldots, -\sigma_{p_i}^i \pm j\omega_{p_i}^i, \ldots, -\sigma_{2p_i+1}^i, \ldots, -\sigma_{n_i}^i$ to each closed–loop subsystem

$$\hat{\mathbf{S}}_i\colon \dot{x}_i = (A_i - B_iK_i)x_i \tag{49.39}$$

by applying decentralized feedback

$$u_i = -K_i x_i\ . \tag{49.40}$$

Using a nonsingular transformation,

$$x_i = T_i\tilde{x}_i, \tag{49.41}$$

we can obtain the closed–loop subsystems as

$$\tilde{\mathbf{S}}_i\colon \dot{\tilde{x}}_i = \Lambda_i\tilde{x}_i\ , \tag{49.42}$$

where the matrix $\Lambda_i = T_i^{-1}(A_i - B_iK_i)T_i$ has the diagonal form

$$\begin{aligned} \Lambda_i \ = \ \text{diag}\Bigg\{ & \begin{bmatrix} -\sigma_1^i & \omega_1^i \\ -\omega_1^i & -\sigma_1^i \end{bmatrix}, \ldots, \begin{bmatrix} -\sigma_{p_i}^i & \omega_{p_i}^i \\ -\omega_{p_i}^i & -\sigma_{p_i}^i \end{bmatrix}, \\ & -\sigma_{2p_i+1}, \ldots, -\sigma_{n_i} \Bigg\}\ . \end{aligned} \tag{49.43}$$

For each transformed subsystem, there exists a suitable Liapunov function $v$: $\mathbb{R}^{n_i} \to \mathbb{R}_+$ of the form

$$v_i(\tilde{x}_i) = (\tilde{x}_i H_i \tilde{x}_i)^{\frac{1}{2}}, \tag{49.44}$$

where $H_i = I_i$ is the solution of the Liapunov matrix equation

$$\Lambda_i H_i + H_i \Lambda_i = -G_i \tag{49.45}$$

for $G_i = \text{diag}\{\sigma_1^i, \sigma_1^i, \ldots, \sigma_{p_i}^i, \sigma_{2p_i+1}^i \ldots, \sigma_{n_i}^i\}$.

To determine the stability of the overall interconnected closed–loop system

$$\tilde{\mathbf{S}}\colon \dot{\tilde{x}}_i = \Lambda_i \tilde{x}_i + \sum_{j \in \mathcal{N}} e_{ij} \Delta_{ij} \tilde{x}_j \tag{49.46}$$

from the stability of the decoupled closed–loop subsystems $\tilde{\mathbf{S}}_i$, we consider subsystem functions $v_i$ as components of a *vector Liapunov function* $v = (v_1, v_2, \ldots, v_N)^T$, and form a candidate Liapunov function $V$: $\mathbb{R}^n \to \mathbb{R}_+$ for the overall system $\tilde{\mathbf{S}}$ as

$$V(\tilde{x}) = \sum_{i \in \mathcal{N}} d_i v_i(\tilde{x}_i), \tag{49.47}$$

where the existence of positive numbers $d_i$ for stability of $\tilde{\mathbf{S}}$ has yet to be established, and $\Delta_{ij} = T_i^{-1} A_{ij} T_j$.

Taking the total time derivative of $V(\tilde{x})$ with respect to $\tilde{\mathbf{S}}$, after lengthy but straightforward computations [30],

$$\dot{V}(\tilde{x}) \leq -d^T \bar{W} z, \tag{49.48}$$

with $d = (d_1, d_2, \ldots, d_N)^T$, $z = (\|\tilde{x}_1\|, \|\tilde{x}_2\|, \ldots, \|\tilde{x}_N\|)^T$, and $\bar{W} = (\bar{w}_{ij})$ is the $N \times N$ aggregate matrix defined as

$$\bar{w}_{ij} = \begin{cases} \frac{1}{2}\sigma_m^i - \bar{e}_{ii}\lambda_M^{1/2}(\Delta_{ii}^T \Delta_{ii}), & i = j \\ -\bar{e}_{ij}\lambda_M^{1/2}(\Delta_{ij}^T \Delta_{ij}), & i \neq j \end{cases} \tag{49.49}$$

where $\sigma_m^i$ is the minimal value of all $\sigma_k^i$, and $\lambda_M(\cdot)$ is the maximal eigenvalue of the indicated matrix.

The elements $\bar{e}_{ij}$ of the fundamental interconnection matrix $\bar{E} = (\bar{e}_{ij})$ are binary numbers defined as

$$\bar{e}_{ij} = \begin{cases} 1, & \mathbf{S}_j \text{ acts on } \mathbf{S}_i \\ 0, & \mathbf{S}_j \text{ does not act on } \mathbf{S}_i . \end{cases} \tag{49.50}$$

In this way, the binary matrix describes the basic interconnection structure of the system S. In the case of two penduli,

$$\bar{E} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{49.51}$$

It has been shown in [30] that stability of $-\bar{W}$ (all eigenvalues of $-\bar{W}$ have negative real parts) implies stability of the closed–loop system $\tilde{\mathbf{S}}$ and, hence, $\hat{\mathbf{S}}$. To explain this fact, we note first that $w_{ii} > 0$, $w_{ij} \leq 0$ $(i \neq j)$, which makes $\bar{W}$ an $M$–matrix (*e.g.*, [30]) if, and only if, there exists a positive vector $d$ $(d_i > 0,\ i \in \mathcal{N})$, so that the vector

$$c^T = d^T \bar{W} \tag{49.52}$$

is a positive vector as well. Positivity of $c$ and $d$ imply $V(\tilde{x}) > 0$ and $\dot{V}(\tilde{x}) < 0$ and, therefore, stability of $\hat{\mathbf{S}}$ by the standard Liapunov argument. Finally, the $M$–matrix property of $\bar{W}$ is equivalent to stability of $-\bar{W}$.

