

# Hessian Tensor and Standard Static Space-times

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ABSTRACT. In this brief survey, we will remark the interaction between the Hessian tensor on a semi-Riemannian manifold and some of the several questions in Lorentzian (and also in semi-Riemannian) geometry where this 2-covariant tensor is involved. In particular, we deal with the characterization of Killing vector fields and the study of a set of consequences of energy conditions in the framework of standard static space-times.

## 1. Introduction

The two central concepts in this note will be the Hessian tensor on a semi-Riemannian manifold and the warped product of semi-Riemannian manifolds (especially standard static space-times). There are many arguments in mathematical physics where these concepts interact.

We briefly recall some basic definitions. Let  $(F, g_F)$  be a semi-Riemannian manifold and  $\varphi \in C^\infty(F)$  be a smooth function on  $F$ . Then the *Hessian of  $\varphi$*  is the  $(0, 2)$ -tensor defined by

$$(1.1) \quad \mathbb{H}_F^\varphi(X, Y) = g_F(\nabla_X^F \text{grad}_F \varphi, Y) = \nabla^F \nabla^F \varphi(X, Y),$$

for any vector fields  $X, Y \in \mathfrak{X}(F)$  where  $\nabla^F$  is the Levi-Civita connection and  $\text{grad}_F$  is the  $g_F$ -gradient operator. The  $g_F$ -trace of  $\mathbb{H}_F^\varphi$  is the Laplace-Beltrami operator,  $\Delta_F \varphi$ . Notice that  $\Delta_F$  is elliptic if  $(F, g_F)$  is Riemannian.

Let  $(B, g_B)$  and  $(F, g_F)$  be pseudo-Riemannian manifolds and also let  $b: B \rightarrow (0, \infty)$  be a smooth function. Then the (singly) *warped product*,  $B \times_b F$  is the product manifold  $B \times F$  furnished with the metric tensor  $g = g_B \oplus b^2 g_F$  defined by

$$(1.2) \quad g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F),$$

where  $\pi: B \times F \rightarrow B$  and  $\sigma: B \times F \rightarrow F$  are the usual projection maps and  $*$  denotes the pull-back operator on tensors. Here, the function  $b$  is called the warping function. Warped product manifolds were introduced in

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general relativity as a method to find general solutions to Einstein's field equations [**B**, **B-E-E**, **O'N**]. Two important examples include generalized Robertson-Walker space-times and standard static space-times (a generalization of the Einstein static universe). Precisely a *standard static space-time* is a Lorentzian warped product where the warping function is defined on a Riemannian manifold called the base and acting on the negative definite metric on an open interval of real numbers, called the fiber. More precisely, a standard static space-time, denoted by  $I_f \times F$ , is a Lorentzian warped product furnished with the metric  $g = -f^2 dt^2 \oplus g_F$ , where  $(F, g_F)$  is a Riemannian manifold,  $f: F \rightarrow (0, \infty)$  is smooth and  $I = (t_1, t_2)$  with  $-\infty \leq t_1 < t_2 \leq \infty$ . In [**O'N**], it was shown that any static space-time is locally isometric to a standard static space-time.

There are many subjects in semi-Riemannian geometry and physics where all these ingredients interact and play a central role. For instance in the study of concircular scalar fields [**Ob**, **Ta**]; in recent studies of Hessian manifolds [**Shi1**]; in several questions of curvature of warped products and the construction of Einstein manifolds [**B**, **D-U1**, **D-U2**, **D-U3**] and in the characterization of Killing vector fields on Robertson-Walker space-times [**San1**], among many others. We will concentrate our attention to the study of Killing vector fields and energy conditions on standard static space-times, where the ingredients mentioned above are involved (see [**D-U4**] and [**D-U5**]). The references mentioned above are mere indications in which the reader can find more specific references for each argument as well as links to alternative approaches and much more.

## 2. Preliminaries and Notation

Throughout the paper  $I$  will denote an open real interval  $I = (t_1, t_2)$ , where  $-\infty \leq t_1 < t_2 \leq \infty$ . Moreover,  $(F, g_F)$  will be a connected Riemannian manifold without boundary with  $\dim F = s$ . Finally, on an arbitrary differentiable manifold  $N$ ,  $C_{>0}^\infty(N)$  denotes the set of all strictly positive  $C^\infty$  functions defined on  $N$  and  $\mathfrak{X}(N)$  will denote the  $C^\infty(N)$ -module of smooth vector fields on  $N$ .

