Chapter 5
Discrete Center Problems

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5.1 Introduction

Our focus in this chapter is on discrete center location problems. This class of problems involves locating one or more facilities on a network to service a set of demand points at known locations in such a way that every demand receives its service from a closest facility, and the maximum distance between a demand and a closest facility is as small as possible. This leads to a minimax type of objective function, which is intrinsically different from the minisum objective that is more widely encountered in location models, for which the primary concern is to minimize the total transportation cost. The term *discrete* in the title refers to a finite set of demand points, while *continuous* versions of center location problems are also possible if the set of demand points to be served constitutes a continuum of points on the network under consideration.

Center location problems most commonly arise in emergency service location, where the concern for saving human life is far more important than any transportation costs that may be incurred in providing that service. Consider, for example, locating a fire station to serve a number of communities interconnected by a road network. If a fire breaks out in any one of these communities, it is crucial for equipment to arrive at the fire as quickly as possible. Similarly, quick delivery of an emergency service is significantly more important in optimally placing, for example, ambulances and police patrol units, than the cost of delivering that service. The common denominator in all of these circumstances is that there is a time delay between the call for service and the actual time of beginning to provide that service that is a direct consequence of the time spent during transportation. All other factors being constant, it makes sense to model such circumstances so that the maximum distance traversed during transportation is as small as possible.

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5.1.1 The Single Facility Case: The Absolute Center Problem

To define the center location problem, let us first consider the single facility problem that involves optimally placing an emergency service facility on a road network that interconnects n communities requiring the services of the facility. It is convenient to represent the road network of interest by an undirected connected network $G=(V', E)$ with vertex set $V' = \{v_1, \ldots, v_n, \ldots, v_{n'}\}$ and edge set E consisting of undirected edges of the form $e_{ij}=[v_i, v_j]$ with edge lengths $L_{ij}>0$. Without loss of generality, we assume that the vertex set includes the $n \leq n'$ communities requiring the services of the facility. We further assume, with re-indexing if necessary, that the first n vertices are the vertices that demand service from the facility. Let $V=\{v_1,\ldots, v_n\} \subseteq V'$ be the demand set. Vertices not in V, if any, may represent, for example, intersections of roads. Edges represent road segments connecting pairs of vertices, and their lengths are positive. We take each edge of the network as an infinite set of points (a continuum) connecting the end-vertices of the edge under consideration and refer to each point along an edge as an *interior* point of that edge if the point is not one of the end-vertices. We take the network G as the union of its edges and write $x \in G$ to mean x is any point along any edge of G.

For any pair of vertices v_i and v_j in the network, a path $P=P(v_i, v_j)$ connecting v_i and v_j is a sequence of alternating vertices and edges that begin at v_i and end at v_j. We define the *length* of a path P to be the sum of the lengths of the edges contained in the path. A *shortest path* connecting v_i and v_j, denoted by $SP(v_i, v_j)$, is a path whose length is the smallest among all paths connecting v_i and v_j. Due to the positivity of edge lengths, every shortest path between a pair of vertices is a simple path; meaning no vertex in the path is repeated. In general, there may be many shortest paths between a pair of vertices, each having the same length. We define $d_{ij}=d(v_i, v_j)$ to be the length of a shortest path connecting v_i and v_j, and refer to d_{ij} as the *distance* between v_i and v_j. Vertex-to-vertex distances are computed via well known all-pairs shortest path algorithms, see, e.g., Floyd (1962) or Dantzig (1967). We extend the definition of the shortest path distance to *any* pair of points $x, y \in G$, vertex or not, by defining the length of a path to be the sum of lengths of edges and subedges contained in the path and defining $d(x, y)$ to be the length of a shortest path connecting x and y. The function $d(\bullet, \bullet)$ satisfies the properties of *nonnegativity*, *symmetry*, and *triangle inequality* which are as follows:

$$\forall x, y \in G,$$

- *Nonnegativity:* $d(x, y) \geq 0$; $d(x, y)=0$ iff $x=y$;
- *Symmetry:* $d(x, y)=d(y, x)$;
- *Triangle Inequality:* $d(x, y) \leq d(x, u)+d(u, y)\ \forall\, u \in G$.

The single facility center location problem is referred to as the *Absolute Center Problem,* a term coined by Hakimi (1964) who introduced this problem to the literature. To define the problem, we associate nonnegative constants w_i and a_i with each vertex v_i, $i=1,\ldots, n$. We refer to each w_i as a *weight* and each a_i as an *addend*. Vertex weights are used as scaling factors to assign relative values of importance to demand vertices based, for example, on population densities. A vertex representing a densely populated business district during work hours may require a more amplified protection against emergency than a vertex representing a rather sparsely populated residential area. Such differences may be reflected into the model by a judicious choice of weights. The addend a_i can be interpreted as preparation time for a fire-fighting squad to get the equipment ready to work at v_i. This preparation time depends in general on the local conditions at a vertex (including access to a fire hydrant, space available for fire engines to position themselves), so that having different addends at different vertices is meaningful. For ambulance services, we may interpret a_i as the time spent transporting the patient from v_i to the closest hospital. If hospital locations are known, this transportation time is a fixed constant that depends only upon the vertex under consideration and a hospital closest to that vertex.

Given w_i, $a_i \geq 0$ ($i=1,\ldots, n$), define the function f for every $x \in G$ by

$$f(x) = \max\{w_i d\,(x, v_i) + a_i\colon i = 1, \ldots, n\} \tag{5.1}$$

and consider the optimization problem

$$r_1 \equiv \min\{f(x)\colon x \in G\}. \tag{5.2}$$

Any point $x^* \in G$ that solves (5.2) is referred to as an *absolute center* of G, and the minimum objective value r_1 is referred to as the *1-radius* of G. If x is restricted to V in (5.1) and (5.2), the resulting problem is called the *vertex-restricted problem*, and its solution is referred to as a *vertex-restricted center*. If the demand set V in relation (5.1) is replaced by the continuum of all points in G, then the definition of $f(\bullet)$ becomes $f(x)=\max\{d(x, y)\colon y \in G\}$ and any point in G that minimizes this function is referred to as a *continuous center* (see Frank 1967). A different continuous demand version of the center problem is also formulated by Minieka (1977). In his formulation, the objective is to minimize the maximum distance from the facility to a farthest point on each edge. A point in G that minimizes this objective function is referred to as a *general center*. Our focus in this chapter is on the absolute center problem. The continuous and general center problems are briefly discussed in Sect. 5.4.

The absolute center problem is referred to as the *weighted problem* if at least one of the weights is different from one and the *weighted problem with addends* if, additionally, at least one addend is nonzero. The case with $w_i=1\ \forall\, i \in I \equiv \{1,\ldots, n\}$ is referred to as the *unweighted problem* or the *unweighted problem with addends,* respectively, depending on if all a_i or not all a_i are zero.

In the unweighted case, the definition of $f(x)$ becomes $f(x)=\min\{d(x, v_i)\colon x \in G\}$ so that $f(x)$ identifies a farthest community and its distance from a facility at x. With $d(x, v_i) \leq f(x)\ \forall\, i \in I$, all communities are covered within a distance of $f(x)$, while there is at least one community whose distance from x is exactly $f(x)$. The optimization in (5.2) seeks to place the facility in such a way that the farthest distance from it to any community is as small as possible. If x^* achieves this, then $f(x^*)$ supplies the value r_1, which is the smallest possible coverage radius from a facility anywhere on the network. Generally, with weights and addends, each community v_i is covered by a facility at x within a distance of $[f(x)-a_i]/w_i$, while at least one community achieves this bound.

5.1.2 The Multi-facility Case: The Absolute p-Center Problem

Multiple facilities are needed in emergency service location when a single facility is not enough to cover all communities within acceptable distance limits. To model the multi-facility version of the problem, let p be a positive integer representing the number of facilities to be placed on the network. Assume that the p facilities under consideration are identical in their service characteristics and that each is uncapacitated so that communities are indifferent as to which particular facility they receive their services from (provided that the service is given in the quickest possible way). Accordingly, if $x_1,\ldots, x_p$ are the locations of the p facilities, then each community prefers to receive its service from the facility closest to it.

Let $X=\{x_1,\ldots,x_p\}$ and define $D(X, v_i)$ to be the distance of vertex v_i to a nearest element of the point set X. That is,

$$D(X, v_i) = \min\left\{d\,(x_1, v_i)\,, \ldots, d(x_p, v_i)\right\}. \tag{5.3}$$

Let $S_p(G)$ be the family of point sets X in G such that $|X|=p$. Hence, $X\in S_p(G)$ implies $X=\{x_1,\ldots, x_p\}$ for some choice of p distinct points $x_1,\ldots, x_p$ of G. We extend the definition of $f(x)$ to the multi-facility case as follows: For each $X\in S_p(G)$, define

$$f(X) = \max\{w_i D(X, v_i) + a_i\colon i = 1, \ldots, n\}. \tag{5.4}$$

The definition in (5.4) reduces to definition (5.2) for the case of $p=1$.

The *Absolute p-Center Problem*, introduced by Hakimi (1965), is the problem of finding a point set $X^*\in S_p(G)$ such that

$$r_p \equiv f\left(X^*\right) = \min\{f(X)\colon X \in S_p(G)\}. \tag{5.5}$$

Any point set $X^*=\{x_1^*,\ldots, x_p^*\}\in S_p(G)$ that solves (5.5) is called an *absolute p-center of G* and each location x_j^* in X^* is referred to as a *center*. The minimum objective value r_p is called the *p-radius* of G. If X is restricted to p-element subsets of V, the resulting problem is referred to as a *vertex restricted p-center problem* and its solution is called a *vertex-restricted p-center*. If each point in the network is a demand point as opposed only to vertices, the resulting problem is called the *continuous p-center problem.* If the maximum distance to a farthest point in each edge is minimized, the resulting problem is *the general p-center problem.* While the continuous and general center problems are equivalent for $p=1$, different problems result for $p>1$.

