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A NIL APPROACH TO SYMMETRICITY OF RINGS

B. Ungor*, H. Kose, Y. Kurtulmaz and A. Harmanci

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We introduce a weakly symmetric ring which is a generalization of a symmetric ring and a strengthening of both a GWS ring and a weakly reversible ring, and investigate properties of the class of this kind of rings. A ring R is called *weakly symmetric* if for any a, b, $c \in R$, abc being nilpotent implies that $Racrb$ is a nil left ideal of R for each $r \in R$. Examples are given to show that weakly symmetric rings need to be neither semicommutative nor symmetric. It is proved that the class of weakly symmetric rings lies also between those of 2-primal rings and directly finite rings. We show that for a nil ideal I of a ring R, R is weakly symmetric if and only if R/I is weakly symmetric. If $R[x]$ is weakly symmetric, then R is weakly symmetric, and $R[x]$ is weakly symmetric if and only if $R[x; x^{-1}]$ is weakly symmetric. We prove that a weakly symmetric ring which satisfies Köthe's conjecture is exactly an NI ring. We also deal with some extensions of weakly symmetric rings such as a Nagata extension, a Dorroh extension.

1. Introduction

Throughout, all rings are associative with identity. In the sequel, the symbols $J(R)$ and nil(R) will stand for the Jacobson radical

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* *Corresponding Author*

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and the set of all nilpotent elements of a ring R, respectively. Symmetric rings are defined by Lambek in [9]. A ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for all a, b, $c \in R$. In [13], Ouyang and Chen discussed weak symmetric rings. A ring R is called *weak symmetric* if $abc \in \text{nil}(R)$ implies $acb \in \text{nil}(R)$ for all a, b, $c \in R$. It is proved in [13] that all symmetric rings are weak symmetric. The class of weak symmetric rings is also studied in [6]. Generalized weakly symmetric rings (or GWS, for short) are studied in [15]. A ring R is called *GWS* if $abc = 0$ implies that bac is nilpotent for all a, b, $c \in R$. Clearly, $abc = 0$ implies bac is nilpotent for all a, b, $c \in R$ if and only if $abc = 0$ implies acb is nilpotent for all a, b, $c \in R$.

In [2], Chakraborty and Das called a ring R *right* (respectively, *left*) *nil-symmetric* if $abc = 0$ (respectively, $cab = 0$) implies $acb = 0$ for all nilpotent a, $b \in R$ and $c \in R$ and the ring R is *nil-symmetric* if it is both right and left nil-symmetric. In [7], nil-symmetric rings are weakend to weak nil-symmetric rings. A ring R is called *weak right nil-symmetric* if $abc = 0$ implies $acb = 0$ for all nilpotent a, b, $c \in R$ and it is called *weak left nil-symmetric* if $abc = 0$ implies $cab = 0$ for all nilpotent a, b, $c \in R$, and R is called *weak nil-symmetric* if it is both weak right nil-symmetric and weak left nil-symmetric. As an another generalization of a symmetric ring, according to [1] and also [12], a ring R is called *semicommutative* if for all a, $b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator over R is an ideal of R. In [3], semicommutativity of rings is generalized to nil-semicommutativity of rings. It is said that a ring R is *nil-semicommutative* if for every a, $b \in R$, ab being nilpotent implies that aRb is a nil subset of R. Every semicommutative ring is nil-semicommutative. According to Cohn [5], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for all a, $b \in R$. In [10], R is said to be *weakly reversible* if $ab = 0$ implies $braR$ is a nil right ideal of R, equivalently, $ab = 0$ implies $Rbra$ is a nil left ideal of R for all a, b, $r \in R$. In [15], a ring R is said to be *quasi-reversible* if $ab = 0$ implies $bRa \subseteq J(R)$ for all a, $b \in R$, and in [8], R is called *central reversible* if being $ab = 0$ implies that ba is central.

Symmetric rings were introduced by Lambek to unify sheaf representations of commutative rings and reduced rings [9], and this concept was generalized by some authors as mentioned above. Motivated by these versions of symmetric rings and reversible rings contexts, in this paper we introduce weakly symmetric rings which are weaker than symmetric rings and stronger than both GWS rings and weakly reversible rings, and investigate their properties. We also prove that the class of weakly symmetric rings has a position between those of 2-primal rings and directly finite rings. Some characterizations of weakly symmetric rings are obtained in terms of nil left ideals and nil right ideals. The set of all nilpotent elements of a corner ring of a weakly symmetric ring is determined, and so weakly symmetricity of a corner ring is investigated. We give a characterization of an NI ring in terms of weakly symmetric rings, that is a ring is NI if and only if it is weakly symmetric and satisfies Köthe's conjecture. We also observe relations between a ring and some of its extensions such as a Nagata extension and a Dorroh extension in terms of weakly symmetricity.