Suppose that  $U \in \mathfrak{X}(I)$  and  $V, W \in \mathfrak{X}(F)$ . If  $\text{Ric}$  and  $\text{Ric}_F$  denote the Ricci tensors of  $I_f \times F$  and  $(F, g_F)$ , respectively, then

$$(2.1) \quad \text{Ric}(U + V, U + W) = \text{Ric}_F(V, W) + f \Delta_F f \, dt^2(U, U) - \frac{1}{f} H_F^f(V, W).$$

If  $\tau$  and  $\tau_F$  denote the scalar curvatures of  $I_f \times F$  and  $F$ , respectively, then

$$(2.2) \quad \tau = \tau_F - 2 \frac{1}{f} \Delta_F f.$$

From now on, for any given  $f \in C_{>0}^\infty(F)$ ,  $Q_F^f$  will denote the 2 covariant tensor

$$(2.3) \quad Q_F^f := \Delta_F f \, g_F - H_F^f.$$

$\text{Ric}_F$  (respectively,  $Q_F^f$ ) denotes the quadratic form associated to  $\text{Ric}_F$  (respectively,  $Q_F^f$ ).

Notice that (2.1) and (2.3) imply that for any  $U \in \mathfrak{X}(I)$  and  $V, W \in \mathfrak{X}(F)$  is

$$(2.4) \quad \text{Ric}(U + V, U + W) = \text{Ric}_F(V, W) + \frac{1}{f}Q_F^f(V, W) - g(U + V, U + W)\frac{1}{f}\Delta_F f.$$

### 3. Killing Vector Fields

To begin with, we recall the concepts of Killing and conformal Killing vector fields on pseudo-Riemannian manifolds. Let  $(N, g_N)$  be a pseudo-Riemannian manifold and  $X \in \mathfrak{X}(N)$ . Then

- $X$  is said to be Killing if  $L_X g_N = 0$ ,
- $X$  is said to be conformal Killing if  $\exists \sigma \in C^\infty(N)$  such that  $L_X g_N = 2\sigma g_N$ ,

where  $L_X$  denotes the Lie derivative with respect to  $X$ . Moreover, for any  $Y$  and  $Z$  in  $\mathfrak{X}(N)$ , we have the following identity (see [O'N, p.250 and p.61])

$$(3.1) \quad L_X g_N(Y, Z) = g_N(\nabla_Y X, Z) + g_N(Y, \nabla_Z X).$$

Notice that any vector field on  $(I, g_I = \pm dt^2)$  is conformal Killing. Indeed, if  $X$  is a vector field on  $(I, g_I)$ , then  $X$  can be expressed as  $X = h\partial_t$  for some smooth function  $h \in C^\infty(I)$ .

For the rest of the paper, let  $M = I_f \times F$  be a standard static space-time with the metric  $g = f^2 g_I \oplus g_F$ , where  $g_I = -dt^2$ . Suppose that  $X, Y, Z \in \mathfrak{X}(I)$  and  $V, W, U \in \mathfrak{X}(F)$ , then (see [U])

$$(3.2) \quad L_{X+V} g(Y + W, Z + U) = f^2 L_X^I g_I(Y, Z) + 2fV(f)g_I(Y, Z) + L_V^F g_F(W, U).$$

Moreover, we also have

$$(3.3) \quad L_{h\partial_t} g_I(Y, Z) = Y(h)g_I(Z, \partial_t) + Z(h)g_I(Y, \partial_t).$$

By combining (3.2) and (3.3), we can state the following result.

**THEOREM 3.1.** [D-U4] *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . Suppose that  $h \in C^\infty(I)$  and  $V \in \mathfrak{X}(F)$ . Then  $h\partial_t + V$  is a conformal Killing vector field on  $M$  with  $\sigma \in C^\infty(M)$  if and only if the following properties are satisfied:*

- (1)  $V$  is conformal Killing on  $F$  with associated  $\sigma \in C^\infty(F)$ ,
- (2)  $h$  is affine, i.e, there exist  $\mu, \nu \in \mathbb{R}$  such that  $h(t) = \mu t + \nu$  for any  $t \in I$ ,
- (3)  $V(f) = (\sigma - \mu)f$ .

Consequently,  $h\partial_t + V$  is a Killing vector field on  $M$  if and only if the following properties are satisfied:

- (1)  $V$  is Killing on  $F$ ,

- (2) there exist  $\mu, \nu \in \mathbb{R}$  such that  $h(t) = \mu t + \nu$  for any  $t \in I$ ,  
(3)  $V(f) = -\mu f$ .