Our focus is on the absolute p-center problem. The definition of $f(X)$ in (5.4) implies that $w_i D(X, v_i)+a_i \leq f(X) \forall i$, so that every community v_i is covered by at least one center in X within a distance of $[f(X)-a_i]/w_i$. Note also that there is at least one community which achieves this bound. The optimization in (5.5) seeks to place the p facilities on the network such that the farthest weighted distance of any community from the nearest facility is as small as possible.

Now that we have a clear idea of the type of location models dealt with in this chapter, we focus next on three classical papers that have had significant impact on the literature in this area of research.

5.2 Three Classical Contributions on Discrete Center Location

We give in this section an overview of three early and fundamental papers that had a significant impact on subsequent research in discrete center location. Each of the Sects. 5.2.1, 5.2.2, and 5.2.3 is devoted to one of these papers. The first work that we investigate is the contribution by Hakimi (1964). This is a seminal paper in that it has led to a whole new area of research that we know of today as *network location*. Hakimi poses two problems in his paper, assuming nonnegative weights and zero addends, and calls them the *absolute median* and the *absolute center* problems. Both problems are posed on a network whose edges are viewed as continua of points. The objective in the absolute median problem is to minimize the weighted sum of distances from the facility to all vertices, while the objective in the absolute center problem is to minimize the maximum of such distances. Hakimi provides an insightful analysis for both problems. One consequence of his analysis is the well known vertex optimality theorem for the absolute median problem. Hakimi's analysis for the absolute center problem has led to a methodology that relies on identifying local minima on edges by inspecting piece-wise linear functions. Hakimi's paper is investigated in Sect. 5.2.1.

A second classical contribution is a paper by Goldman (1972). In his work, Goldman gives a localization theorem for the absolute center problem that helps to localize the search for an optimal location to a subset of the network whenever the network has a certain exploitable structure. Repeated application of the theorem results in an algorithm that either finds an optimal location or reduces the problem to a single cyclic component of the network. Goldman's paper is examined in Sect. 5.2.2.

Minieka (1970) focuses on the multi-facility case and gives a well conceived solution strategy for the unweighted absolute p-center problem, which relies on solving a sequence of set covering problems. Minieka's method is directly extendible to the weighted version. Minieka's paper is covered in Sect. 5.2.3.

5.2.1 Hakimi (1964): The Absolute Center Problem

Hakimi's paper is historically the first paper that considers the absolute center problem on a network. The vertex-restricted version of the 1-center problem is posed as early as 1869 by Jordan (1869), and is directly solved by evaluating the objective function at each vertex. The absolute center problem, on the other hand, requires an infinite search over the continua of points on edges and calls for a deeper analysis than simple vertex enumeration.

Hakimi viewed each edge as a continuum of points. This marks a significant departure from the traditionally accepted view of classical graph theory that takes each undirected edge as an unordered pair of vertices. The kind of network Hakimi had in mind is what we refer to today as an embedded network where each edge $[v_i, v_j]$ is the image of a one-to-one continuous mapping T_{ij} of the unit interval [0, 1] into some space S (e.g. the plane) such that $T_{ij}(0)=v_i$, $T_{ij}(1)=v_j$, and each point x in the interior of $[v_i, v_j]$ is the image $T_{ij}(\alpha)$ of a real number α in the open interval (0,1). A formal definition of an embedded network can be found in Dearing, Francis, and Lowe (1976); for details, also see Dearing and Francis (1974). For our purposes, it suffices to view the network of interest as an embedding in the plane with vertices corresponding to distinct points and edges corresponding to continuous curves connecting pairs of vertices. We assume that, whenever two edges intersect, they intersect only at a vertex. A point x in edge $[v_i, v_j]$ induces two subedges $[v_i, x]$ and $[x, v_j]$ with $[v_i, x]\cup[x, v_j]=[v_i, v_j]$ and $[v_i, x]\cap[x, v_j]=\{x\}$.

Hakimi observed that the optimization problem $\min\{f(x)\colon x\in G\}$, where $f(x)\equiv\max\{w_i d(x, v_i)\colon i\in I\}$, can be solved by minimizing $f(\bullet)$ on each edge separately and then choosing the best of the edge-restricted minima. This is an immediate consequence of the fact that the graph G is the union of its edges. It then suffices to develop a solution procedure for the edge restricted problem.

Let $e=[v_p, v_q]$ be an edge of the network. The *edge restricted problem* regarding this edge can then be written as

$$\mathrm{Min}\{f(\mathrm{x})\colon x \in e\}. \tag{5.6}$$

Let $L=L_{pq}$ be the length of the edge e. Observe that as x varies in the edge $\boldsymbol{e}$, the length of the subedge $[v_p, x]$ varies in the interval $[0, L]$. If we denote by x_λ the unique point x in e for which the subedge $[v_p, x]$ has length λ, then we may redefine the edge restricted problem in the equivalent form

$$\mathrm{Min}\{f(x_\lambda)\colon \lambda \in [0, L]\}. \tag{5.7}$$

The form defined by (5.7) is particularly useful for analyzing the structure of $f(\bullet)$ as a function of the real variable λ. We begin the analysis of $f(\bullet)$ by first examining the distance $d(x_\lambda, v_i)$ for a fixed vertex v_i as λ varies in the interval $[0, L]$. Define the function g_i by $g_i(\lambda)=d(x_\lambda, v_i)\,\forall\lambda\in[0, L]$. Observe that a shortest path from an interior point x_λ to vertex v_i must include either the subedge $[v_p, x_\lambda]$ or the subedge $[x_\lambda, v_q]$. Accordingly, $SP(x_\lambda, v_i)$ is either $[v_p, x_\lambda]\cup SP(v_p, v_i)$ or $[x_\lambda, v_q]\cup SP(v_q, v_i)$. Figure 5.1 illustrates these two possibilities. It follows that $g_i(\lambda)$ is the minimum of

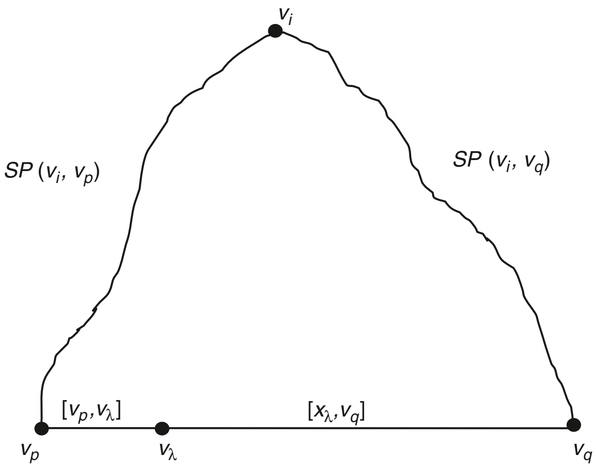


Fig. 5.1 Illustration of shortest path connecting v_i and x_λ

the two path lengths $\lambda + d_{pi}$ and $L - \lambda + d_{qi}$ corresponding, respectively, to the paths $[v_p, x_\lambda] \cup SP(v_p, v_i)$ and $[x_\lambda, v_q] \cup SP(v_q, v_i)$. Accordingly, we have

$$g_i(\lambda) = \min\{\lambda + d_{pi}, L - \lambda + d_{qi}\} \forall \lambda \in [0, L]\,. \tag{5.8}$$

Observe that all quantities in the right side of (5.8) are constants except λ. With this observation, $g_i(\lambda)$ is the pointwise minimum of the two linear functions $\lambda + d_{pi}$ and $L - \lambda + d_{qi}$ in the interval $[0, L]$. The fact that the distance from a fixed vertex v_i to a variable point in a given edge is the pointwise minimum of two linear functions is a key element, observed by Hakimi, that has led to a well-structured theory and solution method.

In general, the pointwise minimum of a finite number of linear functions is a concave piecewise linear function that has, at most, as many pieces as there are linear functions under consideration. Figure 5.2 illustrates a concave piece-wise linear function $h(x)$ that consists of 4 pieces.

Fig. 5.2 A concave piecewise linear function $h(x)$

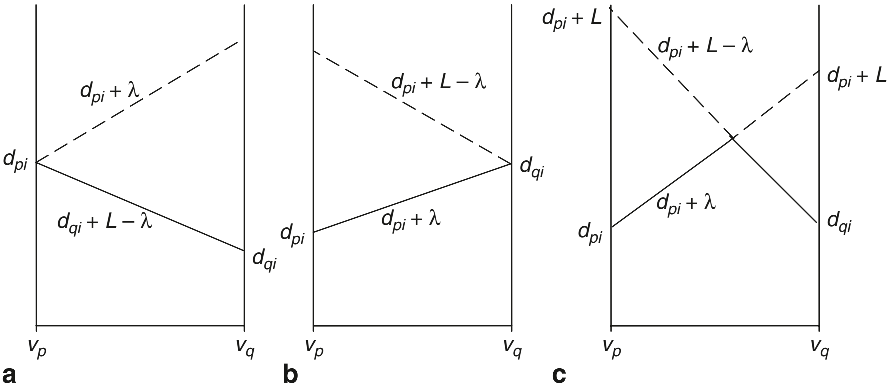


Fig. 5.3 Three possible forms of the function $g_i(\lambda) = \min\{d_{pi} + \lambda, d_{qi} + L - \lambda\}$. **a** Decreasing. **b** Increasing. **c** Two-piece

In our case, g_i is the minimum of two linear functions, so it is either linear or two-piece linear. If g_i is linear, then it is an *increasing* linear function with a slope of $+1$ whenever $\lambda + d_{pi} \leq L - \lambda + d_{qi}\ \forall \lambda \in [0, L]$, while it is a *decreasing* linear function with slope of -1 if the reverse inequality holds. If g_i is a two-piece linear function, then the two linear functions of interest attain the same value at some interior point λ' of the interval $[0, L]$, so that the linear piece in the subinterval $[0, \lambda']$ is increasing while the linear piece in the subinterval $[\lambda', L]$ is decreasing. Figure 5.3 illustrates the three possible forms of g_i. Note that the two linear functions $\lambda + d_{pi}$ and $L - \lambda + d_{qi}$ always intersect at an end-vertex if they do not intersect at an interior point. For example, they intersect at v_p in Fig. 5.3a and at v_q in Fig. 5.3b. In the case of Fig. 5.3a, the linear function $d_{qi} + L - \lambda$ is the smaller of the two linear functions over the entire edge so that $d_{qi} + L \leq d_{pi}$ at $\lambda = 0$. However, d_{pi} is the shortest path length between v_i and v_p while $d_{qi} + L$ is the length of a path connecting v_i and v_p via v_q. This implies that $d_{pi} \leq d_{qi} + L$. The two inequalities result in the equality $d_{qi} + L = d_{pi}$.

