On smooth manifolds with the homotopy type of a homology sphere

Mehmet Akif Erdal

Department of Mathematics, Bilkent University, Ankara, 06800, Turkey

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ABSTRACT

In this paper we study $\mathcal{M}(X)$, the set of diffeomorphism classes of smooth manifolds with the simple homotopy type of X, via a map Ψ from $\mathcal{M}(X)$ into the quotient of $K(X) = [X, BSO]$ by the action of the group of homotopy classes of simple self equivalences of X. The map Ψ describes which bundles over X can occur as normal bundles of manifolds in $\mathcal{M}(X)$. We determine the image of Ψ when X belongs to a certain class of homology spheres. In particular, we find conditions on elements of $K(X)$ that guarantee they are pullbacks of normal bundles of manifolds in $\mathcal{M}(X)$.

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1. Introduction

Unless otherwise stated, by a manifold we mean a smooth, oriented, closed manifold with dimension greater than or equal to 5. Given a simple Poincaré complex X with formal dimension m, a classical problem in topology is to understand the set of diffeomorphism classes of smooth manifolds in the simple homotopy type of X. For such an aim, a fundamental object to study is the smooth simple structure set $\mathcal{S}^s(X)$ (see [1] page 125–126 for notation and details). Elements of $\mathcal{S}^s(X)$ are equivalence classes of simple homotopy equivalences $\omega : M \to X$ from an m-dimensional manifold M. Two such homotopy equivalences $\omega_1 : M_1 \to X$ and $\omega_2 : M_2 \to X$ are said to be equivalent if there is a diffeomorphism $g : M_1 \to M_2$ such that ω_1 is homotopic to the composition $\omega_2 \circ g$. An element of $\mathcal{S}^s(X)$ is called a simple smooth manifold structure on X. Note that composition of an element in $\mathcal{S}^s(X)$ with a simple self equivalence of X gives another element in $\mathcal{S}^s(X)$, although the manifold is still the same. Hence, we need to quotient out simple self equivalences of X in order to get the set of diffeomorphism classes of smooth manifolds in the simple

homotopy type of X. Denote by $\mathrm{Aut}_s(X)$ the group of homotopy classes of simple self equivalences of X. Then $\mathrm{Aut}_s(X)$ acts on $\mathcal{S}^s(X)$ by composition. The set of diffeomorphism classes of smooth manifolds in the simple homotopy type of X, $\mathcal{M}(X)$, is defined as the set of orbits of $\mathcal{S}^s(X)$ under the action of $\mathrm{Aut}_s(X)$, i.e. $\mathcal{M}(X) := \mathcal{S}^s(X)/\mathrm{Aut}_s(X)$.

Let $K(X)$ denote the group of homotopy classes of maps $[X, BSO]$ (here, we abandon the traditional notation $\widetilde{KSO}(X)$ for simplicity). Every simple homotopy equivalence $X \to X$ induces an automorphism on $K(X)$. Let $\overline{\mathrm{Aut}_s}(K(X))$ denote the subgroup of $\mathrm{Aut}(K(X))$ that consist of automorphisms induced by the simple self equivalences of X. There is a canonical action of $\overline{\mathrm{Aut}_s}(K(X))$ on $K(X)$ again given by composition. We denote by $\mathfrak{K}(X)$ the set of orbits of $K(X)$ under the action of $\overline{\mathrm{Aut}_s}(K(X))$.

As pointed out in [1] computations of $\mathrm{Aut}_s(X)$ and $\mathcal{S}^s(X)$ are in general difficult, so does the computation of $\mathcal{M}(X)$. On the other hand, computations of $K(X)$ and $\overline{\mathrm{Aut}_s}(K(X))$ are easier in most cases as $K(-)$ is a (generalized) cohomology group (see [2]). In this paper, we compare $\mathfrak{K}(X)$ with $\mathcal{M}(X)$ where X belongs to a certain class of Poincaré complexes.

There is a map $\Psi : \mathcal{M}(X) \to \mathfrak{K}(X)$ defined by $[\omega : M \to X] \mapsto (\omega^{-1})^*(\nu)$ where $\nu : M \to BSO$ denotes the normal bundle of M (see Proposition 2.2). For a prime q, by a $\mathbb{Z}/q$-homology m-sphere we mean a simple Poincaré complex X of formal dimension m such that $H_*(X; \mathbb{Z}/q) \cong H_*(S^m; \mathbb{Z}/q)$. For a general reference to Poincaré complexes we refer to [3] and [4]. Our purpose is to determine the image of Ψ for certain such homology spheres. Here we also assume that such a homology sphere admits a degree one normal map (equivalently the Spivak normal fibration has a vector bundle reduction), since otherwise the problem is trivial.