Several comments are in order. First, we note that the $M$–matrix property of $\bar{W}$ can be tested by a simple determinantal condition

$$\begin{vmatrix} \bar{w}_{11} & \bar{w}_{12} & \cdots & \bar{w}_{1k} \\ \bar{w}_{21} & \bar{w}_{22} & \cdots & \bar{w}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{w}_{k1} & \bar{w}_{k2} & \cdots & \bar{w}_{kk} \end{vmatrix} > 0, \quad k \in \mathcal{N}. \tag{49.53}$$

Another important feature of the concept of vector Liapunov functions is the *robustness* information about decentrally stabilized interconnected system $\hat{\mathbf{S}}$. The determinantal condition (49.53) is equivalent to the quasidominant diagonal property of $\bar{W}$,

$$\bar{w}_{ii} > d_i^{-1} \sum_{j \neq i}^{N} d_j |\bar{w}_{ij}|, \quad i \in \mathcal{N}. \tag{49.54}$$

where the $d_i$'s are positive numbers. From (49.54), it is obvious that, if $\bar{W}$ is an $M$–matrix, so is $W$ for any $E \leq \bar{E}$, where the inequality is taken element by element; the system $\hat{\mathbf{S}}$ is *connectively stable* [30]. When a system is connectively stabilized by decentralized feedback, stability is robust and can tolerate variations in coupling among the subsystems. When the two penduli are stabilized for any given position $\bar{a}$ of the spring, including the entire length $\ell$ of the penduli, the penduli are stable for any position $a \leq \bar{a}$. In other words, if the penduli are stabilized for the fundamental interconnection matrix $\bar{E}$ of (51), they are stabilized for any interconnection matrix

$$E = \begin{bmatrix} e & e \\ e & e \end{bmatrix}, \tag{49.55}$$

whenever $e \in [0, 1]$.

Finally, the decentrally stabilized system can tolerate nonlinearities in the interconnections among the subsystems. The nonlinear interconnections need not be known since only their size is required to be limited. Once the closed–loop system $\hat{\mathbf{S}}$ is shown to be stable, it follows [30] that a nonlinear time–varying version

$$\hat{\mathbf{S}}_N\colon \dot{\tilde{x}}_i = (A_i - B_i K_i)\tilde{x}_i + h_i(t, \tilde{x}), \quad i \in \mathcal{N} \tag{49.56}$$

of $\hat{\mathbf{S}}$ is connectively stable, provided the conical constraints

$$\|h_i(t, \tilde{x})\| \leq \sum_{j=1}^{N} \bar{e}_{ij} \xi_{ij} \|\tilde{x}_j\|, \quad i \in \mathcal{N} \tag{49.57}$$

on interconnection functions $h_i$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n_i}$ hold, where the nonnegative numbers $\xi_{ij}$ do not exceed $\lambda_M^{1/2}(\Delta_{ij}^T \Delta_{ij})$. This robustness result is useful in practice because, typically, interconnections are poorly known, or they are changing during operation of the controlled system.

## 49.5 Optimization

There is no general method for designing optimal decentralized controls for interconnected systems, even if they are linear and time invariant. For this reason, standard design practice is to optimize each decoupled subsystem using Linear Quadratic (LQ) control laws. Then, suboptimality of the interconnected closed–loop system, which is driven by the union of the locally optimal LQ control laws, is determined with respect to the sum of the quadratic costs chosen for the subsystems. The suboptimal decentralized control design is attractive because, under relatively mild conditions, suboptimality implies stability. Furthermore, the degree of suboptimality can serve as a measure of robustness with respect to a wide spectrum of uncertainties residing in both the subsystems and their interactions.

Consider again the interconnected system

$$\mathbf{S}\text{: } \dot{x}_i = A_i x_i + B_i u_i + \sum_{i \in \mathcal{N}} A_{ij} x_j \, , \quad i \in \mathcal{N} \tag{49.58}$$

in the compact form

$$\mathbf{S}\text{: } \dot{x} = A_D x + B_D u + A_C x \, . \tag{49.59}$$

We assume that the subsystems

$$\mathbf{S}_i\text{: } \dot{x}_i = A_i x_i + B_i u_i \tag{49.60}$$

or, equivalently, their union

$$\mathbf{S}\text{: } \dot{x} = A_D x + B_D u \, , \tag{49.61}$$

is controllable, that is, all pairs $(A_i, B_i)$ are controllable.

With $\mathbf{S}_D$ we associate a quadratic cost

$$J_D(x_0, u) = \int_0^\infty (x^T Q_D x + u^T R_D u) dt \, , \tag{49.62}$$

where $Q_D = \text{diag}\{Q_1, Q_2, \ldots, Q_N\}$ is a symmetric nonnegative definite matrix, $R_D = \text{diag}\{R_1, R_2, \ldots, R_N\}$ is a symmetric positive definite matrix, and the pair $(A_D, Q_D^{1/2})$ is observable. The cost $J_D$ can be considered as a sum of subsystem costs

$$J_i(x_{i0}, u_i) = \int_0^\infty (x_i^T Q_i x_i + u_i^T R u_i) dt \, . \tag{49.63}$$

In order to satisfy the decentralized constraints on the control law, we solve the standard LQ optimal control problem $(\mathbf{S}_D, J_D)$ to get

$$u_D^\odot = -K_D x \, , \tag{49.64}$$

where $K_D = \text{diag}\{K_1, K_2, \ldots, K_N\}$ is given as

$$K_D = R_D^{-1} B_D^T P_D \, ,$$

and $P_D = \text{diag}\{P_1, P_2, \ldots, P_N\}$ is the unique symmetric positive definite solution of the algebraic Riccati equation

$$A_D^T P_D + P_D A_D - P_D B_D R_D^{-1} B_D^T P_D + Q_D = 0 \, . \tag{49.65}$$

The control $u_D^\odot$, when applied to $\mathbf{S}_D$, results in the closed–loop system

$$\hat{\mathbf{S}}_D^\odot\text{: } \dot{x} = (A_D - B_D K_D) x \, , \tag{49.66}$$

which is optimal and produces the optimal cost

$$J_D^\odot(x_0) = x_0^T P_D x_0 \, . \tag{49.67}$$

The important fact about the locally optimal control $u_D^\odot$ is that it is decentralized. Each component

$$u_i^\odot = -K_i x_i \tag{49.68}$$

of $u_D^\odot$ uses only the local state $x_i$. Generally, the proposed control strategy is not globally optimal, but we can proceed to determine if the cost $J_D^\oplus(x_0)$ corresponding to the closed–loop interconnected system

$$\hat{\mathbf{S}}^\oplus\text{: } \dot{x} = (A_D - B_D K_D + A_C) x \tag{49.69}$$

is finite. If it is, then $\mathbf{S}^\oplus$ is suboptimal and a positive number $\mu$ exists such that

$$J_D^\oplus(x_0) \le \mu^{-1} J_D^\odot(x_0) \tag{49.70}$$

for all $x_0 \in \mathbb{R}^n$. The number $\mu$ is called the degree of suboptimality of $u_D^\odot$.