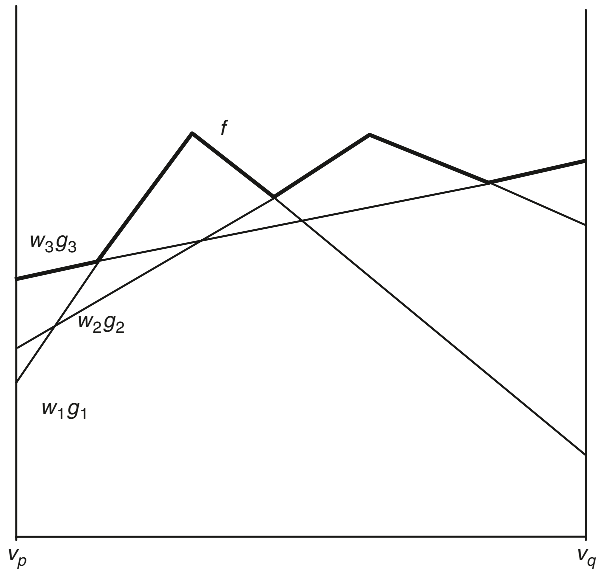
Consider now the weighted distance $w_i d(x_\lambda, v_i) = w_i g_i(\lambda)$ as λ varies in $[0, L]$. Since w_i is positive and g_i is the minimum of two linear functions with slopes ± 1, $w_i g_i(\bullet)$ is a concave piecewise linear function with at most two pieces and with slopes of $\pm w_i$. The only difference from the previous case is that the slopes are now $\pm w_i$ rather than ± 1.

Let us now focus on the analysis of the function $f(\bullet)$ on edge e. By definition, $f(x)$ is the maximum over $i \in I$ of the weighted distances $w_i d(x, v_i)$. Using again the variable point $x_\lambda \in e$ as λ varies in $[0, L]$, we have

$$f(x_\lambda) = \max\{w_i g_i(\lambda) \colon i \in I\}. \tag{5.9}$$

Since each $w_i g_i(\bullet)$ is a concave piecewise linear function with at most two pieces, $f(\bullet)$ is the pointwise maximum of n such functions. Accordingly, the restriction of

Fig. 5.4 $f(\bullet)$ as the maximum of three concave two-piece linear functions $w_i g_i$



$f(\bullet)$ to an edge results in a piecewise linear function. Figure 5.4 illustrates this for the case of $n=3$. In general, the maximum of concave functions is not concave, so the only exploitable property of $f(\bullet)$ is piecewise linearity. It is quite clear that the minimum of a piecewise linear function on a closed interval either occurs at a break point or at an end point of the interval. Hakimi's method searches for break points where the slopes to the left and to the right of the point are oppositely signed. The functions $w_i g_i(\bullet)$ are plotted for $i \in I$ on each edge and $f(\bullet)$ is constructed by taking their pointwise maximum. The minimum of $f(\bullet)$ on a given edge is found by inspecting the qualifying break-points of the resulting graph.

Hakimi demonstrates his method on a network with six vertices and eight edges. We reproduce his network from Hakimi (1964) in Fig. 5.5. The edge lengths are shown next to the edges. The vertex-to-vertex distances are shown in Table 5.1.

Let the edges be numbered $e_1,\ldots,e_8$ as shown in Fig. 5.5. The plots of the functions $g_i(\bullet)$ and $f(\bullet)$ on each edge e_j are shown in Fig. 5.6, assuming that all vertex weights are equal to one. The plots of $f(\bullet)$ are in bold. The edge-restricted optimum on edge $e_1=[v_6, v_5]$ shown in Fig. 5.6a is at point x_1, at a distance of 1.5 from v_6 with $f(x_1)=5.5$. For edge $e_2=[v_5, v_3]$ shown in Fig. 5.6b, there are two local optima, one at v_5 and the other at v_3, with $f(v_5)=f(v_3)=6$. For edge $e_3=[v_1, v_6]$ shown in Fig. 5.6c, there is an edge restricted optimum at point x_3, which is at a distance of 2.5 units from v_1 with $f(x_3)=5.5$.

For edge $e_4=[v_1, v_4]$ shown in Fig. 5.6d, there are two edge-restricted optima, one at v_1 and the other at point x_4, at a distance of 2 units from v_1 with $f(v_1)=f(x_4)=6$. For edge $e_5=[v_1, v_2]$ shown in Fig. 5.6e, there are two edge restricted optima, one at v_1 and the other at x_5, at a distance of 2 units from v_1 with $f(v_1)=f(x_5)=6$. For edge $e_6=[v_2, v_4]$ shown in Fig. 5.6f, the two edge-restricted optima are at end-vertices v_2

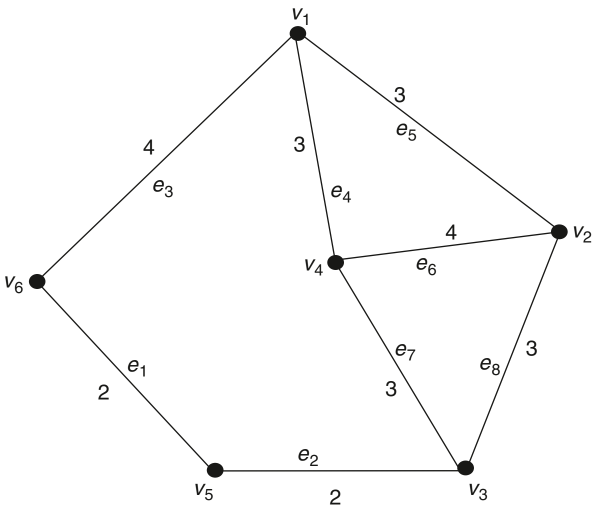


Fig. 5.5 An illustrative network. (Taken from Hakimi 1964)

Table 5.1 Vertex-to-vertex distances for the example network

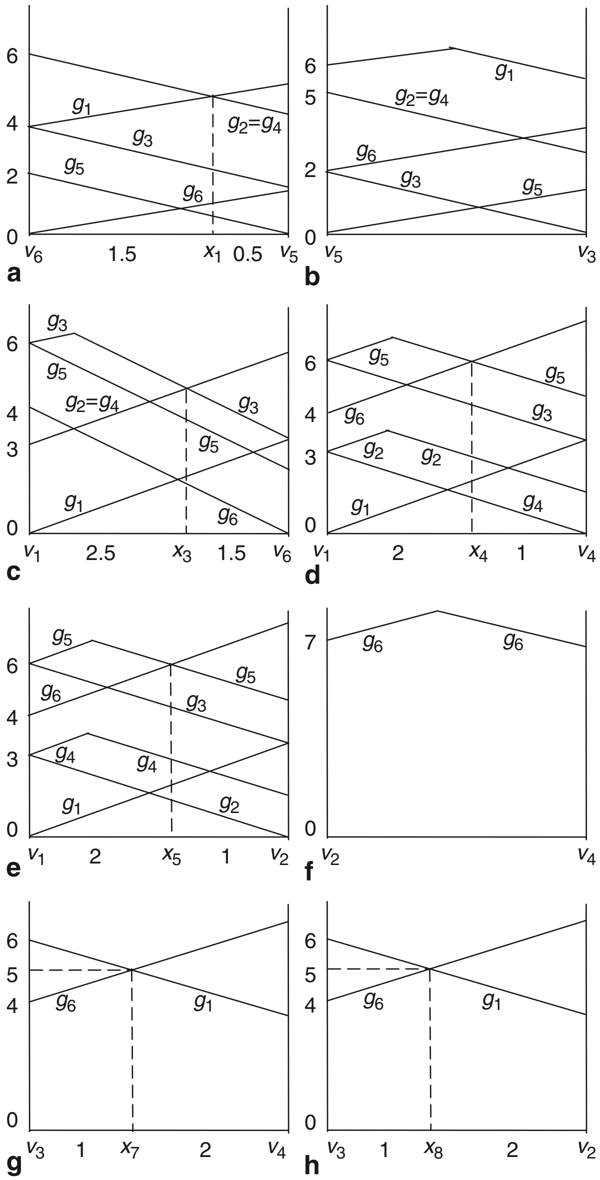
	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	3	6	3	6	4
v_2	3	0	3	4	5	7
v_3	6	3	0	3	2	4
v_4	3	4	3	0	5	7
v_5	6	5	2	5	0	2
v_6	4	7	4	7	2	0

and v_4 with $f(v_2)=f(v_4)=7$. For edge $e_7=[v_3, v_4]$ shown in Fig. 5.6g, the edge-restricted optimum is at point x_7, at a distance of 1 unit from v_3 with $f(x_7)=5$. Finally, for edge $e_8=[v_3, v_2]$ shown in Fig. 5.6h, the edge-restricted optimum is at point x_8, at a distance of 1 unit from v_3 with $f(x_8)=5$. Accordingly, there are two absolute centers for the network of Fig. 5.5, one at x_7 and the other at x_8 with $f(x_7)=f(x_8)=5$.