In [D-U4], to provide a characterization of Killing vector fields on standard static space-times, we modify the procedure used in [San1] (see also [C-d]) to study the structure of Killing and conformal Killing vector fields on warped products. In [San1], the author obtains full characterizations of the Killing and conformal Killing vector fields on generalized Robertson-Walker space-times. Here, we will state some of the main results about the characterization of Killing vector fields obtained in [D-U4].

Let  $(F, g_F)$  be a Riemannian manifold of dimension  $s$  admitting at least one *nonzero* Killing vector field. Thus, there exists a basis  $\{K_{\bar{b}} \in \mathfrak{X}(F) \mid \bar{b} = 1, \dots, \bar{s}\}$  for the set of Killing vector fields on  $F$ . At this point, we would like to emphasize that the dimension of the set of conformal Killing vector fields on  $(I, -dt^2)$  is infinite, so that one cannot apply directly the procedure in [San1] before observing that the form of conformal Killing vector fields on  $(I, -dt^2)$  is trivial (i.e, any vector field on  $(I, -dt^2)$  is conformal). Adapting the Sánchez technique to  $M = I_f \times F$ , a vector field  $K \in \mathfrak{X}(M)$  is a Killing vector field if and only if  $K$  can be written in the form

$$(3.4) \quad K = \psi h \partial_t + \phi^{\bar{b}} K_{\bar{b}},$$

where  $h, \phi^{\bar{b}} \in C^\infty(I)$  for any  $\bar{b} \in \{1, \dots, \bar{m}\}$  and  $\psi \in C^\infty(F)$  satisfy

$$(3.5) \quad \begin{cases} h' \psi + \phi^{\bar{b}} K_{\bar{b}}(\ln f) & = 0 \\ d\phi^{\bar{b}} \otimes g_F(K_{\bar{b}}, \cdot) + g_I(h \partial_t, \cdot) \otimes f^2 d\psi & = 0. \end{cases}$$

Since  $d\phi^{\bar{b}} = (\phi^{\bar{b}})' dt$  with  $\phi^{\bar{b}} \in C^\infty(I)$  and  $g_I(h \partial_t, \cdot) = -h dt$ , (3.5) is equivalent to

$$(3.6) \quad \begin{cases} h' \psi + \phi^{\bar{b}} K_{\bar{b}}(\ln f) & = 0 \\ (\phi^{\bar{b}})' dt \otimes g_F(K_{\bar{b}}, \cdot) & = h dt \otimes f^2 d\psi. \end{cases}$$

The following notation will be useful. Let  $h$  be a continuous function defined on a real interval  $I$ . If there exists a point  $t_0 \in I$  such that  $h(t_0) \neq 0$ , then  $I_{t_0}$  denotes the connected component of  $\{t \in I : h(t) \neq 0\}$  such that  $t_0 \in I_{t_0}$ . By the method of separation of variables and a detailed analysis of system (3.6), one can state the following result.

**THEOREM 3.2.** [D-U4] *Let  $(F, g_F)$  be a Riemannian manifold,  $f \in C_{>0}^\infty(F)$  and  $\{K_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}}$  a basis of Killing vector fields on  $(F, g_F)$ . Let also  $I$  be an open interval of the form  $I = (t_1, t_2)$  in  $\mathbb{R}$ , where  $-\infty \leq t_1 < t_2 \leq \infty$ . Consider the standard static space-time  $I_f \times F$  with the metric  $g = -f^2 dt^2 \oplus g_F$ .*

*Then, any Killing vector field on  $I_f \times F$  admits the structure*

$$(3.7) \quad K = \psi h \partial_t + \phi^{\bar{b}} K_{\bar{b}}$$

*where  $h$  and  $\phi^{\bar{b}} \in C^\infty(I)$  for any  $\bar{b} \in \{1, \dots, \bar{m}\}$  and  $\psi \in C^\infty(F)$ .*

Furthermore, assume that  $K$  is a vector field on  $I_f \times F$  with the structure as in (3.7). Hence,

- (i) if  $h \equiv 0$ , then the vector field  $K = \phi^{\bar{b}} K_{\bar{b}}$  is Killing on  $I_f \times F$  if and only if the functions  $\phi^{\bar{b}}$  are constant and  $\phi^{\bar{b}} K_{\bar{b}}(\ln f) = 0$ .
- (ii) if  $h \equiv h_0 \neq 0$  is constant, then the vector field  $K = \psi h_0 \partial_t + \phi^{\bar{b}} K_{\bar{b}}$  is Killing on  $I_f \times F$  if and only if
 
$$(3.8) \quad \begin{cases} f^2 \text{grad}_F \psi \text{ is a Killing vector field on } (F, g_F) \text{ with} \\ \text{coefficients } \{\tau_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}} \text{ relative to the basis } \{K_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}}; \\ (f^2 \text{grad}_F \psi)(\ln f) = 0 \text{ (i.e., } \text{grad}_F \psi(f) = 0); \\ \forall \bar{b} : \phi^{\bar{b}}(t) = h_0 \tau^{\bar{b}} t + \omega^{\bar{b}} \text{ with } \omega^{\bar{b}} \in \mathbb{R} : \omega^{\bar{b}} K_{\bar{b}}(\ln f) = 0. \end{cases}$$