We can determine the index $\mu$ by first computing the performance index

$$J_D^\oplus(x_0) \;=\; x_0^T H x_0 \, , \tag{49.71}$$

where

$$\begin{aligned} H &= \int_0^\infty \exp(\hat{A}^T t) G_D \exp(\hat{A} t) dt, \\ G_D &= Q_D + P_D B_D R_D^{-1} B_D^T P_D, \end{aligned} \tag{49.72}$$

and the closed–loop matrix is

$$\hat{A} \;=\; A_D - B_D K_D + A_C \, . \tag{49.73}$$

It is important to note that $u_D^\odot$ is suboptimal if, and only if, the symmetric matrix $H$ exists. The existence of $H$ is guaranteed by the stability of $\hat{\mathbf{S}}$, in which case we can compute $H$ as the unique solution of the Liapunov matrix equation

$$\hat{A}^T H + H \hat{A} = -G_D \, . \tag{49.74}$$

The degree of suboptimality, which is the largest we can obtain in this context, is given as

$$\mu^* = \lambda_M^{1/2}(H P_D^{-1}) \, . \tag{49.75}$$

Details of this development, as well as the broad scope of suboptimality, were described in [30], where special attention was devoted to the robustness implications of suboptimality. First, we can explicitly characterize suboptimality in terms the of interconnection matrix $A_C$. The system $\hat{\mathbf{S}}^\oplus$ is suboptimal with degree $\mu$ if the matrix

$$\begin{aligned} F(\mu) \;=\; & A_C^T P_D + P_D A_C - (1-\mu) \\ & (Q_D + P_D B_D R_D^{-1} B_D^T P_D) \end{aligned} \tag{49.76}$$

is nonpositive definite. This is a sufficient condition for suboptimality, but one that implies stability if the pair $\{A_D + A_C, Q_D^{1/2}\}$ is detectable.

Another important aspect of nonpositivity of $F(\mu)$ is that it implies stability even if each control $u_i^{\odot}$ is replaced by a nonlinearity $\phi_i(u_i^{\odot})$, which is contained in a sector, or by a linear time-invariant dynamic element. Furthermore, if the subsystems are single-input systems, then each subsystem feedback loop has infinite gain margin, at least $\pm\cos^{-1}(1-\frac{1}{2}\mu)$ phase margin, and at least $50\mu\%$ gain reduction tolerance. These are the standard robustness characteristics of an optimal LQ control law, which are modified by the degree of suboptimality. It is interesting to note that the optimal robustness characteristics can be recovered by solving the inverse problem of optimal decentralized control. The matching conditions are one of the conditions that guarantee the solution of the problem.

The concept of suboptimality extends to the case of *overlapping subsystems*, when subsystems share common parts, and control is required to conform with the *overlapping information structure constraints.* By expanding the underlying state space, the subsystems become disjoint and decentralized control can be designed for the expanded system by standard techniques. Finally, the control laws obtained are contracted for implementation in the original system. This expansion–contraction framework is known as the Inclusion Principle. For a comprehensive presentation of the Principle, see [30].

## 49.6 Adaptive Decentralized Control

As mentioned in the section on decentrally stabilizable structures, many large scale interconnected systems with a good interconnection structure can be stabilized by a high-gain type decentralized control. How high the gain should be depends on how strong the interconnections are. If a bound on the interconnections is known, then stability can be guaranteed by a fixed high-gain controller. However, if such a bound is not available, then one has to use an adaptive controller which adjusts the gain to a value needed for overall stability.

Consider an interconnected system consisting of single-input subsystems

$$\mathbf{S}\colon\ \dot{x}_i(t) = A_i x_i(t) + b_i[u_i(t) + h_i(t, x(t))]\,, \quad i \in \mathcal{N} \tag{49.77}$$

where, without loss of generality, the pairs $(A_i, b_i)$ are assumed to be in controllable canonical form, and the nonlinear matching interconnections $h_i\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are assumed to satisfy

$$|h_i(t,x)| \le \sum_{j\in\mathcal{N}} \alpha_{ij}\|x_j\| \tag{49.78}$$

for some *unknown* constants $\alpha_{ij} \ge 0$. Let a decentralized state feedback

$$u_i(t) = -\rho(t)k_i^T R_i(\rho(t))\,, \quad i \in \mathcal{N} \tag{49.79}$$

be applied to $\mathbf{S}$, where $R_i(\rho) = \operatorname{diag}\{\rho^{n_i-1}, \dots, \rho, 1\}$, with $\rho(t)$ being a time-varying gain, and $k_i^T$ are such that the matrices $\hat{A}_i = A_i - b_i k_i^T$ have distinct eigenvalues $\lambda_{il}$, $i \in \mathcal{N}$, $l = 1, 2, \dots, n_i$. Let $T_i$ denote the modal matrices of $\hat{A}_i$, i.e., $T_i\hat{A}_iT_i^{-1} = M_i = \operatorname{diag}\{\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in_i}\}$. Then a time-varying coordinate transformation $z_i(t) = T_iR_i(\rho(t))x_i(t)$ transforms the closed-loop system $\hat{\mathbf{S}}$ into

$$\hat{\mathbf{S}}\colon\ \dot{z}_i(t) = \rho(t)M_i z_i(t) + g_i(t, z(t))\,, \quad i \in \mathcal{N}, \tag{49.80}$$

where, provided $0 \le \dot{\rho}(t) \le 1 \le \rho(t)$,

$$\|g_i(t,z)\| \le \sum_{j\in\mathcal{N}} \beta_{ij}\|z_j\| \tag{49.81}$$

for some *unknown* constants $\beta_{ij} \ge 0$. From (49.80) and (49.81) it follows that there exists a $\rho^* > 0$ so that $\hat{\mathbf{S}}$ is stable for all $\rho(t)$ satisfying $0 \le \dot{\rho}(t) \le 1 \le \rho^* \le \rho(t)$, as can be shown by the vector Liapunov approach. However, the crucial point is that $\rho^*$ depends on the unknown bounds $\beta_{ij}$. Fortunately, the difficulty can be overcome by increasing $\rho(t)$ adaptively until it is high enough to guarantee stability of $\hat{\mathbf{S}}$. A simple adaptation rule that serves the purpose is

$$\dot{\rho}(t) = \min\{1, \gamma\|x(t)\|\} \tag{49.82}$$

where $\gamma > 0$ is arbitrary. Although the control law is decentralized, $\rho(t)$ is adjusted based on complete state information.