5.2.2 *Goldman (1972): A Localization Theorem for the Absolute Center*

In this section, we continue with a localization theorem for the absolute center problem studied by Goldman (1972). Goldman's localization theorem for the absolute center problem is motivated by a similar localization theorem introduced earlier by Goldman (1971) for the absolute median problem. Goldman's earlier result for the median problem led to a very efficient tree-trimming algorithm for computing optimal medians of tree networks. His result for the absolute center problem is similarly

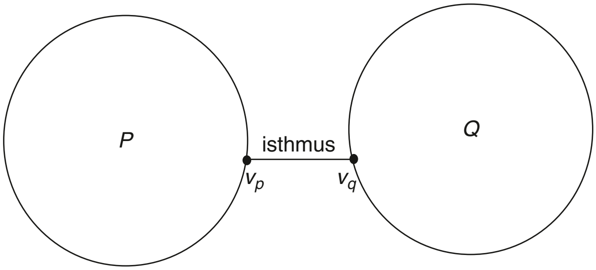
Fig. 5.6 Determining local centers of edges of the network shown in Fig. 5.5



structured and either finds an optimum solution or reduces the problem to a cyclic component of the network.

To begin the analysis, consider the *unweighted* absolute center problem *with addends* on a network $G=(V', E)$. We assume again the first n vertices in V' are the demand vertices and constitute the demand set V. For any point $x \in G$, the objective

Fig. 5.7 An isthmus $[v_p, v_q]$ with subnetworks P and Q



function is defined by $f(x) \equiv \max\{a_i + d(v_i, x): i \in I\}$, and the objective is to find a point $x^* \in G$ for which $f(x^*) \leq f(x)\ \forall x \in G$.

Goldman's localization theorem works best in networks that have edges that are not contained in any simple cycles. Goldman refers to any such edge as an "isthmus." An *isthmus* of G is an edge $[v_p, v_q]$ whereby deleting the interior of this edge results in two disconnected components P and Q. Here, we assume that v_p is in P and v_q is in Q. Figure 5.7 illustrates the definition. An isthmus cannot be contained in any simple cycle of G, otherwise there is a path from a vertex in P to a vertex in Q that does not pass through the edge $[v_p, v_q]$. This, of course, implies that deleting the interior of the edge $[v_p, v_q]$ does not result in two disconnected subsets of G.

Consider an isthmus $e = [v_p, v_q]$ and the associated components P and Q of G where $P \cup e \cup Q = G$, $P \cap e = \{v_p\}$, $Q \cap e = \{v_q\}$, and $P \cap Q = \emptyset$. Let v_i and v_j be a pair of vertices with $v_i \in P$ and $v_j \in Q$. All paths connecting v_i and v_j pass through e so that $d_{ij} = d_{ip} + L + d_{qj}$, where $L \equiv L_{pq}$ is the length of e. Consider a variable point x_λ that moves from v_p to v_q along the edge e as λ varies in the interval $[0, L]$. With λ being the length of the subedge $[v_p, x_\lambda]$ and $L - \lambda$ being the length of the subedge $[x_\lambda, v_q]$, we have $g_i(\lambda) \equiv d(v_i, x_\lambda) = d_{ip} + \lambda$. Hence, $g_i(\bullet)$ is a linear increasing function that begins with value d_{ip} at v_p and ends with value $d_{ip} + L$ at v_q. Similarly, for $v_j \in Q$, we have $g_j(\lambda) = d(v_j, x_\lambda) = d_{jq} + L - \lambda$ so that $g_j(\bullet)$ is a linear decreasing function that begins with the value $d_{jq} + L$ at v_p and ends with the value d_{jq} at v_q.

Consider now the edge restricted problem min $\{f(x_\lambda): x_\lambda \in e\}$. We may partition the demand vertices into the disjoint vertex subsets $V \cap P$ and $V \cap Q$ so that the definition of $f(x_\lambda)$ becomes

$$f(x_\lambda) = \max\left\{f_p(x_\lambda), f_q(x_\lambda)\right\} \tag{5.10}$$

where

$$f_p(x_\lambda) \equiv \max\{a_i + g_i(\lambda): v_i \in V \cap P\} \tag{5.11}$$

and

$$f_q(x_\lambda) \equiv \max\{a_j + g_j(\lambda): v_j \in V \cap Q\}. \tag{5.12}$$

Since each g_i is a linear increasing function with identical slopes for vertices $v_i \in V \cap P$, the functions $a_i + g_i(\lambda)$ are also linear increasing with identical slopes and

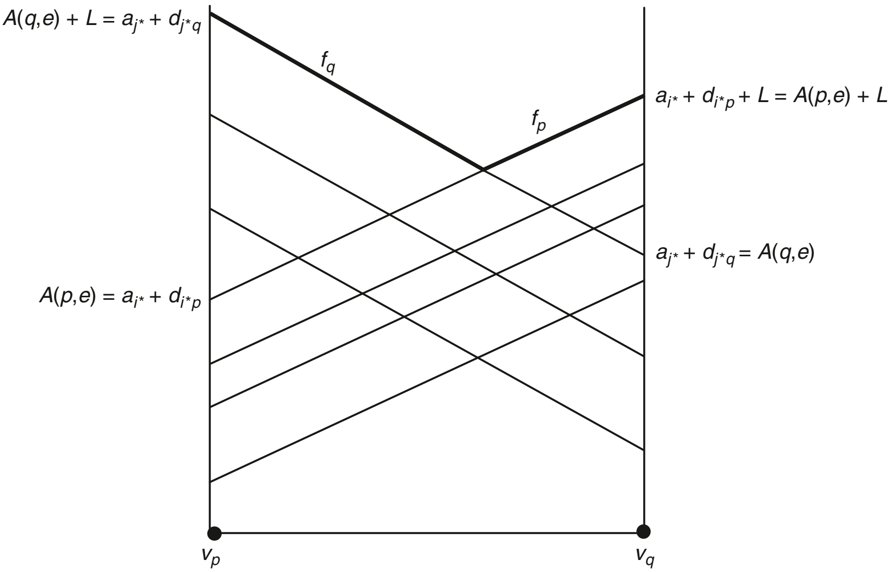


Fig. 5.8 The functions f_p and f_q

with intercepts of a_i+d_{ip} and $a_i+d_{ip}+L$ at v_p and v_q, respectively. Because the slopes are identical, the largest intercept defines $f_p(\bullet)$ on the entire edge. That is, there is a vertex $v_{i^*}\in V\cap P$ such that $a_{i^*}+d_{i^*p}=\max\{a_i+d_{ip}: v_i\in V\cap P\}$ and $f_p(x_\lambda)=a_{i^*}+d_{i^*p}+\lambda$ for $\lambda\in[0,L]$. Similarly, there is a vertex $v_{j^*}\in V\cap Q$ such that $a_{j^*}+d_{j^*q}=\max\{a_j+d_{jq}: v_j\in V\cap Q\}$ and $f_q(x_\lambda)=a_{j^*}+d_{j^*q}+L-\lambda$. Figure 5.8 illustrates the functions $f_p(x_\lambda)$ and $f_q(x_\lambda)$ as the maximum of increasing and decreasing linear functions, respectively, with identical slopes.

Let $A(p, e)$ and $A(q, e)$ be the highest intercepts at v_p and v_q, respectively. That is,

$$A\,(p,e) = \max\{a_i + d_{ip}: v_i \in V \cap P\} \tag{5.13}$$

and

$$A\,(q,e) = \max\{a_j + d_{jq}: v_j \in V \cap Q\}. \tag{5.14}$$

We then have $A(p, e)=a_{i^*}+d_{i^*p}$ and $A(q, e)=a_{j^*}+d_{j^*q}$ where the indices i^* and j^* are as defined before. Additionally, we have

$$f_p(x_\lambda) = A(p,e) + \lambda \;\forall\; \lambda \in [0, L]\,, \tag{5.15}$$

$$f_q\,(x_\lambda) = A(q,e)\, + L - \lambda \;\forall\; \lambda \in [0, L]\,, \tag{5.16}$$

and

$$f\,(x_\lambda) = \max\{A(p,e) + \lambda, A(q,e) + L - \lambda\} \;\forall\; \lambda \in [0, L]\,. \tag{5.17}$$

Goldman's localization theorem can then be stated as follows.