In what follows, $\mathbb{Z}$ and $\mathbb{Z}_n$, where n is a positive integer denote the ring of integers and the ring of integers modulo n, respectively. Also $M_n(R)$ denotes the ring of all $n \times n$ matrices and $T_n(R)$ stands for the ring of all $n \times n$ upper triangular matrices over a ring R for a positive integer n. We write $R[x]$ and $R[x; x^{-1}]$ for the polynomial ring and the Laurent polynomial ring over R, respectively.

2. Weakly Symmetric Rings

This section is devoted to study on weakly symmetric rings. The position of the class of weakly symmetric rings among some classes of rings such as symmetric rings, nil-semicommutative rings, GWS rings, weakly reversible rings, 2-primal rings and directly finite rings is determined. Some properties of weakly symmetric rings are investigated and characterizations of these rings are obtained.

Definition 2.1. A ring R is said to be *weakly symmetric* if for all a, b, c, $r \in R$, abc being nilpotent implies that $Racrb$ is a nil left ideal of R, equivalently, abc being nilpotent implies that $acrbR$ is a nil right ideal of R.

Commutative rings, Boolean rings, symmetric rings are weakly symmetric. It is known that every symmetric ring is reversible. When we deal with the weakly case, every weakly symmetric ring is weakly reversible. In fact, if R is a weakly symmetric ring and $ab = 0$, where $a, b \in R$, then $1ab = 0$ and so $R1bra$ is a nil left ideal of R for all $r \in R$. Hence $Rbra$ is a nil left ideal.

Proposition 2.2. *The class of weakly symmetric rings is closed under isomorphisms, subrings and finite direct products.*

Proof. Follows by definitions. □

Corollary 2.3. *Let R be a ring. If $R[x]$ is weakly symmetric, then R is weakly symmetric.*

Proof. If $R[x]$ is weakly symmetric, then R is weakly symmetric since it is isomorphic to a subring of $R[x]$. □

Note that homomorphic images of weakly symmetric rings need not be weakly symmetric as shown below.

Example 2.4. Let R and $S = R/I$ denote the rings in ([15], Example 2.11). Then R is a noncommutative principal ideal domain, so it is weakly symmetric but S is not GWS. Since weakly symmetric rings are GWS (see Proposition 2.11), S is not a weakly symmetric ring.

Proposition 2.5. *Let R be a ring and I an ideal of R with $I \subseteq nil(R)$. Then R is weakly symmetric if and only if R/I is weakly symmetric.*

Proof. Assume that R is weakly symmetric. Let $\bar{R} := R/I$ and $\bar{a}, \bar{b}, \bar{c} \in \bar{R}$ with $\bar{a}\bar{b}\bar{c}$ nilpotent in $\bar{R}$. There exists a positive integer m such that $(abc)^m \in I$. By hypothesis, abc is a nilpotent element of R. By assumption, $Racrb$ is a nil left ideal of R for all $r \in R$. Hence $\bar{R}\bar{a}\bar{c}\bar{r}\bar{b}$ is a nil left ideal of $\bar{R}$.

Suppose now that $\bar{R}$ is weakly symmetric and a, b, $c \in R$ with abc nilpotent in R. Then $\bar{a}\bar{b}\bar{c}$ is nilpotent in $\bar{R}$. For any $r \in R$, by hypothesis, $\bar{R}\bar{a}\bar{c}\bar{r}\bar{b}$ is a nil left ideal of $\bar{R}$. Hence for any $s \in R$, there exists a positive integer n such that $(sacrb)^n \in I$. By hypothesis, $(sacrb)^n$ is nilpotent, and so $sacrb$ is nilpotent for each $s \in R$. Thus $Racrb$ is a nil left ideal of R. This completes the proof. $\square$

Recall that a ring R is called *NI* if all nilpotent elements of R form an ideal.

