14

Fractional Fourier Transform*

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14.1 Introduction

The ordinary Fourier transform and related techniques are of great importance in many areas of science and engineering. The fractional Fourier transform (FRT) is a generalization of the ordinary Fourier transform with an order (or power) parameter a. This chapter provides an introduction to the fractional Fourier transform and discusses some of its more important properties. The FRT also has a growing list of applications in several areas. An overview of applications that have received interest so far are provided at the end of this chapter. Those interested in learning about the transform and its applications in greater depth are referred to [23,122,123,129].

Mathematically the ath order fractional Fourier transform operator is the ath power of the ordinary Fourier transform operator. (Readers not familiar with functions of operators may think of them in analogy with functions of matrices. In the discrete case, where the discrete ordinary and fractional Fourier transform operators are represented by matrices, this is actually the case.) If we denote the ordinary Fourier transform operator by $\mathcal{F}$, then the ath order fractional Fourier transform operator is denoted by $\mathcal{F}^a$. The zeroth-order fractional Fourier transform operator $\mathcal{F}^0$ is equal to the identity operator $\mathcal{I}$. The first-order fractional Fourier transform operator $\mathcal{F}^1$ is equal to the ordinary Fourier transform operator. Integer values of a correspond to repeated application of the Fourier transform; for instance, $\mathcal{F}^2$ corresponds to the Fourier transform of the Fourier transform. $\mathcal{F}^{-1}$ corresponds to the inverse Fourier transform operator. The a'th order transform of the ath order transform is equal to the $(a' + a)$th order transform; that is $\mathcal{F}^{a'}\mathcal{F}^a = \mathcal{F}^{a'+a}$, a property referred to as index additivity. For instance, the 0.5th fractional Fourier transform operator $\mathcal{F}^{0.5}$, when applied twice, amounts to ordinary Fourier transformation. Or, the 0.4th transform of the 0.3rd transform is the 0.7th transform. The order a may assume any real value, however the operator $\mathcal{F}^a$ is periodic in a with period 4; that is $\mathcal{F}^{a+4j} = \mathcal{F}^a$ where j is any integer. This is because $\mathcal{F}^2$ equals the parity operator $\mathcal{P}$ which maps $f(u)$ to $f(-u)$ and $\mathcal{F}^4$ equals the identity operator. Therefore, the range of a is usually restricted to $(-2, 2]$ or $[0, 4)$. Complex-ordered transforms have also been discussed by some authors, although there remains much to do in this area both in terms of theory and applications.

The same facts can also be thought of in terms of the functions which these operators act on. For instance, the zeroth-order fractional Fourier transform of the function $f(u)$ is merely

* Parts of this chapter appeared in or were adapted from Ozaktas and Kutay [121].

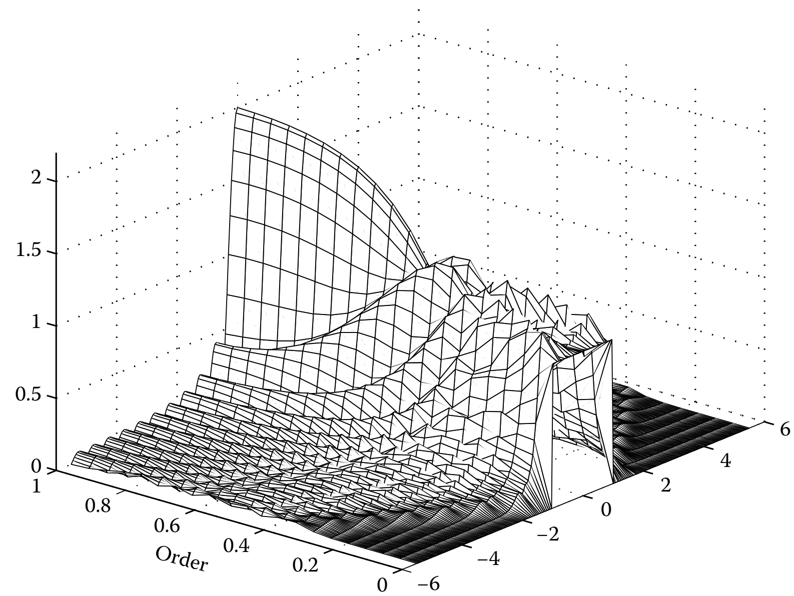

FIGURE 14.1 Magnitude of the fractional Fourier transform of the rectangle function as a function of the transform order. (From Ozaktas, H. M. and Kutay, M. A., *Proceedings of the European Control Conference*. European Union Control Association and University of Porto, Porto, Portugal, 2001. With permission.)

the function itself, and the first-order transform is its ordinary Fourier transform $F(\mu)$, where μ denotes the frequency domain variable. The ath fractional Fourier transform of $f(u)$ is denoted by $f_a(u)$ so that $f_0(u) = f(u)$ and $f_1(\mu) = F(\mu)$ (or $f_1(u) = F(u)$ since the functional equality does not depend on the dummy variable employed).

An example is given in Figure 14.1, where we see the magnitude of the fractional Fourier transforms of the rectangle function for different values of the order $a \in [0, 1]$. We observe that as a varies from 0 to 1, the rectangle function evolves into a sinc function, which is the ordinary Fourier transform of the rectangle function.

The earliest known references dealing with the transform go back to the 1920s and 1930s; since then the transform has been reinvented several times. It has received the attention of a few mathematicians during the 1980s [100,106,109]. However, interest in the transform really grew with its reinvention/reintroduction by researchers in the fields of optics and signal processing, who noticed its relevance for a variety of application areas [8,88,102,117,124,125]. A detailed account of the history of the transform may be found in [129].

Fractionalization of the Fourier transform has led to interest in fractionalization of other transforms [5,91,175] such as the Hilbert transform [137] and the cosine–sine and Hartley transforms [30,134], and extensions to the study of time–frequency distributions [130,132,143]. These will not be dealt with in this chapter.

Throughout this chapter, the imaginary unit is denoted by i and the square root is defined such that the argument of the result lies in the interval $(-\pi/2, \pi/2]$.

The first three to five sections can be read as a tutorial on the fractional Fourier transform, and the other sections can be read or consulted as needed.

14.2 Definition and Essential Properties

The most straightforward way of defining the fractional Fourier transform is as a linear integral transform as follows:

$$f_a(u) = \int_{-\infty}^{\infty} K_a(u, u') f(u')\, du', \tag{14.1}$$

$$K_a(u, u') = \sqrt{1 - i \cot \alpha}\, \exp\left[i\pi(\cot \alpha\, u^2 - 2 \csc \alpha\, uu' + \cot \alpha\, u'^2)\right],$$

$$\alpha = \frac{a\pi}{2},$$

when $a \neq 2j$ for integer j. When $a = 4j$ the transform is defined as $K_a(u, u') = \delta(u - u')$ and when $a = 4j + 2$ the transform is defined as $K_a(u, u') = \delta(u + u')$. It can be shown that the above kernel for $a \neq 2j$ indeed approaches these delta function kernels as a approaches even integers. For $0 < |a| < 2$, the factor $\sqrt{1 - i \cot \alpha}$ can be written as $\exp\{-i[\pi \operatorname{sgn}(\alpha)/4 - \alpha/2]\}/\sqrt{|\sin \alpha|}$ where $\operatorname{sgn}(\cdot)$ is the sign function. It is easy to show that when $a = 1$ the kernel reduces to $\exp(-i2\pi uu')$, corresponding to the ordinary Fourier transform, and that when $a = -1$ the kernel reduces to $\exp(i2\pi uu')$, corresponding to the ordinary inverse Fourier transform.

It is not easy to see from the above definition that the transform is indeed the operator power of the ordinary Fourier transform. In order to find the operator power of the ordinary Fourier transform, we first consider its eigenvalue equation:

$$\mathcal{F}\psi_n(u) = e^{-in\pi/2}\psi_n(u). \tag{14.2}$$

Here the eigenfunctions $\psi_n(u)$, $n = 0, 1, 2 \ldots$ are the Hermite–Gaussian functions defined as $\psi_n(u) = (2^{1/4}/\sqrt{2^n n!})$ $H_n(\sqrt{2\pi}\,u)\exp(-\pi u^2)$, where $H_n(u)$ are the standard Hermite polynomials. $\exp(-in\pi/2)$ is the eigenvalue associated with the nth eigenfunction $\psi_n(u)$. Now, following a standard procedure also used to define functions of matrices, the fractional Fourier transform may be defined such that it has the same eigenfunctions, but the eigenvalues raised to the ath power:

$$\mathcal{F}^a\psi_n(u) = (e^{-in\pi/2})^a\psi_n(u). \tag{14.3}$$

This definition is not unique for at least two reasons. First, it depends on the choice of the Hermite–Gaussian set as the set of eigenfunctions (which is not the only such possible set). Second, it depends on how we resolve the ambiguity in evaluating $[\exp(-in\pi/2)]^a$. The particular definition, which has so far received the greatest attention, has the most elegant properties, and which has found the most applications, follows from choosing $[\exp(-in\pi/2)]^a = \exp(-ian\pi/2)$. With this choice, the fractional Fourier transform of a square-integrable function $f(u)$ can be found by first expanding it in terms of the set of Hermite–Gaussian functions $\psi_n(u)$ as

$$f(u) = \sum_{n=0}^{\infty} C_n\psi_n(u), \tag{14.4}$$

$$C_n = \int_{-\infty}^{\infty} \psi_n(u)f(u)\,du, \tag{14.5}$$

and then applying $\mathcal{F}^a$ to both sides to obtain

$$\mathcal{F}^a f(u) = \sum_{n=0}^{\infty} C_n\mathcal{F}^a\psi_n(u), \tag{14.6}$$

$$f_a(u) = \sum_{n=0}^{\infty} C_n e^{-ian\pi/2}\psi_n(u), \tag{14.7}$$

$$f_a(u) = \int_{-\infty}^{\infty}\left[\sum_{n=0}^{\infty} e^{-ian\pi/2}\psi_n(u)\psi_n(u')\right] f(u')\,du' \tag{14.8}$$

The final form can be shown to be equal to that given by Equation 14.1 through a standard identity (for instance, see Table 2.8.9 in [129]).

Alternative definitions of the transform will arise if we make different choices regarding the eigenfunctions or in taking the fractional powers of the eigenvalues [31,77]. For instance, if the ambiguity in evaluating z^a is resolved by choosing the principal power of z, it turns out that the ath fractional Fourier transform of $f(u)$ can be expressed as a linear combination of the form

$$\beta_0(a)f(u) + \beta_1(a)F(u) + \beta_2(a)f(-u) + \beta_3(a)F(-u), \tag{14.9}$$

where

$F(u)$ is the ordinary Fourier transform of $f(u)$

$\beta_k(a)$ are the order-dependent coefficients of the linear combination (page 139 of [129])

This definition is merely a linear combination of a function and its Fourier transform (and their time-reversed versions). It is worth emphasizing that the definition of the FRT which is the subject of this chapter not only does not correspond to choosing the principal powers, it does not correspond to any unambiguous way of specifying the power function z^a. The special nature of resolving the ambiguity in evaluating $[\exp(-in\pi/2)]^a$ by taking it equal to $\exp(-ian\pi/2)$ is further discussed in [129].

The fractional Fourier transform $f_a(u)$ of a function $f(u)$ also corresponds to the solution of the following differential equation, with $f_0(u) = f(u)$ acting as the initial condition:

$$\left[-\frac{1}{4\pi}\frac{\partial^2}{\partial u^2} + \pi u^2 - \frac{1}{2}\right] f_a(u) = i\frac{2}{\pi}\frac{\partial f_a(u)}{\partial a}. \tag{14.10}$$

The solution to Equation 14.10 can be expressed as

$$f_a(u) = \int_{-\infty}^{\infty} K_a(u, u')f_0(u')\,du', \tag{14.11}$$

where $K_a(u, u')$ is the same kernel as defined in Equation 14.1, a fact which can be shown by direct substitution. Equation 14.10 is the quantum-mechanical harmonic oscillator differential equation, which can be obtained from the classical harmonic oscillator equation through standard procedures [84]. In this interpretation, the order parameter a corresponds to time and $f_a(u)$ gives us the time evolution of the wave function. The kernel $K_a(u, u')$ is sometimes referred to as the harmonic oscillator Green's function: it is the response of the system to $f_0(u) = \delta(u - u')$ [95]. (To be precise, we must note that the harmonic oscillator differential equation differs from equation 10 by the term $-1/2$; see [129].) Further discussion of the relationship of the fractional Fourier transform to harmonic oscillation may be found in [13,84].

The fractional Fourier transform operator can also be expressed in hyperdifferential form:

$$\begin{aligned}\mathcal{F}^a &= e^{-i(a\pi/2)\mathcal{H}},\\ \mathcal{H} &= \pi(\mathcal{D}^2 + \mathcal{U}^2) - \frac{1}{2},\end{aligned} \tag{14.12}$$

where

$\mathcal{U}$ is the coordinate multiplication operator defined as $\mathcal{U}f(u) = uf(u)$

$\mathcal{D}$ is the differentiation operator defined as $\mathcal{D}f(u) = (i2\pi)^{-1}df(u)/du$

With these definitions, Equation 14.12 corresponds to the following expression in the time domain:

$$f_a(u) = \mathcal{F}^a f(u) = \exp\left[-i\left(\frac{a\pi}{2}\right)\left(-\frac{1}{4\pi}\frac{d^2}{du^2} + \pi u^2 - \frac{1}{2}\right)\right] f(u). \tag{14.13}$$

We can convince ourselves that this way of expressing the fractional Fourier transform is equivalent to earlier expressions by noting that the differential equation 10 can be written as $\mathcal{H}f_a(u) = i(2/\pi)\partial f_a(u)/\partial a$. The solution of this equation can be formally expressed as $f_a(u) = \exp(-i(a\pi/2)\mathcal{H})f_0(u)$ where $f_0(u)$ serves as the initial or boundary condition, which is the same as Equation 14.12. In other words, Equation 14.12 is simply the solution of the differential equation given in Equation 14.10, expressed in hyperdifferential form.

We will conclude this section with a derivation that links together several of the concepts presented above. Let us recall the eigenvalue equation (Equation 14.3):

$$\mathcal{F}^a \psi_n(u) = e^{-ian\pi/2}\psi_n(u) = e^{-i\alpha n}\psi_n(u), \tag{14.14}$$

where $\alpha = a\pi/2$ and $\psi_n(u)$ are the Hermite–Gaussian functions satisfying the differential equation (Table 2.8.6 of [129])

$$\left[\frac{d^2}{du^2} + 4\pi^2\left(\frac{2n+1}{2\pi} - u^2\right)\right]\psi_n(u) = 0. \tag{14.15}$$

Now, starting from the last two equations, let us seek a hyperdifferential representation for $\mathcal{F}^a$ of the form $\exp(-i\alpha\mathcal{H})$. Differentiating

$$\exp(-i\alpha\mathcal{H})\psi_n(u) = e^{-i\alpha n}\psi_n(u) \tag{14.16}$$

with respect to α and setting $\alpha = 0$, we obtain

$$\mathcal{H}\psi_n(u) = n\psi_n(u), \tag{14.17}$$

which upon comparison with Equation 14.15 leads to

$$\mathcal{H}\psi_n(u) = \left(-\frac{1}{4\pi}\frac{d^2}{du^2} + \pi u^2 - \frac{1}{2}\right)\psi_n(u). \tag{14.18}$$

By expanding arbitrary $f(u)$ in terms of the $\psi_n(u)$, we obtain

$$\mathcal{H}f(u) = \left(-\frac{1}{4\pi}\frac{d^2}{du^2} + \pi u^2 - \frac{1}{2}\right)f(u), \tag{14.19}$$

by virtue of the linearity of $\mathcal{H}$. Now, in abstract operator form, we may write

$$\mathcal{H} = \pi(\mathcal{D}^2 + \mathcal{U}^2) - \frac{1}{2}, \tag{14.20}$$

precisely corresponding to Equation 14.12.

