NON-OSCILLATING PALEY–WIENER FUNCTIONS

By

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Abstract. A non-oscillating Paley–Wiener function is a real entire function f of exponential type belonging to $L_2(\mathbf{R})$ and such that each derivative $f^{(n)}$, $n = 0, 1, 2, \ldots$, has only a finite number of real zeros. It is established that the class of such functions is non-empty and contains functions of arbitrarily fast decay on $\mathbf{R}$ allowed by the convergence of the logarithmic integral. It is shown that the Fourier transform of a non-oscillating Paley–Wiener function must be infinitely differentiable outside the origin. We also give close to best possible asymptotic (as $n \to \infty$) estimates of the number of real zeros of the n-th derivative of a function f of the class and the size of the smallest interval containing these zeros.

1 Introduction

A Paley–Wiener function (PW-function) is an entire function f of exponential type such that $f \in L_2(\mathbf{R})$. By the Paley–Wiener theorem, the class of all PW-functions coincides with the class of functions f admitting the representation

$$f(z) = \int_{-\infty}^{\infty} e^{-izs} F(s)ds,$$

where F (called the spectral function of f) is an L_2-function with bounded support.

We say that a PW-function f is real if $f(\mathbf{R}) \subset \mathbf{R}$. Oscillatory properties of real PW-functions have been the subject of investigation of a number of works (see, for example, [1], [3] and [8]). J. R. Higgins [3] (p. 77) has constructed a sequence of real PW-functions s_n with the following property. Each s_n has a derivative of order $2n$ with infinitely many real zeros, but each derivative of order less than $2n$ has only a finite number of real zeros. W. J. Walker [8] (p. 1254) and J. R. Higgins [3] (p. 72) posed the question: Does a real PW-function f have for some order n a derivative $f^{(n)}$ which has infinitely many real zeros? For some classes of real PW-functions, a positive answer was obtained by W. J. Walker [8] and J. Clunie et al. [1]. In the latter work, it is mentioned (p. 167) that the answer to this question is still unknown.

In this paper, we answer this question in the negative. Let us call a PW-function f *non-oscillating* if it is real and each derivative $f^{(n)}$ has only a finite number of real zeros. Simple examples of such functions are given in the following result.

Theorem 1. (i) *Suppose a real function $p \in C^{\infty}(\mathbf{R})$ has bounded support and satisfies $p(0) \neq 0$. Then the function*

$$f_1(x) := \int_0^{\infty} p(s) \sin xs \, ds$$

is a non-oscillating PW-function.

(ii) *Suppose $p \in C^{\infty}(\mathbf{R})$ has a bounded support, $p(-s) = \bar{p}(s)$, for all real s, and $\int_{-\infty}^{\infty} p(s)\, ds \neq 0$. Then the function*

$$f_2(x) := \frac{1}{x} \int_{-\infty}^{\infty} (1 - e^{ixs}) p(s) ds$$

is a non-oscillating PW-function.

Proof. Since p has compact support, the function f_1 is a real PW-function. Choose any integer $n \geq 1$. The statement (i) can be proved simply by integrating by parts $2n$ times:

$$f_1^{(2n-1)}(x) = (-1)^{n-1} \int_0^{\infty} s^{2n-1} p(s) \cos xs \, ds$$

$$= \left[\frac{\cos xs}{x^{2n}} \left(s^{2n-1} p(s)\right)^{(2n-1)} \right]_0^{\infty} - \frac{1}{x^{2n}} \int_0^{\infty} \left(s^{2n-1} p(s)\right)^{(2n)} \cos xs \, ds.$$

Since the function $(s^{2n-1}p(s))^{(2n)}$ is continuous and has bounded support, the last integral tends to zero as $|x| \to \infty$. We conclude that

$$|f_1^{(2n-1)}(x)| = (2n-1)! \frac{|p(0)|}{x^{2n}} (1 + o(1)), \quad |x| \to \infty, \quad n = 1, 2, \ldots.$$

This shows that for any $n \geq 1$ the function $f_1^{(2n-1)}$ can have only a finite number of real zeros. Hence, by Rolle's theorem, any derivative $f_1^{(n)}, n = 1, 2, \ldots$ has only a finite number of real zeros.

(ii) It is clear that f_2 is a real PW-function. Observe that

$$f_2^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \int_{-\infty}^{\infty} p(s) ds - \int_{-\infty}^{\infty} p(s) \left(\frac{e^{ixs}}{x}\right)^{(n)} ds.$$

Since the the function p is infinitely differentiable on the real line, for any natural number k, the second integral in the right hand side is $o(|x|^{-k})$ as $x \to \infty$. This shows that

$$|f_2^{(n)}(x)| = \frac{n!}{|x|^{n+1}} \left| \int_{-\infty}^{\infty} p(s)ds \right| (1 + o(1)), \quad |x| \to \infty, \quad n = 1, 2, \dots,$$

so that every derivative $f_2^{(n)}$ can have only a finite number of real zeros.

The functions f_1 and f_2 in Theorem 1 tend to zero like $|x|^{-1}$, and their n-th derivatives tend to zero like $|x|^{-n-1}$ as $|x| \to \infty, x \in \mathbf{R}$. Since the n-th derivative of a non-oscillating PW-function is also a non-oscillating PW-function, for any $n = 1, 2, \dots$ there exist non-oscillating PW-functions f which satisfy $|f(x)| = O(|x|^{-n})$ as $|x| \to \infty$. Below, we show that non-oscillating PW-functions f can have arbitrarily fast decay provided that the logarithmic integral converges (which is true for every PW-function):

$$\int_{-\infty}^{\infty} \frac{|\log |f(x)||}{1 + x^2} dx < \infty.$$

For instance, Theorem 4 below implies that for any $\rho \in (0, 1)$ and any $\sigma > 0$, there exists a real non-oscillating PW-function f of type σ such that

$$|x|^{\rho} \log \frac{1}{|f(x)|}$$

is bounded from below and above by positive constants for all sufficiently large $|x|, x \in \mathbf{R}$.

There remains an open question related to the *slowest* possible rate of decay of a real non-oscillating PW-function on $\mathbf{R}$. For example, we do not know whether there exist non-oscillating PW-functions decreasing on $\mathbf{R}$ like $|x|^{-\alpha}$ with $1/2 < \alpha < 1$.

Observe that the spectral function of the function f_1 in Theorem 1 is $(i/2)p(|s|)\,\operatorname{sign} s,\ s \in \mathbf{R}$. This function is infinitely differentiable outside the origin. One can verify that this is also true for the spectral function of f_2. Our next result shows that the spectral function of any non-oscillating PW-function has this property.