The same idea can also be used in constructing adaptive decentralized dynamic output feedback controllers for various classes of large scale systems with structured nonlinear, time-varying interconnections. A typical example is a system described by

$$\begin{aligned}\mathbf{S}\colon\ \dot{x}_i(t) &= A_i x_i(t) + b_i u_i(t) + h_i(t, x(t))\,,\\ y_i(t) &= c_i^T x_i(t)\,, \quad i \in \mathcal{N}\end{aligned} \tag{49.83}$$

where

1. the decoupled subsystems described by the triples $(A_i, b_i, c_i^T)$ are controllable and observable,
2. the transfer functions $G_i(s) = c_i^T(sI - A_i)^{-1}b_i$ of the decoupled systems are minimum phase, have *known* relative degree $q_i$ and *known* high frequency gain $\kappa_i = \lim_{s\to\infty} s^{q_i}G_i(s)$, and
3. the nonlinear interconnections $h_i\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{n_i}$ are of the form $h_i(t,x) = b_i f_i(t,x) + g_i(t,y)$ where $f_i\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and $g_i\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{n_i}$ satisfy

$$\begin{aligned}|f_i(t,x)| &\le \sum_{j\in\mathcal{N}} \alpha_{ij}^f\|x_j\|\\ \|g_i(t,x)\| &\le \sum_{j\in\mathcal{N}} \alpha_{ij}^g\|y_j\|\end{aligned} \tag{49.84}$$

for some *unknown* constants $\alpha_{ij}^f$, $\alpha_{ij}^g$ where $x(t) = [x_1^T(t), x_2^T(t), \dots x_N^T(t)]^T$ and $y(t) = [y_1(t), y_2^T(t), \dots y_N(t)]^T$ are the state and the output of the overall system.

Finally, suitable adaptive decentralized control schemes can be developed by forcing an interconnected system of the form

(49.83) to track a decoupled stable linear reference model described as

$$\begin{aligned} \mathrm{S}_M\colon\ \dot{x}_{Mi}(t) &= A_{Mi}x_{Mi}(t) + b_{Mi}r_i(t), \\ y_{Mi}(t) &= c_{Mi}^T x_{Mi}(t)\,, \quad i \in \mathcal{N}, \end{aligned} \tag{49.85}$$

under reasonable assumptions on S and $\mathrm{S}_M$.

## 49.7 Discrete and Sampled–Data Systems

Most of the results concerning the stability and stabilization of continuous–time interconnected systems can be carried over to the discrete case with suitable modifications. Yet, there is a distinct approach to the stability analysis of discrete systems, which is to translate the problem in to that of a continuous system for which abundant results are available. For an idea of this approach, consider a system

$$\mathrm{S}_{SD}\colon\ x(t+1) = (A_0 + \sum_{k\in\mathcal{K}} p_k A_k)x(t) \tag{49.86}$$

where $A_0$ is a stable matrix additively perturbed by $p_k A_k$, $k \in \mathcal{K} = \{1, 2, \ldots, K\}$ with $p_k$ standing for one of $K$ perturbation parameters. The purpose is to find the largest region in the parameter space within which $\mathrm{S}_{SD}$ remains stable. By choosing a Liapunov function $v(x) = x^T P x$, where $P$ is the positive definite solution of the discrete Liapunov equation,

$$A_0^T P A_0 - P = -I, \tag{49.87}$$

it can be shown that $\mathrm{S}_{SD}$ is stable, provided $I - W(p)$ is positive definite, where

$$\begin{aligned} W(p) &= \sum_{k\in\mathcal{K}} p_k(A_k^T P A_0 + A_0^T P A_k) \\ &\quad + \sum_{k,l\in\mathcal{K}} p_k p_l A_k^T P A_l. \end{aligned} \tag{49.88}$$

Since the perturbation parameters appear nonlinearly in $W(p)$, characterization of a stability region in the parameter space is not easy. However, $I - W(p)$ is positive definite if the continuous system

$$\dot{\xi}(t) = (-I + \sum_{k\in\mathcal{K}} p_k E_k)\xi(t) \tag{49.89}$$

is stable, where

$$E_k = \begin{bmatrix} 0 & P^{1/2}A_k \\ E_k^T P^{1/2} & A_k^T P A_0 + A_0^T P A_k \end{bmatrix}. \tag{49.90}$$

An analysis of the stability of the perturbed continuous system in (49.89) provides a sufficient condition for the stability of the discrete system in (49.86). This idea can be generalized to the stability analysis of discrete interconnected systems by treating the interconnections as perturbations to nominal stable decoupled subsystems.

A major difference between discrete and continuous systems is that characterizing decentrally stabilizable interconnections for discrete systems is not as easy as for continuous systems. For example, there is no discrete counterpart to the matching conditions. On the other hand, most existing control schemes for continuous systems seem applicable to sampled–data systems provided the sampling rate is sufficiently high. To illustrate this observation, consider the decentralized control of an interconnected system,

$$\begin{aligned} \mathrm{S}\colon\ \dot{x}_i(t) &= A_i x_i(t) + b_i[u_i(t) \\ &\quad + \sum_{j\in\mathcal{N}} d_{ij}^T x_j(t)]\,, \quad i \in \mathcal{N}, \end{aligned} \tag{49.91}$$

using sampled–data feedback of the form

$$\begin{aligned} u_i(t) &= -k_i^T(t - t_m)x_i(t_m), \\ &\quad t_m \le t < t_{m+1}\,, \end{aligned} \tag{49.92}$$

where $t_m$ are the sampling instants, and $k_i(t)$ are time–varying local feedback gains. With $T_m = t_{m+1} - t_m$ denoting the $m$th sampling period, it can be shown that the choice of

$$k_i^T(t) = [\delta^{n_i}(t)\ldots\delta'(t)\,\delta(t)] \tag{49.93}$$

or similar feedback gains having impulsive behavior, stabilize S provided $T_m$ are sufficiently small. How small the sampling periods should be requires knowledge of the bounds on the interconnections. If these bounds are not available, then a simple centralized adaptation scheme, such as

$$T_{m+1}^{-1} = T_m^{-1} + \sum_{j\in\mathcal{N}} \gamma_i \|x_i(t_{m-m_i})\|, \tag{49.94}$$

with $\gamma_i > 0$, decreases $T_m$ to the value needed for stability. Clearly, this is a high–gain stabilization scheme coupled with fast sampling, owing its success to the matching structure of the interconnections [36]. Similar adaptive sampled–data control schemes are available for more general classes of interconnected systems.

## 49.8 Graph–Theoretic Decompositions

Decomposition of large scale systems and their associated problems is often desirable for computational reasons. In such cases, decentralization or any other structural constraints on the controllers, estimators, or the design process itself, is preferred rather than necessary. Depending on the particular problem in hand, one may be interested in obtaining Lower Block Triangular (LBT) decompositions, input and/or output reachable acyclic decompositions, $\epsilon$–decompositions, overlapping decompositions, etc. [30]. In all of these decomposition schemes, the problem is to find a suitable partitioning and reordering of the input, state, or output variables so that the resulting decomposed system has some desirable structural properties. As expected, the system graph plays the key role, with graph–theory providing the tools.