A brief list of the fractional Fourier transforms of common functions is provided in Section 14.4. Many of the elementary and operational properties of the FRT are collected in Section 14.5, which can be recognized as generalizations of the corresponding properties of the ordinary Fourier transform.

14.3 Fractional Fourier Domains

One of the most important concepts in Fourier analysis is the concept of the Fourier (or frequency) domain. This "domain" is understood to be a space where the Fourier transform representation of the signal lives, with its own interpretation and qualities. This naturally leads one to inquire into the nature of the domain where the fractional Fourier transform representation of a function lives. This is best understood by referring to Figure 14.2, which shows the phase space spanned by the axes u (usually time or space) and μ (temporal or spatial frequency). This phase space is also referred to as the time–frequency or space–frequency plane in the signal processing literature. The horizontal axis u is simply the time or space domain, where the original function lives. The vertical axis μ is simply the frequency (or Fourier) domain where the ordinary Fourier transform of the function lives. Oblique axes making angle α constitute domains where the ath order fractional Fourier transform lives, where a and α are related through $\alpha = a\pi/2$. Notice that this description is consistent with the fact that the second Fourier transform is equal to the parity operation (associated with the $-u$ axis), the fact that the -1st transform corresponds to the inverse Fourier transform (associated with the $-\mu$ axis), and the periodicity of $f_a(u)$ in a (adding a multiple of 4 to a corresponds to adding a multiple of 2π to α).

For those familiar with phase spaces from a mechanics—rather than a signal analysis—perspective, we note that the correspondence between spatial frequency and momentum allows one to construct a correspondence between the familiar mechanical phase space of a single degree of freedom (defined by the space axis and the momentum axis), and the phase space of signal analysis (defined by the space axis and the spatial frequency axis). What is important to understand for the present purpose is that the phase space or time– and/or space–frequency planes we are talking about is essentially the same physical construct as the classical phase space of mechanics.

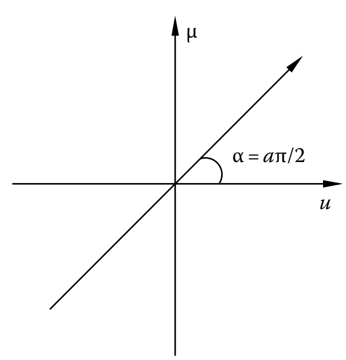

FIGURE 14.2 Phase space and the ath order fractional Fourier domain. (From Ozaktas, H. M. and Kutay, M. A., *Proceedings of the European Control Conference*. European Union Control Association and University of Porto, Porto, Portugal, 2001. With permission.)

Referring to axes making angle $\alpha = a\pi/2$ with the u axis as the "ath fractional Fourier domain" is supported by several of the properties of the fractional Fourier transform to be discussed further in Section 14.5. However, the most substantial justification is based on the fact that

fractional Fourier transformation corresponds to rotation in phase space.

This can be formulated in many ways, the most straightforward being to consider a phase-space distribution (or time/space–frequency representation) of the function $f(u)$, such as the Wigner distribution $W_f(u, \mu)$, which is defined as

$$W_f(u,\mu) = \int_{-\infty}^{\infty} f(u + u'/2) f^*(u - u'/2) e^{-i2\pi\mu u'}\, du'. \quad (14.21)$$

The many properties of the Wigner distribution [37,67] support its interpretation as a function giving the distribution of signal energy in phase space (the time- or space-frequency plane). That is, the Wigner distribution answers the question "How much of the signal energy is located near this time and frequency?" (Naturally, the answer to this question can only be given within limitations imposed by the uncertainty principle.) Three of the important properties of the Wigner distribution are

$$\int_{-\infty}^{\infty} W_f(u,\mu)\, d\mu = \mathcal{R}_0[W_f(u,\mu)] = |f(u)|^2, \quad (14.22)$$

$$\int_{-\infty}^{\infty} W_f(u,\mu)\, du = \mathcal{R}_{\pi/2}[W_f(u,\mu)] = |F(\mu)|^2, \quad (14.23)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(u,\mu)\, du\, d\mu = \|f\|^2 = \text{Signal energy}. \quad (14.24)$$

Here $\mathcal{R}_\alpha$ denotes the integral projection (or Radon transform) operator which takes an integral projection of the two-dimensional function $W_f(u, \mu)$ onto an axis making angle α with the u axis, to produce a one-dimensional function (page 56 of [129]).

Now, it is possible to show that the Wigner distribution $W_{f_a}(u, \mu)$ of $f_a(u)$ is a clockwise rotated version of the Wigner distribution $W_f(u, \mu)$ of $f(u)$. Mathematically,

$$W_{f_a}(u,\mu) = W_f(u\cos\alpha - \mu\sin\alpha, u\sin\alpha + \mu\cos\alpha). \quad (14.25)$$

That is, the act of fractional Fourier transformation on the original function, corresponds to rotation of the Wigner distribution [88,107,117]. An immediate corollary of this result, supported by Figure 14.3, is

$$\mathcal{R}_\alpha[W_f(u,\mu)] = |f_a(u)|^2, \quad (14.26)$$

which is a generalization of Equations 14.22 and 14.23. This equation means that the projection of the Wigner distribution of $f(u)$ onto the axis making angle α gives us $|f_a(u)|^2$, the squared magnitude of the ath fractional Fourier transform of the function. Since projection onto the u axis (the time or space domain) gives $|f(u)|^2$ and projection onto the $\mu = u_1$ axis (the frequency domain) gives $|F(\mu)|^2$, it is natural to refer to the axis making angle α as the ath order fractional Fourier domain.

Closely related to the Wigner distribution is the ambiguity function $A_f(\bar{u}, \bar{\mu})$ of the function $f(u)$, defined as

$$A_f(\bar{u},\bar{\mu}) = \int_{-\infty}^{\infty} f(u' + \bar{u}/2) f^*(u' - \bar{u}/2) e^{-i2\pi\bar{\mu}u'}\, du'. \quad (14.27)$$

Whereas the Wigner distribution is the prime example of an *energetic* time-frequency representation, the ambiguity function

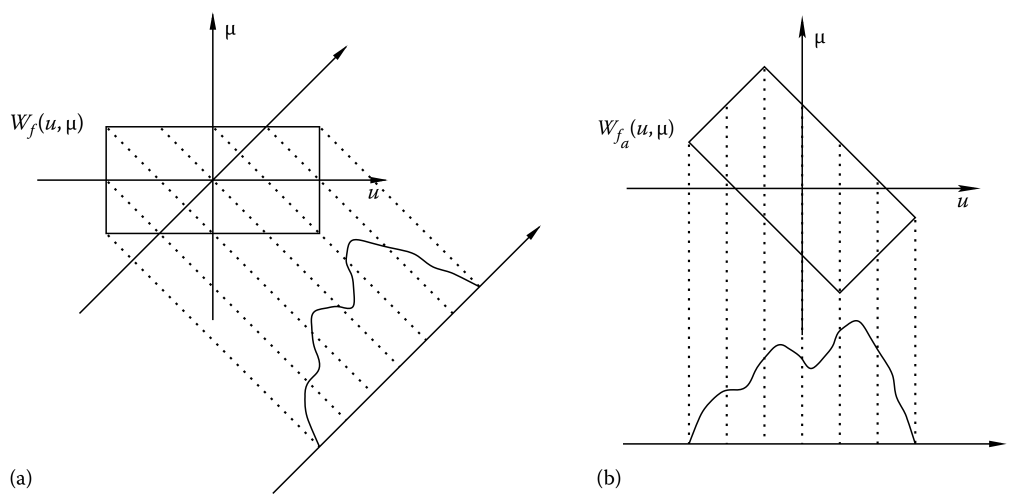

FIGURE 14.3 (a) Projection of $W_f(u, \mu)$ onto the u_a axis. (b) Projection of $W_{f_a}(u, \mu)$ onto the u axis. (From Ozaktas, H. M. and Kutay, M. A., *Proceedings of the European Control Conference*. European Union Control Association and University of Porto, Porto, Portugal, 2001. With permission.)

is the prime example of a *correlative* time–frequency representation. The ambiguity function deserves this by virtue of the following properties [37,67]:

$$A_f(\bar{u},0) = \mathcal{S}_0[A_f(\bar{u},\bar{\mu})] = \int_{-\infty}^{\infty} f(u'+\bar{u})f^*(u')\,du', \tag{14.28}$$

$$A_f(0,\bar{\mu}) = \mathcal{S}_{\pi/2}[A_f(\bar{u},\bar{\mu})] = \int_{-\infty}^{\infty} F(\mu'+\bar{\mu})F^*(\mu')\,d\mu', \tag{14.29}$$

$$A_f(\bar{u},\bar{\mu}) \le A_f(0,0) = \|f\|^2 = \text{En}[f] = \text{Signal energy}, \tag{14.30}$$

which say that the on-axis profiles of the ambiguity function are equal to the autocorrelation of the signal in the time and frequency domains, respectively. Here $\mathcal{S}_\alpha$ denotes the slice operator that returns the slice $A_f(\rho\cos\alpha,\rho\sin\alpha)$ of the two-dimensional function $A_f(\bar{u},\bar{\mu})$ (page 56 of [129]).

Now, it is possible to show that slices of the ambiguity function $A_f(\bar{u},\bar{\mu})$ satisfy

$$\mathcal{S}_\alpha[A_f(\bar{u},\bar{\mu})](\rho) = A_f(\rho\cos\alpha,\rho\sin\alpha) = f_{2\alpha/\pi}(\rho) * f^*_{2\alpha/\pi}(-\rho), \tag{14.31}$$

where * denotes ordinary convolution. Just as oblique projections of the Wigner distribution correspond to the squared magnitudes of the fractional Fourier transforms of the function, the oblique slices of the ambiguity function correspond to the autocorrelations of the fractional Fourier transforms of the function.

Finally, we note that the ambiguity function is related to the Wigner distribution by what is essentially a two-dimensional Fourier transform:

$$A_f(\bar{u},\bar{\mu}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(u,\mu)e^{-i2\pi(\bar{\mu}u-\bar{u}\mu)}\,du\,d\mu. \tag{14.32}$$

14.4 Fractional Fourier Transforms of Some Common Functions

Below we list the fractional Fourier transforms of some common functions. Transforms of most other functions must usually be computed numerically (Section 14.13).

Unit function: The fractional Fourier transform of $f(u)=1$ is

$$\mathcal{F}^a[1] = \sqrt{1+i\tan\alpha}\,e^{-i\pi u^2\tan\alpha}. \tag{14.33}$$

This equation is valid when $a \ne 2j+1$ where j is an arbitrary integer. The transform is $\delta(u)$ when $a=2j+1$.

Delta function: The fractional Fourier transform of a delta function $f(u)=\delta(u-u_0)$ is

$$\mathcal{F}^a[\delta(u-u_0)] = \sqrt{1-i\cot\alpha}\,e^{i\pi(u^2\cot\alpha-2uu_0\csc\alpha+u_0^2\cot\alpha)}. \tag{14.34}$$

This expression is valid when $a \ne 2j$. The transform of $\delta(u-u_0)$ is $\delta(u-u_0)$ when $a=4j$ and $\delta(u+u_0)$ when $a=4j+2$.

Harmonic function: The fractional Fourier transform of a harmonic function $f(u)=\exp(i2\pi\mu_0 u)$ is

$$\mathcal{F}^a[e^{i2\pi\mu_0 u}] = \sqrt{1+i\tan\alpha}\,e^{-i\pi(u^2\tan\alpha-2u\mu_0\sec\alpha+\mu_0^2\tan\alpha)}. \tag{14.35}$$

This equation is valid when $a \ne 2j+1$. The transform of $\exp(i2\pi\mu_0 u)$ is $\delta(u-\mu_0)$ when $a=4j+1$ and $\delta(u+\mu_0)$ when $a=4j+3$.

General chirp function: The fractional Fourier transform of a general chirp function $f(u)=\exp[i\pi(\chi u^2+2\xi u)]$ is

$$\mathcal{F}^a[e^{i\pi(\chi u^2+2\xi u)}] = \sqrt{\frac{1+i\tan\alpha}{1+\chi\tan\alpha}}\,e^{i\pi[u^2(\chi-\tan\alpha)+2u\xi\sec\alpha-\xi^2\tan\alpha]/[1+\chi\tan\alpha]}. \tag{14.36}$$

This equation is valid when $a-(2/\pi)\arctan\chi \ne 2j+1$. The transform of $\exp(i\pi\chi u^2)$ is $\sqrt{1/(1-i\chi)}\,\delta(u)$ when $[a-(2/\pi)\arctan\chi]=2j+1$ and $\sqrt{1/(1-i\chi)}$ when $[a-(2/\pi)\arctan\chi]=2j$.

Hermite–Gaussian functions: The fractional Fourier transform of a Hermite–Gaussian function $f(u)=\psi_n(u)$ is

$$\mathcal{F}^a[\psi_n(u)] = e^{-in\alpha}\,\psi_n(u). \tag{14.37}$$

General Gaussian function: The fractional Fourier transform of a general Gaussian function $f(u)=\exp[-\pi(\chi u^2+2\xi u)]$ is

$$\mathcal{F}^a[e^{-\pi(\chi u^2+2\xi u)}] = \sqrt{\frac{1-i\cot\alpha}{\chi-i\cot\alpha}}\,e^{i\pi\cot\alpha[u^2(\chi^2-1)+2u\chi\xi\sec\alpha+\xi^2]/[\chi^2+\cot\alpha]} \times e^{-\pi\csc^2\alpha(u^2\chi+2u\xi\cos\alpha-\chi\xi^2\sin^2\alpha)/(\chi^2+\cot\alpha)}. \tag{14.38}$$

Here $\chi>0$ is required for convergence.

14.5 Basic and Operational Properties of the Fractional Fourier Transform

Here we present a list of the more important basic and operational properties of the FRT. Readers can easily verify that the operational properties, such as those for scaling, coordinate multiplication, and differentiation, reduce to the corresponding property for the ordinary Fourier transform when $a=1$.

Linearity: Let $\mathcal{F}^a$ denote the ath order fractional Fourier transform operator. Then $\mathcal{F}^a\left[\sum_k b_k f_k(u)\right] = \sum_k b_k\left[\mathcal{F}^a f_k(u)\right]$.

Integer orders: $\mathcal{F}^k = (\mathcal{F})^k$ where $\mathcal{F}$ denotes the ordinary Fourier transform operator. This property states that when a is equal to an integer k, the ath order fractional Fourier transform is equivalent to the kth integer power of the ordinary Fourier transform, defined by repeated application. It also follows that $\mathcal{F}^2 = \mathcal{P}$ (the parity operator), $\mathcal{F}^3 = \mathcal{F}^{-1} = (\mathcal{F})^{-1}$ (the inverse transform operator), $\mathcal{F}^4 = \mathcal{F}^0 = \mathcal{I}$ (the identity operator), and $\mathcal{F}^j = \mathcal{F}^{j \bmod 4}$.

Inverse: $(\mathcal{F}^a)^{-1} = \mathcal{F}^{-a}$. In terms of the kernel, this property is stated as $K_a^{-1}(u, u') = K_{-a}(u, u')$.

