Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 2, p. 243–248

On the Poisson representation of a function harmonic in the upper half-plane

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Received November 21, 2001

New conditions of validity of the Poisson representation (in usual and generalized form) for a function harmonic in the upper half-plane are obtained. These conditions differ from known ones by weaker growth restrictions inside the half-plane and stronger restrictions on the behavior in the neighbourhood of the real axis.

Dedicated to centennial of Naum Il'ich Akhiezer

In the theory of entire and subharmonic functions the following Poisson representation of a real-valued harmonic function u in $\mathbf{C}_+ := \{z : \ \mathrm{Im} z > 0\}$ is very important:

$$u(z) = \frac{y}{\pi} \int\limits_{-\infty}^{\infty} \frac{d\nu(t)}{(x-t)^2 + y^2} + cy, \ z = x + iy, \ y > 0, \tag{1}$$

where c is a real constant and ν is a real-valued σ-finite Borel measure on $\mathbf{R}$ such that

$$\int\limits_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+t^2} dt < \infty.$$

Mathematics Subject Classification 2000: 31A05, 31A10.

We mention applications to the theory of integral transforms [1, Ch. 4]; the entire functions theory [7, Ch. 5]; [8, Part II]; [5, Ch. 3]; the theory of H_p spaces [6, Ch. 6].

It is well-known (see, e.g., [8, p. 100]; [5, p. 39]; [6, p. 107]) that the necessary and sufficient condition for (1) to be true is representability of u in the form $u = u_1 - u_2$ where u_1 and u_2 are non-negative harmonic functions in $\mathbf{C}_+$. Nevertheless, for several applications (see, e.g., [7, Ch. 5]; [8, Part II]; [5, Ch. 3]), conditions that can be expressed in terms of *growth* of u are more useful. For functions u continuous in the closure $\overline{\mathbf{C}}_+$ of $\mathbf{C}_+$ the strongest version of conditions of this kind is contained in the following result of R. Nevanlinna [9]:

Theorem A. ([9]) *Let u be a real-valued function harmonic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$ and satisfying the conditions:*

(i) there exists a sequence $\{r_k\}$, $r_k \to \infty$, such that

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta = O(r),\ r = r_k \to \infty; \tag{2}$$

(ii)

$$\int_{-\infty}^{\infty} \frac{u^+(t)}{1+t^2} dt < \infty. \tag{3}$$

Then u admits representation (1) with $d\nu(t) = u(t)dt$.

In [2] and [3], different conditions of validity of representation (1) have been found:

Theorem B. ([2, 3]) *Let u be a real-valued function harmonic in $\mathbf{C}_+$ and satisfy*

(i) there exists a sequence $\{r_k\}$, $r_k \to \infty$, such that

$$\int_0^\pi u^+(re^{i\theta}) \sin\theta d\theta \le \exp(o(r)),\ r = r_k \to \infty; \tag{4}$$

(ii) there exists $H > 0$ such that

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+t^2} dt < \infty. \tag{5}$$

Then (1) holds. If u is continuous in $\overline{\mathbf{C}}_+$ then $d\nu(t) = u(t)dt$.

Comparing conditions of theorems A and B, we see that (2) is more restrictive than (4) whereas (3) is less restrictive than (5). It should also be mentioned that continuity of u in $\overline{\mathbf{C}}_+$ is not assumed in Theorem B. Assumptions (4) and (5) in Theorem B are sharp in the folllowing sense. Example $u(z) = \mathrm{Re}\{\cos z\}$ shows that "o" cannot be replaced by "O" in (4). Moreover, (5) cannot be replaced by

$$\int_{-\infty}^{\infty} \frac{|u(t+iH)|}{1+t^2} dt < \infty,$$

for some $H > 0$, as the example $u(z) = \mathrm{Im}\{(z-iH)^{2n}\},\ n \in \mathbf{N}$, shows. It is also worth to mention that $|u(t+is)|$ cannot be replaced with $u^+(t+is)$ in (5), as the example $u(z) = -\mathrm{Re}\{z^{2n}\},\ n \in \mathbf{N}$, shows.

Present work is devoted to conditions of the validity of more general representations including (1) as a special case. Further we assume that all harmonic functions and Borel measures are real-valued.