### 49.8.1 LBT Decompositions

LBT decompositions are used to reorder the states of system **S** in (49.6), so that the subsystems have a hierarchical interconnection pattern as

$$\begin{aligned} \mathbf{S}: \dot{x}_i &= \sum_{j=1}^{i} A_{ij}x_j + B_i u\,, \quad i \in \mathcal{N}, \\ y &= \sum_{i\in\mathcal{N}} C_i x_i. \end{aligned} \tag{49.95}$$

Such a decomposition corresponds to transforming the $A$ matrix into a Lower Block–Triangular form by symmetric row and column permutations (hence the name LBT decomposition). In terms of system graph, LBT decomposition is the almost trivial problem of identifying the strong components of the truncated digraph $\mathbf{D}_x = (\mathcal{X}, \mathcal{E}_x)$, where $\mathcal{E}_x \subset \mathcal{E}$ contains only the edges connecting state vertices.

LBT decompositions offer computational simplification in the standard state feedback or observer design problems. For example, the problem of designing a state feedback

$$u = -Kx = -\sum_{i\in\mathcal{N}} K_i x_i \tag{49.96}$$

for arbitrary pole placement, can be reduced to computation of the individual blocks $K_i$ of $K$ in a recursive scheme involving the subsystems only.

### 49.8.2 Acyclic IO Reachable Decompositions

In acyclic Input–Output (IO) reachable decompositions, the purpose is to decompose **S** into the form

$$\begin{aligned} \mathbf{S}: \dot{x}_i &= \sum_{j=1}^{i} A_{ij}x_j + \sum_{j=1}^{i} B_{ij}u_j, \\ y_i &= \sum_{j=1}^{i} C_{ij}x_j\,, \quad i \in \mathcal{N}. \end{aligned} \tag{49.97}$$

That is, in addition to the $A$ matrix, the $B$ and $C$ matrices must have LBT structure. In addition to the desired structure of the system matrices, it is also necessary that the decoupled subsystems represented by $(A_{ii}, B_{ii}, C_{ii})$ are at least structurally controllable and observable, and that none is further decomposable.

Because the LBT structure is concerned with the reachability properties of the system, both this structure and input and/or output reachability requirements for the subsystems, which are necessary for structural controllability and/or observability, can be taken care of by a suitable decomposition scheme based on binary operations on the reachability matrix of the system digraph. The requirement that the subsystems be dilation free, which is the second condition for structural controllability and/or observability, is of a different nature, however, and should be checked separately after the input–output reachability decomposition has been obtained.

When outputs are of no concern, it is easy to identify all possible acyclic, irreducible, input reachable decompositions of a given system. If some of the resulting decoupled subsystems turn out to contain dilations (destroying structural controllability), then they can suitably be combined with one or more subsystems at a higher level of hierarchy to eliminate the dilations without destroying the LBT structure. Provided that the overall system is structurally controllable, this process eventually gives an acyclic, irreducible decomposition in which all subsystems are structurally controllable. Of course, dual statements are valid for acyclic output reachable decompositions.

Once an acyclic decomposition into controllable subsystems is obtained, many design problems can be decomposed accordingly. An obvious example is the state feedback structure in (49.96). A more complicated problem is the suboptimal state feedback design discussed in the section on optimization. For the system in (49.97), the test matrix $F(\mu)$, with the inclusion of the input coupling terms $B_{ij}$, becomes

$$F(M_D) = F_D(M_D) + F_C(M_D) + F_C^T(M_D)\,, \tag{49.98}$$

where $M_D = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_N\}$, allowing different $\mu_i$'s for $\mathbf{S}_i$'s, $F_D(M_D) = [(1-\mu_i^{-1})(Q_i + K_i^T R_i K_i)]$, and $F_C(M_D) = [F_{ij}(\mu_i)]$ with

$$F_{ij}(\mu_i) = \begin{cases} \mu_i^{-1}P_i(A_{ij} - B_{ij}K_j)\,, & i > j \\ 0, & i \le j. \end{cases} \tag{49.99}$$

From the structure of $F(M_D)$ it is clear that the choice $\mu_i = \epsilon^{N+1-i}$, $i \in \mathcal{N}$, results in a negative definite $F(M_D)$ for sufficiently small $\epsilon$. This guarantees existence of a suboptimal state feedback control law with the degree of suboptimality $\mu = \epsilon^N$. In practice, it is possible to achieve a much better $\mu$ by a careful choice of the weight matrices $Q_i$ and $R_i$.

In a similar way, acyclic, structurally observable decompositions can be used to design suboptimal state estimators, which are discussed below in the context of sequential optimization for acyclic IO decompositions.

To illustrate the use of acyclic IO decompositions in a standard LQG optimization problem, it suffices to consider decomposition of a discrete–time system into only two subsystems as

$$\begin{aligned} \mathbf{S}_1: x_1(t+1) &= A_{11}x_1(t) + B_{11}u_1(t) + w_1(t), \\ y_1(t) &= C_{11}x_1(t) + v_1(t), \\ \mathbf{S}_2: x_2(t+1) &= A_{21}x_1(t) + A_{22}x_2(t) \\ &\quad + B_{21}u_1(t) + B_{22}u_2(t) + w_2(t), \\ y_2(t) &= C_{21}x_1(t) + C_{22}x_2(t) + v_2(t), \end{aligned} \tag{49.100}$$

with the usual assumptions on the input and measurement noises $\omega_i$ and $v_i$, $i = 1, 2$. Let each subsystem be associated with a performance criterion

$$\begin{aligned} \mathcal{E}J_i &= \mathcal{E}\left\{\lim_{T\to\infty} T^{-1} \sum_{t=0}^{T-1} \left[x_i^T(t)Q_i x_i(t) \right.\right. \\ &\quad \left.\left. + u_i^T(t)R_i u_i(t)\right]\right\}\,, \quad i = 1, 2 \end{aligned} \tag{49.101}$$

where $\mathcal{E}$ denotes expectation.