Corollary 2.6. *Let R be an NI ring. Then R and $R/nil(R)$ are weakly symmetric.*

Proof. Since R is an NI ring, $R/\text{nil}(R)$ is reduced, and so it is weakly symmetric. On the other hand, R is also weakly symmetric by Proposition 2.5. $\square$

In the next diagram we summarize implications among aforementioned classes of rings, and then we prove some of them, so we determine the position of the class of weakly symmetric rings.

$$\begin{tikzcd}
\text{reduced} \arrow[r, Rightarrow] & \text{symmetric} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{semicommutative} \arrow[d, Rightarrow] & \\
 & & \text{nil-semicommutative} \arrow[d, Rightarrow] & \\
 & \text{GWS} & \text{weakly symmetric} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{weak symmetric} \\
 & & \text{weakly reversible} \arrow[r, Rightarrow] & \text{quasi-reversible}
\end{tikzcd}$$

PROPOSITION 2.7. *Every nil-semicommutative ring is weakly symmetric.*

Proof. Assume that R is a nil-semicommutative ring. Let a, b, $c \in R$ with abc nilpotent. By assumption, $a(bc)$ being nilpotent implies that $aRbc$ is a nil subset of R. Hence $(acb)c$ is nilpotent. Again by assumption, $(acb)Rc$ is nil, and so $(acbac)1$ is nilpotent. It follows that $(acbac)R1$ is nil. Then $acbacb$ is nilpotent. This implies that acb is nilpotent. Thus $acRb$ is nil, so $1(acrb)$ is nilpotent for all $r \in R$. Therefore $1Racrb$ is nil. This completes the proof. $\square$

Since every symmetric ring is semicommutative, we have the following corollary.

COROLLARY 2.8. *Every symmetric ring is weakly symmetric.*

Let R be a ring and s, t be central in R. Consider the ring $H_{(s,t)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}$.
Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$. Note that any element A of $H_{(s,t)}(R)$ has the form $\begin{pmatrix} sc + te + f & 0 & 0 \\ c & te + f & e \\ 0 & 0 & f \end{pmatrix}$.

Lemma 2.9. *Let R be a ring, and let s, t be central in R. Then the set of all nilpotent elements of $H_{(s,t)}(R)$ is $nil(H_{(s,t)}(R)) =$*
$$\left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R) \mid a, d, f \in nil(R), c, e \in R \right\}.$$

Proof. Let $A = \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in \mathrm{nil}(H_{(s,t)}(R))$ be nilpotent. There exists a positive integer n such that $A^n = 0$. Then $a^n = d^n = f^n = 0$. Conversely, assume that $a^n = 0$, $d^m = 0$ and $f^k = 0$ for some positive integers n, m, k. Let $p = \max\{n, m, k\}$. Then $A^{2p} = 0$. □

Example 2.10. Let R be a reduced ring. Since $H_{(0,0)}(R)$ is nil-semicommutative, by Proposition 2.7, $H_{(0,0)}(R)$ is weakly symmetric.

Proposition 2.11. *Every weakly symmetric ring is GWS.*

Proof. Let R be a weakly symmetric ring and a, b, $c \in R$ with $abc = 0$. Then $1a(bc)$ is nilpotent. So $R(bc)ra$ is a nil left ideal for each $r \in R$. In particular, bca is nilpotent. Again, invoking weakly symmetricity, $Rbarc$ is nil for each $r \in R$. It follows that bac is nilpotent. □

Proposition 2.12. *Every weakly symmetric ring is directly finite.*

Proof. Let R be a weakly symmetric ring. Since GWS rings are directly finite, by Proposition 2.11, R is directly finite. □

Recall that a ring is said to be *2-primal* if the prime radical coincides with the set of all nilpotent elements of the ring. By the following theorem, the class of 2-primal rings lies strictly between the classes of symmetric rings and weakly symmetric rings.

THEOREM 2.13. *Every 2-primal ring is weakly symmetric.*

Proof. Every 2-primal ring is NI. By Corollary 2.6, R is weakly symmetric. □

As in [15], recall that a ring R is *nil-regular* if each nilpotent element is regular, that is, for any nilpotent $a \in R$, there exists $b \in R$ such that $a = aba$.

PROPOSITION 2.14. *Let R be a ring. Then the following statements are equivalent.*

(1) *R is a nil-regular weakly symmetric ring.*
(2) *R is a nil-regular GWS ring.*
(3) *R is a reduced ring.*

Proof. (1) $\Longrightarrow$ (2): Clear from the fact that every weakly symmetric ring is GWS.

(2) $\Longleftrightarrow$ (3): It is proved in ([15], Proposition 3.1).

(3) $\Longrightarrow$ (1): Obvious. □

There are weakly symmetric rings but not semicommutative as the following example shows.

EXAMPLE 2.15. Let F be a field. Then the ring $T_3(F)$ is weakly symmetric which is not semicommutative.

In the next we give some characterizations of weakly symmetric rings.