Let m be an odd number and S be a subset of the set of primes between $(m+4)/4$ and $(m+2)/2$. Denote by $\mathfrak{K}(X)_{(S,\mathsf{q}_1)} \subset \mathfrak{K}(X)$ the set of orbits in $\mathfrak{K}(X)$ that can be represented by elements ξ such that for each p in S the first $\mathrm{mod}\, p$ Wu class of ξ satisfy the identity $\mathsf{q}_1^p(\xi) + \mathsf{q}_1^p(X) = 0$ (see [5] or [6]). Note that $K(S^m) \cong 0$ for m odd and $m \neq 1(\mathrm{mod}\, 8)$. The main object of this paper is to prove the following:

Theorem 1.1. *Let m be an odd number and S be a subset of the set of primes between $(m+4)/4$ and $(m+2)/2$. Let X with a given map $f : S^m \to X$ be a $\mathbb{Z}/q$-homology m-sphere, so that f is a $\mathbb{Z}/q$-homology isomorphism for every prime $q < (m+2)/2$ with $q \notin S$. Assume further that $\pi_1(X)$ is of odd order. Then the image of Ψ consists of orbits in $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$ that are represented by elements in the kernel of $f^* : K(X) \to K(S^m)$. In particular, if $m \neq 1(\mathrm{mod}\, 8)$, then the image of Ψ is $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$. Furthermore, if $S = \emptyset$, then Ψ is surjective.*

Observe that as S gets larger, the image of Ψ gets smaller. In particular, if $T = \{q \text{ prime } : q < \frac{m+2}{2}\}\backslash S$, then we do not need to make the assumption of Theorem 1.1 on the $\mathrm{mod}\, q$ Wu classes for the primes $q \in T$. On the other hand, for an odd prime q, X being a $\mathbb{Z}/q$-homology sphere implies there is no q-torsion in $K(X)$. Hence, primes in T also affect the image of Ψ.

It is well-known, due to [7], that a degree one normal map can be surgered to a simple homotopy equivalence if and only if the associated surgery obstruction vanishes. The main result in [8] states that if $\pi_1(X)$ is of odd order, then the odd dimensional surgery obstruction groups, $L_m^s(\mathbb{Z}[\pi_1(X)])$, vanish, i.e. every degree one normal map can be surgered to a simple homotopy equivalence. An essential step in the proof of Theorem 1.1 is that, under the stated conditions, an element in $K(X)$ admits a degree one normal map if an only if it is in the kernel of f^* and it has the same $\mathrm{mod}\, p$ Wu classes as the Spivak normal fibration of X for each $p \in S$. In particular, if $m \neq 1(\mathrm{mod}\, 8)$, then bundles admitting degree one normal maps are completely determined by their $\mathrm{mod}\, p$ Wu classes for $p \in S$. In the case when $S = \emptyset$ and $m \neq 1(\mathrm{mod}\, 8)$ every stable vector bundle over X admits a degree one normal map. The rest is to determine the action of $\overline{\mathrm{Aut}_s}(K(X))$ on $K(X)$, which is given by the restriction of the canonical action of $\mathrm{Aut}(K(X))$.

Some examples of such X come from the smooth spherical space forms. Some applications of our theorem are discussed in Section 4. Smith theory may also provide examples, although we do not mention any such example in this note.

2. Notation and preliminaries

Let X be a simple Poincaré complex with formal dimension m. Given a stable vector bundle $\xi : X \to BSO$, a ξ-manifold is a manifold whose stable normal bundles lifts to X through ξ, we refer [9] for more details (in [9] such objects are called (B, f)-manifolds). We denote by $\Omega_k(\xi)$ the cobordism group of k-dimensional ξ-manifolds. An element of $\Omega_k(\xi)$ is often denoted by $[\rho : M \to X]$, where M is a k-dimensional manifold and ρ is a lifting of its stable normal bundle to X through ξ, and brackets denote the homotopy class of such liftings (see [10] Proposition 2 for this notation). Such a map ρ is called a *normal map*, and if degree of ρ is equal to 1, i.e. $\rho_*[M] = [X] \in H_m(X; \mathbb{Z})$, then it is called a *degree one normal map* (see [11] Definition 3.46). Due to the Pontrjagin–Thom construction, the group $\Omega_k(\xi)$ is isomorphic to k-th homotopy group of $M\xi$, the Thom spectra associated to ξ [12].