Theorem 2. *Let f be a non-oscillating PW-function. Then its spectral function belongs to $C^{\infty}(\mathbf{R} \setminus \{0\})$.*

Let us denote by $r(n, f)$ the maximum of the moduli of real zeros of the n-th derivative of a non-oscillating PW-function f and by $p(n, f)$ the number of its real zeros. Rolle's theorem implies that both $r(n, f)$ and $p(n, f)$ are strictly increasing functions of n and, moreover, $p(n, f) \geq n$. It seems natural to ask how fast $r(n, f)$ and $p(n, f)$ may grow as $n \to \infty$.

Theorem 3. *Let f be a non-oscillating PW-function. Then*

$$\log r(n, f) \geq (1 + o(1)) \log n, \quad n \to \infty. \tag{1}$$

The inequality in (1) is sharp in the sense that there are functions f for which the inequality can be replaced by equality. We deduce Theorem 3 from the more general Theorem 5 below.

Observe that inequality (1) also remains valid for $p(n, f)$. This follows from the trivial inequality $p(n, f) \geq n$ and Corollary 1 below. The problem of finding more precise bounds and any upper bound for $r(n, f)$ and $p(n, f)$ remains open.

2 Rate of decay and real zeros of a non-oscillating PW-function

We assume familiarity with the notion of a *proximate order* ([2], Ch. 2, §2; [4], Ch. 1, sec. 12). Recall that a proximate order is a continuously differentiable positive function $\rho(r)$ on $[0, \infty)$ satisfying the conditions

(i) $\exists \lim_{r\to\infty} \rho(r) =: \rho \geq 0$;

(ii) $\lim_{r\to\infty} (\rho'(r)/\rho(r)) r \log r = 0$.

Put

$$V(r) := r^{\rho(r)}.$$

The simplest example of a proximate order is $\rho(r) \equiv \rho > 0$. In this case, $V(r) = r^{\rho}$; and the considerations below would be significantly simpler and independent of results of [2] and [4] related to proximate orders.

Condition (ii) implies that the function V is strictly increasing for sufficiently large r. Evidently, it is possible to change $\rho(r)$ on a large interval $[0, R]$ in such a way that V becomes strictly increasing on $[0, \infty)$ and $V(0) = 0$. We shall assume that this change has been made.

The following result relates to the rate of decay on $\mathbf{R}$ of non-oscillating PW-functions.

Theorem 4. *Let $\rho(r)$ be a proximate order such that*

$$\lim_{r\to\infty} \rho(r) = \rho > 0, \quad \textit{and} \quad \int_1^\infty \frac{V(r)}{r^2}\, dr < \infty. \tag{2}$$

There exists a real non-oscillating PW-function f such that

$$C_1 < \frac{1}{V(|x|)} \log \frac{1}{|f(x)|} < C_2, \quad |x| > r_0,\ x \in \mathbf{R}, \tag{3}$$

where r_0, C_1 and C_2 are positive constants.

Let f be a non-oscillating PW-function and $r(n, f)$ the maximum of the moduli of real zeros of $f^{(n)}$. Our next result estimates the rate of the increase of $r(n, f)$.

Theorem 5. (i) *For any proximate order $\rho(r)$ satisfying (2), there exists a non-oscillating PW-function f such that*

$$r(n, f) \leq v(Cn \log n), \quad n = 0, 1, 2, \ldots, \tag{4}$$

where v is the inverse function for V and C is a positive constant.

(ii) *For any non-oscillating PW-function f of type* 1,

$$r(n, f) \geq \frac{\pi}{2e} n(1 + o(1)), \quad n \to \infty. \tag{5}$$

We derive from Theorem 5 (i) the following fact, related to the possible growth of the number $p(n, f)$ of real zeros of the n-th derivative of a non-oscillating PW function f.

Corollary 1. *For any proximate order $\rho(r)$ satisfying (2), there exists a non-oscillating PW-function f such that*

$$p(n, f) \leq v(Cn \log n), \quad n = 0, 1, 2, \ldots, \tag{6}$$

where v is the inverse function for V and C is a positive constant.

Evidently, Theorem 5 (ii) implies inequality (1). Applying Theorem 5 (i) to the proximate order $\rho(r)$ such that $V(r) = r^{\rho(r)} = r/\log^2(1 + r)$, we see that there exists a non-oscillating PW-function f for which $r(n, f) = O(n \log^2 n)$, $n \to \infty$. For this function, the inequality in (1) becomes equality. Similarly, Corollary 1 implies that there exists a non-oscillating PW-function such that $\log p(n, f) = (1 + o(1)) \log n$, $n \to \infty$.

The rest of this paper is organized as follows. Some auxiliary results are proved in the next section. Section 4 is then devoted to the proof of Theorem 4, and Sections 5 and 6 to the proof of Theorem 5 and Corollary 1. An auxiliary result which is used in the proof of Theorem 5 is proved in Section 7. Finally, Theorem 2 is proved in Section 8.

Theorems 4 and 5 were announced in [7] with a short description of the method of proof.

3 Auxiliary lemmas

Lemma 1. *Let $\rho_1(r)$ be a proximate order such that* $\lim_{r \to \infty} \rho_1(r) = \rho_1 \in (0, 1)$ *and let $V_1(r) = r^{\rho_1(r)}$. Define an increasing sequence of positive numbers a_k as* *the solutions of the equations*

$$V_1(r) = k, \quad k = 1, 2, 3, \ldots, \tag{7}$$

and set

$$h_1(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right).$$

Then the circles

$$C_k(\epsilon) = \{z : |z - a_k| \le \epsilon a_k / V_1(a_k)\}, \quad k = 1, 2, 3, \ldots$$

do not intersect each other provided that number ϵ is small enough. Further, the estimate

$$\log|h_1(re^{i\varphi})| = \frac{\pi \cos \rho_1(\varphi - \pi)}{\sin \pi\rho_1} V_1(r) + o(V_1(r)), \quad r \to \infty, \quad 0 \le \varphi \le 2\pi \tag{8}$$

holds outside the union of the cicrles $C_k(\epsilon)$.