Unitarity: $(\mathcal{F}^a)^{-1} = (\mathcal{F}^a)^{\mathrm{H}} = \mathcal{F}^{-a}$ where $()^{\mathrm{H}}$ denotes the conjugate transpose of the operator. In terms of the kernel, this property can be stated as $K_a^{-1}(u, u') = K_a^*(u', u)$.

Index additivity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2+a_1}$. In terms of kernels this can be written as $K_{a_2+a_1}(u, u') = \int K_{a_2}(u, u'')K_{a_1}(u'', u')\, du''$.

Commutativity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_1}\mathcal{F}^{a_2}$.

Associativity: $\mathcal{F}^{a_3}(\mathcal{F}^{a_2}\mathcal{F}^{a_1}) = (\mathcal{F}^{a_3}\mathcal{F}^{a_2})\mathcal{F}^{a_1}$.

Eigenfunctions: $\mathcal{F}^a[\psi_n(u)] = \exp(-ian\pi/2)\psi_n(u)$. Here $\psi_n(u)$ are the Hermite–Gaussian functions defined in Section 14.2.

Parseval: $\int f^*(u)g(u)du = \int f_a^*(u)g_a(u)du$. This property is equivalent to unitarity. Energy or norm conservation ($\mathrm{En}[f] = \mathrm{En}[f_a]$ or $\|f\| = \|f_a\|$) is a special case.

Time reversal: Let $\mathcal{P}$ denote the parity operator: $\mathcal{P}[f(u)] = f(-u)$, then

$$\mathcal{F}^a\mathcal{P} = \mathcal{P}\mathcal{F}^a \tag{14.39}$$

$$\mathcal{F}^a[f(-u)] = f_a(-u) \tag{14.40}$$

Transform of a scaled function: Let $\mathcal{M}(M)$ and $\mathcal{Q}(q)$ denote the scaling $\mathcal{M}(M)[f(u)] = |M|^{-1/2}f(u/M)$ and chirp multiplication $\mathcal{Q}(q)[f(u)] = e^{-i\pi qu^2}f(u)$ operators, respectively. Here the notation $\mathcal{M}(M)[f(u)]$ means that the operator $\mathcal{M}(M)$ is applied to the function $f(u)$. Then

$$\mathcal{F}^a\,\mathcal{M}(M) = \mathcal{Q}(-\cot\alpha\,(1 - (\cos^2\alpha')/(\cos^2\alpha))) \times \mathcal{M}(\sin\alpha/M\sin\alpha')\,\mathcal{F}^{a'}, \tag{14.41}$$

$$\mathcal{F}^a[|M|^{-1/2}f(u/M)] = \sqrt{\frac{1 - i\cot\alpha}{1 - iM^2\cot\alpha}}\,e^{i\pi u^2\cot\alpha(1-(\cos^2\alpha')/(\cos^2\alpha))} \times f_{a'}\left(\frac{Mu\sin\alpha'}{\sin\alpha}\right). \tag{14.42}$$

Here $\alpha' = \arctan(M^{-2}\tan\alpha)$ and α' is taken to be in the same quadrant as α. This property is the generalization of the ordinary Fourier transform property stating that the Fourier transform of $f(u/M)$ is $|M|F(M\mu)$. Notice that the fractional Fourier transform of $f(u/M)$ cannot be expressed as a scaled version of $f_a(u)$ for the same order a. Rather, the fractional Fourier transform of $f(u/M)$ turns out to be a scaled and chirp modulated version of $f_{a'}(u)$ where $a' \neq a$ is a different order.

Transform of a shifted function: Let $\mathcal{SH}(u_0)$ and $\mathcal{PH}(\mu_0)$ denote the shift $\mathcal{SH}(u_0)[f(u)] = f(u + u_0)$ and the phase shift $\mathcal{PH}(\mu_0)[f(u)] = \exp(i2\pi\mu_0 u)f(u)$ operators, respectively. Then

$$\mathcal{F}^a\,\mathcal{SH}(u_0) = e^{i\pi u_0^2\sin\alpha\cos\alpha}\mathcal{PH}(u_0\sin\alpha)\,\mathcal{SH}(u_0\cos\alpha)\mathcal{F}^a, \tag{14.43}$$

$$\mathcal{F}^a[f(u + u_0)] = e^{i\pi u_0^2\sin\alpha\cos\alpha}e^{i2\pi uu_0\sin\alpha}f_a(u + u_0\cos\alpha). \tag{14.44}$$

We see that the $\mathcal{SH}(u_0)$ operator, which simply results in a translation in the u domain, corresponds to a translation followed by a phase shift in the ath fractional domain. The amount of translation and phase shift is given by cosine and sine multipliers which can be interpreted in terms of “projections” between the axes.

Transform of a phase-shifted function:

$$\mathcal{F}^a\,\mathcal{PH}(\mu_0) = e^{-i\pi\mu_0^2\sin\alpha\cos\alpha}\mathcal{PH}(\mu_0\cos\alpha)\,\mathcal{SH}(-\mu_0\sin\alpha)\mathcal{F}^a, \tag{14.45}$$

$$\mathcal{F}^a[\exp(i2\pi\mu_0 u)f(u)] = e^{-i\pi\mu_0^2\sin\alpha\cos\alpha}e^{i2\pi u\mu_0\cos\alpha} f_a(u - \mu_0\sin\alpha). \tag{14.46}$$

Similar to the shift operator, the phase-shift operator, which simply results in a phase shift in the u domain, corresponds to a translation followed by a phase shift in the ath fractional domain. Again the amount of translation and phase shift are given by cosine and sine multipliers.

Transform of a coordinate multiplied function: Let $\mathcal{U}$ and $\mathcal{D}$ denote the coordinate multiplication $\mathcal{U}[f(u)] = uf(u)$ and differentiation $\mathcal{D}[f(u)] = (i2\pi)^{-1}df(u)/du$ operators, respectively. Then

$$\mathcal{F}^a\,\mathcal{U}^n = [\cos\alpha\,\mathcal{U} - \sin\alpha\,\mathcal{D}]^n\mathcal{F}^a, \tag{14.47}$$

$$\mathcal{F}^a[u^n f(u)] = [\cos\alpha\,u - \sin\alpha\,(i2\pi)^{-1}d/du]^n f_a(u). \tag{14.48}$$

When $a = 1$, the transform of a coordinate multiplied function $uf(u)$ is the derivative of the transform of the original function $f(u)$, a well-known property of the Fourier transform. For arbitrary values of a, we see that the transform of $uf(u)$ is a linear combination of the coordinate-multiplied transform of the original function and the derivative of the transform of the original function. The coefficients in the linear combination are $\cos\alpha$ and $-\sin\alpha$. As a approaches 0, there is more $uf(u)$ and less $df(u)/du$ in the linear combination. As a approaches 1, there is more $df(u)/du$ and less $uf(u)$.

Transform of the derivative of a function:

$$\mathcal{F}^a\,\mathcal{D}^n = [\sin\alpha\,\mathcal{U} + \cos\alpha\,\mathcal{D}]^n\mathcal{F}^a, \tag{14.49}$$

$$\mathcal{F}^a[[(i2\pi)^{-1}d/du]^n f(u)] = [\sin\alpha\,u + \cos\alpha\,(i2\pi)^{-1}d/du]^n f_a(u). \tag{14.50}$$

When $a = 1$ the transform of the derivative of a function $df(u)/du$ is the coordinate-multiplied transform of the original function. For arbitrary values of a, we see that the transform is again a linear combination of the coordinate-multiplied transform of the original function and the derivative of the transform of the original function.

Transform of a coordinate divided function:

$$\mathcal{F}^a[f(u)/u] = -i \csc\alpha\, e^{i\pi u^2 \cot\alpha} \int_{-\infty}^{2\pi u} f_a(u') e^{(-i\pi u'^2 \cot\alpha)}\, du'. \tag{14.51}$$

Transform of the integral of a function:

$$\mathcal{F}^a\left[\int_{u_0}^{u} f(u')\, du'\right] = \sec\alpha\, e^{-i\pi u^2 \tan\alpha} \int_{u_0}^{u} f_a(u') e^{i\pi u'^2 \tan\alpha}\, du'. \tag{14.52}$$

A few additional properties are

$$\mathcal{F}^a[f^\star(u)] = f^\star_{-a}(u), \tag{14.53}$$

$$\mathcal{F}^a[(f(u) + f(-u))/2] = (f_a(u) + f_a(-u))/2, \tag{14.54}$$

$$\mathcal{F}^a[(f(u) - f(-u))/2] = (f_a(u) - f_a(-u))/2. \tag{14.55}$$

It is also possible to write convolution and multiplication properties for the fractional Fourier transform, though these are not of great simplicity (page 157 of [129] and [9,174]).

A function and its ath order fractional Fourier transform satisfy an "uncertainty relation," stating that the product of the spread of the two functions, as measured by their standard deviations, cannot be less than $|\sin(a\pi/2)|/4\pi$ [116].

We may finally note that the transform is continuous in the order a. That is, small changes in the order a correspond to small changes in the transform $f_a(u)$. Nevertheless, care is always required in dealing with cases where a approaches an even integer, since in this case the kernel approaches a delta function.

14.6 Dual Operators and Their Fractional Generalizations

The dual of the operator $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathrm{D}}$ and satisfies

$$\mathcal{A}^{\mathrm{D}} = \mathcal{F}^{-1}\mathcal{A}\mathcal{F}. \tag{14.56}$$

$\mathcal{A}^{\mathrm{D}}$ performs the same action on the frequency-domain representation $F(\mu)$, that $\mathcal{A}$ performs on the time-domain representation $f(u)$. For instance, if $\mathcal{A}$ represents the operation of multiplying with the coordinate variable u, then the dual $\mathcal{A}^{\mathrm{D}}$ represents the operation of multiplying $F(\mu)$ with μ, which in the time domain corresponds to the operator $(i2\pi)^{-1}d/du$. The fractional operators we deal with in this section perform the same action in a fractional domain:

$$\mathcal{A}_a = \mathcal{F}^{-a}\mathcal{A}\mathcal{F}^a. \tag{14.57}$$

This equation generalizes Equation 14.56 and reduces to it when $a = 1$ with $\mathcal{A}_1 = \mathcal{A}^{\mathrm{D}}$. If again $\mathcal{A}$ corresponds to the multiplication of $f(u)$ with u, then $\mathcal{A}_a$ corresponds to the multiplication of $f_a(u_a)$ with u_a, where u_a denotes the coordinate variable associated with the ath fractional Fourier domain. The effect of $\mathcal{A}_a$ in the ordinary time domain can be expressed as $\cos\alpha\, uf(u) + \sin\alpha\,(i2\pi)^{-1}df(u)/du$ (see "Transform of a coordinate multiplied function" in Section 14.5).

To distinguish the kind of fractional operators discussed in this section from the ath operator power of $\mathcal{A}$ which is denoted by $\mathcal{A}^a$, we are denoting them by $\mathcal{A}_a$. The FRT is the ath operator power of the ordinary Fourier transform, but the fractional operators here are operators that perform the same action, such as coordinate multiplication, in different fractional Fourier domains. To further emphasize the difference, we note that for $a = 0$, $\mathcal{A}_0 = \mathcal{A}$ while $\mathcal{A}^0 = \mathcal{I}$; and for $a = 1$, $\mathcal{A}_1 = \mathcal{A}^{\mathrm{D}}$ while $\mathcal{A}^1 = \mathcal{A}$. In other words, $\mathcal{A}_a$ interpolates between the operator $\mathcal{A}$ and its dual $\mathcal{A}^{\mathrm{D}}$, gradually evolving from one member of the dual pair to the other as the fractional order goes from zero to one. On the other hand, $\mathcal{A}^a$ interpolates between the identity operator and the operator $\mathcal{A}$.

The first pair of dual operators we will consider are the coordinate multiplication $\mathcal{U}$ and differentiation $\mathcal{D}$ operators, whose effects in the time domain are to take a function $f(u)$ to $uf(u)$ and $(i2\pi)^{-1}df(u)/du$, respectively. The fractional forms of these operators $\mathcal{U}_a$ and $\mathcal{D}_a$ are defined so as to have the same functional effect in the ath domain; they take $f_a(u_a)$ to $u_a f_a(u_a)$ and $(i2\pi)^{-1}df_a(u_a)du_a$, respectively. In the time domain these operations correspond to taking $f(u)$ to $\cos\alpha\, uf(u) + \sin\alpha\,(i2\pi)^{-1} df(u)/du$ and $-\sin\alpha\, uf(u) + \cos\alpha\,(i2\pi)^{-1}df(u)/du$, respectively. (These and similar results are a consequence of the operational properties presented in Section 14.5.) These relationships can be captured elegantly in the following operator form:

$$\begin{aligned} \mathcal{U}_a &= \cos\alpha\, \mathcal{U} + \sin\alpha\, \mathcal{D}, \\ \mathcal{D}_a &= -\sin\alpha\, \mathcal{U} + \cos\alpha\, \mathcal{D}. \end{aligned} \tag{14.58}$$

The phase shift operator $\mathcal{PH}(\eta)$ and the translation operator $\mathcal{SH}(\xi)$ are also duals which are defined in terms of the $\mathcal{U}$ and $\mathcal{D}$ operators as $\mathcal{PH}(\eta) = \exp(i2\pi\eta\mathcal{U})$ and $\mathcal{SH}(\xi) = \exp(i2\pi\xi\mathcal{D})$. (Such expressions are meant to be interpreted in terms of their series expansions.) These operators take $f(u)$ to $\exp(i2\pi\eta u)f(u)$ and $f(u + \xi)$, respectively. The fractional forms of these operators are defined as $\mathcal{PH}_a(\eta) = \exp(i2\pi\eta\mathcal{U}_a)$ and $\mathcal{SH}_a(\xi) = \exp(i2\pi\xi\mathcal{D}_a)$ and satisfy

$$\begin{aligned} \mathcal{PH}_a(\eta) &= \exp(i\pi\eta^2 \sin\alpha\cos\alpha)\mathcal{PH}(\eta\cos\alpha)\mathcal{SH}(\eta\sin\alpha), \\ \mathcal{SH}_a(\xi) &= \exp(-i\pi\xi^2 \sin\alpha\cos\alpha)\mathcal{PH}(-\xi\sin\alpha)\mathcal{SH}(\xi\cos\alpha). \end{aligned} \tag{14.59}$$

The scaling operator $\mathcal{M}(M)$ can be defined as $\mathcal{M}(M) = \exp[-i\pi(\ln M)(\mathcal{UD} + \mathcal{DU})]$ where $M > 0$. It takes $f(u)$ to $\sqrt{1/M}f(u/M)$. This operator is its own dual in the sense that scaling in the time domain corresponds to descaling in the frequency domain: the Fourier transform of $\sqrt{1/M}f(u/M)$ is $\sqrt{M}F(M\mu)$. The fractional form is defined as $\mathcal{M}_a(M) = \exp[-i\pi(\ln M)(\mathcal{U}_a\mathcal{D}_a + \mathcal{D}_a\mathcal{U}_a)]$ and satisfies

$$\mathcal{M}_a(M) = \mathcal{F}^{-a}\mathcal{M}(M)\mathcal{F}^{a}. \tag{14.60}$$