Our primary tool is the James spectral sequence, which is a variant of the Atiyah–Hirzebruch spectral sequence (see [10], Section II). Let h be a generalized homology theory represented by a connective spectrum, $F \to X \xrightarrow{f} B$ be an h-orientable fibration with fiber F and $\xi : X \to BSO$ be a stable vector bundle. The James spectral sequence for h, f and ξ has E_2-page $E^2_{s,t} = H_s(B; h_t(M\xi|_F))$ and converges to $h_{s+t}(M\xi)$. In the case when h is the stable homotopy, the edge homomorphism of this spectral sequence coming from the base line is as follows:

Proposition 2.1 *(see [10] Proposition 2). The edge homomorphism of the James spectral sequence for stable homotopy, $f : X \to B$ and $\xi : X \to BSO$ is a homomorphism $\mathfrak{ed} : \Omega_n(\xi) \to H_n(B, \mathbb{Z})$ given by*

$$\mathfrak{ed}[\rho : M \to X] = f_* \circ \rho_*[M]$$

for every element $[\rho : M \to X] \in \Omega_n(\xi)$.

The Atiyah–Hirzebruch spectral sequences for $M\xi$ is isomorphic to the James spectral sequence for stable homotopy, $id : X \to X$ and $\xi : X \to BSO$. This follows from the fact that $M\xi|_*$ is the sphere spectrum. This isomorphism is given by the Thom isomorphism (see proof of Proposition 1 in [10]). In this paper, we will only use this edge homomorphism of the James spectral sequence for the stable homotopy, the identity map $id : X \to X$ and a given stable vector bundle $\xi : X \to BSO$. In this case $\mathfrak{ed} : \Omega_n(\xi) \to H_n(X, \mathbb{Z})$ is the map given by $[\rho : M \to X] \mapsto \rho_*[M]$ for every element $[\rho : M \to X] \in \Omega_n(\xi)$. The other edge homomorphism for this spectral sequence will be denoted by $\bar{\mathfrak{ed}} : \pi_*(\mathbb{S}) \to \pi_*(M\xi)$, where $\mathbb{S}$ denotes the sphere spectrum.

Recall that $\mathfrak{K}(X)$ denotes the quotient $K(X)/\overline{\mathrm{Aut}_s}(K(X))$ (see Section 1). Let $\Phi : K(X) \to \mathfrak{K}(X)$ be the quotient map. We define a map Ψ from $\mathcal{M}(X)$ to $\mathfrak{K}(X)$ as follows: Let M be a smooth manifold equipped with a simple homotopy equivalence $\omega : M \to X$ and let ν be the stable normal bundle of M. Let $g : X \to M$ be the homotopy inverse of ω. Then the pullback bundle $g^*(\nu)$ defines an element in $K(X)$. If $[M]$ is the diffeomorphism class of M in $\mathcal{M}(X)$, we define $\Psi[M] := \Phi(g^*([\nu]))$.

Proposition 2.2. *Ψ is well defined.*

Proof. Let K be another manifold in the orbit $[M]$ with normal bundle κ, with a diffeomorphism $t : K \to M$ and with a simple homotopy equivalence $h : X \to K$. Since $t^*([\nu]) = [\kappa]$, we have $h^*t^*([\nu]) = h^*([\kappa])$. Since $\omega : M \to X$ is the homotopy inverse of g, we have $h^*([\kappa]) = h^*t^*([\nu]) = h^*t^*\omega^*g^*([\nu])$. Hence, $h^*([\kappa])$ and $g^*([\nu])$ differ by an automorphism in $\overline{\mathrm{Aut}_s}(K(X))$ as the composition $\omega \circ t \circ h$ is homotopic to a simple self homotopy equivalence of X. By definition, in $\mathfrak{K}(X)$ they are the same. $\square$

Let $p = 2b + 1$ be an odd prime. For a vector bundle, or in general a spherical fibration, ξ over X, there exist cohomology classes $\mathsf{q}_k^p(\xi)$ in $H^{4bk}(X; \mathbb{Z}/p)$, known as mod p Wu classes, introduced in [5]. We write q_k

instead of q_k^p if the prime we consider is clear from the context. These classes are defined by the identity $\mathsf{q}_k(\xi) = \theta^{-1}\mathcal{P}^k\theta(1)$. Here, $\mathcal{P}^n$ denotes the Steenrod's reduced p-th power operation and $\theta : H^*(X; \mathbb{Z}/p) \to H^*(T\xi; \mathbb{Z}/p)$ denotes the Thom isomorphism. For more details on $\mathrm{mod}\, p$ Wu classes we refer to [6], Ch.19.