Proof. It follows immediately from the definition (7) and the properties of proximate order that the circles $C_k(\epsilon)$ do not intersect for all sufficiently small $\epsilon > 0$. This means that the roots a_k of the function h_1 form a so-called R-set (for the definition see [4], p. 95). The asymptotic formula (8) for $re^{i\varphi} \notin \bigcup_k C_k(\epsilon)$ now follows from a known result on entire functions of completely regular growth ([4], p. 96, Theorem 5). □

Corollary 2. *Set*

$$h_2(z) = h_1(-z^2) = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{a_k}\right).$$

Then $h_2(z)$ is positive on $\mathbf{R}$*, real on* $i\mathbf{R}$ *and*

(i)
$$\log h_2(x) = \frac{\pi}{\sin \pi\rho_1} V_1(x^2) + o(V_1(x^2)), \quad |x| \to \infty;$$

(ii) *on every interval $(\sqrt{a_k}, \sqrt{a_{k+1}})$, there is a point $b_k, k = 1, 2, \ldots,$ such that*

$$\log|h_2(\pm ib_k)| = \pi \cot \pi\rho_1 \cdot V_1(b_k^2) + o(V_1(b_k^2)), \quad b_k \to \infty.$$

Lemma 2. *Let $\rho(r)$, $\lim_{r\to\infty} \rho(r) = \rho \in (0, 1]$ be a proximate order. Then there exists an entire function*

$$g(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{c_k^2}\right), \quad 0 < c_1 < c_2 < \cdots, \quad \sum_k c_k^{-2} < \infty, \tag{9}$$

which is positive on $\mathbf{R}$*, real on* $i\mathbf{R}$ *and such that*

(i)

$$\log g(x) = \frac{\pi}{\sin(\pi\rho/2)} V(|x|) + o(V(|x|)), \quad |x| \to \infty; \tag{10}$$

(ii) *every interval* (c_k, c_{k+1}) *contains a point* $d_k, k = 1, 2, \ldots$, *such that*

$$\log|g(\pm id_k)| = \pi \cot \frac{\pi\rho}{2} \cdot V(d_k) + o(V(d_k)), \quad d_k \to \infty. \tag{11}$$

To establish this lemma, it suffices to apply Corollary 2 with $\rho_1 = \frac{1}{2}\rho$ and set $g(z) = h_2(z)$, $c_k = \sqrt{a_k}$, $d_k = b_k$.

Lemma 3. *Suppose* $\rho(r)$ *is a proximate order satisfying (2). Then there exists an even real PW-function* h *with real zeros such that*

$$\log|h(x)| \le -V(|x|) + O(1), \quad |x| \to \infty, \quad x \in \mathbf{R}. \tag{12}$$

Proof. This lemma is a special case of the Beurling–Malliavin multiplier theorem. We give the proof below for the reader's convenience.

Assume additionally that $\lim_{r\to\infty} \rho(r) = 1$. Let $\{a_k\}$ be a sequence of positive numbers defined by the equations

$$V(a_k) = k, \quad k = 1, 2, \ldots.$$

Set

$$h_1(z) = \prod_{k=1}^{\infty} \frac{\sin(z/a_k)}{z/a_k}.$$

It is easy to verify that (2) yields $\sum_k (1/a_k) < \infty$. Hence, this infinite product converges and represents an entire function of exponential type.

Let $n(r)$ be the number of points a_k to be found in the disc $\{z : |z| \le r\}$. We have, for every real x and integer n, that

$$|h_1(x)| \le \frac{1}{|x|^n} \prod_{k=1}^{n} a_k.$$

Now set $n = n(|x|)$ and observe that

$$|h_1(x)| \le \exp\left\{ -\int_0^{|x|} \frac{n(t)}{t}\, dt \right\}.$$

It is easily seen from (2) that $n(t)$ is asymptotically equivalent to $t^{\rho(t)}$. Hence, by known properties of the proximate order ([4], p. 34), we conclude that

$$\int_0^{|x|} \frac{n(t)}{t}\, dt = V(|x|)(1 + o(1)).$$

This gives

$$\log|h_1(x)| \le -V(|x|)(1+o(1)) \le -\frac{1}{2}V(|x|) + C.$$

Hence the function $h(z) = h_1(z^2)$ satisfies the conclusions of the Lemma.

In the case $\lim_{r\to\infty}\rho(r) = \rho < 1$, one can construct $h_1(z)$ by taking a constant $\rho_1 \in (\rho, 1)$ instead of the function $\rho(r)$. Then one gets

$$\log|h_1(x)| \le -\frac{1}{\rho_1}|x|^{\rho_1}(1+o(1)) \le -V(|x|)(1+o(1)),$$

and again one can set $h(z) = h_1^2(z)$.

Lemma 4. *Suppose $\rho(r)$ is a proximate order, and $q(z)$ is an entire function of completely regular growth with respect to $\rho(r)$. Assume that*

(i) *$q(z)$ does not vanish in $\{z : |\arg z| \le \alpha < \pi\}$,*

(ii) $\exists \lim_{x\to+\infty}(\log|q(x)|)/V(x) \ne 0$.

Then the derivative $q'(z)$ does not vanish in $\{z : |\arg z| \le \alpha/3, |z| > R\}$ for all large R, and the asymptotic equation

$$\left(\frac{d}{dz}\right)^n \frac{1}{q(z)} = (-1)^n \left(\frac{q'(z)}{q(z)}\right)^n \frac{1}{q(z)}\left(1 + O\left(\frac{1}{V(|z|)}\right)\right), \tag{13}$$

$$z \to \infty, \quad |\arg z| \le \alpha/3,$$

holds for all $n = 1, 2, \ldots$

Proof. We require the following fact (see [2], p. 99, Theorem 5.1). Let $\rho(r)$, $\lim_{r\to\infty}\rho(r) = \rho$ be a proximate order. Then there exists a function $L(z)$ analytic in $\{z : |\arg z| < \pi\}$ such that

$$V(r)(= r^{\rho(r)}) = r^\rho L(r)(1 + o(1)), \quad r \to \infty, \tag{14}$$

$$L(re^{i\varphi}) = L(r)(1 + o(1)), \quad r \to \infty, \tag{15}$$

where (15) holds uniformly in $\varphi \in (-\pi + \delta, \pi - \delta)$ for every $\delta \in (0, \pi)$.