The dual chirp multiplication $\mathcal{Q}(q)$ and chirp convolution $\mathcal{R}(r)$ operators are defined as $\mathcal{Q}(q) = \exp(-i\pi\, q\mathcal{U}^2)$ and $\mathcal{R}(r) = \exp(-i\pi\, rD^2)$. In the time domain they take $f(u)$ to $\exp(-i\pi\, qu^2)f(u)$ and $\exp(-i\pi/4)\sqrt{1/r}\exp(i\pi\, u^2/r)*f(u)$, respectively. Their fractional forms are defined as $\mathcal{Q}_a(q) = \exp(-i\pi q\mathcal{U}_a^2)$ and $\mathcal{R}_a(r) = \exp\left(-i\pi r\mathcal{D}_a^2\right)$ and satisfy

$$\begin{aligned}\mathcal{Q}_a(q) &= \mathcal{R}(-\tan\alpha)\,\mathcal{Q}(q\cos^2\alpha)\,\mathcal{R}(\tan\alpha),\\ \mathcal{R}_a(r) &= \mathcal{Q}(-\tan\alpha)\,\mathcal{R}(r\cos^2\alpha)\,\mathcal{Q}(\tan\alpha).\end{aligned} \tag{14.61}$$

We now turn our attention to the final pair of dual operators we will discuss. The discretization $\mathcal{DI}(\Delta\mu)$ and periodization $\mathcal{PE}(\Delta u)$ operators can be defined in terms of the phase shift and translation operators: $\mathcal{DI}(\Delta\mu) = \sum_{k=-\infty}^{\infty}\mathcal{PH}(k\Delta\mu)$ and $\mathcal{PE}(\Delta u) = \sum_{k=-\infty}^{\infty}\mathcal{SH}(k\Delta u)$. The parameters $\Delta u > 0$ and $\Delta\mu > 0$ correspond to the period of replication in the time and frequency domains, respectively. Unlike the other operators defined above, these operators do not in general have inverses. Since sampling in the time domain corresponds to periodic replication in the frequency domain and vice versa, we also define $\delta u = 1/\Delta\mu$ and $\delta\mu = 1/\Delta u$, denoting the sampling interval in the time and frequency domains, respectively. It is possible to show that the discretization and periodization operators take $f(u)$ to $\delta u\sum_{k=-\infty}^{\infty}\delta(u - k\delta u)f(k\delta u)$ and $\sum_{k=-\infty}^{\infty}f(u - k\Delta u)$, respectively. In the time domain, the discretization operator corresponds to multiplication with an impulse train, and the periodization operator corresponds to convolution with an impulse train (and vice versa in the frequency domain). Discretization in the time domain corresponds to periodization in the frequency domain and periodization in the time domain corresponds to discretization in the frequency domain. This is what is meant by the duality of these two operators. The fractional versions of these operators can be defined as $\mathcal{DI}_a(\Delta\mu) = \sum_{k=-\infty}^{\infty}\mathcal{PH}_a(k\Delta\mu)$ and $\mathcal{PE}_a(\Delta u) = \sum_{k=-\infty}^{\infty}\mathcal{SH}_a(k\Delta u)$ and satisfy

$$\begin{aligned}\mathcal{DI}_a(\Delta\mu) &= \mathcal{R}(-\tan\alpha)\,\mathcal{DI}(\Delta\mu\cos\alpha)\,\mathcal{R}(\tan\alpha),\\ \mathcal{PE}_a(\Delta u) &= \mathcal{Q}(-\tan\alpha)\,\mathcal{PE}(\Delta u\cos\alpha)\,\mathcal{Q}(\tan\alpha).\end{aligned} \tag{14.62}$$

Equations 14.58 through 14.62 all express the fractional operators in terms of their non-fractional counterparts. Equations 14.58 through 14.60 are directly related to the corresponding operational properties presented in Section 14.5, and may be considered abstract ways of expressing them (transform of a coordinate multiplied or differentiated function, transform of a phase-shifted or shifted function, transform of a scaled function, respectively).

The fractional operators in Equation 14.62 interpolate between periodicity and discreteness with the smooth transition being governed by the parameter a. However, this is not the only significance of the fractional periodicity and discreteness operators. In practice, one cannot realize infinite periodic replication; any periodic replication must be limited to a finite number of periods. This corresponds to multiplying the infinite periodic replication operator with a window function, and will be referred to as partial periodization. Likewise, one cannot realize discretization with true impulses; any discretization will involve finite-width sampling pulses. This corresponds to convolving a true impulse sampling operator with a window function, and will be referred to as partial discretization. Thus, the partial periodization and discretization operations represent practical real-life replication and sampling operations. It has been shown that fractional periodization and discretization operators can be expressed in terms of partial periodization and discretization operators [128]. Therefore, the fractional periodization and discretization operators are also related to real-life sampling and periodic replication.

The subject matter of this section is further discussed in [128,156].

14.7 Time-Order and Space-Order Representations

Interpreting the fractional Fourier transforms $f_a(u)$ of a function $f(u)$ for different values of the order a as a two-dimensional function of u and a leads to the concept of time-order (or space-order) signal representations. Just like other time-frequency and time-scale (or space-frequency and space-scale) signal representations, they constitute an alternative way of displaying the content of a signal. These representations are redundant in that the information of a one-dimensional signal is displayed in two dimensions. There are two variations of the time-order representation, the rectangular time-order representation and the polar time-order representation.

For the rectangular time-order representation, $f_a(u)$ is interpreted as a two-dimensional function, with u the horizontal coordinate and a the vertical coordinate. As such, the representations of the signal $f(u)$ in all fractional domains are displayed simultaneously. Mathematically, the rectangular time-order representation $T_f(u, a)$ of a signal f is defined as

$$T_f(u, a) = f_a(u). \tag{14.63}$$

Figure 14.4 illustrates the definition of the rectangular time-order representation. Such a display of the fractional Fourier transforms of the rectangle function is shown in Figure 14.1.

For the polar time-order representation, $f_a(u) = f_{2\alpha/\pi}(\rho)$ is interpreted as a polar two-dimensional function where ρ is the

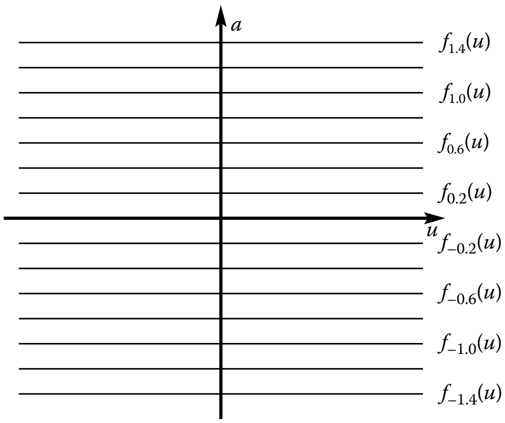

FIGURE 14.4 The rectangular time-order representation. (From Ozaktas, H. M. and Kutay M. A., Technical Report BU-CE-0005, Bilkent University, Department of Computer Engineering, Ankara, January 2000; Ozaktas, H. M., Zalevsky, Z., and Kutay, M. A., *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. John Wiley & Sons, New York, 2001. With permission.)

radial coordinate and α is the angular coordinate. As such, all the fractional Fourier transforms of $f(u)$ are displayed such that $f_a(\rho)$ lies along the radial line making angle $\alpha = a\pi/2$ with the horizontal axis. Mathematically, the polar time-order representation $T_f(\rho, \alpha)$ of a signal f is defined as

$$T_f(\rho, \alpha) = f_{2\alpha/\pi}(\rho). \tag{14.64}$$

$T_f(\rho, \alpha)$ is periodic in α with period 2π as a result of the fact that $f_a(\rho)$ is periodic in a with period 4. $T_f(\rho, \alpha)$ can be consistently defined for negative values of ρ as well by using the property

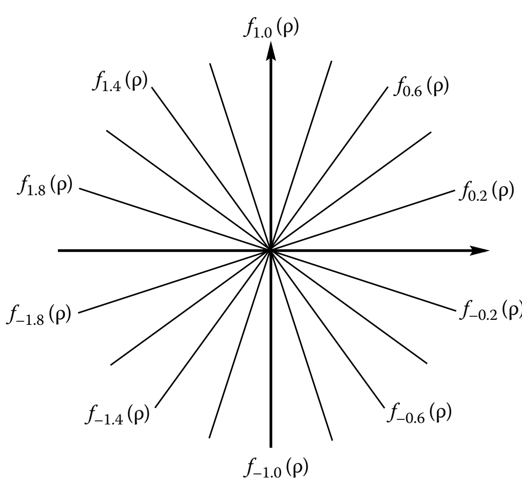

FIGURE 14.5 The polar time-order representation. (From Ozaktas, H. M. and Kutay M. A., Technical Report BU-CE-0005, Bilkent University, Department of Computer Engineering, Ankara, January 2000; Ozaktas, H. M., Zalevsky, Z., and Kutay, M. A., *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. John Wiley & Sons, New York, 2001. With permission.)

$f_{a\pm 2}(\rho) = f_a(-\rho)$, from which it also follows that $T_f(\rho, \alpha) = T_f(-\rho, \alpha \pm \pi)$. Figure 14.5 illustrates the definition of the polar time-order representation.

As a consequence of its definition, there is a direct relation between the polar time-order representation and the concept of fractional Fourier domains. Each fractional Fourier transform $f_a(\rho)$ of the signal f "lives" in the ath domain, defined by the radial line making angle $\alpha = a\pi/2$ with the u axis. The polar time-order representation can be considered as a time–frequency space since the horizontal and vertical axes correspond to time and frequency. The oblique slices of the polar representation are simply equal to the fractional Fourier transforms. The slice at $\alpha = 0$ is the time-domain representation $f(\rho)$, the slice at $\alpha = \pi/2$ is the frequency-domain representation $F(\rho)$, and other slices correspond to fractional transforms of other orders.

We now discuss a number of properties of the polar time-order representation. The original function is obtained from the distribution as

$$f(u) = f_0(u) = T_f(u, 0). \tag{14.65}$$

The time-order representation of the a'th fractional Fourier transform of a function is simply a rotated version of the time-order representation of the original function

$$T_{f_{a'}}(\rho, \alpha) = T_f(\rho, \alpha + \alpha'), \tag{14.66}$$

where $\alpha' = a'\pi/2$. Since the time-order representation is linear, the representation of any linear combination of functions is the same as the linear combination of their representations.

We now discuss the relationship of time-order representations with the Wigner distribution and the ambiguity function. We had already encountered the Radon transform of the Wigner distribution:

$$\mathcal{R}_\alpha[W_f(u, \mu)](\rho) = |f_{2\alpha/\pi}(\rho)|^2 = |T_f(\rho, \alpha)|^2. \tag{14.67}$$

Thus, the Radon transform of the Wigner distribution, interpreted as a polar function, corresponds to the absolute square of the polar time-order representation. We also already encountered the following result, which is a consequence of the projection-slice theorem (page 56 of [129]):

$$\begin{aligned} \mathcal{S}_\alpha[A_f(\bar{u}, \bar{\mu})](\rho) &= A_f(\rho \cos \alpha, \rho \sin \alpha) \\ &= T_f(\rho, \alpha) * T_f^*(-\rho, \alpha) = f_{2\alpha/\pi}(\rho) * f^*_{2\alpha/\pi}(-\rho), \end{aligned} \tag{14.68}$$

where $*$ denotes ordinary convolution. The Radon transforms and slices of the Wigner distribution and the ambiguity function are summarized in Table 14.1. For both the Wigner distribution and the ambiguity function, the Radon transform is of product form and the slice is of convolution form. The essential difference between the Wigner distribution and the ambiguity function lies in the scaling of ρ by 2 or $1/2$ on the right-hand side.

TABLE 14.1 Radon Transforms and Slices of the Wigner Distribution and the Ambiguity Function

$\mathcal{RDN}_\alpha[W_f(u,\mu)](\rho) = f_{2\alpha/\pi}(\rho) f^*_{2\alpha/\pi}(\rho) = T_f(\rho,\alpha) T^*_f(\rho,\alpha)$
$\mathcal{RDN}_\alpha[A_f(\bar{u},\bar{\mu})](\rho) = f_{2\alpha/\pi}(\rho/2) f^*_{2\alpha/\pi}(-\rho/2) = T_f(\rho/2,\alpha) T^*_f(-\rho/2,\alpha)$
$\mathcal{SLC}_\alpha[W_f(u,\mu)](\rho) = 2f_{2\alpha/\pi}(2\rho) * 2f^*_{2\alpha/\pi}(2\rho) = 2T_f(2\rho,\alpha) * 2T^*_f(2\rho,\alpha)$
$\mathcal{SLC}_\alpha[A_f(\bar{u},\bar{\mu})](\rho) = f_{2\alpha/\pi}(\rho) * f^*_{2\alpha/\pi}(-\rho) = T_f(\rho,\alpha) * T^*_f(-\rho,\alpha)$

Sources: From Ozaktas, H. M. and Kutay, M. A., Technical Report BU-CE-0005, Bilkent University, Department of Computer Engineering, Ankara, January 2000; Ozaktas, H. M., et al., *The Fractional Fourier Transforms with Applications in Optics and Signal Processing*. John Wiley & Sons, New York, 2001. With permission.)

Note: The upper row can also be expressed as $|f_{2\alpha/\pi}(\rho)|^2 = |T_f(\rho,\alpha)|^2$.

Analogous expressions for the Radon transforms and slices of the polar time-order representation $T_f(\rho,\alpha)$ and its two-dimensional Fourier transform $\tilde{T}_f(\bar{\rho},\bar{\alpha})$ are given in Table 14.2. The slice of $T_f(\rho,\alpha)$ at a certain angle is simply equal to the fractional Fourier transform $f_a(\rho)$ by definition (with $\alpha = a\pi/2$). The Radon transform of $\tilde{T}_f(\bar{\rho},\bar{\alpha})$ at an angle ϕ is given by $f_{b+1}(\rho)$ or $T_f(\rho,\phi+\pi/2)$, a $\pi/2$ rotated version of $T_f(\rho,\alpha)$ (with $\phi = b\pi/2$). We already know that the time-frequency representation whose projections are equal to $|f_a(u)|^2$ is the Wigner distribution. We now see that the time–frequency representation whose projections are equal to $f_a(u)$ is the two-dimensional Fourier transform of the polar time-order representation (within a rotation).

Thus in Tables 14.1 and 14.2 we present a total of eight expressions for the Radon transforms and slices of the Wigner distribution and its two-dimensional Fourier transform (the ambiguity function), and the Radon transforms and slices of the polar time-order representation and its two-dimensional Fourier transform.

The polar time-order representation is a linear time–frequency representation, unlike the Wigner distribution and ambiguity function which are quadratic. Its importance stems from the fact that the Radon transforms (integral projections) and slices of the Wigner distribution and the ambiguity function can be expressed in terms of products or convolutions of various scaled forms of the time-order representation and its two-dimensional Fourier transform. These representations are discussed in greater detail in Chapter 5 of [129].

TABLE 14.2 Radon Transforms and Slices of the Polar Time-Order Representation and Its Two-Dimensional Fourier Transform

$\mathcal{RDN}_\phi[T_f(\rho,\alpha)](\varrho) = \int_{-\pi/2}^{\pi/2} f_{2(\phi+\theta)/\pi}(\varrho \sec\theta)\, \varrho \sec^2\theta\, d\theta$
$\mathcal{RDN}_\phi[\tilde{T}_f(\bar{\rho},\bar{\alpha})](\varrho) = f_{2\phi/\pi+1}(\varrho)$
$\mathcal{SLC}_\phi[T_f(\rho,\alpha)](\varrho) = f_{2\phi/\pi}(\varrho)$
$\mathcal{SLC}_\phi[\tilde{T}_f(\bar{\rho},\bar{\alpha})](\varrho) = \frac{i}{2\pi}\int_{-\pi/2}^{\pi/2} f'_{2(\phi+\theta)/\pi+1}(\varrho\cos\theta)\sec\theta\, d\theta$

Sources: From Ozaktas, H. M. and Kutay, M. A., Technical Report BU-CE-0005, Bilkent University, Department of Computer Engineering, Ankara, January 2000; Ozaktas, H. M., et al., *The Fractional Fourier Transforms with Applications in Optics and Signal Processing*. John Wiley & Sons, New York, 2001. With permission.)