- (iii) if  $K$  is a Killing vector field on  $I_f \times F$  with the nonconstant function  $h$ , then the set of functions  $h$ ,  $\psi$  and  $\{\phi_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}}$  satisfy

$$(3.9) \quad \left\{ \begin{array}{l} (a) \left\{ \begin{array}{l} \psi \equiv 0; \\ \phi^{\bar{b}}(t) = \omega^{\bar{b}} \text{ on } I_{t_0} \text{ where } \omega^{\bar{b}} \in \mathbb{R} : \omega^{\bar{b}} K_{\bar{b}}(\ln f) = 0 \end{array} \right. \\ \text{or} \\ (b) \left\{ \begin{array}{l} f^2 \text{grad}_F \psi \text{ is a Killing vector field on } (F, g_F) \\ \text{with coefficients } \{\tau_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}} \text{ relative to the} \\ \text{basis } \{K_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}}; \\ (f^2 \text{grad}_F \psi)(\ln f) = \nu \psi \text{ where } \nu \text{ is constant;} \\ h(t) = \begin{cases} ae^{\sqrt{-\nu}t} + be^{-\sqrt{-\nu}t} & \text{if } \nu \neq 0 \\ at + b & \text{if } \nu = 0, \end{cases} \\ \text{with } a, b \in \mathbb{R}; \\ \forall \bar{b} : \phi^{\bar{b}}(t) = \tau^{\bar{b}} \int_{t_0}^t h(s) ds + \omega^{\bar{b}} \text{ with } \omega^{\bar{b}} \in \mathbb{R} : \\ h'(t_0)\psi + \omega^{\bar{b}} K_{\bar{b}}(\ln f) = 0 \text{ on } I_{t_0} \end{array} \right. \end{array} \right.$$

for any  $t_0 \in I$  with  $h(t_0) \neq 0$ .

Conversely, if a set of functions  $h$ ,  $\psi$  and  $\{\phi_{\bar{b}}\}_{1 \leq \bar{b} \leq \bar{m}}$ , satisfy (3.9) with an arbitrary  $t_0$  in  $I$  and the entire interval  $\bar{I}$  (instead of  $I_{t_0}$ ) and  $\psi \in C^\infty(F)$ , then the vector field  $\tilde{K}$  on the standard static space-time  $I_f \times F$  associated to the set of functions as in (3.7) is Killing on  $I_f \times F$ .

For clarity, we also state the following lemma which covers the case where the Riemannian manifold  $(F, g_F)$  admits no nonidentical zero Killing vector field.

**LEMMA 3.3.** *Let  $(F, g_F)$  be a Riemannian manifold of dimension  $s$  and  $f \in C_{>0}^\infty(F)$ . Let also  $I$  be an open interval of the form  $I = (t_1, t_2)$  in  $\mathbb{R}$ , where  $-\infty \leq t_1 < t_2 \leq \infty$ . Suppose that the only Killing vector field on  $(F, g_F)$  is the zero vector field. Then all the Killing vector fields on the standard static space-time  $I_f \times F$  are given by  $h_0 \partial_t$  where  $h_0$  is a constant.*

*Theorem 3.2* is relevant to the problem given by:

$$(3.10) \quad \begin{cases} f \in C_{>0}^\infty(F), \psi \in C^\infty(F); \\ f^2 \text{grad}_F \psi \text{ is a Killing vector field on } (F, g_F); \\ (f^2 \text{grad}_F \psi)(\ln f) = \nu \psi, \nu \in \mathbb{R}. \end{cases}$$

We are interested in the existence of nontrivial solutions for (3.10). To study this, for any  $Z \in \mathfrak{X}(F)$  and  $\varphi \in C^\infty(F)$  we define the (0,2)-tensor on  $(F, g_F)$  given by

$$(3.11) \quad B_Z^\varphi(\cdot, \cdot) := d\varphi(\cdot) \otimes g_F(Z, \cdot) + g_F(\cdot, Z) \otimes d\varphi(\cdot).$$

A central role in our study of (3.10) is played by the next proposition which also shows up the relevance of the Hessian tensor in all these questions.