Let us verify that (15) gives

$$L'(re^{i\varphi}) = o\left(\frac{L(r)}{r}\right), \quad r \to \infty, \tag{16}$$

uniformly with respect to $\varphi \in [-\pi/2, \pi/2]$. Indeed, since ([2], p. 73, [4], p. 32)

$$\lim_{r\to\infty} \frac{L(kr)}{L(r)} = 1$$

for every $k > 0$, then, in view of (15),

$$L(\zeta) = L(r) + o(L(r)), \quad r \to \infty,$$

uniformly with respect to ζ in the sector $\{\zeta : |\arg\zeta| < 2\pi/3, r/2 < |\zeta| < 3r/2\}$. It now follows from the Cauchy integral formula that

$$L'(re^{i\varphi}) = \frac{1}{2\pi i}\int_{|\zeta - re^{i\varphi}|=r/2} \frac{L(\zeta)}{(\zeta - re^{i\varphi})^2}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta - re^{i\varphi}|=r/2} \frac{L(\zeta) - L(r)}{(\zeta - re^{i\varphi})^2}\, d\zeta = o\left(\frac{L(r)}{r}\right), \quad r \to \infty,$$

which establishes (16).

By known results in the theory of functions of completely regular growth ([4], pp. 94–95, Theorem 4), we have

$$\log q(z) = Az^\rho L(r) + o(V(r)), \quad r \to \infty,\ z = re^{i\varphi},\ |\varphi| \le \alpha/2,$$

where A is the limit in assumption (ii) of the lemma. This and (15) give

$$\log q(z) = Az^\rho L(z) + o(V(r)), \quad r \to \infty,\ z = re^{i\varphi},\ |\varphi| \le \alpha/2.$$

Let us now differentiate the formula. Using (16) and estimating the derivative of the remainder term with the help of the Cauchy integral, we get

$$\text{(17)} \qquad \frac{q'(z)}{q(z)} = \rho A z^{\rho-1} L(z) + o\left(\frac{V(r)}{r}\right), \quad r \to \infty,\ z = re^{i\varphi},\ |\varphi| \le \alpha/3,$$

$$\left(\frac{q'(z)}{q(z)}\right)' = \rho(\rho-1)Az^{\rho-2}L(z) + o\left(\frac{V(r)}{r^2}\right), \quad r \to \infty,\ z = re^{i\varphi},\ |\varphi| \le \alpha/3,$$

$$\text{(18)} \qquad \left(\frac{q'(z)}{q(z)}\right)' \left(\frac{q(z)}{q'(z)}\right)^2 = O\left(\frac{1}{V(r)}\right), \quad r \to \infty,\ z = re^{i\varphi},\ |\varphi| \le \alpha/3.$$

It is clear from (14), (15) and (17) that the derivative $q'(z)$ does not vanish in $\{z : |\arg z| \le \alpha/3, |z| > R\}$ for all large R.

To verify (13), we use induction. When $n = 1$, (13) is evident. Assume that (13) holds for some n. Then we have

$$\left(\frac{d}{dz}\right)^{n+1} \frac{1}{q(z)} = \frac{d}{dz}\left[(-1)^n \left(\frac{q'(z)}{q(z)}\right)^n \frac{1}{q(z)} \left(1 + O\left(\frac{1}{V(r)}\right)\right)\right]$$

$$= (-1)^n \left[n \left(\frac{q'(z)}{q(z)} \right)^{n-1} \left(\frac{q'(z)}{q(z)} \right)' \frac{1}{q(z)} - \left(\frac{q'(z)}{q(z)} \right)^{n+1} \frac{1}{q(z)} \right] \left(1 + O\left(\frac{1}{V(r)} \right) \right)$$

$$+(-1)^n \left(\frac{q'(z)}{q(z)} \right)^n \frac{1}{q(z)} \cdot O\left(\frac{1}{rV(r)} \right) = (-1)^{n+1} \left(\frac{q'(z)}{q(z)} \right)^{n+1} \frac{1}{q(z)}$$

$$\cdot \left\{ \left[1 - n \left(\frac{q'(z)}{q(z)} \right)' \left(\frac{q(z)}{q'(z)} \right)^2 \right] \left(1 + O\left(\frac{1}{V(r)} \right) \right) - \left(\frac{q'(z)}{q(z)} \right)^{-1} O\left(\frac{1}{rV(r)} \right) \right\}.$$

This, together with (18) and (17), establishes (13).

4 Proof of Theorem 4

Suppose that g is a function whose existence is established in Lemma 2. Let us show that there exists a real even entire function G of exponential type with real zeros such that

$$\lim_{|x| \to \infty \, x \in \mathbf{R}} x^n G(x) g(x) = 0, \quad n = 1, 2, \ldots. \tag{19}$$

Let h be a function from Lemma 3 and let $m \geq 2 + 2\pi/\sin(\pi\rho/2)$ be an even number. Then, by (10) and (12), there is a constant $D > 0$ such that for all real x,

$$\log |h^m(x) g(x)| \leq -V(|x|) + D.$$

Set $G(z) = h^m(z)$. We see that an even stronger statement than (19) holds:

$$|G(x) g(x)| \leq \exp\{-V(|x|) + D\}.$$

Since h is even, $h(iy)$ is real, so that G is positive on $i\mathbf{R}$. Moreover, G is bounded on $\mathbf{R}$ and all its zeros are real. A well-known result on entire functions of exponential type (see, e.g., [4], p. 240, Theorem 5) then implies the asymptotic equality

$$\log G(iy) = A|y| + o(|y|), \quad |y| \to \infty,$$

where A is some positive constant. In view of (10) and (11), this gives

$$|G(\pm i d_k) g(\pm i d_k)| \to \infty, \quad k \to \infty,$$

$$\text{sign } (G(\pm i d_k) g(\pm i d_k)) = (-1)^k, \quad k = 1, 2, \ldots.$$

We see that there exists a small number $c > 0$ that every interval (c_{2k-1}, c_{2k}) contains at least two points, say p_{2k-1} and p_{2k}, which are roots of the equation

$$G(iy) g(iy) = -c.$$

Set

$$q(z)=\prod_{k=1}^{\infty}\left(1+\frac{z^2}{p_k^2}\right),$$

and define

$$f(z)=\frac{G(z)g(z)+c}{q(z)}. \tag{20}$$

We claim that this function satisfies the conclusion of Theorem 4. Since f is entire and is a ratio of two functions of exponential type, it is itself of exponential type.