Theorem 1 (Localization Theorem): *Exactly one of three cases applies:*

(a) $A(q, e)-A(p, e)\geq L$: *Then the problem can be reduced to* Q, *with* a_q *replaced by* $max\{a_q, A(p, e)+L\}$.
(b) $A(p, e)-A(q, e)\geq L$: *Then the problem can be reduced to* P, *with* a_p *replaced by* $max\{a_p, A(q, e)+L\}$.
(c) $|A(p, e)-A(q, e)|<L$. *Then the optimal location is in the interior of edge e.*

In case (a), the lowest value $A(q, e)$ of the linear decreasing function is at least as large as the highest value $L+A(p, e)$ of the linear increasing function so that the value of $f(x_\lambda)$ is defined by the linear decreasing function $A(q, e)+L-\lambda$ on the entire edge. This is sufficient to conclude that any point in $P\cup e-\{v_q\}$cannot be an optimal location. A more formal justification for this is as follows. Suppose $x\in P\cup e-\{v_q\}$. Then, we have:

$$\begin{aligned}
f(x) &= \max\{a_j + d(v_j,x): j \in I\} \\
&\geq \max\{a_j + d(v_j,x): v_j \in V \cap Q\} \\
&= \max\{a_j + d_{jq} + L + d(v_p,x): v_j \in V \cap Q\} \\
&= d(v_p,x) + L + A(q,e) \\
&> A(q,e) \\
&= f(v_q).
\end{aligned}$$

This proves that $f(x)>f(v_q)$ for all x in $P\cup e-\{v_q\}$, so this set cannot contain an optimum. We confine the search for an optimum to the subset Q by deleting all points in P and all points in e except v_q. Replacing the addend a_q by the larger of a_q or $A(p, e)+L$ is needed because for any candidate point $x\in Q$, if $f(x)$ is defined by a vertex in P, then $f(x)=d(x, v_q)+L+A(p, e)$ where the quantity $L+A(p, e)$ is the new value of a_q. Note that if $a_q>A(p, e)+L$, then no demand vertex in P can supply the value of $f(x)$ for $x\in Q$ (since $f(x)\geq w_q d(x, v_q)+a_q>w_q d(x, v_q)+L+A(p, e)\geq\max\{w_i d(x, v_i)+a_i: v_i\in V\cap P\}$).

Case (b) is similar to case (a) with function $A(p, e)+\lambda$ being at least as large as the function $A(q, e)+L-\lambda$ on the entire edge so that $f(x_\lambda)$ is defined now by $A(p, e)+\lambda$ for $\lambda\in[0, L]$. Using similar arguments as in case (a), it is apparent in case (b) that $f(x)>f(v_p)=A(p, e)\,\forall x\in Q\cup e-\{v_p\}$ implying that no point in $Q\cup e-\{v_p\}$ qualifies as an optimal location. Replacing a_p by $\max\{a_p, A(q, e)+L\}$ is needed to account for the largest a_j+d_{jq} value that can be supplied by demand vertices v_j in $Q\cup e-\{v_p\}$ which is the deleted portion of the network.

In case (c), the linear functions $A(p, e)+\lambda$ and $A(q, e)+L-\lambda$ intersect at an interior point $x_{\lambda*}$ of the edge with λ^* defined by $\lambda^*=0.5[A(q, e)+L-A(p, e)]$. Evaluating f at $x_{\lambda*}$, we obtain $f(x_{\lambda*})=0.5[A(p, e)+A(q, e)+L]=f_p(\lambda^*)=f_q(\lambda^*)$ and letting i^* and j^* be the indices of the two critical vertices in $V\cap P$ and $V\cap Q$, respectively, such that $A(p, e)=a_{i*}+d_{i*p}$ and $A(q, e)=a_{j*}+d_{j*q}$, we obtain $f(x_{\lambda*})=0.5[a_{i*}+d_{i*p}+a_{j*}+d_{j*q}]$. Whenever case (c) occurs, $x_{\lambda*}$ is the unique optimal location.

The localization theorem offers a direct computational advantage for tree networks because every edge in a tree network is an isthmus. Let T be a tree network. Any vertex v_t of the tree that is adjacent to exactly one vertex v_s is referred to as a *tip*. It is well known that every tree has at least two tip vertices. The following algorithm uses the localization theorem repeatedly, "trimming" the tree successively by deleting each time a selected tip and the interior of the edge that connects it to the unique adjacent vertex, unless the localization theorem concludes that the optimal location occurs at the selected tip or in the interior of the connecting edge (cases (a) or (c) in the theorem). The process is described in the procedure below.

Algorithm 1: Tree Trimming Procedure

Step 1: If T consists of a single vertex, stop; that vertex is an optimal solution.

Step 2: Select a tip v_p and let v_q be the vertex adjacent to v_p. Let $e=[v_p, v_q]$ and L be the length of e. Take $A(p, e)=a_p$ and calculate $A(q, e)=\max\{w_j d_{qj}+a_j\colon j\in I, j\neq p\}$. If $A(p, e)\geq A(q, e)+L$, then tip v_p is optimal; stop. If $|A(q, e)-a_p|<L$, then the optimal solution is the interior point $x_{\lambda *}$ of e with the length of subedge $[v_p, x_{\lambda *}]$ given by $\lambda^*=0.5[A(q, e)+L-a_p]$; stop.

Step 3: Delete tip v_p and the interior of edge e from T. Delete p from I. Replace a_q with $\max\{a_q, a_p+L\}$ and return to Step 1.

If the network G under consideration is not a tree, then the localization theorem can be repeatedly used for each isthmus of G, one at a time. Termination occurs when either an optimal location is found or the problem is reduced to a single cyclic component. In the latter case, Hakimi's method is used to solve the reduced problem on the last cyclic component that has persisted. The only computational gain in this case is the reduction of the problem from the initial network with many cycles to a single cyclic component. The number of edge restricted problems that need to be solved is smaller than would have resulted from a direct application of the method on the original network.

An extension of the localization theorem to the weighted case with addends is possible, but its algorithmic utility is limited, because the computational advantages gained in the unweighted case from the updating of the addends do not occur in the weighted case. To outline the weighted version, consider an isthmus $e=[v_p, v_q]$ with associated components P and Q as defined before. In the weighted case, for $x_\lambda\in e$, we have $f(x_\lambda)=\max\{f_p(x_\lambda), f_q(x_\lambda)\}$ where $f_p(x_\lambda)=\max\{w_i(d_{ip}+\lambda)+a_i\colon v_i\in V\cap P\}$ and $f_q(x_\lambda)=\max\{w_j(d_{jq}+L-\lambda+a_j\colon v_j\in V\cap Q\}$. It follows that $f_p(x_\lambda)$ is the maximum of *increasing* linear functions with slopes w_i corresponding to demand vertices v_i in P and $f_q(x_\lambda)$ is the maximum of *decreasing* linear functions with slopes $-w_j$ corresponding to demand vertices v_j in Q. It follows that $f_p(x_\lambda)$ is a convex piecewise linear increasing function and $f_q(x_\lambda)$ is a convex piecewise linear decreasing function. Define $A(p, e)=\max\{a_i+w_i d_{ip}\colon v_i\in V\cap P\}$ and $A'(p, e) = \max\{a_i+w_i(d_{ip}+L)$:

$v_i \in V \cap P\}$. Because it is monotone increasing, $f_p(\bullet)$ has its lowest value at v_p and its highest value at v_q with $f_p(v_p) = A(p, e)$ and $f_p(v_q) = A'(p, e)$. Similarly, define $A(q, e) = \max\{a_j + w_j d_{jq}: v_j \in V \cap Q\}$ and $A'(q, e) = \max\{a_j + w_j(d_{jq} + L): v_j \in V \cap Q\}$. Because it is monotone decreasing, f_q has its highest value at v_p with $f(v_p) = A'(q, e)$ and its lowest value at v_q with $f(v_q) = A(q, e)$. The analogous version of the localization theorem for the weighted case is as follows:

Theorem 2 (Localization Theorem for Weighted Case): *Exactly one of the three cases apply:*

(a) $A(q, e) \geq A'(p, e)$: *Then, the optimum lies in* Q.
(b) $A(p, e) \geq A'(q, e)$: *Then, the optimum lies in* P.
(c) $A(q, e) < A'(p, e)$ *and* $A(p, e) < A'(q, e)$: *Then the optimum is located in the interior of* e.

The assertion in part (a) is a direct consequence of the fact that $f_q(x_\lambda) \geq f_p(x_\lambda)$ on the entire edge because the lowest value of the decreasing function $f_q(\bullet)$ is at least as high as the highest value of the increasing function $f_p(\bullet)$. Part (b) is similar, with $f_p(x_\lambda)$ being at least as large as $f_q(x_\lambda)$ on the entire edge. In part (c), the functions $f_p(\bullet)$ and $f_q(\bullet)$ intersect at an interior point of the edge, and the point of intersection is the minimizer of f. The power of the theorem is partly lost now due to the fact that, even though the optimum can be localized to subsets Q or P, respectively, in parts (a) or (b), the computational advantages available in the unweighted case are no longer available in the weighted case, as the computations of the parameters $A(\bullet, \bullet)$ and $A'(\bullet, \bullet)$ now require the data of the entire network.

5.2.3 Minieka (1970): Solving p-Center Problems via a Sequence of Set Covering Problems

We now focus on the absolute p-center problem where $1 < p < n$. The case $p \geq n$ is trivially solved by placing a center at each of the n demand vertices. Minieka (1970) has solved this problem in a clever way by solving a sequence of set covering problems.

With $S_p(G) \equiv$ set of all subsets of G consisting of p points, $X \in S_p(G)$, $D(X, v_i) \equiv \min\{d(x_j, v_i): x_j \in X\}$, and $f(X) \equiv \max\{w_i D(X, v_i): i \in I\}$, to solve the absolute p-center problem, we look for a point set $X^* \in S_p(G)$ such that $f(X^*) \leq f(X)\ \forall X \in S_p(G)$. Because each facility can be located anywhere on the network, this calls for an infinite search.

Minieka (1970) considers the unweighted version of the problem, but his approach can be directly extended to the weighted version; see, e.g., Kariv and Hakimi (1979). Minieka reduces the infinite search in $S_p(G)$ to a finite search, by observing that the absolute 1-center of the network occurs at one of a finite number of break points of $f(\bullet)$. Consider an edge $e = [v_p, v_q]$. If $x_{\lambda*}$ is an edge-restricted minimum of $f(\bullet)$ in the interior of e, then $x_{\lambda*}$ is a break point of $f(\bullet)$ defined by the intersection of two piecewise linear functions associated with a pair of vertices. With this motivation, we define U to be the set of all points u in G that qualify for en edge-restricted minimum. That is, U is the set of points $u \in G$ such that u is the unique point in its edge for which $d(v_i, u) = d(u, v_j)$ for a pair of vertices $v_i, v_j \in V$ with $i \neq j$. Because the piecewise linear functions have slopes of ± 1, the uniqueness requirement in the definition implies that the slopes of the two intersecting linear pieces are oppositely signed. There exists an absolute 1-center in the set $P \equiv V \cup U$. Clearly, there can be at most $n(n-1)/2$ intersection points in an edge, implying that U has at most $|E|n(n-1)/2$ elements in it. Hence, P is a finite dominating set (i.e., a finite set that supplies an optimum solution) for the unweighted absolute 1-center problem.