The sequential optimization procedure consists of minimizing $\mathcal{E}J_1$ and $\mathcal{E}J_2$ subject to the dynamic equations for the systems $\mathbf{S}_1$ and $(\mathbf{S}_1, \mathbf{S}_2)$, respectively. The first problem has the standard solution $u_1^*(t) = -K_1\hat{x}_1(t)$, where $K_1$ is the optimal control gain found from the solution of the associated Riccati equation, and $\hat{x}_1(t)$ is the best estimate of $x_1(t)$ given the output information $\mathcal{Y}_1^{t-1} = \{y_1(0), \ldots, y_1(t-1)\}$. The estimate $\hat{x}_1(t)$ is generated by the Kalman filter

$$\begin{aligned} \hat{x}_1(t+1) &= A_{11}\hat{x}_1(t) + B_{11}u_1^*(t) \\ &\quad + L_1[y_1(t) - c_{11}\hat{x}_1(t)] \qquad (49.102) \end{aligned}$$

where $L_1$ is the steady–state estimator gain. With the control $u_1^*$ applied to $\mathbf{S}_1$, the overall system becomes

$$\mathbf{S}: \begin{bmatrix} \hat{x}_1(t+1) \\ x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \qquad (49.103)$$

$$\begin{bmatrix} A_{11} - B_{11}K_1 - L_1C_{11} & L_1C_{11} & 0 \\ -B_{11}K_1 & A_{11} & 0 \\ -B_{21}K_1 & A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ B_{22} \end{bmatrix} u_2(t) + \begin{bmatrix} L_1 v_1(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}$$

which preserves the LBT structure of the original system. Assuming that both $\mathcal{Y}_1^{t-1}$ and $\mathcal{Y}_2^{t-1} = \{y_2(0), \ldots, y_2(t-1)\}$ are available for constructing the control $u_2^*$ (which is consistent with the idea of sequential optimization), the problem reduces to minimization of $\mathcal{E}J_2$ subject to (103). An analysis of the standard solution procedure reveals that the optimal control law can be expressed as

$$u_2^*(t) = -K_{21}\hat{x}_1(t) - K_{22}\hat{\xi}(t) \qquad (49.104)$$

where $K = [K_{21}\ K_{22}]$ is the optimal control gain, and $\hat{\xi}(t)$ is the optimal estimate of $x(t) = [x_1^T(t)\ \ x_2^T(t)]^T$, given $\mathcal{Y}_1^{t-1}$ and $\mathcal{Y}_2^{t-1}$. Furthermore, the $2n_1 + n_2$–dimensional Riccati equation, from which $K$ is constructed, can be decomposed into an $n_2$–dimensional Riccati equation involving the parameters of the second isolated subsystem and a Liapunov equation corresponding to an $n_2 \times 2n_1$ dimensional matrix. This results in considerably simplifying the solution of the optimal control gain. However, the Kalman filter for $\hat{\xi}(t)$ still requires the solution of an $(n_1 + n_2)$–dimensional Riccati equation.

Other sequential optimization schemes based on various information structure constraints can be analyzed similarly; for details, see [30].

## 49.8.3 Nested Epsilon Decompositions

Epsilon decomposition of a square matrix $M$ is concerned with transforming $M$ by symmetric row and column permutations into a form

$$P^T M P = M_D + \epsilon M_C \qquad (49.105)$$

where $M_D$ is block diagonal, and $\epsilon$ is a prescribed small number [27]. The problem is equivalent to identifying the connected components of a subgraph $\mathbf{D}^\epsilon$ of the digraph $\mathbf{D}$ associated with $M$, which is obtained by deleting all edges of $\mathbf{D}$ corresponding to those elements of $M$ with magnitude smaller than $\epsilon$. All of the vertices of a connected component of $\mathbf{D}^{\epsilon_1}$ appear in the same connected component of $\mathbf{D}^{\epsilon_2}$ for any $\epsilon_2 < e_1$. Thus one can identify a number of distinct values $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_K$ such that

$$\begin{aligned} P^T M P &= (\ldots((M_0 + \epsilon_1 M_1) + \epsilon_2 M_2) \\ &\quad + \ldots + \epsilon_K M_K), \qquad (49.106) \end{aligned}$$

which is a *nested epsilon decomposition* of $M$ as illustrated in Figure 49.3.

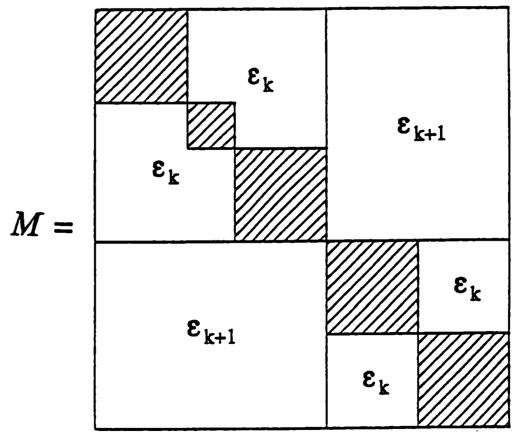


**Figure 49.3** Nested epsilon decompositions.

As seen from the figure, a large $\epsilon$ results in a finer decomposition than a small $\epsilon$ does. Thus the choice of $\epsilon$ provides a compromise between the size and the number of components and the strength of the interconnections among them. A nice property of nested epsilon decompositions is that once the decomposition corresponding to some $\epsilon_k$ is obtained, the decomposition corresponding to $\epsilon_{k+1}$ can be found by working with a smaller digraph obtained by condensing $\mathbf{D}^{\epsilon_k}$ with respect to its components.

An immediate application of the nested epsilon decompositions is the stability analysis of a large scale system via vector Liapunov functions, where the matrix $M$ is identified with the matrix $A$ of the system in (6). Provided the subsystems resulting from the decomposition are stable, the stability of the overall system can easily be established by means of the aggregate matrix $W$ in (49), whose off–diagonal elements are of the order of $\epsilon$.

The nested epsilon decomposition algorithm can also be applied with some modifications to decompose a system with inputs as

$$\begin{aligned} \dot{x}_i &= A_{ii}x_i + B_{ii}u_i + \epsilon \sum_{j \neq i}^{N} (A_{ij}x_j + B_{ij}u_j), \\ & i \in \mathcal{N}. \qquad (49.107) \end{aligned}$$

If each decoupled subsystem identified by a pair $(A_{ii}, B_{ii})$ is stabilized by a local state feedback of the form $u_i = -K_i x_i$, $i \in \mathcal{N}$, with the local gains not excessively high, then the closed–loop system preserves the weak–coupling property of the open–loop system, providing an easy way to stabilize the overall system. The same idea can also be employed in designing decentralized estimators [30] based on a suitable epsilon decomposition of the pair $(A, C)$.