Let us show that f satisfies condition (3). Using (19), we get

$$\log|f(x)|=\log\left|\frac{c+o(1)}{q(x)}\right|=\log\frac{1}{q(x)}+O(1)$$

$$=\log\frac{1}{|g(x)|}+\log\left|\frac{g(x)}{q(x)}\right|+O(1),\quad |x|\to\infty.$$

Since $c_{2k-1}<p_{2k-1}<p_{2k}<c_{2k}$, we see that

$$\log\left|\frac{g(x)}{q(x)}\right|=\sum_{k=1}^{\infty}\log\frac{1+x^2/c_{2k-1}^2}{1+x^2/p_{2k-1}^2}+\sum_{k=1}^{\infty}\log\frac{1+x^2/c_{2k}^2}{1+x^2/p_{2k}^2}$$

$$<\sum_{k=1}^{\infty}\log\frac{1+x^2/c_{2k-1}^2}{1+x^2/p_{2k-1}^2}=\log\left(1+\frac{x^2}{c_1^2}\right)+\sum_{k=1}^{\infty}\log\frac{1+x^2/c_{2k+1}^2}{1+x^2/p_{2k-1}^2}<\log\left(1+\frac{x^2}{c_1^2}\right).$$

One can get a similar estimate of $\log|g(x)/q(x)|$ from below. This gives

$$\log|f(x)|=\log\frac{1}{|g(x)|}+O\left(\log|x|\right),\quad |x|\to\infty.$$

Now (3) follows from (10).

It remains to verify that f is non-oscillating, that is every derivative of f has only a finite number of real zeros. Set $F(z)=G(z)g(z)$. It follows from (20) that

$$\left(\frac{d}{dz}\right)^n f(z)=c\left(\frac{d}{dz}\right)^n\frac{1}{q(z)}+\left(\frac{d}{dz}\right)^n\frac{F(z)}{q(z)}. \tag{21}$$

By construction, function q is an entire function with purely imaginary roots at $\pm ip_k$. Denote by $n_q^{\pm}(r)$ and $n_g^{\pm}(r)$ the number of roots of q and g in $\{z:|z|\le r,\ \pm\Im z>0\}$, respectively. It follows from (9) and the construction of q that

$$|n_q^{\pm}(r)-n_g^{\pm}(r)|=O(1),\quad r\to\infty.$$

Since $n_g^{\pm}(r)=V(r)(1+o(1))$, $n_q^{\pm}(r)$ has the same asymptotics. It follows that q is an entire function of completely regular growth with respect to the proximate order $\rho(r)$. Observe that $|q(z)| \le q(\pm|z|)$, so that q satisfies the assumption (ii) of Lemma 4. Thus, formula (13) holds. By Lemma 4, q' has only a finite number of real zeros; therefore, the first term on the right-hand side of (21) has only a finite number of real zeros. By (13) and (20), to finish the proof it now suffices to establish that

$$\left(\frac{d}{dx}\right)^n \frac{F(x)}{q(x)} = o\left(\left|\frac{q'(x)}{q(x)}\right|^n \frac{1}{q(x)}\right), \quad |x| \to \infty. \tag{22}$$

We use Lemma 4 to get

$$\left(\frac{d}{dx}\right)^n \frac{F(x)}{q(x)} = \sum_{j=0}^{n} C_n^j F^{(j)}(x) \left(\frac{d}{dx}\right)^{n-j} \frac{1}{q(x)}$$

$$= \sum_{j=0}^{n} C_n^j F^{(j)}(x) \left[(-1)^{n-j} \left(\frac{q'(x)}{q(x)}\right)^{n-j} \frac{1}{q(x)} \left(1 + O\left(\frac{1}{V(|x|)}\right)\right)\right]$$

$$= \left(\frac{q'(x)}{q(x)}\right)^n \frac{1}{q(x)} \sum_{j=1}^{n} C_n^j F^{(j)}(x) \left[(-1)^{n-j} \left(\frac{q(x)}{q'(x)}\right)^j \left(1 + O\left(\frac{1}{V(|x|)}\right)\right)\right],$$

$$|x| \to \infty.$$

It follows from (17) and (14) that

$$\frac{q(x)}{q'(x)} = O\left(\frac{|x|}{V(|x|)}\right) = o(|x|), \quad |x| \to \infty.$$

Since F is an entire function of exponential type and $F(x) = O(|x|^{-m})$ for any natural number m,

$$F^{(j)}(x) = O(|x|^{-m}), \quad |x| \to \infty$$

for any natural numbers j and m. This establishes (22) and completes the proof of Theorem 4.

5 Proof of Theorem 5 (i)

Let h, $h(0) = 1$, be a real entire function of exponential type whose existence is established by Lemma 3. We shall need the estimate.

$$|h(x+iy)| \le c\exp[a|y| - bV(|x|)], \quad |y| \le |x|, \tag{23}$$

where a, b, c are positive constants.

Let a be any number strictly greater than the type of h, and L be a function analytic in $\{z : |\arg z| < \pi\}$ satisfying (14) and (15). Set

$$h_1(z) = h(z)\exp[iaz + (1/2)z^\rho L(z)].$$

Clearly, h_1 is analytic in the quadrant $Q = \{z : 0 \le \arg z \le \pi/2\}$. Formulas (14) and (12) show that h_1 is bounded on the positive half-axis. Further, it follows from (14), (15) and (2) that

$$\Re(z^\rho L(z)) = r^\rho \cos(\rho\varphi) \cdot V(r)(1 + o(1)) = o(r), \quad r \to \infty. \tag{24}$$

We see that h_1 is of order ≤ 1 in Q and bounded on the positive imaginary half-axis. By the Phragmén–Lindelöf principle, h_1 is bounded in Q. Hence

$$|h(z)| \le c|\exp[-iaz - (1/2)z^\rho L(z)]|, \quad z \in Q,$$

c being a positive constant. Recalling that (by (2)) $\rho \le 1$ and using (24), we obtain (23) for the angle $\{z = x + iy : 0 \le y \le x\}$. For the other three angles of the form $\{z : 0 \le \pm y \le \pm x\}$, the proof of (23) is similar.

Set

$$f(z) = \frac{1 - h(z)}{z}.$$

We show that (4) holds for this function (and hence it is a non-oscillating PW-function). Clearly,

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} + \left(\frac{h(x)}{x}\right)^{(n)}.$$

We estimate the second term in the right-hand side for large x. Let

$$M_x(r) = \max_{|z-x|=r} |h(z)|.$$

Inequality (23) shows that for $x > r$,

$$M_x(r) \le c\exp\{ar - bV(x - r)\}. \tag{25}$$

The Cauchy integral formula gives

$$\left|\left(\frac{h(x)}{x}\right)^{(n)}\right| \le n!\frac{M_x(r)}{(x - r)r^n}.$$