For each prime p, let $\mathsf{q}_1^p(X)$ denote the negative of the $\mathrm{mod}\, p$ Wu class of ν_X, the Spivak normal fibration of X (which exists since X is a finite complex, see [4]). Given S a set of primes, let $K(X)_{(S,\mathsf{q}_1)}$ denote the subset of $K(X)$ that consist of elements ξ such that for each p in S the first $\mathrm{mod}\, p$ Wu class of ξ satisfies the identity $\mathsf{q}_1^p(\xi) + \mathsf{q}_1^p(X) = 0$ (or equivalently $\mathsf{q}_1^p(\xi) = \mathsf{q}_1^p(\nu_X)$). Since the class $\mathsf{q}_1^p(X)$ is a homotopy type invariant of X (see [6] Ch. 19), the subset $K(X)_{(S,\mathsf{q}_1)}$ is invariant under the action of $\overline{\mathrm{Aut}_s}(K(X))$. We denote the quotient of this action by $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$. In particular, if $S = \emptyset$, then $\mathfrak{K}(X)_{(S,\mathsf{q}_1)} = \mathfrak{K}(X)$.

Notation 2.3. $E_{*,*}^*(\xi)$ will denote the James spectral sequence for the stable homotopy as the generalized homology theory, identity fibration $id : X \to X$ and the stable bundle $\xi : X \to BSO$. The abbreviations JSS and AHSS will be used for the James and Atiyah–Hirzebruch Spectral sequences respectively. For any finite spectrum E, $E_q^\wedge$ will denote the q-nilpotent completion of E at the prime q (also called localization at $\mathbb{Z}/q$, corresponding to localization at the Moore spectrum $M\mathbb{Z}_q$ of $\mathbb{Z}/q$), see [13].

3. Main results

Let X be a simple Poincaré complex with formal dimension m. We impose the following condition on a stable vector bundle $\xi : X \to BSO$:

Condition 3.1. For each $r \le m$ the differential $d^r : E_{m,0}^r(\xi) \to E_{m-r,r-1}^r(\xi)$ in the James spectral sequence is zero.

Observe that the image of the edge homomorphism of $E_{*,*}^*(\xi)$ in $H_m(X, \mathbb{Z})$ is the intersection of the kernels of all of the differentials with source $E_{m,0}^*(\xi)$, i.e. $im(\mathfrak{ed}) = \bigcap_r \ker(d^r)$. Thus, Condition 3.1 implies that the group $E_{m,0}^2(\xi) = H_m(X; \mathbb{Z})$ is equal to $E_{m,0}^\infty(\xi)$, i.e. edge homomorphism is surjective. For a given class $[\rho : M \to X]$ in $\Omega_m(\xi)$ we have $\mathfrak{ed}[\rho : M \to X] = \rho_*[M]$. Therefore, we can find a class $[\rho : M \to X]$ in $\Omega_m(\xi)$ such that $\mathfrak{ed}[\rho : M \to X] = \rho_*[M]$ is a generator of $H_m(X; \mathbb{Z})$ with the preferred orientation. As a result, we get a degree one normal map $\rho : M \to X$, i.e. we have a surgery problem.

If Condition 3.1 does not hold for ξ, i.e. we have a non-trivial differential $d^r : E_{m,0}^r(\xi) \to E_{m-r,r-1}^r(\xi)$ for some r, then the edge homomorphism can not be surjective. This means $\rho_*[M]$ can not be a generator of $H_m(X; \mathbb{Z})$, i.e. ρ can not be a degree one map. Hence, there is not a degree one normal map that represents a class in $\Omega_m(\xi)$. As a result, there is not a manifold simple homotopy equivalent to X whose stable normal bundle lifts to X through ξ. Hence, we have the following lemma:

Lemma 3.2. *A stable vector bundle ξ admits a degree one normal map if and only if Condition 3.1 holds for ξ.*

For the JSS for ξ, $E_{*,*}^*(\xi)$, there is a corresponding (isomorphic) AHSS for the Thom spectrum $M\xi$, i.e. the AHSS whose E^2-page is $H_*(M\xi, \pi_*^S(*))$ which converges to $\pi_*(M\xi)$, with the isomorphism given by the Thom isomorphism. For a given prime q, it is well known that the q-primary part of π_k^S is zero whenever $0 < k < 2q - 3$ (see [14]). We use finiteness of π_k^S [15]. On each $\mathrm{mod}\, q$ torsion part, the first non-trivial differentials of the AHSS are given by the duals of the stable primary cohomology operations. Due to Wu formulas, when we pass to the JSS we need to know the action of Steenrod algebra on the Thom class. For $p = 2$ the action of Steenrod squares on the Thom class $U \in H^*(M\xi; \mathbb{Z}/2)$ is determined by the Stiefel–Whitney classes. In fact the $(\mathrm{mod}\, 2)$ Wu formula asserts that $Sq^i(U) = U \cup w_i$ (see [6], p. 91).