### 49.8.4 Overlapping Decompositions

Consider a system

$$\tilde{\mathbf{S}}: \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) \tag{49.108}$$

with an $\tilde{n}$–dimensional state vector $\tilde{x}$. Let columns of the matrix $V \in \mathbb{R}^{\tilde{n}\times n}$ form a basis for an $n$–dimensional $A$–invariant subspace of $\mathbb{R}^{\tilde{n}}$, and let $A$ be the restriction of $\tilde{A}$ to $\mathrm{Im}V \simeq \mathbb{R}^n$, that is, $\tilde{A}V = VA$. Then the smaller order system

$$\mathbf{S}: \dot{x}(t) = Ax(t) \tag{49.109}$$

is called a restriction of $\tilde{\mathbf{S}}$. Conversely, starting with the system $\mathbf{S}$, one can obtain an expansion $\tilde{\mathbf{S}}$ of $\mathbf{S}$ by defining $\tilde{A} = VAV^L + M$, where $V^L$ is any left inverse of $V$, and $M$ is any complementary matrix satisfying $MV = 0$. The very definition of a restriction implies that $\mathbf{S}$ is stable if $\tilde{\mathbf{S}}$ is.

In many problems associated with large scale systems, it may be desirable to expand a system $\mathbf{S}$ to a larger dimensional one which possess some nice structural properties. The increase in dimensionality of the problem may very well be offset by the nice structure of the expansion. As an example, consider a system $\mathbf{S}$ with

$$A = \left[\begin{array}{cc|c} A_{11} & A_{12} & \epsilon A_{13} \\ \hline \epsilon A_{21} & \multicolumn{1}{|c|}{A_{22}} & \epsilon A_{23} \\ \hline \epsilon A_{31} & \multicolumn{1}{|c}{A_{32}} & A_{33} \end{array}\right] \tag{49.110}$$

where $\epsilon$ is a small parameter. Letting

$$V = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{49.111}$$

where $I_k$ denotes an identity matrix of order $n_k$, one obtains an expansion $\tilde{\mathbf{S}}$ with

$$\tilde{A} = \left[\begin{array}{cc|cc} A_{11} & A_{12} & 0 & \epsilon A_{13} \\ \epsilon A_{21} & A_{22} & 0 & \epsilon A_{23} \\ \hline \epsilon A_{21} & 0 & A_{22} & \epsilon A_{23} \\ \epsilon A_{31} & 0 & A_{32} & A_{33} \end{array}\right]. \tag{49.112}$$

Since $\tilde{\mathbf{S}}$ has an obvious decomposition into two weakly coupled subsystems, one can take advantage of this structural property in stability analysis, which is not available for the original system $\mathbf{S}$.

One can easily notice from the structure of $V$ in (114) that the expansion $\tilde{\mathbf{S}}$ of $\mathbf{S}$ is obtained simply by repeating the equation for the middle part $x_2$ of the state vector $x = [x_1^T\ x_2^T\ x_3^T]^T$. In some sense, $x_2$ is treated as common to two overlapping components $\tilde{x}_1 = [x_1^T\ x_2^T]^T$ and $\tilde{x}_2 = [x_2^T\ x_3^T]^T$ of $x$. Thus the partitioning of the $A$ matrix in (113) is termed the *overlapping decomposition.*

Although the expansion matrix $V$ can be any matrix with full column rank, if it is restricted to contain one and only one unity element in each row (which corresponds, as in the case above, to repeating some of the state equations in the expanded domain), then one can develop a suitable graph–theoretic algorithm to find the smallest expansion which has a disjoint decomposition (into decoupled or $\epsilon$–coupled components) with the property that no component is further decomposable.

The idea of overlapping decompositions via expansions can be extended to systems with inputs. A system

$$\tilde{\mathbf{S}}: \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \tag{49.113}$$

is said to be an expansion of

$$\mathbf{S}: \dot{x}(t) = Ax(t) + Bu(t) \tag{49.114}$$

if $\tilde{B} = VB$ in addition to $\tilde{A}V = VA$. Consider the optimal control problems of minimizing the performance criteria

$$\begin{aligned} J &= \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \\ \tilde{J} &= \int_0^\infty [\tilde{x}^T(t)\tilde{Q}\tilde{x}(t) + u^T(t)Ru(t)]dt \end{aligned} \tag{49.115}$$

associated with $\mathbf{S}$ and $\tilde{\mathbf{S}}$. The optimal solutions are

$$u(t) = -Kx(t), \text{ and } u(t) = -\tilde{K}\tilde{x}(t), \tag{49.116}$$

respectively, resulting in closed–loop systems

$$\begin{aligned} \hat{\mathbf{S}}: \dot{x}(t) &= (A - BK)x(t), \\ \hat{\tilde{\mathbf{S}}}: \dot{\tilde{x}}(t) &= (\tilde{A} - \tilde{B}\tilde{K})\tilde{x}(t). \end{aligned} \tag{49.117}$$

Thus, $\hat{\mathbf{S}}$ is a restriction of $\hat{\tilde{\mathbf{S}}}$ if $(\tilde{A} - \tilde{B}\tilde{K})V = V(A - BK)$, or equivalently, if $\tilde{K} = KV$. The last condition is satisfied if $\tilde{Q}$ and $Q$ are related as $Q = V^T\tilde{Q}V$, in which case the optimal cost matrices are also related as $P = V^T\tilde{P}V$. This analysis shows that, if the cost matrices $\tilde{Q}$ and $R$ of the expanded system are chosen to be block diagonal with diagonal blocks associated with the decoupled expanded subsystems, then its optimal (in case of complete decoupling) or suboptimal (in case of weak decoupling) solution can be contracted back to an optimal or suboptimal solution of the original system with respect to a suitably chosen performance criterion.

## References


[1] Anderson, B. D. O. and Clements, D. J., Algebraic characterization of fixed modes in decentralized control, *Automatica,* 17, 703–712, 1981.

[2] Brusin, V. A. and Ugrinovskaya, E. Ya., Decentralized adaptive control with a reference model, *Avtomatika i Telemekhanika,* 10, 29–36, 1992.

[3] Chae, S. and Bien, Z., Techniques for decentralized control for interconnected systems, in *Control and Dynamic Systems,* C. T. Leondes, Ed., Academic Press, Boston, 41, 273–315, 1991.