It is now clear that $f^{(n)}(x)$ does not vanish, provided that

$$\left|\left(\frac{h(x)}{x}\right)^{(n)}\right| \le \frac{n!}{x^{n+1}}.$$

Hence, $f^{(n)}(x)$ does not vanish if

$$n!\frac{M_x(r)}{(x - r)r^n} \le \frac{n!}{x^{n+1}}.$$

For $r < x/2$, this can be rewritten as

$$M_x(r) \leq \frac{1}{2}\left(\frac{r}{x}\right)^n.$$

By (25), the last inequality holds if

$$c\exp\{ar - bV(x-r)\} \leq \frac{1}{2}\left(\frac{r}{x}\right)^n. \tag{26}$$

Observe that there is a constant $q > 0$ such that

$$V(x-r) \geq V(x/2) \geq qV(x), \quad 0 < x < r/2.$$

The first inequality holds because V is increasing; the second follows from a well-known property of proximate order ([2], p. 73; [4], p. 33). Therefore, (26) holds if

$$ar - bqV(x) \leq n\log(r/x) + c_1,$$

where c_1 is a positive constant. Set

$$r = \frac{bq}{2a}V(x).$$

Then we see that $f^{(n)}(x)$ cannot vanish provided that

$$-\frac{bq}{2}V(x) \leq n\log\left(\frac{bq}{2a}\cdot\frac{V(x)}{x}\right) + c_1$$

or

$$V(x) \geq \frac{2n}{bq}\log\left(\frac{2a}{bq}\cdot\frac{x}{V(x)}\right) - \frac{2c_1}{bq}. \tag{27}$$

Since $V(x) \geq \delta x^{\rho/2}$ for all $x \geq 1$ if δ is sufficiently small, $v(u) \leq (u/\delta)^{2/\rho}$ for all sufficiently large u. Taking this into account, one easily checks that $x = v(Cn\log n)$ satisfies (27) for $n = 1, 2, \ldots$, provided that C is large enough. Hence (4) holds.

6 Proof of Theorem 5 (ii) and Corollary 1

For any natural number n, we denote by $\nu_n(t)$ the number of zeros of $f^{(n)}$ in the disc $\{z : |z| \leq t\}$ (counting multiplicities). If $f^{(n)}(0) \neq 0$, then by Jensen's formula,

$$\int_0^r \frac{\nu_n(t)}{t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^{(n)}(re^{i\varphi})|d\varphi - \log|f^{(n)}(0)|, \quad r \geq 0. \tag{28}$$

We choose $r = er(n, f)$ and estimate the left-hand (right-hand) side of (28) from below (above) for this value of r.

Observe that, by the definition of $r(n,f)$ and $p(n,f)$, we have

$$\nu_n(r(n,f)) \geq p(n,f).$$

Since

$$\int_0^{er} \frac{\nu_n(t)}{t} dt \geq \int_r^{er} \frac{\nu_n(t)}{t} dt \geq \nu_n(r),$$

we get

$$\int_0^{er(n,f)} \frac{\nu_n(t)}{t} dt \geq p(n,f). \tag{29}$$

To estimate the first term of the right-hand side of (28), we observe that f admits the representation

$$f(z) = \int_{-1}^{1} e^{izt}\psi(t)dt,$$

where $\psi \in L_2(-1,1)$; therefore,

$$|f^{(n)}(z)| = \left| \int_{-1}^{1} e^{izt}(it)^n \psi(t)dt \right|$$

$$\leq \int_{-1}^{1} e^{-yt}|t|^n|\psi(t)|dt \leq \int_{-1}^{1} e^{-yt}|\psi(t)|dt \leq \|\psi\|_2 e^{|y|}, \quad n = 0,1,2,\ldots.$$

Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^{(n)}(re^{i\varphi})|d\varphi \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\|\psi\|_2 e^{r|\sin\varphi|}\right) d\varphi = \frac{2}{\pi}r + \log\|\psi\|_2$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^{(n)}(er(n,f)e^{i\varphi})|d\varphi \leq \frac{2}{\pi}er(n,f) + \log\|\psi\|_2. \tag{30}$$

To estimate the second term of the right-hand side of (28), we need the following lemma, whose proof will be given in the next section of the paper.

Lemma 5. *Let f be an entire function admitting the representation*

$$f(z) = \int_{-1}^{1} e^{izt}d\mu(t), \tag{31}$$

where μ is a complex-valued Borel measure on $[-1,1]$ of finite total variation such that $\{-1,1\} \subset \operatorname{supp}\mu$. Then there exists an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of natural numbers such that

(i) $\lim_{j\to\infty} n_{j+1}/n_j = 1$;

(ii) $\log|f^{(n)}(0)| = o(n), \quad n = n_j \to \infty.$

By Lemma 5, we have

$$\log|f^{(n)}(0)| = o(n), \quad n = n_j \to \infty.$$

Substituting this, (29) and (30) into (28), we obtain

$$p(n, f) \le \frac{2}{\pi} er(n, f) + o(n), \quad n = n_j \to \infty. \tag{32}$$

By Rolle's theorem, $p(n, f) \ge n$; therefore, (32) yields

$$r(n, f) \ge \frac{\pi}{2e} n + o(n), \quad n = n_j \to \infty.$$

Taking into account that $r(n, f)$ increases in n and the sequence $\{n_j\}_{j=1}^{\infty}$ satisfies Lemma 5(i), we get (5).

Let f be the function whose existence has been established in Theorem 5 (i). Evidently, (32) is applicable to f. Using the inequality (4), we get

$$p(n, f) \le \frac{2}{\pi} ev(Cn \log n) + o(n), \quad n = n_j \to \infty.$$

Since $p(n, f)$ increases in n and the sequence $\{n_j\}_{j=1}^{\infty}$ satisfies Lemma 5(i), we can increase the constant C in such a way that (6) will hold.

7 Proof of Lemma 5

Note that the equality

$$\limsup_{n\to\infty} \frac{1}{n} \log|f^{(n)}(0)| = 0 \tag{33}$$

is trivially true because f is of exponential type 1. Therefore, the existence of some sequence $\{n_j\}_{j=1}^{\infty}$ satisfying (ii) is evident. We show that such a sequence can be chosen to be rather dense, namely that (i) is fulfilled. First we prove this under the additional assumption that the measure μ is even, that is the function f admits the representation

$$f(z) = \int_0^1 \cos(zt) d\mu(t), \tag{34}$$

where $1 \in \operatorname{supp} \mu$. In this case,

$$f^{(2k)}(0) = (-1)^k \int_0^1 t^{2k} d\mu(t), \quad f^{(2k+1)}(0) = 0, \quad k = 0, 1, \ldots.$$