Let S/q denote the homology theory given by $\mathbb{S}_q^\wedge$. Let E be a spectrum. Consider the AHSS for the homology theory S/q, i.e. the coefficient groups will be $\pi_*(\mathbb{S}_q^\wedge)$. Due to naturality of the AHSS, the first

non-zero differentials have to be stable primary cohomology operations independent of the generalized cohomology theory, see pp. 208 [2]. For each i with $0 < i < 2q - 3$ we have $\pi_i(\mathbb{S}_q^\wedge) = 0$ and $\pi_{2q-3}(\mathbb{S}_q^\wedge) = \mathbb{Z}/q$. Thus, the first non-trivial differential in this AHSS appears at the $(2q - 2)$-th page. This differential has to be a stable primary cohomology operation. The only $\bmod q$ operations in this range are 0 and dual of the $\bmod q$ Steenrod operation $\mathcal{P}^1$. As in the proof of Lemma in [10, pp. 751], letting $E = \Sigma^{2q-2} H\mathbb{Z}/p$ as a test case, one can see that d^{2q-2} is not always zero. The d^2 differential in $E^*_{*,*}(\xi)$ is given by the dual of the map $x \mapsto Sq^2(x) + w_2(\xi) \cup x$, see [10] Proposition 1. Let us write q_1 for q_1^q, where q is a fixed odd prime. We obtain a similar formula for the first non-zero differentials in $E^*_{*,*}(\xi)$ acting on $\bmod q$ torsion part.

Lemma 3.3. *For each $n \geq 2q - 2$ the differential on the $\bmod q$ part $d^{2q-2} : E_{n,0}^{2q-2}(\xi) \to E_{n-2q+2,2q-3}^{2q-2}(\xi)$ is equal to the dual of the map*

$$\delta : H^{n-2q+2}(X; \mathbb{Z}/q) \to H^n(X; \mathbb{Z}/q)$$

defined as $x \mapsto \mathcal{P}^1(x) + \mathsf{q}_1(\xi) \cup x$, composed with $\bmod q$ reduction.

Proof. Consider the AHSS for $M\xi$ and S/q. In this case the coefficient groups of the AHSS will be $\pi_*(\mathbb{S}_q^\wedge)$ and it will converge to $\pi_*(M\xi_q^\wedge)$. From above remarks, the differential d^{2q-2} in the AHSS for $M\xi$ and S/q, is the dual of the $\bmod q$ Steenrod operation

$$\mathcal{P}^1 : H^{n-2q+2}(M\xi, \mathbb{Z}/q) \to H^n(M\xi, \mathbb{Z}/q).$$

By the Thom isomorphism theorem an element of $H^*(M\xi, \mathbb{Z}/q)$ is of the form $U \cup \sigma$ where $\sigma \in H^*(X; \mathbb{Z}/q)$ and U is the Thom class. On the passage to the JSS, Cartan's formula implies

$$\mathcal{P}^1(U \cup \sigma) = U \cup \mathcal{P}^1(\sigma) + \mathcal{P}^1(U) \cup \sigma = U \cup \mathcal{P}^1(\sigma) + U \cup \mathsf{q}_1(\xi) \cup \sigma$$

hence in the James spectral sequence these differentials become duals of the maps $\sigma \mapsto \mathcal{P}^1(\sigma) + \mathsf{q}_1(\xi) \cup \sigma$ composed with $\bmod q$ reduction. □

We have the following lemma for the differential d^m with source $E_{m,0}^m(\xi)$:

Lemma 3.4. *Let q be a prime and m be an odd number. Let X be a $\mathbb{Z}/q$-homology sphere with a given $\mathbb{Z}/q$-homology isomorphism $f : S^m \to X$. Then for any stable vector bundle $\xi : X \to BSO$ that is in the kernel of $f^* : K(X) \to K(S^m)$, the image of the differential d^m in $E_{0,m-1}^m(\xi)$ has trivial q-torsion.*