This representation has the form

$$u(z) = \int_{-\infty}^{\infty} P_q(z,t) d\nu(t) + \mathrm{Im} P(z), \tag{6}$$

where

$$P_q(z,t) = \mathrm{Im}\left\{\frac{1}{\pi} \cdot \frac{(1+tz)^q}{(t-z)(1+t^2)^q}\right\}, \quad q \in \mathbf{N} \cup \{0\},$$

ν is a σ-finite Borel measure on $\mathbf{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^{q+1}} < \infty,$$

and P is a real polynomial of degree at most q.

For u harmonic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$ and satisfying conditions

$$\int_0^{\pi} u^+(re^{i\theta}) \sin\theta d\theta = O(r^q),\ r \to \infty;$$

$$\int_{-\infty}^{\infty} \frac{u^+(t)}{1+|t|^{q+1}} dt < \infty, \tag{7}$$

representation (6) (with $d\nu(t) = u(t)dt$) belongs to R. Nevanlinna [9]. Without the continuity assumption on u in $\overline{\mathbf{C}}_+$ and under the growth condition

$$\max_{0<\theta<\pi} u^+(re^{i\theta}) = O(r^\alpha), \quad \alpha < q, \tag{8}$$

it belongs to N.V. Govorov [4, p. 25].

Our first result is the following:

Theorem 1. *Let u be a function harmonic in $\mathbf{C}_+$, satisfying condition (4) of Theorem B and the following condition:*
there exists $\alpha > 0$ such that

$$\liminf_{s\to 0^+} \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+|t|^\alpha} dt < \infty. \tag{9}$$

Then u admits representation (6), where $q = \max\{n \in \mathbf{N} \cup \{0\} : \ n < \alpha\}$, ν is a σ-finite Borel measure on $\mathbf{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^\alpha} < \infty,$$

and P is a real polynomial of degree at most q.

Note that Nevanlinna's [9] and Govorov's [4] results, mentioned above, are not contained in Theorem 1 because condition (9) is more restrictive than both (7) and (8).

It is easy to see that Theorem B is contained in Theorem 1: if $\alpha = 2$ then $q = 1$,

$$P_1(z,t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}, \quad z = x + iy,$$

condition (9) with $\alpha = 2$ is less restrictive than (5).

Our next result is related to the question: "Can we replace (9) with a condition requiring convergence of the integrals in (9) only over two horizontal lines?" In some sense, the answer is affirmative. To formulate our result we need:

Lemma 1. *Let $u(z)$ be a function harmonic in $\mathbf{C}_+$ and satisfying the condition*

$$\exists H > 0, \ \forall R > 0, \ \sup_{0<y<H} \int_{-R}^{R} |u(x+iy)| dx < \infty. \tag{10}$$

Then there exists a Borel measure ν on $\mathbf{R}$ such that for all $R > 0$ it satisfies $|\nu|([-R, R]) < \infty$, and the function

$$u(z) - \int_{-R}^{R} P_q(z,t)d\nu(t), \quad q \in \mathbf{N} \cup \{0\},$$

is harmonic in $\mathbf{C}_+$, continuous in $\mathbf{C}_+ \cup (-R, R)$ and vanishes on $(-R, R)$.

Our second result is the following:

Theorem 2. *Let u be a function harmonic in $\mathbf{C}_+$ satisfying condition (10) of Lemma 1 and (4) of Theorem B. Assume additionally that u satisfies the following condition:*
there exist $H > 0$ and $\alpha > 0$ such that

$$\int_{-\infty}^{\infty} \frac{|u(t+iH)|}{1+|t|^\alpha}dt + \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^\alpha} < \infty,$$

where ν is the σ-finite Borel measure defined in Lemma 1. Then u admits representation (6), where q, ν and P are as in Theorem 1.

The following corollary to Theorem 2 is immediate.

Corollary 1. *Let u be a function harmonic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$ and satisfying (4) of Theorem B. Suppose there exist $H > 0$ and $\alpha > 0$ such that*

$$\int_{-\infty}^{\infty} \frac{|u(t)| + |u(t+iH)|}{1+|t|^\alpha}dt < \infty.$$

Then u admits representation (6) with $d\nu(t) = u(t)dt$.

Remind that, for $\alpha = 2$, representation (6) reduces to (1).

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