Minieka (1970) observed that P is also a finite dominating set for the unweighted absolute p-center problem. To justify this, suppose we have an absolute p-center $X^* = \{x_1^*, \ldots, x_p^*\}$. If not all points of X^* are in P, we may construct an absolute p-center X' from X^* that fulfills this requirement. To do so, partition the demand set V into subset $V_1, \ldots, V_p$ such that all vertices in subset V_i have the i-th element x_{i^*} of X^* as their closest center (ties are broken arbitrarily). Let x_i' be an optimal solution in P to the absolute 1-center problem defined with respect to the demand set V_i. This implies that $\max\{d(x_i', v_r)\colon v_r \in V_i\} \leq \max\{d(x_i^*, v_r)\colon v_r \in V_i\}$. Define $X' = \{x_1', \ldots, x_p'\}$. Since $D(X', v_r) \leq d(x_i', v_r)\ \forall\, v_r \in V$ and $\forall\, i \in \{1, \ldots, p\}$, we have

$$
\begin{aligned}
f(X') &= \max\{D(X', v_r)\colon v_r \in V\} \\
&\leq \max\{\max\{d(x_1', v_r)\colon v_r \in V_1\}, \ldots, \\
&\qquad \max\{d(x_p', v_r)\colon v_r \in V_p\}\} \\
&\leq \max\{\max\{d(x_1^*, v_r)\colon v_r \in V_1\}, \ldots, \\
&\qquad \max\{d(x_p^*, v_r)\colon v_r \in V_p\}\} \\
&= \max\{D(X^*, v_r)\colon v_r \in V\} \\
&= f(X^*)
\end{aligned}
$$

which proves that X' is an absolute p-center solution with $X' \subset P$.

With P supplying an optimal solution to the absolute p-center problem, we may now transform it to a sequence of set covering problems. Given a zero-one matrix $\mathbf{A}$ and a cost vector $\mathbf{c}$, the binary program

$$\text{Min } \mathbf{cy} \tag{5.18}$$

$$\text{s.t. } \mathbf{Ay} \geq \mathbf{1} \tag{5.19}$$

$$\mathbf{y} \in \{0, 1\}^n \tag{5.20}$$

is known to be the *set covering problem*. This problem arises when a given set needs to be covered by the union of a collection of its subsets at minimum cost. Let S be a given set with h elements and let $S_1, \ldots, S_k$ be a collection of nonempty subsets of S. Suppose given costs c_i, $i \in K \equiv \{1, \ldots, k\}$, where c_i is the cost of using subset S_i. If we choose a subset K' of K, the corresponding subcollection $\{S_i\colon i \in K'\}$ is said to *cover* S if $\cup\{S_i\colon i \in K'\} = S$. The object is to choose a subset K^* of K, such that the corresponding subcollection $\bigcup_{i \in K^*} S_i$ covers S, and its cost, $\sum_{i \in K^*} c_i$, is as small as possible among all subcollections that cover S. To convert the problem to the binary program defined by (5.18)–(5.20), define the h by k matrix $\mathbf{A}$ with elements $a_{ij}=1$, if the i-th element of S is an element of the subset S_j, and $a_{ij}=0$ if not. Let y_j be a binary variable with $y_j=1$ if subset S_j is selected and $y_j=0$ if not. To cover all elements of S, we impose the constraint

$$\sum_{j=1}^{k} a_{ij} y_j \geq 1 \qquad \forall\, i = 1, ..., h \tag{5.21}$$

which requires at least one y_j for which $a_{ij}=1$ is set equal to 1. This ensures that at least one subset S_j, which contains the i-th element of S is selected by the i-th constraint. The summation on the left side of (5.21) is the dot product of the i-th row of $\mathbf{A}$ with the column vector $\mathbf{y}$ and, accordingly, (5.19) is nothing but a more compact form of the h constraints in (5.21).

In the above formulation, the h rows of $\mathbf{A}$ correspond to the h elements of S. These are the elements that need to be covered. The columns of $\mathbf{A}$ correspond to the k given subsets of S. To make the connection of the set covering problem to the p-center problem, we take S to be V. That is, the elements that need to be covered are the demand vertices $v_1,\ldots, v_n$. The subsets S_j of S are determined on the basis of the finite dominating set P that we identified. Let $p_1,\ldots, p_k$ be an enumeration of the elements of P and let $r>0$ be a selected radius of coverage. Define S_j, $j=1,\ldots, |P|$, to be the set of vertices $v_i \in V$ for which $d(p_j, v_i)\leq r$. Accordingly, the matrix $\mathbf{A}$ in our case has n rows and $k\equiv|P|$ columns and the subsets S_j are defined by the set of demand vertices that are accessible by a facility at p_j within a distance of at most r units. We define the costs $c_j=1\ \forall j\in \{1,\ldots, k\}$ and define $a_{ij}=1$ if $d(v_i, p_j)\leq r$ and $a_{ij}=0$ if $d(v_i, p_j)>r$.

The resulting set covering problem with $\mathbf{A}=[a_{ij}]$, $\mathbf{c}=(1,\ldots, 1)$, and $\mathbf{y}=(y_1,\ldots, y_k)^T$ selects the fewest possible points from P such that every demand vertex has at least one selected point within a distance of at most r units. If the resulting number of points from the set covering solution for a given value of r is at most p while it is strictly greater than p relative to a new radius $r'<r$, then r is, in fact, the p-radius r_p and any optimal solution to the set covering problem relative to this r identifies an absolute p-center solution (by appending as many arbitrarily selected points from P as needed if the set covering solution outputs less than p points). One major question that remains unanswered is how to pick the correct value for r (i.e., the value of r that results in a set covering solution of at most p while any reduction in r results in a set covering solution of more than p points). Minieka has given a well conceived method for accomplishing this. His method relies on modifying $\mathbf{A}$ appropriately and is described in the next paragraph.

Consider a set $X=\{x_1,\ldots, x_p\}$ of p points from P. Put $r=f(X)$ and construct the matrix $\mathbf{A}$ with respect to this choice of r. The resulting set covering problem has a feasible solution $\mathbf{y}=(y_1,\ldots, y_k)$ with $y_j=1$ if $p_j\in X$ and $y_j=0$ if $p_j \notin X$. The objective value defined by $\sum_j y_j$ is equal to p. Suppose now we modify the matrix $\mathbf{A}$ by redefining a_{ij} to be equal to 1 if $d(v_i, p_j)<r$ and $a_{ij}=0$, otherwise. The new version of

A is identical to the old version except that all entries a_{ij} that were one before due to $d(v_i, p_j)$ being equal to r are now replaced by zeroes while $a_{ij}=1$ are retained for all index pairs ij for which $d(v_i, p_j)<r$. Let $\mathbf{A}'$ be the modified version of **A**. Clearly, the new matrix $\mathbf{A}'$ is defined relative to a new radius $r'<r$, but the value of r' is not specified. Even though Minieka does not discuss this issue, some reflection on it reveals that r' is any real number such that $\alpha \leq r' < r$ where α is the largest entry in the list of distances $\{d(p_j, v_i): p_j \in P, v_i \in V\}$ that is smaller than r. Solve the set covering problem with matrix $\mathbf{A}'$. Let $\mathbf{y}'$ be an optimal solution and p' be the optimal objective value. If $p'>p$, then clearly X is an absolute p-center since more than p points from P are required to cover each demand vertex within a distance of less than r. This is equivalent to saying that there does not exist a point set X' in P such that $|X'| \leq p$ and $f(X')<r=f(X)$.

The same conclusion is also valid if there is no feasible solution to the set covering problem with matrix $\mathbf{A}'$. In the remaining case, there is an optimal solution $\mathbf{y}'$ to the set covering problem of matrix $\mathbf{A}'$ with optimal objective value of $p' \leq p$. In this case, X is not optimal because $\mathbf{y}'$ induces a solution $X' \subset P$ with $|X'|=p' \leq p$ and $f(X')<r=f(X)$. When this happens, we repeat the process once again with X', $\mathbf{A}'$, and $r' \equiv f(X')$, replacing the roles of X, **A**, and $r=f(X)$, respectively. That is, we modify $\mathbf{A}'$ to obtain a new matrix $\mathbf{A}''$, such that the elements a_{ij} are set equal to 1 if $d(v_i, p_j)<r'$, and 0 otherwise. The set covering problem is re-solved with the new matrix $\mathbf{A}''$ to obtain an optimal solution $\mathbf{y}''$, if it exists, with optimal objective value p''. The optimality of X' is concluded if the set covering problem admits no feasible solution or if it has an optimal solution $\mathbf{y}''$ with optimal value $p''>p$. In the remaining case, $\mathbf{y}''$ induces a new solution $X'' \subset P$ with $|X''|=p'' \leq p$, and the procedure must be repeated. The process must eventually terminate with an optimal p-center solution when either an infeasible set covering problem is encountered or a feasible set covering problem, whose optimal objective value is strictly greater than p, is encountered. The number of repetitions that can occur until termination is at most $n|P|$, since the set of ones in each modified version of **A** is a proper subset of the immediately preceding version of **A**.

5.3 The Impact of the Classical Contributions

Among the three classical papers discussed in the previous section, Hakimi's (1964) contribution is viewed by many, including this author, as a seminal work that has led to the birth and growth of the research area known today as network location.