Proof. Let $\epsilon : S^m \to BSO$ be the stable vector bundle given by the composition $\xi \circ f$. The map $f : S^m \to X$ induces a map of spectra $Mf : M\epsilon \to M\xi$. Since f is a $\mathbb{Z}/q$-homology isomorphism, the induced map is also $\mathbb{Z}/q$-homology isomorphism, due to Thom isomorphism. Both $M\xi$ and $M\epsilon$ are connective spectra and of finite type. The space X is q-good (see Definition I.5.1 in [16]) due to 5.5 of [17]. Thus, the induced map $Mf_q^\wedge : M\epsilon_q^\wedge \to M\xi_q^\wedge$ is a homotopy equivalence (see [13] Proposition 2.5 and Theorem 3.1). We have $f^*(\xi) = 0$ in $K(S^m)$. Hence, ϵ is a trivial bundle. The Thom spectrum $M\epsilon$ is then homotopy equivalent to the wedge of spectra $\mathbb{S} \vee \Sigma^m \mathbb{S}$, as it is the suspension spectrum of $S^m \vee S^0$. Recall that $E_{*,*}^*(\epsilon)$ is the JSS for the identity fibration $S^m \to S^m$ and the trivial stable vector bundle. Hence, $E_{*,*}^*(\epsilon)$ collapses on the second page. As a result, q-torsion in $E_{0,m-1}^m(\epsilon)$ survives to the $E_{0,m-1}^\infty(\epsilon)$. The result follows by comparing the q-torsion in $E_{0,m-1}^m(\epsilon)$, via Mf, with the q-torsion in $E_{0,m-1}^m(\xi)$. □

The following lemma is a partial converse of Lemma 3.4:

Lemma 3.5. *Assume $q = 2$ and $m = 1(\mathrm{mod}\, 8)$ in Lemma 3.4. Then, if $\xi \notin \ker(f^*)$, then Condition 3.1 does not hold for $E_{*,*}^*(\xi)$.*

Proof. Suppose that $\xi \notin \ker(f^*)$ and Condition 3.1 holds for $E_{*,*}^*(\xi)$. Let $\xi \circ f = \mu : S^m \to BSO$ be the nontrivial element in $K(S^m)$. Then, by a comparison as in the proof of Lemma 3.4, Condition 3.1 holds for $E_{*,*}^*(\mu)$. Thus, there is a degree one normal map $\rho : M \to S^m$ representing a class in $\Omega_m(\mu)$. Since $L_m^s(\mathbb{Z}) = 0$, see [18], every such map can be surgered to a simple homotopy equivalence, $\tilde{\rho} : \tilde{M} \to S^m$. However, it is well known that every homotopy sphere is stably parallelizable (see [18]). Hence, we get a contradiction, as $\mu \circ \tilde{\rho}$ is nontrivial in $K(S^m)$. $\square$

Proof of Theorem 1.1. Since both $|\pi_1(X)|$ and m are odd, due to Theorem 1 in [8], the surgery obstruction groups vanish. Hence, every degree one normal map can be surgered to a homotopy equivalence. We will show that elements in $K(X)_{(S,\mathsf{q}_1)}$ are the ones that admit a degree one normal map.

Let $[\xi]$ be an orbit in $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$ represented by $\xi : X \to BSO$. Consider the James spectral sequence, $E_{*,*}^*(\xi)$. Let q be a prime with $q < (m+2)/2$ and $q \notin S$. Then X is a $\mathbb{Z}/q$-homology sphere. By Lemma 3.4 the image of d^m has trivial q-torsion. Since $E_{m-r,r-1}^r(\xi) = H_{m-r}(X; \pi_{r-1}^S)$ does not contain q-torsion when $r < m$, image of the differentials $d^r : E_{m,0}^r(\xi) \to E_{m-r,r-1}^r(\xi)$ have trivial q-torsion as well. Hence, all of the differentials based at $E_{m,0}^r(\xi)$ have trivial q-torsion in their images.

Now, let $p \in S$. Then $2p - 2 < m < 4p - 4$. It is well known that for $t < 4p - 5$ the p-torsion in π_t^S vanishes except when $t = 2p - 3$ (see for example [14], III Theorem 3.13, B). Since $2p - 2 < m < 4p - 4$, $E_{0,m-1}^m$ has trivial p-torsion, we have $d^m = 0$. Hence, the only differential whose image can contain mod-p torsion appears at degree $2p - 2$. By Lemma 3.3, the differential d^{2p-2} on the p-torsion part is equal to the dual of the map

$$\delta : H^{m-2p+2}(X; \mathbb{Z}/p) \to H^m(X; \mathbb{Z}/p)$$

defined as $x \mapsto \mathcal{P}^1(x) + \mathsf{q}_1(\xi) \cup x$, composed with $(\mathrm{mod}\, p)$ reduction. Let x be an element in $H^{m-2p+2}(X; \mathbb{Z}/p)$. By Poincaré duality, there exists an element s in $H^{2p-2}(X; \mathbb{Z}/p)$ such that $\mathcal{P}^1(x) = s \cup x$ (see [19] Section 2). By definition $s = \mathsf{q}_1(X)$. Then d^{2p-2} is trivial on mod-p torsion as ξ is an element in $K(X)_{(S,\mathsf{q}_1)}$, i.e. as $\mathsf{q}_1^p(X) + \mathsf{q}_1^p(\xi) = 0$.