14.8 Linear Canonical Transforms

Linear canonical transforms (LCTs) are a three-parameter family of linear integral transforms. Many important operations and transforms including the FRT are special cases of linear canonical transforms. Readers wishing to learn more than we can cover here are referred to [129,164].

The linear canonical transform $f_{\mathbf{M}}(u)$ of $f(u)$ with parameter $\mathbf{M}$ is most conveniently defined as

$$f_{\mathbf{M}}(u) = \int_{-\infty}^{\infty} C_{\mathbf{M}}(u,u') f(u')\, du', \tag{14.69}$$

$$C_{\mathbf{M}}(u,u') = \sqrt{\beta}\, e^{-i\pi/4} \exp\left[i\pi(\alpha u^2 - 2\beta u u' + \gamma u'^2)\right],$$

where α, β, and γ are real parameters. The label $\mathbf{M}$ represents the three parameters α, β, and γ which completely specify the transform. Linear canonical transforms are unitary; that is, the inverse transform kernel is the Hermitian conjugate of the original transform kernel: $\mathcal{C}^{-1}_{\mathbf{M}}(u,u') = \mathcal{C}^*_{\mathbf{M}}(u',u)$.

The composition of any two linear canonical transforms is another linear canonical transform. In other words, the effect of consecutively applying two linear canonical transforms with different parameters is equivalent to applying another linear canonical transform whose parameters are related to those of the first two. (Actually this is strictly true only within a $\pm$ sign factor [129,164].) Such compositions are not in general commutative, but they are associative.

Finding the parameters of the composite transform is made easier if we define a 2×2 unit-determinant matrix to represent the parameters of the transform. We let the symbol $\mathbf{M}$ (which until now denoted the three parameters α, β, γ) now be defined as a matrix of the form

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \gamma/\beta & 1/\beta \\ -\beta+\alpha\gamma/\beta & \alpha/\beta \end{bmatrix} = \begin{bmatrix} \alpha/\beta & -1/\beta \\ \beta-\alpha\gamma/\beta & \gamma/\beta \end{bmatrix}^{-1}, \tag{14.70}$$

with determinant $AD - BC = 1$. The three original parameters can be expressed in terms of the matrix elements as $\alpha = D/B$, $\beta = 1/B$, and $\gamma = A/B$, and the definition of linear canonical transforms can be rewritten as

$$f_{\mathbf{M}}(u) = \int_{-\infty}^{\infty} C_{\mathbf{M}}(u,u') f(u')\, du', \tag{14.71}$$

$$C_{\mathbf{M}}(u,u') = \sqrt{1/B}\, e^{-i\pi/4} \exp\left[i\pi\left(\frac{D}{B}u^2 - 2\frac{1}{B}uu' + \frac{A}{B}u'^2\right)\right].$$

Now, it is easy to show the following results: The matrix $\mathbf{M}_3$ corresponding to the composition of two systems is the matrix product of the matrices $\mathbf{M}_2$ and $\mathbf{M}_1$ corresponding to the individual systems. That is,

$$\mathbf{M}_3 = \mathbf{M}_2\mathbf{M}_1, \tag{14.72}$$

where

$\mathbf{M}_1$ is the matrix of the transform that is applied first
$\mathbf{M}_2$ is the matrix of the transform that is applied next

Furthermore, the matrix corresponding to the inverse of a linear canonical transform is the inverse of the matrix corresponding to the original transform.

The set of linear canonical transforms satisfies all the axioms of a noncommutative group (closure, associativity, existence of identity, inverse of each element), just like the set of all unit-determinant 2×2 matrices (again within a $\pm$ sign). Certain subsets of the set of linear canonical transforms are groups in themselves and thus are subgroups. Some of them will be discussed below. For example, the fractional Fourier transform is a subgroup with one real parameter.

The effect of linear canonical transforms on the Wigner distribution of a function can be expressed quite elegantly in terms of the elements of the matrix $\mathbf{M}$:

$$W_{f_\mathbf{M}}(Au + B\mu, Cu + D\mu) = W_f(u, \mu), \tag{14.73}$$

$$W_{f_\mathbf{M}}(u, \mu) = W_f(Du - B\mu, -Cu + A\mu). \tag{14.74}$$

A similar relationship holds for the ambiguity function as well. The above result means that the Wigner distribution of the transformed function is simply a linearly distorted form of the Wigner distribution of the original function, with the value of the Wigner distribution at each time/space–frequency point being mapped to another time/space–frequency point. Since the determinant of $\mathbf{M}$ is equal to unity, this pointwise geometrical distortion or deformation is area preserving; it distorts but does not concentrate or deconcentrate the Wigner distribution.

We now discuss several special cases of linear canonical transforms that correspond to specific forms of the matrix $\mathbf{M}$. The last of these special cases will be the fractional Fourier transform which corresponds to the case where $\mathbf{M}$ is the rotation matrix.

The scaling operation takes $f(u)$ to $\sqrt{1/M}f(u/M)$. The inverse of a scaling operation with parameter $M > 0$ is a scaling operation with parameter $1/M$. The $\mathbf{M}$ matrix is of the form

$$\begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix} \tag{14.75}$$

and the Wigner distribution of the scaled function is $W_f(u/M, M\mu)$ (Figure 14.6b shows how the Wigner distribution is scaled for $M = 2$).

Let us now consider chirp multiplication which takes $f(u)$ to $e^{-i\pi q u^2} f(u)$. The inverse of this operation with parameter q has the same form but with parameter $-q$. Its $\mathbf{M}$ matrix is

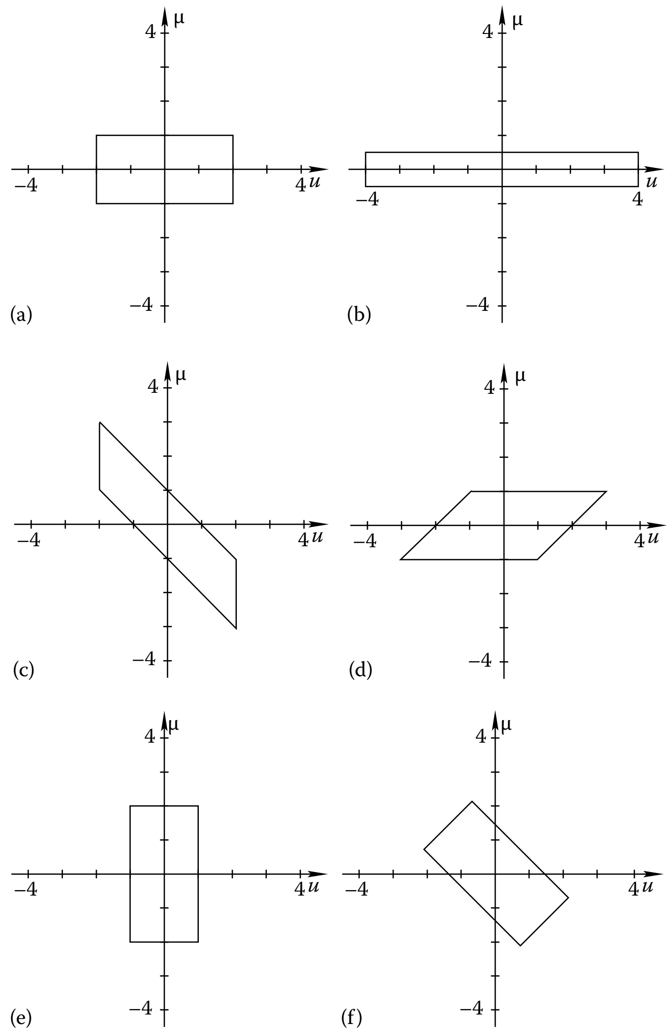

FIGURE 14.6 (a) Rectangular region in the time/space-frequency plane, in which most of the signal energy is assumed to be concentrated. Effect of (b) scaling with $M = 2$, (c) chirp multiplication with $q = 1$, (d) chirp convolution with $r = 1$, (e) Fourier transformation, (f) fractional Fourier transformation with $a = 0.5$. (From Ozaktas, H. M., Zalevsky, Z., and Kutay, M. A., *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. John Wiley & Sons, New York, 2001. With permission.)

$$\begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix} \tag{14.76}$$

and the Wigner distribution of the chirp multiplied function is $W_f(u, \mu + qu)$ (Figure 14.6c shows this vertical shearing for $q = 1$).

Now consider chirp convolution which takes $f(u)$ to $e^{-i\pi/4}\sqrt{1/r}\,\exp(i\pi u^2/r) * f(u)$. The inverse of this operation with parameter r has the same form but with parameter $-r$. Its $\mathbf{M}$ matrix is

$$\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \tag{14.77}$$

and the Wigner distribution of the chirp convolved function is $W_f(u - r\mu, \mu)$ (Figure 14.6d shows this horizontal shearing for $r = 1$).

The ordinary Fourier transform takes $f(u)$ to $\int_{-\infty}^{\infty} f(u')e^{-i2\pi uu'}\,du'$. However, the Fourier transform that is a special case of linear canonical transforms has a slightly modified definition, taking $f(u)$ to $e^{-i\pi/4}\int f(u')e^{-i2\pi uu'}\,du'$. The **M** matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{14.78}$$

and the Wigner distribution of the Fourier transformed function is $W_f(-\mu, u)$ (Figure 14.6e shows this $\pi/2$ rotation).

Finally, we turn our attention to the fractional Fourier transform, which takes $f(u)$ to $f_a(u)$ as defined in Equation 14.1. The inverse of the ath order FRT is the $-a$th order FRT. The **M** matrix is

$$\begin{bmatrix} \cos(a\pi/2) & \sin(a\pi/2) \\ -\sin(a\pi/2) & \cos(a\pi/2) \end{bmatrix} \tag{14.79}$$

and the Wigner distribution of the Fourier transformed function is

$$W_f[\cos(a\pi/2)\,u - \sin(a\pi/2)\,\mu,\ \sin(a\pi/2)\,u + \cos(a\pi/2)\,\mu]. \tag{14.80}$$

We have already encountered this expression before in Equation 14.25 (Figure 14.6f shows this rotation by angle $\alpha = a\pi/2$ when $a = 0.5$).

To summarize, we see that fractional Fourier transforms constitute a one-parameter subgroup of linear canonical transforms corresponding to the case where the **M** matrix is the rotation matrix, and the fractional order parameter corresponds to the angle of rotation. Fractional Fourier transformation corresponds to rotation of the Wigner distribution in the time/space–frequency plane (phase space). The ordinary Fourier transform is a special case of the fractional Fourier transform, which is in turn a special case of linear canonical transforms.

The matrix formalism not only allows one to easily determine the parameters of the concatenation (composition) of several LCTs, it also allows a given LCT to be decomposed into more elementary operations such as scaling, chirp multiplication and convolution, and the fractional Fourier transform. This is often useful for both analytical and numerical purposes. Of the many such possible decompositions here we list only a few (see page 104 of [129]):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & (A-1)/C \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}\begin{bmatrix} 1 & (D-1)/C \\ 0 & 1 \end{bmatrix} \tag{14.81}$$

$$= \begin{bmatrix} 1 & 0 \\ (D-1)/B & 1 \end{bmatrix}\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ (A-1)/B & 1 \end{bmatrix}. \tag{14.82}$$

Such decompositions usually show how an arbitrary LCT can be expressed in terms of its special cases. Specifically, the above two decompositions show how any unit-determinant matrix can be written as the product of lower and upper triangular matrices, which we have seen correspond to chirp multiplication and convolution operations.

Another important decomposition is the decomposition of an arbitrary LCT into a fractional Fourier transformation followed by scaling followed by chirp multiplication:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \tag{14.83}$$

where

$$\alpha = \operatorname{arccot}(A/B), \tag{14.84}$$

$$M = \operatorname{sgn}(A)\sqrt{A^2+B^2}, \tag{14.85}$$

$$q = \frac{A}{B(A^2+B^2)} - \frac{D}{B}, \tag{14.86}$$

where $\operatorname{sgn}(A)$ is the sign of A. The ranges of the square root and the arccotangent both lie in $(-\pi/2, \pi/2]$. Equation 14.83 can be interpreted geometrically as follows: any linear distortion in the time/space–frequency plane can be realized as a rotation followed by scaling followed by shearing. This decomposition is important because it forms the basis of a fast and accurate algorithm for digitally computing arbitrary linear canonical transforms [76,119]. These algorithms compute LCTs with a performance similar to that of the fast Fourier transform (FFT) algorithm in computing the Fourier transform, both in terms of speed and accuracy. Further discussion of decompositions of the type of Equation 14.83 may be found in [4]. Other works on the computation of LCTs include [64,65].

Many of the elementary and operational properties of LCTs are collected in Section 14.9, which can be recognized as generalizations of the corresponding properties of the fractional Fourier transform.

14.9 Basic and Operational Properties of Linear Canonical Transforms

Here we present a list of the more important basic and operational properties of the LCTs. Readers can easily verify that the operational properties reduce to the corresponding property for the fractional Fourier transform when **M** is the rotation matrix.

Linearity: Let $\mathcal{C}_{\mathbf{M}}$ denote the linear canonical transform operator with parameter matrix **M**. Then $\mathcal{C}_{\mathbf{M}}[\sum_k b_k f_k(u)] = \sum_k b_k[\mathcal{C}_{\mathbf{M}} f_k(u)]$.

Inverse: $(\mathcal{C}_{\mathbf{M}})^{-1} = \mathcal{C}_{\mathbf{M}^{-1}}$.

Unitarity: $(\mathcal{C}_{\mathbf{M}})^{-1} = (\mathcal{C}_{\mathbf{M}})^{\mathrm{H}} = \mathcal{C}_{\mathbf{M}^{-1}}$ where $()^{\mathrm{H}}$ denotes the conjugate transpose of the operator.

Associativity: $(\mathcal{C}_{\mathbf{M}_1}\mathcal{C}_{\mathbf{M}_2})\mathcal{C}_{\mathbf{M}_3} = \mathcal{C}_{\mathbf{M}_1}(\mathcal{C}_{\mathbf{M}_2}\mathcal{C}_{\mathbf{M}_3})$.

Eigenfunctions: Eigenfunctions of linear canonical transforms are discussed in [133].

Parseval: $\int f^*(u)g(u)du = \int f^*_{\mathbf{M}}(u)g_{\mathbf{M}}(u)du$. This property is equivalent to unitarity. Energy or norm conservation ($\text{En}[f] = \text{En}[f_{\mathbf{M}}]$ or $\|f\| = \|f_{\mathbf{M}}\|$) is a special case.

Time reversal: Let $\mathcal{P}$ denote the parity operator: $\mathcal{P}[f(u)] = f(-u)$, then

$$\mathcal{C}_{\mathbf{M}}\mathcal{P} = \mathcal{P}\mathcal{C}_{\mathbf{M}}, \tag{14.87}$$

$$\mathcal{C}_{\mathbf{M}}[f(-u)] = f_{\mathbf{M}}(-u). \tag{14.88}$$

Transform of a scaled function:

$$\mathcal{C}_{\mathbf{M}}[|K|^{-1}f(u/K)] = \mathcal{C}_{\mathbf{M}'}[f(u)] = f_{\mathbf{M}'}(u). \tag{14.89}$$

Here $\mathbf{M}'$ is the matrix that corresponds to the parameters $\alpha' = \alpha$, $\beta' = K\beta$, and $\gamma' = K^2\gamma$.