**PROPOSITION 3.4. [D-U4]** *Let  $(F, g_F)$  be a Riemannian manifold,  $f \in C_{>0}^\infty(F)$  and  $\psi \in C^\infty(F)$ . Then the vector field  $f^2 \text{grad}_F \psi$  is Killing on  $(F, g_F)$  if and only if*

$$(3.12) \quad \mathbf{H}_F^\psi + \frac{1}{f} B_{\text{grad}_F \psi}^f = 0.$$

The latter proposition and the identity  $f g_F(\text{grad}_F \psi, \text{grad}_F f) = (f \text{grad}_F \psi)(f)$  allow to express (3.10) in the equivalent form

$$(3.13) \quad \begin{cases} f \in C_{>0}^\infty(F), \psi \in C^\infty(F); \\ \mathbf{H}_F^\psi + \frac{1}{f} B_{\text{grad}_F \psi}^f = 0; \\ f g_F(\text{grad}_F \psi, \text{grad}_F f) = \nu \psi, \nu \in \mathbb{R}. \end{cases}$$

By *Proposition 3.4*, if the dimension of the Lie algebra of Killing vector fields of  $(F, g_F)$  is zero, then the system (3.13) has only the trivial solution given by a constant  $\psi$  (this constant is not 0 only if  $\nu = 0$ ). This happens, for instance when  $(F, g_F)$  is a compact Riemannian manifold of negative-definite Ricci curvature without boundary, indeed it is sufficient to apply the vanishing theorem due to Bochner (see for instance [Bo], [B, Theorem 1.84]).

The next *Lemma 3.5* allows to prove that the system (3.13) is still *equivalent* to

$$(3.14) \quad \begin{cases} f \in C_{>0}^\infty(F), \psi \in C^\infty(F); \\ \mathbf{H}_F^\psi + \frac{1}{f} B_{\text{grad}_F \psi}^f = 0; \\ -\Delta_{g_F} \psi = \nu \frac{2}{f^2} \psi \text{ where } \nu \text{ is a constant.} \end{cases}$$

**LEMMA 3.5. [D-U4]** *Let  $(F, g_F)$  be a Riemannian manifold and  $f \in C_{>0}^\infty(F)$ . If  $(\nu, \psi)$  satisfies (3.13), then  $\nu$  is an eigenvalue and  $\psi$  is an associated  $\nu$ -eigenfunction of the elliptic problem:*

$$(3.15) \quad -\Delta_{g_F} \psi = \nu \frac{2}{f^2} \psi \text{ on } (F, g_F).$$

Thus, by arguments of critical points and maximum principle, we obtain the following characterization results.

**PROPOSITION 3.6.** *Let  $(F, g_F)$  be a compact Riemannian manifold and  $f \in C_{>0}^\infty(F)$ . Then  $(\nu, \psi)$  satisfies (3.13) if and only if  $\nu = 0$  and  $\psi$  is constant.*

**THEOREM 3.7.** *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . If  $(F, g_F)$  is compact then, the set of all Killing vector fields on the standard static space-time  $(M, g)$  is given by*

$$\{a\partial_t + \tilde{K} \mid a \in \mathbb{R}, \tilde{K} \text{ is a Killing vector field on } (F, g_F) \text{ and } \tilde{K}(f) = 0\}.$$

**EXAMPLE 3.8.** (*Killing vector fields in the Einstein static universe*) In [Sha], the author studied Killing vector fields of a closed homogeneous and isotropic universe (for related questions in quantum field theory and cosmology see [F, L-L]). Theorem 6.1 of [Sha] corresponds to *Theorem 3.7* for the spherical universe  $\mathbb{R} \times \mathbb{S}^3$  with the pseudo-metric  $-(R^2 dt^2 - R^2 h_0)$ , where the sphere  $\mathbb{S}^3$  endowed with the usual metric  $h_0$  induced by the canonical Euclidean metric of  $\mathbb{R}^4$  and  $R$  is a real constant (i.e., a stable universe).

As we have already mentioned, any Killing vector field of a compact Riemannian manifold of negative-definite Ricci tensor is equal to zero. Thus, one can easily state the following result.

**COROLLARY 3.9.** *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . Suppose that  $(F, g_F)$  is a compact Riemannian manifold of negative-definite Ricci tensor. Then, any Killing vector field on the standard static space-time  $(M, g)$  is given by  $a\partial_t$  where  $a \in \mathbb{R}$ .*

In [San-Sen, Theorem 5], it is shown that the decomposition of a space-time as a standard static one is essentially unique when the fiber  $F$  is compact. We observe that *Corollary 3.9* enables us to establish a stronger conclusion (i.e., nonexistence of a nontrivial strictly stationary<sup>1</sup> field) under a stronger assumption involving the definiteness of the Ricci tensor.