Hakimi was the first researcher to pose and analyze the absolute center and median problems in the context of a transportation/communication network, where each edge is a continuum of points. Travel occurs in a network along paths composed of sequences of edges, which is intrinsically different from travel paths available in analogous planar location problems. This feature leads to distances on a network defined by shortest path lengths. Hakimi's first fundamental contribution is his concise analysis of the shortest path distance from a fixed point in the network to a variable point in an edge. The fact that this distance is the minimum of two linear functions results in a concave one or two-piece linear function in a network context, while normed distances in analogous planar location problems are convex. Convexity is a desirable property that leads to strong theory and efficient algorithms in many optimization problems, but it fails in the context of network location unless the network is a tree, as pointed out by Dearing et al. (1976). The theory and algorithms in network location, with certain exceptions of tree location problems, had to be developed with new viewpoints not readily available in analogous planar problems and Hakimi's concave two-piece linear characterization of the edge-restricted distance has provided a foundation for subsequent work.

An immediate consequence of concave piecewise linearity is that multiplication by a positive weight preserves this property. The sum of convex functions is also convex which leads to the well known vertex-optimality theorem for median problems by Hakimi (1964). For the absolute center problem, however, the objective function is defined by the maximum of concave piecewise linear functions and this does not preserve concavity as in the case of the median problem. Even though concavity is lost, piecewise linearity is still retained. This leads to a large, but finite, number of candidate points for local optima on any edge, defined by intersections of pairs of linear pieces with oppositely signed slopes (i.e., directional derivatives). The restriction of local optima to finitely many breakpoints is a fundamental result, initially conceived and used by Hakimi (1964), and exploited later by Minieka (1970) for solving the multi-facility unweighted problem through the solution of finitely many set covering problems. Extensions are given later by Kariv and Hakimi (1979) for the weighted case and by Hooker et al. (1991) for convex nonlinear cost functions.

All subsequent work on 1-centers and p-centers have used this result in one way or another. Most of the focus for solving the 1-center problem has been on developing more efficient computational methods that eliminate unnecessary breakpoints or edges during the search for local optima; some pertinent results can be found in Kariv and Hakimi (1979), Handler (1974), Odoni (1974), Halpern (1979), and Sforza (1990). Algorithms for solving p-center problems are in one of two categories: set covering based or enumeration base. The set covering approach of Minieka (1970) has initiated a series of contributions on the same or related themes by other researchers including Christofides and Viola (1971), Garfinkel et al. (1977), Toregas et al. (1971), and Elloumi et al. (2004). Enumeration based methods enumerate in different ways p-element subsets of the set P; see, e.g., Kariv and Hakimi (1979), Moreno (1986), Tamir (1988), and Hooker (1989).

Goldman's paper, discussed in Sect. 5.2.2, focuses on exploiting the structure of the network under consideration. The particular topological element Goldman (1972) has focused on is the type of edge whose removal from the network, except its end-points, results in two disconnected components. Such an edge is referred to as an isthmus by Goldman. An isthmus has a very special feature: Every path originating in one of the resulting components and terminating in the other component must pass through the isthmus. This has an important consequence for the unweighted case. The longest of the shortest paths connecting a pair of vertices, one in each component, passes through the isthmus under consideration, and its mid-point is either in the isthmus, in which case it is optimum, or in one of the components, in which case the search can be reduced to that component.

The most visible impact of Goldman's paper is that it has drawn attention to special structures in solving location problems on networks, primarily trees. Every edge in a tree is an isthmus. Goldman's algorithm for unweighted trees requires a quadratic number of arithmetic operations in the number of vertices. Handler (1973) and Halfin (1974) developed more efficient linear time algorithms for the unweighted case. The weighted case for tree networks is analyzed and efficiently solved by Dearing and Francis (1974), Hakimi et al. (1978), Hedetniemi et al. (1981), Kariv and Hakimi (1979), and Megiddo (1983). Dearing (1977) and Francis (1977) have extended the problem to incorporate nonlinear monotonic functions of distances and have described efficient solutions methods for tree networks. Goldman's paper has also directed attention to more general structures than trees, but not much can be done unless the cyclic portions of a network (blocks) induce a tree structure when each such component is represented by a single node; see, e.g., the work by Chen et al. (1988), and Kincaid and Lowe (1990). Special structure in multi-facility minimax problems have also led to many elegant results and efficient algorithms for tree networks. Some of the contributions are those by Handler (1978), Hakimi et al. (1978), Kariv and Hakimi (1979), Tansel et al. (1982), Megiddo and Tamir (1983), Frederickson and Johnson (1983), Megiddo et al. (1983), Jaeger and Kariv (1985), and Shaw (1999). As Dearing et al. (1976) point out, convexity of distance is an important property for tree networks and has a significant part in developing theory and efficient algorithms for the single facility case. Convexity does not explain, however, why absolute p-center location problems are so efficiently solvable on tree networks, since the p-center objective function is not convex even on a tree network.

5.4 Subsequent Work in Discrete Center Location

In this section, we survey the subsequent work in discrete center location. We first focus on the single facility case on general networks, followed by problems on tree networks, and finally we consider other specially structured networks. Then the multi-facility problem is covered, again, first on general networks, and then on trees.

5.4.1 The Absolute 1-Center on General Networks

Hakimi's (1964) method requires solving an edge-restricted problem on each edge by inspecting break points that are oppositely signed in either direction. There are at most $n(n-1)/2$ breakpoints per edge which requires evaluating the objective function at $O(n^2|E|)$ points. This makes Hakimi's method an $O(n^3m)$ algorithm where $m \equiv |E|$. Later, Hakimi et al. (1978) presented an $O(mn^2 \log n)$ version of the same algorithm. This bound is improved to $O(mn \log n)$ for the unweighted case. Kariv and Hakimi (1979) solved the weighted case in $O(mn \log n)$ and the unweighted case in $O(mn)$ time. This is the best known bound for the absolute 1-center problem. The $O(mn \log n)$ bound for the unweighted case is also achieved by Sforza (1990), whose algorithm for the weighted case is $O(kmn \log n)$, where k is a factor that depends on the precision level and weight distribution. This bound does not improve the bound of Kariv and Hakimi (1979), but Sforza's algorithm is more effective in CPU time.

Edge elimination techniques rely on devising lower bounds for each edge and eliminating those edges whose lower bounds are larger than the best objective value attained during the search for optimum. Handler (1974), Odoni (1974), Christofides (1975), and Halpern (1979) made use of edge elimination techniques that have resulted in improved CPU times, where Halpern's bound is stronger than the others. Sforza's (1990) edge elimination technique has been found to be quite successful in practice due to its ability to eliminate 80% of edges in many problems.

All the algorithms mentioned above are improved versions of Hakimi's original technique. Minieka (1981)'s $O(n^3)$ algorithm, on the other hand, only makes use of the distance matrix without using the vertex-to-point cost functions.

An important theoretical contribution is due to Hooker (1986) who analyzed the nonlinear version of the 1-center problem for the problem with convex cost functions and proposed a general purpose algorithm. His analysis is based on decomposing the network into tree-like segments and solving a convex programming problem on each segment. The objective function defined by maximum of convex functions of distances is convex on any tree-like segment, and a local minimum can be found by solving a convex programming problem. Hooker (1986) proved that there are $O(n)$ tree-like segments on an edge.

Shier and Dearing (1983) made another important theoretical contribution in their study of a family of nonlinear single facility location problems on a network that includes, as special cases, the absolute 1-center and absolute 1-median problems. They characterize locally optimal solutions by means of directional derivatives. This characterization is equivalent, in the case of the absolute 1-center problem, for the point under consideration to be a breakpoint of $f(\bullet)$ such that f increases in every "moveable" direction at that point. If the point under consideration is an interior point of some edge, then there are only two directions of movement out of that point. Hence, an interior point is a local optimum if and only if it is a break point of f defined by the intersection of two weighted distance functions associated with a pair of distinct vertices, such that the increase in one of the functions is accompanied by a decrease in the other one if one moves slightly away from the point in either direction.

Continuous demand versions of the absolute 1-center problem are also considered. There are two versions. Minieka (1970) defines the *general absolute center* of a network G as a point whose maximum distance to a farthest point on each edge is minimized. In contrast, Frank (1967) defines a *continuous center* of a network as a point whose maximum distance to any point on the network is minimized. The two definitions are equivalent for the case of 1-centers. Minieka (1977) showed that Hakimi's algorithm for the absolute 1-center can be used to find the general absolute 1-center if one replaces the distance function $d(x, y)$ with a new distance function $d'(x, e)$ which denotes the distance between x and a farthest point in edge e. Frank (1967) defined the continuous 1-center problem and showed that it can be solved via Hakimi's algorithm.

5.4.2 The Absolute 1-Center on Trees and Other Special Structured Networks

Beginning with Goldman's localization theorem, considerable attention has been given to tree networks. Other special structures have also received some attention.

An important property that has led to efficient algorithms for trees has to do, at least in good part, with the convexity of distance on tree networks. Dearing et al. (1976) generalized in a theoretical framework the earlier convexity observations of Goldman and Witzgall (1970) and Handler (1973), as well as nonconvexity observations of Goldman (1971) and Hakimi (1964). Dearing et al. (1976) prove that the function $d(x, y)$ as a function of x alone, or as a function of x and y, is convex if and only if the network is a tree network. This implies that the objective function in the absolute p-center problem is convex on a tree and nonconvex on a cyclic network. Convexity implies that any local minimum on a tree network is also a global minimum.