Transform of a shifted function:

$$\mathcal{C}_{\mathbf{M}}[f(u-u_0)] = \exp[i\pi(2uu_0C - u_0^2AC)]f_{\mathbf{M}}(u-Au_0). \tag{14.90}$$

Here u_0 is real.

Transform of a phase-shifted function:

$$\mathcal{C}_{\mathbf{M}}[\exp(i2\pi\mu_0 u)f(u)] = \exp[i\pi\mu_0 D(2u-\mu_0 B)]f_{\mathbf{M}}(u-B\mu_0). \tag{14.91}$$

Here μ_0 is real.

Transform of a coordinate multiplied function:

$$\mathcal{C}_{\mathbf{M}}[u^n f(u)] = [Du - B(i2\pi)^{-1}d/du]^n f_{\mathbf{M}}(u). \tag{14.92}$$

Here n is a positive integer.

Transform of the derivative of a function:

$$\mathcal{C}_{\mathbf{M}}[[(i2\pi)^{-1}d/du]^n f(u)] = [-Cu + A(i2\pi)^{-1}d/du]^n f_{\mathbf{M}}(u). \tag{14.93}$$

Here n is a positive integer.

A few additional properties are

$$\mathcal{C}_{\mathbf{M}}[f^*(u)] = f^*_{\mathbf{M}^{-1}}(u), \tag{14.94}$$

$$\mathcal{C}_{\mathbf{M}}[(f(u)+f(-u))/2] = (f_{\mathbf{M}}(u)+f_{\mathbf{M}}(-u))/2, \tag{14.95}$$

$$\mathcal{C}_{\mathbf{M}}[(f(u)-f(-u))/2] = (f_{\mathbf{M}}(u)-f_{\mathbf{M}}(-u))/2. \tag{14.96}$$

A function and its linear canonical transform satisfy an "uncertainty relation," stating that the product of the spread of the two functions, as measured by their standard deviations, cannot be less than $|B|/4\pi$ [129].

14.10 Filtering in Fractional Fourier Domains

Filtering, as conventionally understood, involves taking the Fourier transform of a signal, multiplying it with a Fourier-domain transfer function, and inverse transforming the result (Figure 14.7a). Here, we consider filtering in fractional Fourier domains, where we take the fractional Fourier transform, apply a filter function in the fractional Fourier domain, and inverse transform to the original domain (Figure 14.7b). Formally the filter output is written as

$$f_{\text{single}}(u) = [\mathcal{F}^{-a}\,\Lambda_h\,\mathcal{F}^a]f(u) = \mathcal{T}_{\text{single}}f(u), \tag{14.97}$$

where

$\mathcal{F}^a$ is the ath order fractional Fourier transform operator

Λ_h denotes the operator corresponding to multiplication by the filter function $h(u)$

$\mathcal{T}_{\text{single}}$ is the operator representing the overall filtering configuration

To understand the basic motivation for filtering in fractional Fourier domains, consider Figure 14.8, where the Wigner distributions of a desired signal and an undesired noise term are superimposed. We observe that the signal and noise overlap in both the 0th and 1st domains, but they do not overlap in the 0.5th domain (consider the projections onto the $u_0 = u$, $u_1 = \mu$, and $u_{0.5}$ axes). Although it is not possible to eliminate the noise in the time or frequency domains, we can eliminate it easily by using a simple amplitude mask in the 0.5th domain.

Fractional Fourier domain filtering can be applied to the problem of signal recovery or estimation from observations, where the signal to be recovered has been degraded by a known distortion or blur, and the observations are noisy. The problem is to reduce or eliminate these degradations and noise. The solution of such problems depends on the observation model and the prior knowledge available about the desired signal, degradation process, and noise. A commonly used observation model is

$$g(u) = \int h_{\text{d}}(u,u')f(u')\,du' + n(u), \tag{14.98}$$

where

$h_{\text{d}}(u,u')$ is the kernel of the linear system that distorts or blurs the desired signal $f(u)$

$n(u)$ is an additive noise term

The problem is to find an estimation operator represented by the kernel $h(u,u')$, such that the estimated signal

$$f_{\text{est}}(u) = \int h(u,u')g(u')\,du' \tag{14.99}$$

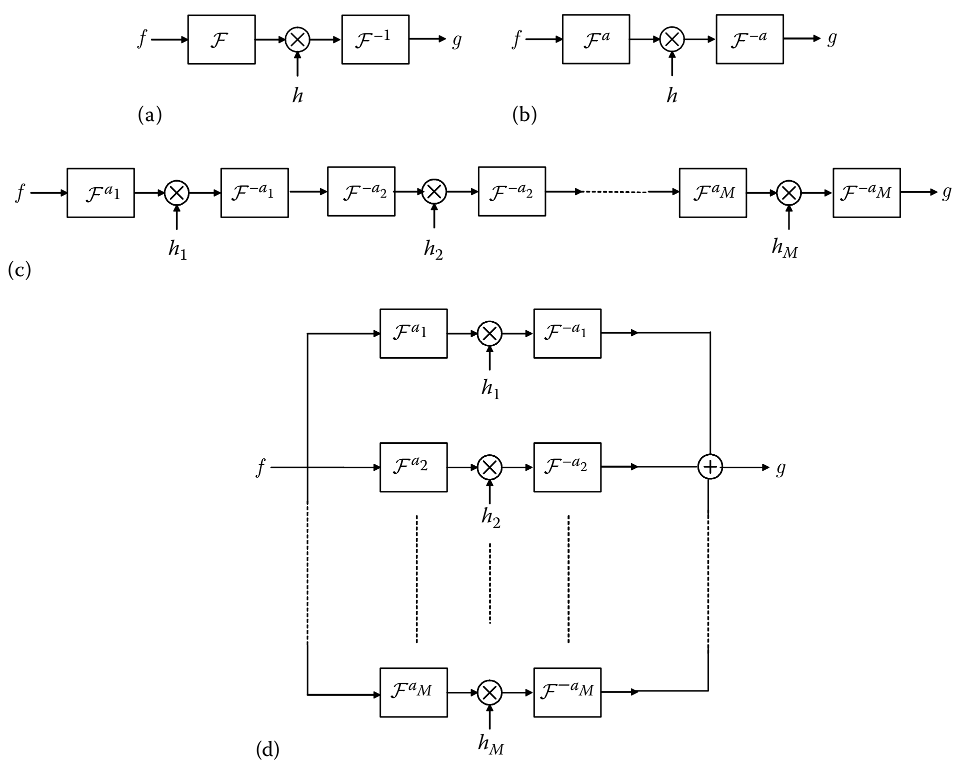

FIGURE 14.7 (a) Filtering in the frequency domain; (b) filtering in the *a*th order fractional Fourier domain; (c) multi-stage (series) filtering; (d) multi-channel (parallel) filtering.

optimizes some criteria. Despite its limitations, one of the most commonly used objectives is to minimize the mean square error σ^2_{err} defined as

$$\sigma^2_{\text{err}} = \left\langle \int |f_{\text{est}}(u) - f(u)|^2 \, du \right\rangle, \tag{14.100}$$

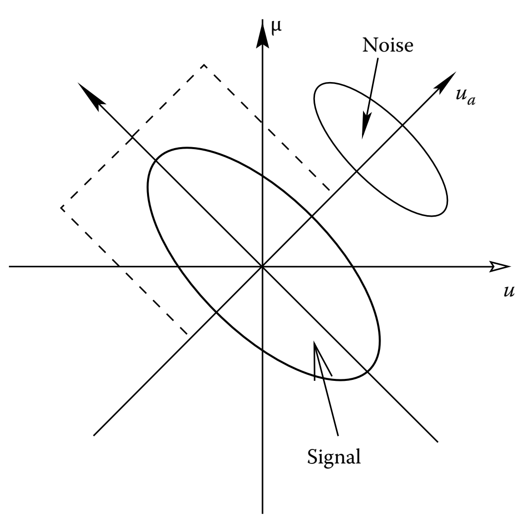

FIGURE 14.8 Filtering in a fractional Fourier domain as observed in the time- or space-frequency plane. $a = 0.5$ as drawn. (From Ozaktas, H. M., et al., *J Opt Soc Am A-Opt Image Sci Vis*, 11:547–559, 1994. With permission.)

where the angle brackets denote an ensemble average. The estimation or recovery operator minimizing σ^2_{err} is known as the optimal Wiener filter. The kernel $h(u, u')$ of this optimal filter satisfies the following relation [87]:

$$R_{fg}(u, u') = \int h(u, u'')R_{gg}(u'', u')\, du'' \quad \text{for all } u, u', \tag{14.101}$$

where

$R_{fg}(u, u')$ is the statistical cross-correlation of $f(u)$ and $g(u)$

$R_{gg}(u, u')$ is the statistical autocorrelation of $g(u)$

In the general case $h_{\text{d}}(u, u')$ represents a time varying system, and there is no fast algorithm for obtaining $f_{\text{est}}(u)$.

We can formulate the problem of obtaining an estimate $f_{\text{est}}(u) = f_{\text{single}}(u)$ of $f(u)$ by using the *a*th order fractional Fourier domain filtering configuration (Equation 14.97). As we will see in Section 14.13, the fractional Fourier transform can be efficiently computed with an $\sim N \log N$ algorithm similar to the fast Fourier transform algorithm used to compute the ordinary Fourier transform. Therefore, the fractional Fourier transform can be implemented nearly as efficiently as the ordinary Fourier transform, and the cost of fractional Fourier domain filtering is approximately the same as the cost of ordinary Fourier domain filtering. The optimal multiplicative filter function $h(u)$ for a given order a that minimizes the mean square error defined in Equation 14.100

for the filtering configuration represented by Equation 14.97 is given by [85]:

$$h(u_a) = \frac{\iint K_a(u_a, u) K_{-a}(u_a, u') R_{fg}(u, u')\, du'\, du}{\iint K_a(u_a, u) K_{-a}(u_a, u') R_{gg}(u, u')\, du'\, du}, \quad (14.102)$$

where the statistical cross-correlation and autocorrelation functions $R_{fg}(u, u')$ and $R_{gg}(u, u')$ can be obtained from the functions $R_{ff}(u, u')$ and $R_{nn}(u, u')$, which are assumed to be known. The corresponding mean square error can be calculated from Equation 14.100 for different values of a, and the value of a resulting in the smallest error can be determined.

Generalizations of the ath order fractional Fourier domain filtering configuration are the multistage (repeated or serial) and the multichannel (parallel) filtering configurations. These systems consist of M single-domain fractional Fourier filtering stages in series or in parallel (Figure 14.7). $M = 1$ corresponds to single-domain filtering in both cases. In the multistage system shown in Figure 14.7c, the input is first transformed into the a_1th domain where it is multiplied by a filter $h_1(u)$. The result is then transformed back into the original domain and the same process is repeated M times consecutively. This amounts to sequentially visiting the domains $a_1, a_2, a_3, \ldots$, and applying a filter in each. On the other hand, the multichannel system consists of M single-domain blocks in parallel (Figure 14.7d). For each channel k, the input is transformed to the a_kth domain, multiplied with a filter $h_k(u)$, and then transformed back. If these configurations are used to obtain an estimate $f_{\text{ser}}(u)$ or $f_{\text{par}}(u)$ of $f(u)$ in terms of $g(u)$, we have

$$f_{\text{ser}}(u) = [\mathcal{F}^{-a_M} \Lambda_{h_M} \cdots \mathcal{F}^{a_2 - a_1} \Lambda_{h_1} \mathcal{F}^{a_1}]\, g(u) = \mathcal{T}_{\text{ser}}\, g(u), \quad (14.103)$$

$$f_{\text{par}}(u) = \left[\sum_{k=1}^{M} \mathcal{F}^{-a_k} \Lambda_{h_k} \mathcal{F}^{a_k} \right] g(u) = \mathcal{T}_{\text{par}}\, g(u), \quad (14.104)$$

where

- $\mathcal{F}^{a_k}$ represents the a_kth order fractional Fourier transform operator
- Λ_{h_k} denotes the operator corresponding to multiplication by the filter function $h_k(u)$
- $\mathcal{T}_{\text{ser}}$, $\mathcal{T}_{\text{par}}$ are the operators representing the overall filtering configurations

Both of these equations reduce to Equation 14.97 for $M = 1$.

Multistage and multichannel filtering systems as described above are a subclass of the class of general linear systems whose input–output relation is given in Equation 14.99. Such linear systems have in general N^2 degrees of freedom, where N is the time-bandwidth product of the signals. Obtaining the output from the input normally takes $\sim N^2$ time, unless the system kernel $h(u, u')$ has some special structure which can be exploited. Shift-invariant (time- or space-invariant) systems are also a subclass of general linear systems whose system kernels $h(u, u')$ can always be expressed in the form $h(u, u') = h(u - u')$. They are a restricted subclass with only N degrees of freedom, but can be implemented in $\sim N \log N$ time in the ordinary Fourier domain.

We may think of shift-invariant systems and general linear systems as representing two extremes in a cost–performance trade-off. Shift-invariant systems exhibit low cost and low performance, whereas general linear systems exhibit high cost and high performance. Sometimes use of shift-invariant systems may be inadequate, but at the same time use of general linear systems may be an overkill and prohibitively costly. Multistage and multichannel fractional Fourier domain filtering configurations interpolate between these two extremes, offering greater flexibility in trading off between cost and performance.

Both filtering configurations have at most $MN + M$ degrees of freedom. Their digital implementation will take $O(MN \log N)$ time since the fractional Fourier transform can be implemented in $\sim N \log N$ time. These configurations interpolate between general linear systems and shift-invariant systems both in terms of cost and flexibility. If we choose M to be small, cost and flexibility are both low; $M = 1$ corresponds to single-stage filtering. If we choose M to be larger, cost and flexibility are both higher; as M approaches N, the number of degrees of freedom approaches that of a general linear system.

Increasing M allows us to better approximate a given linear system. For a given value of M, we can approximate this system with a certain degree of accuracy (or error). For instance, a shift-invariant system can be realized with perfect accuracy with $M = 1$. In general, there will be a finite accuracy for each value of M. As M is increased, the accuracy will usually increase (but never decrease). In dealing with a specific application, we can seek the minimum value of M which results in the desired accuracy, or the highest accuracy that can be achieved for given M. Thus these systems give us considerable freedom in trading off efficiency and greater accuracy, enabling us to seek the best performance for a given cost, or the least cost for a given performance. In a given application, this flexibility may allow us to realize a system which is acceptable in terms of both cost and performance.

The cost-accuracy trade-off is illustrated in Figure 14.9, where we have plotted both the cost and the error as functions of the number of filters M for a hypothetical application. The two plots show how the cost increases and the error decreases as we increase M. Eliminating M from these two graphs leads us to a graph of error versus cost.

The multistage and multichannel configurations may be further extended to *generalized filtering configurations* or *generalized filter circuits* where we combine the serial and parallel filtering configurations in an arbitrary manner (Figure 14.10).