THEOREM 2.16. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is weakly symmetric.*
(2) *For any a, b, $c \in R$, whenever abc is nilpotent, $Rcba$ is a nil left ideal.*

(3) *For any a, b, $c \in R$, whenever abc is nilpotent, $Rbrac$ is a nil left ideal for $r \in R$.*
(4) *For any a, b, $c \in R$, whenever abc is nilpotent, $Rcrba$ is a nil left ideal for $r \in R$.*
(5) *For any a, b, $c \in R$, whenever abc is nilpotent, $Rcbra$ is a nil left ideal for $r \in R$.*

Proof. (1) $\Longrightarrow$ (3): Let $a, b, c \in R$ with abc nilpotent. By (1), $Racrb$ is a nil left ideal for each $r \in R$. Hence $1(acr)b$ is nilpotent. By (1) again, $Rbsacr$ is a nil left ideal for each $r, s \in R$. In particular, $Rbsac$ is a nil left ideal for each $s \in R$.

(3) $\Longrightarrow$ (1): Let $a, b, c \in R$ such that abc is nilpotent. The element abc being nilpotent implies that $Rbrac$ is a nil left ideal for each $r \in R$. Hence $(br)(ac)1$ is nilpotent. Thus $R(ac)t(br)$ is a nil left ideal. So $1(ac)b$ is nilpotent. By (3), $Racrb$ is a nil left ideal for any $r \in R$.

(1) $\Longrightarrow$ (2): Assume that R is weakly symmetric. Let a, b, $c \in R$ with abc nilpotent. Then $1(ab)c$ being nilpotent implies that $Rcr(ab)$ is a nil left ideal for each $r \in R$. By letting $r = 1$, cab is nilpotent. By assumption, $Rcbsa$ is a nil left ideal for each $s \in R$. Hence $Rcba$ is a nil left ideal.

(2) $\Longrightarrow$ (4): Assume that a, b, $c \in R$ with abc nilpotent. Then $Rcba$ is a nil left ideal of R. So $rcba$ is nilpotent for each $r \in R$. By (2), nilpotency of $rc(ba)$ implies that $Rbacr$ is a nil left ideal of R. Again by (2), nilpotency of $1(ba)(cr)$ implies that $Rcrba$ is a nil left ideal of R.

(4) $\Longrightarrow$ (5): Let a, b, $c \in R$ with abc nilpotent. By (4), $Rcrba$ is a nil left ideal for each $r \in R$. Then cba is nilpotent. By (4), nilpotency of $(cb)1a$ implies $Rar(cb)$ is a nil left ideal for each $r \in R$.

Then $a1(cb)$ is nilpotent. By (4), $Rcbra$ is a nil left ideal for each $r \in R$.

(5) $\Longrightarrow$ (1): Let a, b, $c \in R$ with abc nilpotent. By (5), nilpotency of $1a(bc)$ implies $Rbcar$ is a nil left ideal for each $r \in R$. In particular, bca is nilpotent. The condition (5) implies that $Racrb$ is a nil left ideal for each $r \in R$. This completes the proof. □

THEOREM 2.17. *Let R be a ring. Then the following are equivalent.*

(1) *R is weakly symmetric.*
(2) *If $a \in R$ is nilpotent, then $Rara$ is a nil left ideal for all $r \in R$.*
(3) *If $a \in R$ is nilpotent, then $araR$ is a nil right ideal for all $r \in R$.*

Proof. (1) $\Longrightarrow$ (2): Let $a \in R$ with $a^n = 0$ for some positive integer n. For any $r \in R$, we have $0 = r^n a^n = r^{n-1}(ra)a^{n-1}$. By (1), $r^{n-1}a^{n-1}(sa)(ra)$ is nilpotent for each $s \in R$. Hence $r^{n-2}(ra)a^{n-2}(sara)$ is nilpotent. By (1), it implies that $r^{n-2}(sara)(sa)(ra)a^{n-2}$ is nilpotent for each $s \in R$. Again by (1), $r^{n-2}a^{n-2}(sara)^2$ is nilpotent. Nilpotency of $r^{n-3}(ra)a^{n-3}(sara)^2$ and (1) imply that $r^{n-3}(sara)^2(sa)(ra)a^{n-3}$ is nilpotent. By invoking (1) again, we have that $r^{n-3}a^{n-3}(sara)^3$ is nilpotent. Continuing in this way, we may have that $sara$ is nilpotent for each $s \in R$. Hence $Rara$ is a nil left ideal for each $r \in R$.

(2) $\Longleftrightarrow$ (3): Clear.

(2) $\Longrightarrow$ (1): Let a, b, $c \in R$ with abc nilpotent in R. By (2), $R(abc)r(abc)$ is a nil left ideal for all $r \in R$. In particular, $R(rb)(abc)(rb)(abc)$ is a nil left ideal for all $r \in R$. Then $(abcrb)^2$ is nilpotent for all $r \in R$. Hence $crbab$, $acrbab$, $abacrb$ and $bacrba$ are nilpotent.