Now, assume that $\mathsf{q}_1^p(X) + \mathsf{q}_1^p(\xi) \neq 0$. Then by Poincaré duality there exist an element $a \in H^{m-2p+2}(X; \mathbb{Z}/p)$ such that $a \cup (\mathsf{q}_1^p(X) + \mathsf{q}_1^p(\xi))$ is nonzero in $H^m(X; \mathbb{Z}/p)$, i.e. $d^{2p-2} \neq 0$.

As a result, Condition 3.1 holds for ξ if and only if $\xi \in K(X)_{(S,\mathsf{q}_1)} \cap \ker(f^*)$. By Lemma 3.2 $\xi \in K(X)_{(S,\mathsf{q}_1)} \cap \ker(f^*)$ if and only if ξ admits a degree one normal map. Hence, we can do surgery and obtain a smooth manifold M with a simple homotopy equivalence $\omega : M \to X$ that represent a class in $\Omega_m(\xi)$ if and only if $\xi \in K(X)_{(S,\mathsf{q}_1)} \cap \ker(f^*)$. By Lemma 3.4, the image of Ψ consists of orbits in $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$ represented by elements in the kernel of $f^* : K(X) \to K(S^m)$.

In the case when $m \neq 1(\mathrm{mod}\, 8)$ we have $K(S^m) = 0$ by Bott periodicity, i.e. image of Ψ is $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$. If $S = \emptyset$, then $\mathfrak{K}(X)_{(S,\mathsf{q}_1)} = \mathfrak{K}(X)$, i.e. Ψ is a surjection. This completes the proof. $\square$

The following corollary is essentially stronger than the main result.

Corollary 3.6. *Let m and S be as in Theorem 1.1. Suppose that X and X' are $\mathbb{Z}/q$-homology m-spheres fitting into a zigzag $S^m \xrightarrow{f} X' \xleftarrow{g} X$, so that both f and g are $\mathbb{Z}/q$-homology isomorphisms for every prime $q < (m+2)/2$ with $q \notin S$. If $\pi_1(X)$ is a free product of finitely many odd order groups, then the image of Ψ contains all orbits in $\mathfrak{K}(X)_{(S,\mathsf{q}_1)}$ which are represented by elements in $g^*(\ker(f^*))$, where f^* and g^* are the induced maps $K(S^m) \xleftarrow{f^*} K(X') \xrightarrow{g^*} K(X)$.*

Proof. Assume $q \notin S$ with $q < (m+2)/2$. For a bundle ξ' over X', Condition 3.1 holds for ξ' if and only if $\xi' \in K(X')_{(S,\mathfrak{q}_1)} \cap \ker(f^*)$ for each q. One can compare spectral sequences for ξ' and $g^*(\xi')$ as in the proof of Lemma 3.4, and show that d^m differential is trivial on $E^r_{m,0}(g^*(\xi'))$ whenever $\xi' \in \ker(f^*)$ for each such prime q. Thus, Condition 3.1 holds for $g^*(\xi')$, whenever $\xi' \in K(X')_{(S,\mathfrak{q}_1)} \cap \ker(f^*)$. Repeating the arguments of Lemma 3.5, one can show that if $\xi' \notin \ker(f^*)$, then Condition 3.1 does not hold for $g^*(\xi')$. The result follows from Theorem 5 in [20], together with Lemma 3.2 above. □

Corollary 3.6, for example, allows us to do similar estimations for connected sums of manifolds satisfying the assumptions of Theorem 1.1.

Remark 3.7. It can be seen from the proof of Theorem 1.1 (resp. Corollary 3.6) that we do not need a single map f (or g) which is simultaneously a $\mathbb{Z}/q$-homology isomorphism for every prime $q < (m+2)/2$ with $q \notin S$. It is enough that for every prime $q < (m+2)/2$ with $q \notin S$ there exist maps f_q and g_q (depending on q) which are $\mathbb{Z}/q$-homology isomorphisms. In this case, we need to replace $\ker(f^*)$ (or $g^*(\ker(f^*))$) by intersection over q of all $\ker(f_q^*)$ (or $g_q^*(\ker(f_q^*))$).