Having discussed quite generally the subject of filtering in fractional Fourier domains, we now discuss the closely related concepts of fractional convolution and fractional multiplication [108,117]. The convolution of two signals h and f in the

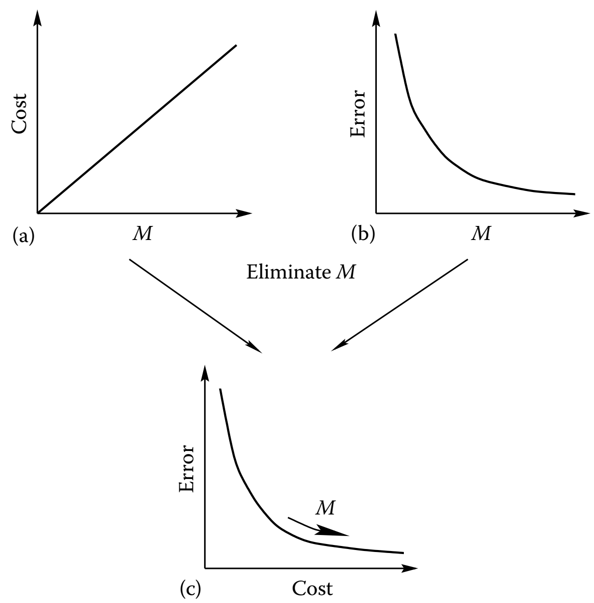

FIGURE 14.9 (a) Cost versus M, (b) error versus M, (c) error versus cost. (From Kutay, M. A., PhD thesis, Bilkent University, Ankara, 1999; Ozaktas, H. M., et al., *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. John Wiley & Sons, New York, 2001. With permission.)

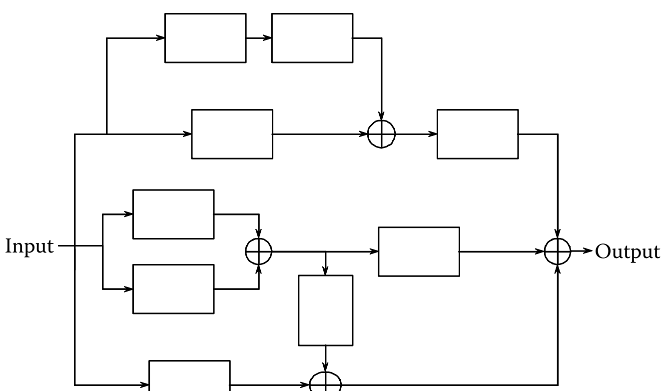

FIGURE 14.10 Generalized filter circuits; each block is of the form $F^{-a_k}\Lambda_{h_k}F^{a_k}$

ath fractional Fourier domain is defined such that their ath order fractional Fourier domain representations $h_a(u_a)$ and $f_a(u_a)$ are convolved to give the corresponding representation of some new signal g:

$$g_a(u_a) = h_a(u_a) * f_a(u_a), \tag{14.105}$$

where $*$ denotes ordinary convolution. Likewise, multiplication of two signals in the ath fractional Fourier domain is defined as

$$g_a(u_a) = h_a(u_a) f_a(u_a). \tag{14.106}$$

Of course, convolution (or multiplication) in the $a = 0$th domain is ordinary convolution (or multiplication) and convolution (or multiplication) in the $a = 1$st domain is ordinary multiplication (or convolution). More generally, convolution (or multiplication) in the ath domain is multiplication (or convolution) in the $(a \pm 1)$th domain (which is orthogonal to the ath domain), and convolution (or multiplication) in the ath domain is again convolution (or multiplication) in the $(a \pm 2)$th domain (the sign-flipped version of the ath domain). Convolution or multiplication in an arbitrary ath domain is an operation "interpolating" between the ordinary convolution and multiplication operations [129]. In light of these definitions, filtering in the ath fractional Fourier domain corresponds to the multiplication of two signals in the ath fractional Fourier domain or equivalently the convolution of two signals in the $a \pm 1$th fractional Fourier domain.

14.11 Fractional Fourier Domain Decompositions

The fractional Fourier domain decomposition (FFDD) [86] is closely related to multichannel filtering and is analogous to the singular-value decomposition in linear algebra [68,154].

The SVD of an arbitrary $N_{\text{out}} \times N_{\text{in}}$ complex matrix $\mathbf{H}$ is

$$\mathbf{H}_{N_{\text{out}} \times N_{\text{in}}} = \mathbf{U}_{N_{\text{out}} \times N_{\text{out}}} \, \Sigma_{N_{\text{out}} \times N_{\text{in}}} \, \mathbf{V}^{\text{H}}_{N_{\text{in}} \times N_{\text{in}}}, \tag{14.107}$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices whose columns are the eigenvectors of $\mathbf{H}\mathbf{H}^{\text{H}}$ and $\mathbf{H}^{\text{H}}\mathbf{H}$, respectively. The superscript H denotes Hermitian transpose. Σ is a diagonal matrix whose elements λ_k (the singular values) are the nonnegative square roots of the eigenvalues of $\mathbf{H}\mathbf{H}^{\text{H}}$ and $\mathbf{H}^{\text{H}}\mathbf{H}$. The number of strictly positive singular values is equal to the rank R of $\mathbf{H}$. The SVD can also be written in the form of an outer product (or spectral) expansion

$$\mathbf{H} = \sum_{k=1}^{R} \lambda_k \mathbf{u}_k \mathbf{v}_k^{\text{H}}, \tag{14.108}$$

where $\mathbf{u}_k$ and $\mathbf{v}_k$ are the columns of $\mathbf{U}$ and $\mathbf{V}$. It is common to assume that the λ_k are ordered in decreasing value.

Let $\mathbf{F}_N^a$ denotes the N-point ath order discrete fractional Fourier transform matrix. The discrete fractional Fourier transform will be defined in Section 14.12. For the purpose of this section, it will suffice to think of this transform in analogy with the ordinary discrete Fourier transform. The discrete Fourier transform of a discrete signal represented by a vector of length N can be obtained by multiplying the vector by the N-point discrete Fourier transform matrix $\mathbf{F}_N$. Likewise, the ath order discrete fractional Fourier transform of a vector is obtained by multiplying it by $\mathbf{F}_N^a$. The discrete transforms can be used to approximately compute the continuous transforms.

The columns of the inverse discrete fractional Fourier transform matrix $\mathbf{F}_N^{-a}$ constitute an orthonormal basis for the ath domain, just as the columns of the identity matrix constitute a basis for the time domain and the columns of the ordinary inverse DFT matrix constitute a basis for the frequency domain. Now, let **H** be a complex $N_{\text{out}} \times N_{\text{in}}$ matrix and $\{a_1, a_2, \ldots, a_N\}$ a set of $N = \max(N_{\text{out}}, N_{\text{in}})$ distinct real numbers such that $-1 < a_1 < a_2 < \cdots < a_N \leq 1$. For instance, a_ks may be chosen uniformly spaced in this interval. We define the FFDD of **H** as [86]

$$\mathbf{H}_{N_{\text{out}} \times N_{\text{in}}} = \sum_{k=1}^{N} \mathbf{F}_{N_{\text{out}}}^{-a_k} \, (\Lambda_{\mathbf{h}_k})_{N_{\text{out}} \times N_{\text{in}}} \, \left(\mathbf{F}_{N_{\text{in}}}^{-a_k}\right)^{\text{H}}, \qquad (14.109)$$

where the $\Lambda_{\mathbf{h}_k}$ are $N_{\text{out}} \times N_{\text{in}}$ diagonal matrices with $N' = \min(N_{\text{out}}, N_{\text{in}})$ complex elements. Starting from the upper left corner, the lth diagonal element of $\Lambda_{\mathbf{h}_k}$ is denoted as h_{kl}, $l = 1, 2, \ldots, N'$ (the lth element of the column vector $\mathbf{h}_k$). When **H** is Hermitian (skew Hermitian), $\mathbf{h}_k$ is real (imaginary). We also recall that $\left(\mathbf{F}_{N_{\text{in}}}^{-a_k}\right)^{\text{H}} = \mathbf{F}_{N_{\text{in}}}^{a_k}$. The FFDD always exists and is unique [129].

If we compare one term in the summation on the right-hand side of Equation 14.109 with the right-hand side of Equation 14.107, we see that they are similar in that they both consist of three terms of corresponding dimensionality, the first and third being unitary matrices and the second being a diagonal matrix. Whereas the columns of **U** and **V** constitute orthonormal bases specific to **H**, the columns of $\mathbf{F}_{N_{\text{out}}}^{-a_k}$ and $\mathbf{F}_{N_{\text{in}}}^{-a_k}$ constitute orthonormal bases for the a_kth fractional Fourier domain. Customization of FFDD is achieved through the coefficients h_{kl} and/or perhaps also the orders a_k.

When **H** is a square matrix of dimension N, the FFDD becomes

$$\mathbf{H} = \sum_{k=1}^{N} \mathbf{F}^{-a_k} \, \Lambda_{\mathbf{h}_k} (\mathbf{F}^{-a_k})^{\text{H}}, \qquad (14.110)$$

where all matrices are of dimension N. The continuous counterpart of the FFDD is similar to this equation, with the summation being replaced by an integral over a [167].

Equation 14.109 represents a decomposition of a matrix **H** into N terms. Each term corresponds to filtering in the a_kth fractional Fourier domain (see Equation 14.97). All terms taken together, the FFDD can be interpreted as the decomposition of a matrix into fractional Fourier domain filters of different orders. An arbitrary matrix **H** will in general not correspond to multiplicative filtering in the time or frequency domain or in any other single fractional Fourier domain. However, **H** can always be expressed as a combination of filtering operations in different fractional domains.

> *A sufficient number of different-ordered fractional Fourier domain filtering operations "span" the space of all linear operations.*

The fundamental importance of the FFDD is that it shows how an arbitrary linear system can be decomposed into this complete set of domains in the time-frequency plane.

Truncating some of the singular values in SVD of **H** has many applications [68,154]. Similary we can eliminate domains for which the coefficients $h_{k1}, h_{k2}, \ldots, h_{kN'}$ are small. This procedure, which we refer to as *pruning* the FFDD, is the counterpart of truncating the SVD. An alternative to this procedure will be referred to as *sparsening*, in which one simply employs a more coarsely spaced set of domains. In any event, the resulting smaller number of domains will be denoted by $M < N$. The upper limit of the summation in equation 109 is replaced by M and the equality is replaced by approximate equality. The equation $\underline{\mathbf{H}} = \tilde{\mathbf{P}}\underline{\mathbf{h}}$ is likewise replaced by $\underline{\mathbf{H}} \approx \tilde{\mathbf{P}}\underline{\mathbf{h}}$. If we solve this in the least-squares sense, minimizing $\|\underline{\mathbf{H}} - \tilde{\mathbf{P}}\underline{\mathbf{h}}\|$, we can find the filter coefficients resulting in the best *M-domain approximation* to **H**. (This procedure amounts to projecting **H** onto the subspace spanned by the MN' basis matrices, which now do not span the whole space.) The correspondence between the pruned FFDD and multichannel filtering configurations is evident; it is possible to interpret multichannel filtering configurations as pruned FFDDs. These concepts have found application in to image compression [166].

14.12 Discrete Fractional Fourier Transforms

Ideally, a discrete version of a transform should exhibit a high level of analogy with its continuous counterpart. This analogy should include basic structural similarity and analogy of operational properties. Furthermore, it is desirable for the discrete transform to usefully approximate the samples of the continuous transform, so that it can provide a basis for digital computation of the continuous transform. The following can be posed as a minimal set of properties that we would like to see in a definition of the discrete fractional Fourier transform (DFRT):

1. Unitarity
2. Index additivity
3. Reduction to the ordinary discrete Fourier transform (DFT) when $a = 1$
4. Approximation of the samples of the continuous FRT

Several definitions of the DFRT have been proposed in the literature. Some of these correspond to totally distinct continuous transforms. For example, one proposal was based on the power series expansion of the DFT matrix and employed the Cayley–Hamilton theorem [147]. If we let $\mathbf{F}^a$ be the $N \times N$ matrix representing the discrete fractional Fourier transform, this definition can be stated as follows:

$$\mathbf{F}^a = \sum_{n=0}^{3} \exp\left(j\frac{3}{4}\pi(n-a)\right) \frac{\sin \pi(n-a)}{4 \sin \frac{1}{4}\pi(n-a)} \, \mathbf{F}^n, \qquad (14.111)$$

where $\mathbf{F}^n$ is the nth (integer) power of the DFT matrix. This definition satisfies all the desired properties listed above, except the critical fourth one: it can not be used to approximate the samples of the continuous fractional Fourier transform which is the subject of this chapter. Rather, it corresponds to the continuous fractional Fourier transform based on principal powers of the eigenvalues discussed in Equation 14.9.

In the rest of this section, we focus on discrete fractional Fourier transforms that correspond to the continuous FRT defined in this chapter. The main task is to first find an eigenvector set of the DFT matrix which can serve as discrete versions of the Hermite–Gaussian functions. Such Hermite–Gaussian vectors have been defined in [26] based on [136]. It can be shown that [26] as $h \to 0$ the difference equation

$$\frac{f(u+h) - 2f(u) + f(u-h)}{h^2} + \frac{2(\cos(2\pi hu) - 1)}{h^2} f(u) = \lambda f(u) \tag{14.112}$$

approximates the Hermite–Gaussian generating differential equation

$$\frac{d^2 f(t)}{dt^2} - 4\pi t^2 f(t) = \lambda f(t). \tag{14.113}$$

When $h = \frac{1}{\sqrt{N}}$ the difference equation (Equation 14.112) has periodic coefficients. Therefore the solutions of the difference equation is also periodic and can be written as the eigenvectors of the following matrix, denoted by **S**:

$$\mathbf{S} = \begin{bmatrix} 2 & 1 & 0 & \dots & 0 & 1 \\ 1 & 2\cos(2\pi/N) & 1 & \dots & 0 & 0 \\ 0 & 1 & 2\cos(2\pi 2/N) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 1 & 2\cos(2\pi(N-1)/N) \end{bmatrix}. \tag{14.114}$$

In other words, the difference equation can be written as $\mathbf{Sf} = \lambda\mathbf{f}$. It can also been shown that **S** commutes with DFT matrix. Since two commuting matrices share a common eigenvector set [154], the eigenvectors of **S** are also eigenvectors of the DFT matrix. Thus the eigenvectors of **S** constitute an orthogonal eigenvector set of the DFT matrix which are analogous to and which approximate the Hermite–Gaussian functions. Further details such as the distinctness of the eigenvectors and enumeration of the eigenvectors with respect to the continuous Hermite–Gaussian functions are discussed in [26].

Having obtained an appropriate set of eigenvectors, the discrete fractional Fourier transform matrix can now be defined as follows:

$$\mathbf{F}^a = \begin{cases} \sum_{k=0,\, k\neq N-1}^{N} \mathbf{u_k} e^{-j\frac{\pi}{2}ka} \mathbf{u_k}^{\mathrm{T}}, & \text{when } N \text{ even} \\ \sum_{k=0,\, k\neq N}^{N} \mathbf{u_k} e^{-j\frac{\pi}{2}ka} \mathbf{u_k}^{\mathrm{T}}, & \text{when } N \text{ odd} \end{cases} \tag{14.115}$$

where $\mathbf{u_k}$ corresponds to the eigenvector of the **S** matrix with k zero-crossings [26]. The necessity of separately writing the summation in equation 14.115 for even and odd dimensions N is a consequence of the eigenvalue multiplicity of the ordinary DFT matrix [26]. This definition of the fractional DFT satisfies all four of the desirable properties we had set out at the beginning. A complementary perspective to this line of development may be found in [13].