At this point, we would like to make some comments about the case where the Riemannian part of a standard static space-time is not compact. While the *Theorem 3.2* does not require the compactness of the Riemannian manifold  $(F, g_F)$ , this assumption is the central idea for a complete characterization similar to the one in *Theorem 3.7*. The key question in our approach is the full characterization of the solutions of (3.14) (or the equivalent problems (3.10) and (3.13)) which is reached if  $(F, g_F)$  is compact. In the noncompact case, the latter question is more difficult. It is possible to obtain partial nonexistence results for (3.14), but the global question is still open. However, there are particular situations, like Example 3.8, where the application of *Theorem 3.2* is sufficient for a complete classification.

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<sup>1</sup>Here, a stationary field means that it is Killing and time-like at the same time (see [San-Sen]).

Other relevant and related problem is the full classification of the conformal Killing vector fields of a standard static space-time. There are partial recent results in this direction (see for instance [A-C, Sha-Iq] and the references therein).

#### 4. Energy Conditions

Recall that a space-time is said to satisfy the *strong energy condition*, briefly SEC, if  $\text{Ric}(X, X) \geq 0$  for all causal tangent vectors  $X$  and the *time-like* (respectively, *null*) *convergence condition*, briefly TCC (respectively, NCC), if  $\text{Ric}(X, X) \geq 0$  for all time-like (respectively, null) tangent vectors  $X$ . Notice that the SEC implies the NCC. Furthermore the TCC is equivalent to the SEC, by continuity. The actual difference between TCC and SEC follows from the fact that while TCC is just a geometric condition imposed on the Ricci tensor, SEC is a condition on the stress-energy tensor. They can be considered equivalent due to the Einstein equation (see below (4.1)).

Moreover, a space-time is said to satisfy the *weak energy condition*, briefly WEC, if  $T(X, X) \geq 0$  for all time-like vectors, where  $T$  is the energy-momentum tensor, which is determined by physical considerations.

In this article, when we consider the energy-momentum tensor, we will assume that the Einstein equation holds (see [H-E, O'N]). More explicitly,

$$(4.1) \quad \text{Ric} - \frac{1}{2}\tau g = 8\pi T.$$

The WEC has many applications in general relativity theory such as nonexistence of closed time-like curve (see [C-P]) and the problem of causality violation ([O-S]). But its fundamental usage still lies in Penrose's singularity theorem (see [P]).

Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . By (2.4),  $\text{Ric}(\partial_t, \partial_t) = f \Delta_F f$ . So, since  $g(\partial_t, \partial_t) = -f^2 < 0$ , the warping function  $f$  is necessarily subharmonic, i.e.  $\Delta_F f \geq 0$ , if the standard static space-time satisfies the SEC (or equivalently, the TCC) [A1]. On the other hand, it is well known that there is no nonconstant subharmonic functions on compact Riemannian manifolds [Bo], and hence  $f$  is a positive constant if  $(F, g_F)$  is compact. Furthermore, applying a family of Liouville type results of Li, Schoen and Yau, in [D-U5] we give a set of sufficient conditions implying the warped function is a positive constant under the hypothesis that  $(F, g_F)$  is complete and noncompact.

Below, we state a set of necessary conditions for a standard static space-time to satisfy the NCC and other Ricci curvature conditions which are useful to study conformal hyperbolicity through the studies of Markowitz, more precisely Theorems 5.1 and 5.8 in [M1]. We also observe that there are more accurately analogous results for a generalized Robertson-Walker space-time given in [E-S, Proposition 4.2] (see also [D-U5]).

**THEOREM 4.1.** [D-U5, A1] *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ , where  $s = \dim F \geq 2$ .*



- (1) If  $\mathcal{R}ic_F$  and  $\mathcal{Q}_F^f$  are positive semi-definite, then  $M$  satisfies the TCC and the NCC.
- (2) If  $\mathcal{R}ic_F$  and  $\mathcal{Q}_F^f$  are negative semi-definite, then  $\mathcal{R}ic(w, w) \leq 0$  for any causal vector  $w \in \mathfrak{X}(M)$ .
- (3) If  $(F, g_F)$  is Ricci flat, then  $\mathcal{Q}_F^f$  is positive semi-definite if and only if  $M$  satisfies the NCC.