By (2), $R(bacrba)s(bacrba)$ is a nil left ideal for all r, $s \in R$. We replace s by cr to get $(bacrba)(cr)(bacrba)$ is nilpotent. It follows from the equality $(bacrba)(cr)(bacrba) = b(acrb)(acrb)(acrb)a$ that $acrb$ is nilpotent. By (2), $R(acrb)s(acrb)$ is a nil left ideal for all $s \in R$. Thus $R(acrb)$ is a nil left ideal of R for all $r \in R$. $\square$

In [14], feckly reduced rings are introduced and studied. A ring R is called *feckly reduced* if $R/J(R)$ is a reduced ring. Note that if R is feckly reduced, then all nilpotent elements of R belong to $J(R)$. In [4], a ring R is called *J-reduced* if all nilpotent elements of R belong to $J(R)$. So every feckly reduced ring is J-reduced. In the next result we show that the class of weakly symmetric rings is also a source of examples for J-reduced rings.

Proposition 2.18. *Every weakly symmetric ring is J-reduced.*

Proof. Let R be a weakly symmetric ring and $a \in R$ with a nilpotent. By Theorem 2.17, $Rara$ is a nil left ideal for each $r \in R$. In particular ra is nilpotent for each $r \in R$. Then $1 - ra$ is invertible for each $r \in R$. Hence $a \in J(R)$. $\square$

In [16], a ring R is defined to be *quasi-normal* if $ae = 0$ implies $eaRe = 0$ for nilpotent a and idempotent e in R. It is proved that R is quasi-normal if and only if $eR(1 - e)Re = 0$ for each idempotent e and, in [15], R is said to be *weakly quasi-normal* if $eR(1-e)Re \subseteq J(R)$ for each $e^2 = e \in R$.

Proposition 2.19. *Every weakly symmetric ring is weakly quasi-normal.*

Proof. Let $e^2 = e \in R$. Then $(1 - e)e$ is nilpotent. By hypothesis, $R(1 - e)re$ is a nil left ideal of R for every $r \in R$. So $R(1 - e)re \subseteq J(R)$. Since $J(R)$ is an ideal, $eR(1 - e)Re \subseteq J(R)$. $\square$

In the next, we determine the set of all nilpotent elements of a weakly symmetric ring, and then we deal with corner rings of weakly symmetric rings.

Proposition 2.20. *Let R be a ring and e an idempotent of R. If R is weakly symmetric, then the following conditions are satisfied.*

(1) *For all $a \in \text{nil}(R)$, $eae \in \text{nil}(eRe)$.*
(2) $\text{nil}(eRe) = e\text{nil}(R)e$.

Proof. (1) We first note that for any x and $y \in R$, xy being nilpotent implies yx is nilpotent. For any x, y, $z \in R$, by hypothesis, xyz being nilpotent implies $Rxzry$ is a nil left ideal for every $r \in R$. In particular, xzy is nilpotent. Consider the following cases:

Case 1: Let $n = 2$. Let $a \in R$ with $a^2 = 0$. Then $Rearea$ is a nil left ideal for each $r \in R$. In particular, $(eRe)(eae)(ere)(eae)$ is a nil left ideal of eRe.

Case 2: Let $n = 3$. So $a^3 = 0$ and $0 = ea^3e = (ea)(aa)e$. Then $(ea)(ea)a$ is nilpotent. Hence $(eae)(ea)a$ being nilpotent implies that $(eae)a(ea)$ is nilpotent. We have $(eae)(eae)(ea)$ is nilpotent. Thus $e(eae)^2(ea)$ being nilpotent implies $eae(eae)^2$ is nilpotent. It follows that eae is nilpotent in eRe.

Let $a \in \text{nil}(R)$ with $a^n = 0$, where $n > 3$. Then $ea^ne = (ea)a^{n-1}e$ is nilpotent. Hence $(ea)e(a^{n-1}) = (eae)(ea^{n-2})a$ is nilpotent. By hypothesis,

$$(eae)a(ea^{n-2}) = (eae)(eae)(ea^{n-3})a$$

is nilpotent. Similarly, $(eae)(eae)(eae)(ea^{n-3})$ is nilpotent. Continuing on this way, eae is nilpotent in eRe.