A MATLAB® routine "dFRT" for the calculation of the discrete fractional Fourier transform matrix defined above is available [25]. The following steps show how to use the routine to compute and plot the samples of the ath order FRT of a continuous function $f(u)$:

1. $h = 1/\sqrt{N}$; tsamples = $(-N/2 * h)$:h: $(N/2 - 1) * h$;
2. f0 = f(tsamples);
3. f0shifted = fftshift(f0);
4. Fa = dFRT(N,a,order); {order can be any number in [2,$N-1$]}
5. fashifted = Fa*f0shifted;
6. fa = fftshift(fashifted);
7. plot(tsamples,fa);

The "fftshift" operations are needed since the DFRT matrix follows the well-known circular indexing rule of the DFT matrix. Normally the approximation "order" is set to 2; higher values correspond to higher-order approximations to the continuous transform than have been discussed here. The approximation order should not be confused with the fractional Fourier transform order a. Figure 14.11 compares the $N = 64$ samples calculated with this routine with the continuous fractional Fourier transform of the example function $f(u) = \sin(2\pi u)\mathrm{rect}(u)$. This function can be interpreted as the windowed version of a single period of the sine waveform between -0.5 and 0.5. As can be seen, the discrete transform fairly closely approximates the continuous one.

A number of other definitions of the discrete FRT which are still compatible with the continuous FRT discussed in this chapter have been proposed. In [138], the authors start with vectors formed by sampling the continuous Hermite–Gaussian functions. These are neither orthogonal nor eigenvectors of the DFT matrix. The authors orthogonalize these through a Gram–Schmidt process involving the **S** matrix. These orthogonal vectors are then used to define the fractional DFRT. We find this method less desirable in that it is based on a numerical rather than an analytical approach. The approach of [101] is similar, but here the eigenvectors of the DFT matrix are constructed by sampling periodically replicated versions of the Hermite–Gaussian functions (which are not orthogonal either).

In [12] another finite dimensional approximation to the Fourier transform similar to the DFT is proposed. This transform has strong connections with the Fourier transform within a group theoretical framework [165]. Furthermore, analytical expressions for the transform can be written in terms of the so-called Kravchuk polynomials, which are known to approximate the Hermite polynomials. A major disadvantage of this approach

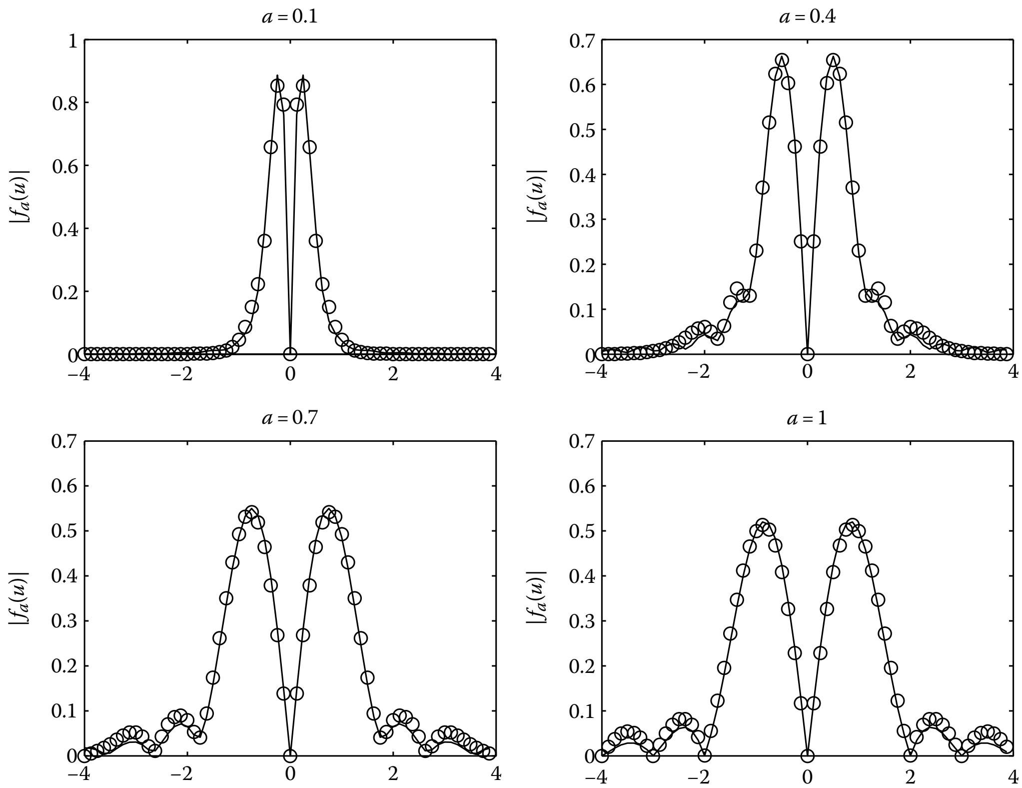

FIGURE 14.11 Approximation of the continuous fractional Fourier transform of $f(u) = sin(2\pi u)rect(u)$ with the discrete fractional Fourier transform.

is that the discrete FRT thus defined does not reduce to the ordinary DFT when $a = 1$.

In [26,61,135,148] yet other definitions are proposed based on the commuting matrices approach already discussed in relation to the **S** matrix. These matrices can be interpreted as higher-order approximation matrices which can be used to obtain increasingly accurate approximations to the continuous transform. A comparison of such matrices is given in [27].

14.13 Digital Computation of the Fractional Fourier Transform

The FRT of a continuous function whose time- or space-bandwidth product is N can be computed in the order of $N \log N$ time [115], similar to the ordinary Fourier transform. Therefore, if in some application any improvements can be obtained by using the FRT instead of the ordinary Fourier transform, these improvements come at no additional cost.

The following formula allows one to compute the samples of the fractional Fourier transform $f_a(u)$ of a function $f(u)$, in terms of the samples of $f(u)$, in $\sim N \log N$ time where $N = \Delta u^2$, under the assumption that Wigner distribution of $f(u)$ is approximately confined to a circle of diameter Δu:

$$f_a\left(\frac{k}{2\Delta u}\right) = \frac{A_\alpha}{2\Delta u} e^{i\pi(\cot\alpha - \csc\alpha)(k/2\Delta u)^2} \times \sum_{l=-N}^{N-1} e^{i\pi\csc\alpha((k-l)/2\Delta u)^2} e^{i\pi(\cot\alpha-\csc\alpha)(l/2\Delta u)^2} f\left(\frac{l}{2\Delta u}\right). \tag{14.116}$$

The summation is recognizable as a convolution, which can be computed in $\sim N \log N$ time by using the fast Fourier transform (FFT). The result is then obtained by a final chirp multiplication. The overall procedure takes $\sim N \log N$ time. A MATLAB code based on this formula may be found in [79]. A broader discussion of computational issues may be found in [115].

Note that this method is distinct from that discussed in Section 12. There, the discrete fractional Fourier transform was defined. The samples of the fractional Fourier transform of a function are then found by multiplying the discrete fractional Fourier transform matrix with the sample vector of the function to be transformed. Since a method for calculating this matrix product in $\sim N \log N$ time is presently not available, the operation will take $\sim N^2$ time. The approach in this section does not involve a definition of the discrete fractional Fourier transform, and can be viewed as a method to numerically compute the fractional Fourier transform integral.

14.14 Applications

The purpose of this section is to highlight some of the applications of the fractional Fourier transform which have received greater interest so far. The reader may consult [23,122,123,129] for further references.

The fractional Fourier transform is of potential usefulness in every area in which the ordinary Fourier transform is used. The typical pattern of discovery of a new application is to concentrate on an application where the ordinary Fourier transform is used and ask if any improvement or generalization might be possible by using the fractional Fourier transform instead. The additional order parameter often allows better performance or greater generality because it provides an additional degree of freedom over which to optimize.

Typically, improvements are observed or are greater when dealing with time/space-variant signals or systems. Furthermore, very large degrees of improvement often becomes possible when signals of a chirped nature or with nearly linearly increasing frequencies are in question, since chirp signals are the basis functions associated with the fractional Fourier transform (just as harmonic functions are the basis functions associated with the ordinary Fourier transform). Fractional Fourier transforms are also of special use when dealing with integral transforms whose kernels are of quadratic-exponential type, the diffraction integral being the most common example.

14.14.1 Applications in Signal and Image Processing

The FRT has found widespread application in signal and image processing, some of which are reviewed here (also see [157]).

One of the most striking applications is that of filtering in fractional Fourier domains, whose foundations have been discussed in Section 14.10 [117]. In traditional filtering, one takes the Fourier transform of a signal, multiplies it with a Fourier-domain transfer function, and inverse transforms the result. Here, one takes the fractional Fourier transform, applies a filter function in the fractional Fourier domain, and inverse transforms to the original domain. It has been shown that considerable improvement in performance is possible by exploiting the additional degree of freedom coming from the order parameter a. This improvement comes at no additional cost since computing the fractional Fourier transform is not more expensive than computing the ordinary Fourier transform (Section 14.13). The concept has been generalized to multistage and multichannel filtering systems which employ several fractional Fourier domain filters of different orders [81,82]. These schemes provide flexible and cost-efficient means of designing time/space-variant filtering systems to meet desired objectives. Fractional Fourier domain filtering has been useful in optical signal separation [43] and signal and image recovery and restoration in the presence of time/space-varying distortions such as space-varying blurs and nonstationary noise, with application to compensation of nonconstant velocity camera motion and atmospheric turbulence [48,50,83,85].

The FRT has also found many applications in pattern recognition and detection. Correlation is the underlying operation in matched filtering, which is used to detect signals. Fractional correlation has been defined in a number of different ways [71,92,103,180]. It has been shown how to control the degree of shift-invariance by adjusting the order a, which in turn allows one to design systems which detect objects within a certain region but reject them otherwise [92]. Joint-transform correlation is a well-known optical correlation technique, whose fractional version has received considerable attention [78,90]. The FRT has been studied as a preprocessing unit for neural network object recognition [14]. Some other applications in the pattern recognition area are face recognition [72] and realization and improvement of navigational tasks [149].

The windowed fractional Fourier transform has been studied in [29,46,104]. The possibility of changing the fractional order as the window is moved and/or choosing different orders in the two dimensions makes this a very flexible tool suited for various pattern recognition tasks, such as fingerprint recognition [171] or detection of targets in specific locations [59]. A review of applications of the FRT to pattern recognition as of 1998 is presented in [105].

The FRT has found a number of applications in radar signal processing. In [2], detection of linear frequency modulated signals is studied. In [69], radar return transients are analyzed in fractional domains. In [35,155], detection of moving targets for airborne radar systems is studied. In [11,10], synthetic aperture radar image reconstruction algorithms have been developed using the fractional Fourier transform.

The transform has found application to interpolation [53,150] and superresolution of multidimensional signals [32,62,151], phase retrieval from two or more intensity measurements [6,7,41,54,55], system and transform synthesis [50], processing of chirplets [22], signal and image compression [117,162,166], watermarking [40,112], speech processing [176], acoustic signal processing [60,173], ultrasound imaging [17], and antenna beamforming [168]. A large number of publications discuss the application of the FRT to encryption; for instance, see [33,63,66,111,161].

14.14.2 Applications in Communications

The FRT has found applications in spread spectrum communications systems [1], multicarrier communications systems [97], in the processing of time-varying channels [110], and beamforming for next generation wireless communication systems [75].

The concept of multiplexing in fractional Fourier domains, which generalizes time-domain and frequency-domain multiplexing, has been proposed in [117].

14.14.3 Applications in Optics and Wave Propagation

The fractional Fourier transform has received a great deal of interest in the area of optics and especially optical signal processing (also known as Fourier optics or information optics) [5,18,91,118,127,129,139,159]. Optical signal processing is an analog signal processing method which relies on the representation of signals by light fields and their manipulation with optical elements such as lenses, prisms, transparencies, holograms, and so forth. Its key component is the optical Fourier transformer which can be realized using one or two lenses separated by certain distances from the input and output planes. It has been shown that the fractional Fourier transform can be optically implemented with equal ease as the ordinary Fourier transform [88,124,127,146], allowing a generalization of conventional approaches and results to their more flexible or general fractional counterparts.

The fractional Fourier transform has also been shown to be intimately related to wave and beam propagation and diffraction. The process of diffraction of light in free space (or any other disturbance satisfying a similar wave equation) has been shown to be nothing but a process of continual fractional Fourier transformation; the distribution of light becomes fractional Fourier transformed as it propagates, evolving through continuously increasing orders [118,126,127,139].

More generally, it is well known that a rather broad class of optical systems can be modeled as linear canonical transforms, which were discussed in Section 14.8 [15,129]. These include optical systems consisting of arbitrary concatenations of thin lenses and sections of free space, as well as sections of quadratic graded-index media. It has been shown that all such systems can be expressed in the form of a fractional Fourier transform operation followed by appropriate scaling and a residual chirp factor, which can be interpreted as a change in the radius of curvature of the output plane (Equation 14.83) [118,119]. Therefore, all such optical systems can be interpreted as fractional Fourier transformers [16,118,127], and the propagation of light through such systems can be viewed as a process of continual fractional Fourier transformation with the fractional transform order monotonically increasing as light propagates through the system. The case of free-space optical diffraction in the Fresnel approximation, discussed in the previous paragraph, is a special case of this more general result, and rests on expressing the Fresnel integral in terms of the FRT. Similar results hold for other wave and beam propagation modalities that satisfy a similar wave equation as the optical wave equation, or an equation similar to that of the quantum-mechanical harmonic oscillator, including electromagnetic and acoustic waves [47].

As noted above, the fractional Fourier transform plays a central role in the study of optical systems consisting of arbitrary sequences of lenses. Also of interest are systems in which thin optical filters (masks) are inserted at various points along the optical axis. Such systems can be modeled as multistage fractional Fourier domain filtering systems with multiplicative filters inserted between fractional Fourier transform stages, which were discussed in Section 14.10.

The fractional Fourier transform has also found application in the study of laser resonators and laser beams. The order of the fractional transform has been shown to be proportional to the Gouy phase shift accumulated during Gaussian beam propagation [49,126] and also to be related to laser resonator stability [126,140,179]. Other laser applications have also been reported [94].

The FRT has also found use in increasing the resolution of low-resolution wave fields [32], optical phase retrieval from two or more intensity measurements [41,54,55], coherent and partially coherent wave field reconstruction using phase-space tomography [99,144,145], optical beam characterization and shaping [3,38,44,172,177], synthesis of mutual intensity functions [52], and the study of partially coherent light [20,24,51,153,160,163].

It has found further use in quantum optics [170], studies of the human eye [141,142], lens design problems [42], diffractive optics [58,158], optical profilometry [181], speckle photography [131] and metrology [73], holographic interferometry [152], holographic data storage [70], digital holography [34,36,178], holographic three-dimensional television [113,114], temporal pulse processing [21,45,89], solitons [39], and fiber Bragg gratings [98].

14.14.4 Other Applications

The fractional Fourier transform has found several other applications not falling under the above categories. We discuss some of these here.

The FRT has been employed in quantum mechanics [56,57,93,96]. It has been shown that certain kinds of time-varying second-order differential equations (with nonconstant coefficients) can be solved by exploiting the additional degree of freedom associated with the fractional order parameter a [74,100,109]. Based on the relationship of the fractional Fourier transform to harmonic oscillation (Section 14.2), it may be expected to play an important role in the study of vibrating systems [84]. It has so far received only limited attention in the area of control theory and systems [28], but we believe it has considerable potential for use in this field. The FRT has been shown to be related to perspective projections [169]. The transform has been employed to realize free-space optical interconnection architectures [50].

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