Now we state a small result about energy conditions in terms of the energy-momentum tensor  $T$ . It is easy to obtain from (4.1), (2.4) and (2.2) that for any  $U \in \mathfrak{X}(I)$  and  $V \in \mathfrak{X}(F)$  is

$$(4.2) \quad 8\pi T(U + V, U + V) = \mathcal{R}ic_F(V) + \frac{1}{f} \mathcal{Q}_F^f(V) - \frac{1}{2} \tau_F g(U + V, U + V).$$

So, as above, since  $g(\partial_t, \partial_t) = -f^2 < 0$  implies that if a standard static space-time satisfies the WEC, then  $\tau_F \geq 0$  and as consequence

**THEOREM 4.2. [D-U5]** *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ , where  $s = \dim F \geq 2$ .*

- (1) If  $\mathcal{R}ic_F$  and  $\mathcal{Q}_F^f$  are positive (respectively negative) semi-definite, then  $\mathsf{T}(w, w) \geq 0$  (respectively  $\leq 0$ ) for any causal vector  $w \in \mathfrak{X}(M)$ .
- (2) If  $(F, g_F)$  is Ricci flat, then for any  $u \in \mathfrak{X}(I)$  and  $v \in \mathfrak{X}(F)$  is  $8\pi T(u + v, u + v) = \mathcal{Q}_F^f(v)$ . Thus,  $\mathcal{Q}_F^f$  is positive semi-definite if and only if  $\mathsf{T}(w, w) \geq 0$  for any vector  $w \in \mathfrak{X}(M)$ .

In [M1] the intrinsic Lorentzian pseudo-distance  $d_M: M \times M \rightarrow [0, \infty)$  was defined by

$$(4.3) \quad d_M(p, q) = \inf_{\alpha} L(\alpha),$$

where the infimum is taken over all the chains of null geodesic segments joining  $p$  and  $q$  and  $L(\alpha)$  means the length of the chain  $\alpha$ . Such a chain  $\alpha$  is a sequence of points  $p = p_0, p_1, \dots, p_k = q$  in  $M$ , pairs of points  $(a_1, b_1), \dots, (a_k, b_k)$  in  $(-1, 1)$  and projective maps (i.e., a projective map is simply a null geodesic with the projective parameter as the natural parameter)  $f_1, \dots, f_k$  from  $(-1, 1)$  into  $M$  such that  $f_i(a_i) = p_{i-1}$  and  $f_i(b_i) = p_i$  for  $i = 1, \dots, k$ . Besides, the length of  $\alpha$  is

$$L(\alpha) = \sum_{i=1}^k \rho(a_i, b_i),$$

where  $\rho$  is the Poincaré distance in  $(-1, 1)$  (see [M1, M2] for details). Notice that  $d_M$  is really a pseudo-distance, i.e., it is non-negative, symmetric and satisfies the triangle inequality. A Lorentzian manifold  $(M, g)$  where  $d_M$  is a distance is called *conformally hyperbolic*.

In [M1, Theorem 5.1], it is proved that if  $(M, g)$  is a null geodesically complete Lorentz manifold satisfying the reverse NCC condition, i.e.  $\mathcal{R}ic(X, X) \leq 0$  for all null vectors  $X$ , then it has a trivial Lorentzian pseudo-distance, i.e.,

$d_M \equiv 0$ . Moreover, in [M1, Theorem 5.8], it is obtained that if  $(M, g)$  is an  $n(\geq 3)$ -dimensional Lorentzian manifold satisfying the NCC and the null generic condition, briefly NGC, (i.e.,  $\text{Ric}(\gamma', \gamma') \neq 0$ , for at least one point of each inextendible null geodesic  $\gamma$ ) then, it is conformally hyperbolic.

Under the light of these theorems, one can easily conclude that

- Complete Einstein space-times (in particular, Minkowski, de-Sitter and the anti-de Sitter space-times) have all trivial Lorentzian pseudo-distances because of Theorem 5.1 of [M1].
- The Einstein static universe has also trivial Lorentzian pseudo-distance since the space-times in the previous item can be conformally imbedded in the Einstein static universe.
- A Robertson-Walker space-time (i.e., an isotropic homogeneous space-time) is conformally hyperbolic due to Theorem 5.9 of [M1].
- The Einstein-de Sitter space  $M$  is conformally hyperbolic and (see Theorem 5 in [M2] for details and a precise formula for the Lorentzian pseudo-distance on this class of space-time).

Applying *Theorem 4.1*, (2.4) and the previous Markowitz results, we obtain the theorems that follow.