(2) It is clear from (1). $\square$

Corollary 2.21. *If R is a weakly symmetric ring, then eRe is weakly symmetric for all $e^2 = e \in R$.*

Consider the following subring $V_n(R)$ of $M_n(R)$, where n is a positive integer:

$$V_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{bmatrix} \mid a_i \in R, 1 \leq i \leq n \right\}.$$

Proposition 2.22. *Let R be a ring and n a positive integer. Then the following statements are equivalent.*

(1) *R is weakly symmetric.*
(2) *$T_n(R)$ is weakly symmetric.*
(3) *$V_n(R)$ is weakly symmetric.*

Proof. (1) $\Longrightarrow$ (2): Suppose that R is weakly symmetric. To prove $T_n(R)$ is weakly symmetric, let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij}) \in T_n(R)$ with $ABC = (d_{ij})$ nilpotent. Then $d_{ii} = a_{ii}b_{ii}c_{ii}$ is nilpotent for $1 \leq i \leq n$. By hypothesis, $Ra_{ii}c_{ii}rb_{ii}$ is a nil left ideal for every $r \in R$. Let $(t_{ij})(a_{ij})(c_{ij})(e_{ij})(b_{ij}) \in T_n(R)ACEB$, where $T = (t_{ij})$, $E = (e_{ij}) \in T_n(R)$. Let $(t_{ii}a_{ii}c_{ii}e_{ii}b_{ii})^{s_{ii}} = 0$ and $s = max\{s_{ii}\}$. Then $\big((t_{ij})(a_{ij})(c_{ij})(e_{ij})(b_{ij})\big)^{sn} = 0$. Hence $T_n(R)ACEB$ is a nil left ideal of $T_n(R)$ for any $E \in T_n(R)$.

(2) $\Longrightarrow$ (1): If $T_n(R)$ is a weakly symmetric ring, then R is weakly symmetric since R is isomorphic to a subring of $T_n(R)$ and weakly symmetric property is preserved under isomorphism of rings.

(2) $\Longleftrightarrow$ (3): Clear by the fact that weakly symmetric property of rings are preserved under subrings and isomorphisms. □

Let R be a ring and M an R-R-bimodule. The *trivial extension of R by M* is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication $(r,m)(s,m') = (rs, rm' + ms)$. This ring is isomorphic to the matrix ring $\left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \mid r \in R, m \in M \right\}$ with the usual matrix operations.

COROLLARY 2.23. *Let R be a ring. Then $T(R,R)$ is weakly symmetric if and only if so is R.*

Proof. Assume that $T(R,R)$ is weakly symmetric. Then R is weakly symmetric since weakly symmetric property for rings are preserved under subrings and isomorphisms. The converse is clear from Proposition 2.22. □

Recall that a ring is called *abelian* if all idempotents are central. Although symmetric rings are abelian, there are weakly symmetric rings that are not abelian and vice versa as shown below.

EXAMPLE 2.24. (1) Let $R = T_2(\mathbb{Z}_2)$. Since $\mathbb{Z}_2$ is a commutative ring, it is weakly symmetric. By Proposition 2.22, R is weakly symmetric. But R is not abelian because $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is an idempotent element of R which is not central.

(2) Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{Z}, a \equiv d \ (mod\ 2), b \equiv c \equiv 0 \ (mod\ 2) \right\}$.

Note that zero and identity are only idempotents of R and R is not a GWS ring by ([15], Example 2.12). Therefore R is not weakly symmetric by Proposition 2.11.

THEOREM 2.25. *Let D be a domain and S a subring of $M_2(D)$. Assume that S is weakly symmetric.*

(1) *If* $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in S$, *then* S *is the ring of upper triangular matrices over* D *or every element of* S *has the form* $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$, *where* $x + z = y + t$.

(2) *If* $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \in S$, *then* S *is the ring of lower triangular matrices over* D *or every element of* S *has the form* $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$, *where* $x + z = y + t$.

Proof. (1) Assume that S is not a ring of upper triangular matrices over D. Let $S' = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in S \mid x + z = y + t \right\}$.We first note that S' is a subring of S.

If $S' = S$, there is nothing to do. Otherwise assume that $S' \neq S$. Let $\begin{bmatrix} x & y \\ z & t \end{bmatrix} \in S$ with $a = x + z - (y + t)$ nonzero. Then $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x+z & y+t \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} x+z & y+t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x+z & x+z \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x+z & x+z \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} x+z & y+t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \in S$. Let $\begin{bmatrix} r & s \\ u & v \end{bmatrix} \in S$ be an arbitrary element. Assume that $u \neq 0$. Then by hypothesis,

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} r & s \\ u & v \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r & s \\ u & v \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ra(r+u) \\ 0 & ua(r+u) \end{bmatrix}$$

is nilpotent. So $ua(r + u) = 0$. Since ua is nonzero, $r = -u$. On the other hand, $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}^2 = 0$.