We prove the lemma by contradiction. Let us assume that Lemma 5 is wrong. Then there exist numbers $q > 1$, $\varepsilon > 0$ and a sequence of disjoint intervals $[a_j, b_j]$ such that a_j and b_j are natural numbers satisfying the condition

$$b_j/a_j \ge q, \quad j = 0, 1, 2, \ldots, \tag{35}$$

$$\limsup_{k\to\infty,\, k\in A} \frac{1}{k} \log |f^{(2k)}(0)| < -\varepsilon, \tag{36}$$

where

$$A = \bigcup_{j=1}^{\infty} [a_j, b_j].$$

Consider the function

$$F(z) = \int_0^1 t^{2z+2} d\mu(t).$$

This function is analytic and bounded in the closed half-plane $\{z : \Re z \geq 0\}$. Since $F(k-1) = (-1)^k f^{(2k)}(0),\ k = 0, 1, \ldots$, (36) implies

$$\limsup_{k\to\infty} \frac{1}{k} \log |F(k)| = 0.$$

Hence, since F is bounded on $\mathbf{R}$, we have

$$\limsup_{x\to+\infty} \frac{1}{x} \log |F(x)| = 0. \tag{37}$$

On the other hand, (36) implies

$$\limsup_{k\to\infty,\, k\in A} \frac{1}{k} \log |F(k)| < -\varepsilon. \tag{38}$$

By a well-known result ([5], p. 104, Theorem 3),

$$\begin{aligned} \log |F(z)| &= -\frac{x}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(s)}{x^2 + (y-s)^2} + \sum_l \log \left| \frac{z - a_l}{z + \bar{a}_l} \right| - cx \\ &=: -u_1(z) + u_2(z) - cx, \end{aligned} \tag{39}$$

where ν is a non-negative Borel measure on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} \frac{d\nu(s)}{1 + s^2} < \infty, \tag{40}$$

the a_l's are points in the half-plane $\{z : \Re z > 0\}$ satisfying the condition

$$\sum_l \frac{\cos(\arg a_l)}{|a_l|} < \infty \tag{41}$$

and c is a non-negative constant.

Observe that (37) yields $c = 0$, and (40) implies $u_1(x) = o(x),\quad x \to +\infty$. Therefore, by (38) we see that

$$\limsup_{k\to\infty\, k\in A} \frac{u_2(k)}{k} < -\varepsilon. \tag{42}$$

To prove the lemma, we show that this inequality cannot hold.

By the Hayman–Azarin theorem (see, e.g., [5], p. 109, Theorem 1),

$$u_2(z) \geq -(1/2)\varepsilon|z|, \quad \Re z > 0, \tag{43}$$

outside some exceptional set of discs $C_m = \{z : |z - z_m| < \delta_m\}$ of finite view, that is, such that

$$\sum_m (\delta_m/|z_m|) < \infty.$$

Since

$$\sum_{|z_m|>R} (\delta_m/|z_m|) \to 0 \quad \text{as } R \to +\infty,$$

we can choose R and the rays $\Lambda_\pm := \{z : \arg z = \pm\theta\}$, $0 < \theta < \pi/4$, in such a way that

$$(\Lambda_\pm \cap \{z : |z| > R\}) \cap \bigcup_m C_m = \emptyset.$$

The Hayman–Azarin theorem ([5], p. 109) also implies that there is a sequence of segments

$$\{[d_p, d_p + \eta_p]\}_{p=1}^{\infty}, \quad 0 < d_p < d_p + \eta_p < d_{p+1} \uparrow +\infty,$$

satisfying the condition

$$\sum_{p=1}^{\infty} (\eta_p/d_p) < \infty$$

and such that (43) holds outside the half-annuli

$$\{z : d_p < |z| < d_p + \eta_p, \ \Re z > 0\}.$$

Let us consider the system of sectors

$$K_p := \{z : d_p < |z| < d_p + \eta_p, \ |\arg z| < \theta\}, \quad p \geq p_0,$$

where p_0 is so large that the circumference $\{z : |z| = R\}$ does not intersect K_p for $p \geq p_0$. Evidently, the sectors K_p are pairwise disjoint and their union covers the set $A \cap \{z : |z| > R\}$. Moreover, we have

$$u_2(z) \geq -(\varepsilon/2)|z|, \quad z \in \partial K_p, \ p \geq p_0.$$

We split the system of sectors $\{K_p\}_{p=p_0}^{\infty}$ into two groups. The first (second) one consists of those, for which $\eta_p \geq 1$ ($\eta_p < 1$). We denote the sectors of the first (second) group by K_p' (K_p'').

Setting

$$A' = A \cap \bigcup_p K_p',$$

we obtain

$$\sum_{k\in A'}\frac{1}{k}\le\sum_{\eta_p\ge 1}\sum_{d_p\le k\le d_p+\eta_p}\frac{1}{k}\le\sum_{\eta_p\ge 1}\frac{\eta_p+1}{d_p}\le\sum_{\eta_p\ge 1}\frac{2\eta_p}{d_p}\le 2\sum_{p=1}^{\infty}\frac{\eta_p}{d_p}<\infty.$$

Since (33) implies

$$\sum_{k\in A}\frac{1}{k}=\infty,$$

using the notation

$$A''=A\cap\bigcup_p K_p'',$$

we have

$$\sum_{k\in A''}\frac{1}{k}=\infty. \tag{44}$$

Each point $k\in A''$ belongs at least to one sector K_p''. Moreover, each sector K_p'' contains at most one point from A'' because $\eta_p<1$. Thus, to each $k\in A''$, there corresponds a unique sector K_p'', which will be denoted by $K_{p(k)}''$.

Now observe that each sector $K_{p(k)}''$ must contain at least one point a_l from the representation (39). Indeed, otherwise, the function u_2 would be harmonic in $K_{p(k)}''$, and so

$$u_2(k)\ge\min_{z\in\partial K_{p(k)}''}u_2(z)\ge-(\varepsilon/2)(k+\eta_{p(k)})\ge-(\varepsilon/2)(k+1).$$

This contradicts condition (42) for sufficiently large k.

Thus, to each sufficiently large $k\in A''$, there corresponds at least one point $a_l=a_{l(k)}\in K_{p(k)}''$; and to different values of k, there correspond different values $a_{l(k)}$. Since both k and $a_{l(k)}$ belong to the same $K_{p(k)}''$,

$$|k-|a_{l(k)}||<\eta_{p(k)}<1.$$

Therefore, (44) implies

$$\sum_{k\in A''}\frac{1}{|a_{l(k)}|}=\infty.$$

On the other hand,

$$\sum_{k\in A''}\frac{1}{|a_{l(k)}|}\le\frac{1}{\cos(\pi/4)}\sum_{k\in A''}\frac{\cos(\arg a_{l(k)})}{|a_{l(k)}|}\le\frac{1}{\cos(\pi/4)}\sum_{l}\frac{\cos(\arg a_l)}{|a_l|}.$$

Clearly, this inequality and (41) contradict each other, which proves Lemma 5 under the additional assumption that the function f admits the representation (34) and $1\in\operatorname{supp}\mu$.