Goldman's (1972) localization theorem, when applied to a tree, finds an optimum in $O(n^2)$ time. Handler (1973) proves for the unweighted case that the absolute center of a tree is the midpoint of a longest path in the tree and gave an $O(n)$ algorithm. Halfin (1974) modifies Goldman's algorithm and turns it into an $O(n)$ algorithm for trees with unit weights and any addends. Lin (1975) shows that the unweighted problem on a network with addends is equivalent to the unweighted problem on a new network with no addends, where the new network has the same structure as the old one except that for every vertex v_i for which the addend $a_i > 0$, a new vertex v_i' and a new edge $[v_i, v_i']$ is added with length a_i. Hence, addends do not increase the time bounds of proposed algorithms.

Dearing and Francis (1974) analyze the weighted problem on trees and prove that the maximum of the $n(n+1)/2$ numbers $\alpha_{ij} \equiv (d_{ij} + a_i/w_i + a_j/w_j)/(1/w_i + 1/w_j)$, $1 \le i \le j \le n$ is a lower bound for the optimum value of the objective function for any network, and is an attainable lower bound for tree networks. The absolute 1-center of a tree occurs at the point x on the path $P(v_s, v_t\}$, identified by $\alpha_{st} = \max\{\alpha_{ij}: 1 \le i \le j \le n\}$, such that $w_s d(v_s, x) + a_s = w_t d(v_t, x) + a_t$. The computation of α_{st} takes $O(n^2)$ time. Hakimi et al. (1978) propose an $O(n(r+1))$ algorithm for this problem where $r \le n$. Kariv and Hakimi (1979) describe an algorithm that reduces the tree to subtrees until a single edge remains. The local center on the last edge solves the weighted absolute 1-center problem while one of its end-vertices solves the vertex restricted 1-center problem. The time bound is $O(n \log n)$ for weighted trees. Megiddo (1983) solves the weighted absolute 1-center problem on tree networks in $O(n)$ time, which is the best known time bound for this problem. The nonlinear version of the absolute 1-center problem on tree networks, where each weighted distance is replaced by a monotone increasing loss function is considered by Dearing (1977) and Francis (1977). They prove that the maximum of the $n(n+1)/2$ numbers $\beta_{ij} \equiv (f_i^{-1} + f_j^{-1})^{-1}[d_{ij}]$, $1 \le i \le j \le n$, is a lower bound for the minimum objective value of any network, and that this bound is attainable for tree networks. A maximizing pair v_s and v_t identify a path $P(v_s, v_t)$, such that the optimum point is the point x on the path where $f_s[d(v_s, x)] = f_t[d(x, v_t)]$. If $s = t$, this implies v_s is the optimum point.

For special structured networks that are more general than trees, Chen et al. (1988) propose an algorithm, similar in spirit to Goldman's, for linear and nonlinear cost functions. They construct the block diagram of the network in which each block (a maximally connected subgraph that cannot be made disconnected by removing a vertex with its adjacent edges) is represented by a vertex. A block diagram is always a tree. The algorithm either finds an optimum or reduces the problem to a single block. The time bound of the algorithm is $O(n \min\{b, \alpha \log b\})$ for the linear case, where α is the maximum number of cut points in any block and b is the number of blocks. If the algorithm ends with a block, the algorithms of Kariv and Hakimi (1979) or Sforza (1990) may be used to locate the absolute 1-center in the block for the case of linear costs. For nonlinear monotonically increasing convex cost functions, Hooker's (1986 algorithm based on tree-like segments may be used. The time bound of Chen et al.'s (1988) reduction algorithm is $O(n \log n)$ for cactus networks. A more efficient $O(n)$ algorithm is devised for cactus networks that are homemorphic to a 3-cactus by Kincaid and Lowe (1990) that transforms these special networks to trees in which point-to-point distances are preserved.

5.4.3 Absolute *p*-Centers of General Networks

Kariv and Hakimi (1979) prove that the absolute p-center problem on a network is **NP**-Hard even if the network is planar, unweighted, with unit edge lengths and a maximum vertex degree of 3. Solution approaches are based on the existence of a finite dominating set, initially motivated by Hakimi's (1964) method, formalized by Minieka (1970) for the unweighted case, and extended directly to the weighted case by Kariv and Hakimi (1979). This result is further generalized by Hooker et al. (1991) and a unifying approach is given for identifying finite dominating sets in a rather general setting.

All solution approaches proposed for the absolute p-center problem on general networks are based on the existence of a finite dominating set $P = V \cup U$, where the definition of U is revised for the weighted case to include every interior point x such that $w_i d(v_i, x)=w_j d(x, v_j)$ for a pair of vertices v_i and v_j, $i \neq j$, and moving in either direction increases one of the functions while decreasing the other one. Solution approaches are either based on solving a sequence of set covering problems, suggested first by Minieka (1970), or enumerating p element subsets of the set P.

Minieka's (1970) algorithm solves the unweighted version by solving a sequence of set covering problems with successively decreasing values of the covering radius r. Garfinkel et al. (1977) also solve a sequence of set covering problems, but they first reduced the search space by finding a heuristic solution X and eliminating from P all those points whose relative radius is greater than $r \equiv f(X)$. This reduces the number of variables in the set covering problem. Christofides and Viola (1971) solve the weighted problem by first generating the set of all feasible regions in the network reachable by at least one vertex within a distance of r, where r is a fixed parameter, and solving a set covering problem that selects the smallest number of points from these regions. This approach is essentially the same as that of Minieka (1970), except that Christofides and Viola do not make use of the finite dominating set P. Toregas et al. (1971) solve the vertex restricted p-center problem by solving the linear programming relaxation of the associated set covering problem, and adding a cut whenever termination occurred with a fractional solution. Elloumi et al. (2004) devise a new integer programming formulation of the problem based on the set covering idea. The linear programming relaxation of their formulation generates better lower bounds for the problem than those of previous models.

Kariv and Hakimi (1979) propose an $O(m^p n^{2p-1} \log n)$ enumeration algorithm for weighted networks. Their algorithm uses the fact that each center in an optimal solution is the 1-center of a subnetwork. They choose $p-1$ arbitrary centers and solve for the p-th one using a 1-center approach. Moreno (1986) provides an algorithm of time bound $O(m^p n^{p+1} \log n)$. Tamir (1988) combines the algorithms of Kariv and Hakimi (1979) and Moreno (1986) to obtain an algorithm with improved time bounds of $O(m^p n^p \log^2 n)$ for the weighted case and $O(m^p n^p \log n)$ for the unweighted case. Further improvements are made by using dynamic data structures resulting in a time bound of $O(m^p n^{p-1} \log^3 n)$. For the case of nonlinear convex cost functions, Hooker (1989) proposes an enumeration algorithm based on tree-like segments, which is practical for small values of p. The algorithm locates p centers for each combination of p tree-like segments by solving linear and convex problems on each segment. The algorithm becomes intractable when p exceeds 4.

5.4.4 Absolute p-Centers of Tree Networks

The minimum distance $D(X, v_i)$ from a vertex to a collection of p points is not convex even though the distance $d(x, v_i)$ is convex for each $x \in X$. Despite the loss of convexity, the absolute p-center problem in tree networks is solved in polynomial time by various algorithms.

Handler (1978) solves the absolute 2-center and the continuous absolute 2-center problems in $O(n)$ time by solving three 1-center problems. His algorithm does not seem to be extendible to larger values of p. Hakimi et al. (1978) describe an $O(n^{p-1})$ algorithm for unweighted tree networks. Kariv and Hakimi (1979) propose an $O(n^2 \log n)$ algorithm for absolute and vertex restricted weighted and unweighted problems on trees. For the unweighted case, they also develop an $O(n \log^{p-2} n)$ algorithm for the absolute p-center problem and $O(n \log^{p-2} n)$ algorithm for the vertex restricted case. Megiddo and Tamir (1983) propose an $O(n \log^2 n \log\log n)$ algorithm for the weighted absolute p-center problem on trees. An $O(n \log n)$ algorithm is presented by Frederickson and Johnson (1983) for the unweighted case. Megiddo et al. (1981) solve the vertex restricted problem in $O(n \log^2 n)$ time. Jaeger and Kariv (1985) devise an $O(pn \log n)$ algorithm for vertex restricted and absolute p-centers on weighted trees for relatively small values of p. If $p < \log n$ for the vertex restricted p-center and $p < \log n \log\log n$ for the absolute p-center, then this time bound is better than the previous ones. Shaw (1999) presents a unified column generation approach for a general class of facility location problems on trees that includes the absolute p-center problem as a special case. The complexity of his algorithm for the weighted p-center problem on trees is $O(n^2 \log n)$. A nonlinear version of the problem, in which each weighted distance is replaced by a monotonic increasing function of the distance, is considered by Tansel et al. (1982) and solved in $O(n^4 \log n)$ time, which is improved to $O(n^3 \log n)$ by the modification given in Chap. 8 of Mirchandani and Francis (1990). For various duality results, see also Shier (1977).

5.5 Future Directions

Tree network absolute p-center problems are well solved in polynomial time both for linear and nonlinear monotonic cost functions. Bozkaya and Tansel (1998) show that there exists a spanning tree of every connected network, such that solving the absolute p-center problem on the tree solves the p-center problem on the network. Trying to find such a tree is a worthwhile undertaking, since solving the problem on it also solves the problem on the original network.

Nonlinear versions with monotonic increasing functions of distances are more realistic versions of p-center problems that may find applications in a wide variety of contexts. Multi-facility versions of such models on general networks have not been considered in the literature and demand attention.

There is an acute need for more realistic models of emergency or covering type of location problems that address major issues of terrorism, pollution, disaster fighting, and fast depletion of natural resources (such as water). Present models seem to be quite short of capturing important aspects of such problems.

Most often, we assume that data for our problems are available in a nice and clean form whereas, large scale realistic applications often require massive amounts of data that are difficult to obtain and process. Methods need to be developed for constructing and maintaining accurate data bases for large-scale applications.

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