[4] Chen, Y. H., Decentralized robust control design for large–scale uncertain systems: The uncertainty is time–varying, *J. Franklin Institute,* 329, 25–36, 1992.
[5] Chen, Y. H. and Han, M. C., Decentralized control design for interconnected uncertain systems, in *Control and Dynamic Systems,* C. T. Leondes, Ed., Academic Press, Orlando, FL, 56, 219–266, 1993.
[6] Cheng, C. F., Wang, W. J., and Lin, Y. P., Decentralized robust control of decomposed uncertain interconnected systems, *Trans. ASME,* 115, 592–599, 1993.
[7] Cho, Y. J. and Bien, Z., Reliable control via an additive redundant controller, *Int. J. Control,* 50, 385–398, 1989.
[8] Date, R. A. and Chow, J. H., A parametrization approach to optimal $H_2$ and $H_\infty$ decentralized control problems, *Automatica,* 29, 457–463, 1993.
[9] Datta, A., Performance improvement in decentralized adaptive control: A modified model reference scheme, *IEEE Trans. Automatic Control,* 38, 1717–1722, 1993.
[10] Gajić, Z. and Shen, X., *Parallel Algorithms for Optimal Control of Large Scale Linear Systems,* Springer–Verlag, Berlin, Germany, 1993.
[11] Geromel, J. C., Bernussou, J., and Peres, P. L. D., Decentralized control through parameter space optimization, *Automatica,* 30, 1565–1578, 1994.
[12] Gündes, A. N. and Kabuli, M. G., Reliable decentralized control, *Proc. Am. Control Conf.,* Baltimore, MD, pp. 3359–3363, 1994.
[13] Iftar, A., Decentralized estimation and control with overlapping input, state, and output decomposition, *Automatica,* 29, 511-516, 1993.
[14] Ikeda, M., Decentralized Control of Large Scale Systems, in *Three Decades of Mathematical System Theory,* H. Nijmeiyer and J. M. Schumacher, Eds., Springer–Verlag, New York, 1989, 219–242.
[15] Jamshidi, M., *Large–Scale Systems. Modeling and Control,* North–Holland, New York, 1983.
[16] Lakshmikantham, V. and Liu, X. Z., *Stability Analysis In Terms of Two Measures,* World Scientific, Singapore, 1993.
[17] Lakshmikantham, V., Matrosov, V. M., and Sivasundaram, S., *Vector Liapunov Functions and Stability Analysis of Nonlinear Systems,* Kluwer, The Netherlands, 1991.
[18] Leitmann, G., One approach to the control of uncertain systems, *ASME J. Dyn. Syst., Meas., and Control,* 115, 373–380, 1993.
[19] Leondes, C. T., Ed., *Control and Dynamic Systems,* Vols. 22-24, Decentralized/Distributed Control and Dynamic Systems, Academic Press, Orlando, FL, 1985.
[20] Lin, C. T., Structural controllability, *IEEE Trans. Auto. Control,* AC–19, 201–208, 1974.
[21] Lyon, J., Note on decentralized adaptive controller design, *IEEE Trans. Auto. Control,* 40, 89–91, 1995.
[22] Michel, A. N., On the status of stability of interconnected systems, *IEEE Trans. Circuits Syst.,* CAS–30, 326–340, 1983.
[23] Mills, J. K., Stability of robotic manipulators during transition to and from compliant motion, *Automatica,* 26, 861–874, 1990.
[24] Sandell, N. R., Jr., Varaiya, P., Athans, M., and Safonov, M. G., Survey of decentralized control methods for large scale systems, *IEEE Trans. Auto. Control,* AC–23, 108–128, 1978.
[25] Savastuk, S. V. and Šiljak, D. D., Optimal decentralized control, *Proc. Am. Control Conf.,* Baltimore, MD, pp. 3369–3373, 1994.
[26] Sezer, M.E. and Šiljak, D. D., Structurally fixed modes, *Syst. Control Lett.,* 1, 60–64, 1981.
[27] Sezer, M. E. and Šiljak, D. D., Nested $\epsilon$–decomposition and clustering of complex systems, *Automatica,* 22, 321–331, 1986.
[28] Shi, L. and Singh, S. K., Decentralized adaptive controller design for large–scale systems with higher order interconnections, *IEEE Trans. Auto. Control,* 37, 1106–1118, 1992.
[29] Šiljak, D. D., *Large–Scale Dynamic Systems: Stability and Structure,* North–Holland, New York, 1978.
[30] Šiljak, D. D., *Decentralized Control of Complex Systems,* Academic Press, Cambridge, MA, 1991.
[31] Tamura, H. and Yoshikawa, T., *Large–Scale Systems Control and Decision Making,* Marcel Dekker, New York, 1990.
[32] Ünyelioğlu, K. A. and Özgüler, A. B., Reliable decentralized stabilization of feed–forward and feedback interconnected systems, *IEEE Trans. Auto. Control,* 37, 1119–1132, 1992.
[33] Voronov, A. A., Present state and problems of stability theory, *Automatika i Telemekhanika,* 5, 5–28, 1982.
[34] Wang, S. H. and Davison, E. J., On the stabilization of decentralized control systems, *IEEE Trans. Auto. Control,* AC–18, 473–478, 1973.
[35] Wu, W. and Lin, C., Optimal reliable control system design for steam generators in pressurized water reactors, *Nuclear Technology,* 106, 216–224, 1994.
[36] Yu, R., Ocali, O., and Sezer, M. E., Adaptive robust sampled–data control of a class of systems under structured perturbations, *IEEE Trans. Auto. Control,* 38, 1707–1713, 1993.

## Further Reading

There is a number of survey papers on decentralized control and large scale systems [3], [14], [24]. The books on the subject are [10], [15], [19], [29], [31]. For a comprehensive treatment of decentralized control theory, methods, and applications, with a large number of references, see [30].

For further information on vector Liapunov functions and stability analysis of large scale interconnected systems, see the survey papers [22], [33], and books [16], [17].

Adaptive decentralized control has been of widespread recent interest, see [2], [9], [21], [23], [28], [30], [36].

Robustness of decentralized control to both structured and unstructured perturbations has been one of the central issues in the control of large scale systems. For the background of robustness issues in control, which are relevant to decentralized control, see [18], [30]. For new and interesting results on the subject, see [4], [5], [6], [8].

There is a number of papers devoted to design of decentralized control via parameter space optimization, which rely on powerful convex optimization methods. For recent results and references, see [11].

Overlapping decentralized control and the Inclusion Principle are surveyed in [30]. Useful extensions were presented in [13]. The concept of overlapping is basic to reliable control under controller failures using multiple decentralized controllers [30]. For more information about this area, see [7], [12], [32], [35].

In a recent development [25], it has been shown how optimal decentralized control of large scale interconnected systems can be obtained in the classical optimization framework of Lagrange. Both sufficient and necessary conditions for optimality are derived in the context of Hamilton-Jacobi equations and Pontryagin's maximum principle.