If f admits the representation

$$f(z) = \int_0^1 \sin(tz)d\mu(t), \quad 1 \in \operatorname{supp}\mu,$$

then the proof is similar. In general, when f satisfies (31), we set

$$f_1(z) = \int_0^1 \cos(tz)d\mu_1(t), \quad f_2(z) = \int_0^1 \sin(tz)d\mu_2(t),$$

where

$$\mu_1(E) = \mu(E) + \mu(-E), \quad i\mu_2(E) = \mu(E) - \mu(-E)$$

and observe that $1 \in \operatorname{supp}\mu_1 \bigcup \operatorname{supp}\mu_2$. Therefore, we can apply what we have already proved to either f_1 or f_2. Noting that

$$f^{(2k)}(0) = f_1^{(2k)}(0), \quad f^{(2k+1)}(0) = f_2^{(2k+1)}(0), \quad k = 0, 1, 2, \ldots,$$

we obtain the desired assertion.

8 Proof of Theorem 2

We begin this section with the following theorem, which we think has independent interest.

Theorem 6. *Suppose $k \geq 0$ is an integer, f is a real PW-function, and F is its spectral function. Assume that the derivative $f^{(k+3)}(x)$ has only a finite number of changes of sign on the real line. Then $F \in C^k(\mathbf{R} \setminus \{0\})$.*

Theorem 2 follows immediately from this result and the definition of a non-oscillating PW-function.

Another immediate corollary of Theorem 6 is the following result, which for any $k \geq 0$ gives a description of a wide class of real PW-functions f for which the n-th derivative of f, $n \geq 3$, must have infinitely many real zeros.

Corollary 3. *Let f be a real PW-function with spectral function F. Assume $F \notin C^k(\mathbf{R} \setminus \{0\})$ for some integer k. Then $f^{(k+3)}$ has infinitely many changes of sign.*

We deduce Theorem 6 from

Lemma 6. *Let $g \in L_1(\mathbf{R})$ and $g(x) \geq 0$ for all large $|x|$, and let G be the spectral function of g. Suppose there is an even integer $n \geq 2$ such that G is n times differentiable at the origin. Then $G \in C^n(\mathbf{R})$.*

Proof. When g is non-negative on $\mathbf{R}$, this lemma is contained in the well-known Lévy theorem (see Theorem 2.1.1 and its Corollary 1 in [6], p. 21).

In the general case, there exists $a > 0$ such that $g(x) \geq 0$ for $|x| \geq a$. Write

$$g_1(x) := \chi_{(-a,a)}(x)g(x), \quad g_2(x) := g(x) - g_1(x),$$

where $\chi_{(-a,a)}$ is the characteristic function of the interval $(-a, a)$. Then $G = G_1 + G_2$, where G_j is the spectral function of $g_j, j = 1, 2$. Clearly, G_1 can be continued to the complex plane as an entire function, so G_1 is infinitely differentiable. The lemma now follows from Corollary 1 in [6], according to which $G_2^{(n)}$ exists and is continuous on $\mathbf{R}$. □

Proof of Theorem 6. Let f be a real PW-function with spectral function F. We begin with the observation that if f' has only a finite number of real zeros, then $f' \in L_1(\mathbf{R})$. Indeed, there is a number $a > 0$ such that f' does not change the sign (i.e., is either non-positive or non-negative) in (a, ∞) and in $(-\infty, -a)$. This gives

$$\int_{-\infty}^{\infty} |f'(x)|\, dx = \left| \int_{-\infty}^{-a} f'(s)\, ds \right| + \int_{-a}^{a} |f'(s)|\, ds + \left| \int_{a}^{\infty} f'(s)\, ds \right|$$
$$\leq |f(-a)| + \sqrt{2a}\|f'\|_{L_2} + |f(a)| < \infty.$$

The spectral function of f' is $isF(s)$. Since it is the inverse Fourier transform of f', which belongs to $L_1(\mathbf{R})$, it follows that $sF(s)$ is continuous on the real line. Hence, for any $n = 1, 2, \ldots$, the function $s^n F(s)$ is $n - 1$ times differentiable at the origin.

In what follows, we assume that $f(x) \geq 0$ for all large negative x (otherwise, consider the function $-f(x)$). Then f satisfies one of the conditions

(i) $f(x) \geq 0$ for all large positive x,

(ii) $f(x) \leq 0$ for all large positive x.

(i) By Rolle's theorem, for any integer s, $0 \leq s \leq k + 2$, the derivative $f^{(s)}(x)$ has only a finite number of real zeros. It is then clear that for any even number s, $f^{(s)}(x) \geq 0$ for all large $|x|$.

Assume that k is even. Then $f^{(k+2)}(x) \geq 0$ for all large $|x|$. The spectral function of $f^{(k+2)}$ is $(is)^{k+2}F(s)$. It is $k + 1$ times differentiable at the origin. Hence, by Lemma 6 with $g = f^{(k+2)}$, we conclude that $((is)^{k+2}F(s))^{(k)}$ exists and is continuous on $\mathbf{R}$. It follows that $F \in C^k(\mathbf{R} \setminus \{0\})$.

Assume now that k is odd. Then $f^{(k+3)}(x) \geq 0$ for all large x. The same argument shows that the derivative $F \in C^{k+1}(\mathbf{R} \setminus \{0\})$.

(ii) Clearly, for any odd integer s, $1 \leq s \leq k+3$, we have $f^{(s)}(x) \geq 0$ for all large $|x|$. The same argument as in (i) establishes that $F \in C^k(\mathbf{R} \setminus \{0\})$ if k is odd, and $F \in C^{k+1}(\mathbf{R} \setminus \{0\})$ if k is even.

Remark. The observation at the beginning of the proof implies that, for any non-oscillating PW-function f, one has $f^{(k)} \in L_1(\mathbf{R})$, $k = 1, 2, \ldots$. For $k = 0$, this is not always true, as the example of functions f_1, f_2 in Theorem 1 shows. It can be shown that $f^{(k)} \in L_1(\mathbf{R})$, $k = 1, 2, \ldots$ for any real PW-function f such that f' has only finitely many real zeros.

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