Again by hypothesis,

$$\begin{bmatrix} r & s \\ -r & v \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r & s \\ -r & v \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -r^2a^2 \\ 0 & r^2a^2 \end{bmatrix}$$

is nilpotent. Hence $ra = 0$ and so $ua = 0$. Since $a \neq 0$, $u = 0$. That is $(2, 1)$ entry of every matrix in S is zero. This is a contradiction. Thus S has the stated form.

(2) A similar discussion as in (1) completes the proof. □

Even though some subrings of full matrix rings over weakly symmetric rings are weakly symmetric, full matrix rings need not be weakly symmetric.

EXAMPLE 2.26. Consider the ring R in Example 2.24(2). Then R is a subring of $M_2(\mathbb{Z})$. If $M_2(\mathbb{Z})$ is weakly symmetric, then R must be weakly symmetric as a subring of $M_2(\mathbb{Z})$ and so it must be GWS. This is not the case by ([15], Example 2.12). Hence $M_2(\mathbb{Z})$ is not weakly symmetric.

We close this section by a result related to Köthe's Conjecture. Recall the Köthe's conjecture: The sum of two nil right ideals in any ring R is nil.

THEOREM 2.27. *The following hold for a ring R.*

(1) *If R is a weakly symmetric ring and $a \in R$ is nilpotent, then aR and Ra are nil.*
(2) *R is weakly symmetric and Köthe's conjecture holds for R if and only if R is an NI ring.*

Proof. (1) It follows from Theorem 2.17.
(2) $\Longleftarrow$: This implication is clear by definitions and Corollary 2.6.
$\Longrightarrow$: Let a, $b \in R$ be nilpotent elements. By (1), aR and bR are nil. Hence $aR + bR$ is nil as aR and bR are contained in the ideal of nilpotent elements. Thus $a + b$ is nilpotent. □

3. Extensions of Weakly Symmetric Rings

In this section, we consider some extensions of weakly symmetric rings and characterize this class of rings in terms of its extensions. Let R be a commutative ring, M be an R-module, and α be an endomorphism of R. Let $R \oplus M$ be a direct sum of R and M. Define componentwise addition and multiplication given by $(r_1; m_1)(r_2; m_2) = (r_1r_2; \alpha(r_1)m_2 + r_2m_1)$, where r_1, $r_2 \in R$ and m_1, $m_2 \in M$. This extension is called *Nagata extension of R by M and α*, and denoted by $N[R; M; \alpha]$ (see [11]).

Theorem 3.1. *If R is a commutative ring, then the Nagata extension $N[R; R; \alpha]$ is weakly symmetric.*

Proof. Assume that R is a commutative ring. Note that for any $a \in R$, a is nilpotent in R if and only if $(a; x)$ is nilpotent in $N[R; R; \alpha]$ for any $x \in R$. Let $(a; n)$, $(b; m)$, $(c; k) \in N[R; R; \alpha]$ with $(a; n)(b; m)(c; k)$ nilpotent in $N[R; R; \alpha]$. Then $abc \in R$ is nilpotent. Since R is commutative, $Racrb$ is a nil left ideal for every $r \in R$. Hence $N[R; R; \alpha](a; n)(c; k)(x; y)(b; m)$ is a nil left ideal of $N[R; R; \alpha]$ for all $(x; y) \in N[R; R; \alpha]$. $\square$

Given rings R and S, where S has an R-R-bimodule structure and all $a \in R$ and x, $y \in S$ are satisfied the following conditions: $(ax)y = a(xy)$, $(xy)a = x(ya)$, $(xa)y = x(ay)$. The *ideal extension* of R by S (also known as the *Dorroh extension*) $D(R, S)$ is the ring that has the abelian group structure of $R \oplus S$ and multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2, r_1s_2 + r_2s_1 + s_1s_2)$, where r_1, $r_2 \in R$ and $s_1, s_2 \in S$.

Theorem 3.2. *A ring R is weakly symmetric if and only if $D(\mathbb{Z}, R)$ is weakly symmetric.*

Proof. Necessity: Let $(n, r), (m, s), (k, t) \in D(\mathbb{Z}, R)$ with (n, r)

$(m, s)(k, t)$ nilpotent. Then

$$(n, r)(m, s)(k, t) = (nmk, nmt + (ns + mr + rs)(k1 + t)) \qquad (*)$$

is nilpotent. Hence $nmk = 0$. It follows that $n = 0$ or $m = 0$ or $k = 0$. To get rid of confusion, for any integer x, in the proof, x will also denote the element $x1$ of the ring R. We divide the proof some cases as the following.