**THEOREM 4.3. [D-U5]** *Let  $M = \mathbb{R}_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . Suppose that  $\text{Ric}_F$  and  $\mathcal{Q}_F^f$  are negative semi-definite.*

- (1) *If  $(F, g_F)$  is compact, then the Lorentzian pseudo-distance  $d_M$  on the standard static space-time  $(M, g)$  is trivial, i.e.,  $d_M \equiv 0$ .*
- (2) *If  $(F, g_F)$  is complete and  $0 < \inf f$ , then the Lorentzian pseudo-distance  $d_M$  on the standard static space-time  $(M, g)$  is trivial, i.e.,  $d_M \equiv 0$ .*

In *Theorem 4.3*, the general hypothesis ensure the *reversed* NCC, in order to apply [M1, Theorem 5.1]. The additional hypothesis in item (2) of the same theorem implies the null geodesic completeness of  $M$  by [A2, Theorem 3.12]. We observe that there are more general hypotheses which imply null geodesic completeness, see for instance [RS, Th. 3.9(ii b)].

**THEOREM 4.4. [D-U5]** *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . Suppose that  $\text{Ric}_F$  is positive semi-definite and  $\mathcal{Q}_F^f$  is positive definite. Then the standard static space-time  $(M, g)$  is conformally hyperbolic.*

Now we state some results joining the conformal hyperbolicity and causal conjugate points of a standard static space-time by using [B, B-E1, B-E2, C-E] and also [B-E-E]. In [C-E, Theorem 2.3], it was shown that if the line integral of the Ricci tensor along a complete causal geodesic in a Lorentzian manifold is positive, then the complete causal geodesic contains a pair of conjugate points.

Assume that  $\gamma = (\alpha, \beta)$  is a complete causal geodesic in a standard static space-time of the form  $M = I_f \times F$  with the metric  $g = -f^2 dt^2 \oplus g_F$ . Then by using  $g(\gamma', \gamma') \leq 0$  and (2.4) we have

$$\text{Ric}(\gamma', \gamma') = \text{Ric}_F(\beta', \beta') + \frac{1}{f} Q_F^f(\beta', \beta') - \underbrace{g(\gamma', \gamma')}_{\leq 0} \frac{1}{f} \Delta_F f.$$

We can easily state the following existence result for conjugate points of complete causal geodesics in a conformally hyperbolic standard static space-time by *Theorem 4.4* and [C-E, Theorem 2.3].

**THEOREM 4.5.** *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . Suppose that  $\text{Ric}_F$  is positive semi-definite. If  $Q_F^f$  is positive definite, then  $(M, g)$  is conformally hyperbolic and any complete causal geodesic in  $(M, g)$  has a pair of conjugate points.*

By using (2.4), [B-E-E, Propositions 11.7, 11.8 and Theorem 11.9] and [A2, Corollary 3.17], we can establish an existence result for conjugate points of time-like geodesics in a standard static space-time which by *Theorem 4.5* is also conformally hyperbolic.

In the next theorem,  $\mathbf{L}$  denotes the usual time-like Lorentzian length and  $\text{diam}_{\mathbf{L}}$  denotes the corresponding time-like diameter (see [B-E-E, Chapters 4 and 11]).

**THEOREM 4.6.** *Let  $M = I_f \times F$  be a standard static space-time with the metric  $g = -f^2 dt^2 \oplus g_F$ . Suppose that  $\text{Ric}_F$  and  $Q_F^f$  are positive semi-definite. If there exists a constant  $c$  such that  $\frac{1}{f} \Delta_F f \geq c > 0$ , then*

- (1) *any time-like geodesic  $\gamma: [r_1, r_2] \rightarrow M$  in  $(M, g)$  with  $\mathbf{L}(\gamma) \geq \pi \sqrt{\frac{n-1}{c}}$  has a pair of conjugate points,*
- (2) *for any time-like geodesic  $\gamma: [r_1, r_2] \rightarrow M$  in  $(M, g)$  with  $\mathbf{L}(\gamma) > \pi \sqrt{\frac{n-1}{c}}$ ,  $r = r_1$  is conjugate along  $\gamma$  to some  $r_0 \in (r_1, r_2)$ , and as a consequence  $\gamma$  is not maximal,*
- (3) *if  $I = \mathbb{R}$ ,  $(F, g_F)$  is complete and  $\sup f < \infty$ , then  $\text{diam}_{\mathbf{L}}(M, g) \leq \pi \sqrt{\frac{n-1}{c}}$ .*

In the final part of [D-U5] we show some examples and results connecting the tensor  $Q_F^f$ , conformal hyperbolicity, concircular scalar fields and Hessian manifolds, where the role of the Hessian tensor is central.

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