CASE I. Let $n = 0$. By (*), we have $r(m1 + s)(k1 + t) \in R$ is nilpotent. By hypothesis, $Rr(k1 + t)a(m1 + s)$ is a nil left ideal of R for each $a \in R$. Note that an element $(x, y) \in D(\mathbb{Z}, R)$ is nilpotent if and only if y is nilpotent and $x = 0$. We claim that $D(\mathbb{Z}, R)(0, r)(k, t)(u, a)(m, s)$ is a nil left ideal of $D(\mathbb{Z}, R)$. Because, for any $(v, b), (u, a) \in D(\mathbb{Z}, R)$,

$$(v, b)(0, r)(k, t)(u, a)(m, s) = \big(0, (v1 + b)r(k1 + t)(u1 + a)(m1 + s)\big)$$

is nilpotent.

CASE II. $m = 0$ and CASE III. $k = 0$. In either cases, we reach similarly that $D(\mathbb{Z}, R)(n, r)(k, t)(u, a)(m, s)$ is a nil left ideal for each $(u, a) \in D(\mathbb{Z}, R)$.

Sufficiency: Assume that $D(\mathbb{Z}, R)$ is weakly symmetric. Then R is weakly symmetric since R is isomorphic to the subring $\{(0, r) \mid r \in R\} \subseteq D(\mathbb{Z}, R)$ which is weakly symmetric as a subring of weakly symmetric ring $D(\mathbb{Z}, R)$. □

Let R be a ring and S a multiplicatively closed subset of R consisting of the identity 1 and some central regular elements, that is, for any element $s \in S$ and $r \in R$, $sr = 0$ implies that $r = 0$ and s is in the center of R. Consider the ring $S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$. We end this paper by obtaining a characterization of weakly symmetricity of the ring $S^{-1}R$.

Theorem 3.3. *Let R be a ring. Then R is weakly symmetric if and only if $S^{-1}R$ is weakly symmetric.*

Proof. Necessity: Let $s_1^{-1}r_1$, $s_2^{-1}r_2$, $s_3^{-1}r_3 \in S^{-1}R$ with $(s_1^{-1}r_1)(s_2^{-1}r_2)(s_3^{-1}r_3)$ nilpotent. Then $(s_1^{-1}r_1)(s_2^{-1}r_2)(s_3^{-1}r_3) = (s_1s_2s_3)^{-1}(r_1r_2r_3)$ is nilpotent. So $r_1r_2r_3$ is nilpotent. By hypothesis, $Rr_1r_3tr_2$ is a nil left ideal of R for each $t \in R$. Let $s^{-1}a \in S^{-1}R$. For any $s_4^{-1}r_4 \in S^{-1}R$, $(s_4^{-1}r_4)(s_1^{-1}r_1)(s_3^{-1}r_3)(s^{-1}a)(s_2^{-1}r_2) = (s_4s_1s_3ss_2)^{-1}(r_4r_1r_3ar_2)$ is nilpotent since $(s_4s_1s_3ss_2)^{-1}$ is central and $r_4r_1r_3ar_2$ is nilpotent. Hence $S^{-1}R(s_1^{-1}r_1)(s_3^{-1}r_3)(s^{-1}a)(s_2^{-1}r_2)$ is a nil left ideal of $S^{-1}R$.

Sufficiency: Assume that $S^{-1}R$ is weakly symmetric. Then R is weakly symmetric since R is isomorphic to the subring $\{1r \mid r \in R\}$ of $S^{-1}R$ which is weakly symmetric.

Corollary 3.4. *Let R be a ring. Then $R[x]$ is weakly symmetric if and only if $R[x;x^{-1}]$ is weakly symmetric.*

Proof. Assume that $R[x;x^{-1}]$ is weakly symmetric. This way is clear since $R[x]$ is isomorphic to a subring of weakly symmetric ring $R[x;x^{-1}]$. Conversely, let $S = \{1, x, x^2, \dots\}$. Clearly S is a multiplicatively closed subset of $R[x]$. Note that $R[x;x^{-1}] = S^{-1}R[x]$. It follows that $R[x;x^{-1}]$ is weakly symmetric by Theorem 3.3. $\square$

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Department of Mathematics
Ankara University
Ankara
Turkey.

E-mail: bungor@science.ankara.edu.tr

Department of Mathematics
Ahi Evran University
Kirsehir
Turkey.

E-mail: handan.kose@ahievran.edu.tr

Department of Mathematics
Bilkent University
Ankara
Turkey.

E-mail: yosum@fen.bilkent.edu.tr

Department of Mathematics
Hacettepe University
Ankara
Turkey.

E-mail: harmanci@hacettepe.edu.tr