Let Θ_m denote the group of homotopy m-spheres. For any smooth m-manifold M, there is a subgroup $I(M)$ of Θ_m called the inertia group of M, defined as $\{\Sigma \in \Theta_m : \Sigma \# M \cong M\}$, where $\cong$ here means diffeomorphic (see [21]). Two manifolds M_1 and M_2 are said to be almost diffeomorphic if there is a $\Sigma \in \Theta_m$ such that $M_1 \# \Sigma^m \cong M_2$. It is known that almost diffeomorphic manifolds have isomorphic stable normal bundles, as homotopy spheres are stably parallelizable (see [18]). Thus, their images are the same under Ψ. In order to determine the set of manifolds that are almost diffeomorphic to M, one needs to compute $I(M)$. Hence, to determine $\mathcal{M}(X)$, it is necessary to know $I(M)$ for every $[M]$ in $\mathcal{M}(X)$. It is known that $I(M)$ is not a homotopy type invariant, in fact it is not even a PL-homeomorphism type invariant, see for example [22]. As a result, complete determination of $\mathcal{M}(X)$ may not be possible in this generality.

The following corollary says that framed manifolds do not bound in $\Omega_m(\xi)$ for some $\xi : X \to BSO$.

Corollary 3.8. *Under the assumptions of Theorem 1.1 together with $m \neq 1 (\mathrm{mod}\, 8)$ and $S = \emptyset$, the edge map $\bar{\mathfrak{ed}} : \pi_*(\mathbb{S}) \to \pi_*(M\xi)$ is an inclusion for any stable vector bundle $\xi : X \to BSO$.*

Proof. As in the proof of Theorem 1.1 for a prime q the first differential in $E^*_{*,*}(\xi)$ that acts non-trivially on q-torsion appears in dimension $2q - 2$. Since $S = \emptyset$, X is a $\mathbb{Z}/q$-homology sphere for every prime q with $2q - 2 < m$. Hence, $d^r = 0$ for every $r < m$. As in the proof of Lemma 3.4, by comparing with $E^*_{*,*}(\epsilon)$ we get $d^m = 0$ (as $E^*_{*,*}(\epsilon)$ collapses at the second page, due to degree reasons). Therefore, the first nontrivial differential appears when $r \geq m + 1$. But then the target of d^r should be zero. Hence, $E^*_{*,*}(\xi)$ collapses at the second page, and we get that $\bar{\mathfrak{ed}} : \pi_*(\mathbb{S}) \to \pi_*(M\xi)$ is an inclusion. □

Observe that the degree of f (as in Theorem 1.1) plays the important role here, as it is co-prime to smaller primes. One can ask what the necessary and sufficient conditions are on the pair (X, ξ) so that the natural map $\bar{\mathfrak{ed}} : \pi_*^s \to \pi_*(M\xi)$ induced by the inclusion of point is injective. It is well known that such is not the case for classical Thom spectra like MO or MSO (see [12]). In the case when ξ is a trivial bundle, there are examples for which this is true. Another possible question is: For which spaces X, this natural map $\pi_*^s \to \pi_*(M\xi)$ is injective for every stable vector bundle $\xi : X \to BSO$. Corollary 3.8 provides just one such example.

Suppose that $q = 2$ and $m = 1 (\mathrm{mod}\, 8)$ in Lemma 3.4. In this case f induces the identity on cohomology with coefficients $\mathbb{Z}/2$. Bott periodicity theorem asserts that $K(S^m) = K^{-m}(S^0) = \mathbb{Z}/2$. The map f induces a map on the Atiyah–Hirzebruch spectral sequences. At the second page we have $f^* : H^m(X; \mathbb{Z}/2) \to H^m(S^m; \mathbb{Z}/2)$, which is an isomorphism. The mod-2 class in $H^m(S^m; \mathbb{Z}/2)$ survives to the infinity page of the Atiyah–Hirzebruch spectral sequence for $K(S^m)$. Hence, f^* is a surjection on the infinity page by the

naturality of the AHSS. It follows that $f^* : K(X) \to K(S^m)$ is surjective (for the case when X is a spherical space form, this follows from [23], Theorem 1-(b)). Hence, we have the following remark:

Remark 3.9. In the case when $q = 2$ and $m = 1 (\mathrm{mod}\, 8)$ in Lemma 3.4, we have $[K(X) : \ker(f^*)] = 2$.

Since $2 \notin S$ in Theorem 1.1, we have $[K(X) : \ker(f^*)] = 2$ as well. Thus, we can determine the image in the case when $m = 1 (\mathrm{mod}\, 8)$ as well.

4. Examples

Let $L^k(n)$ denote the quotient space $S^{2k+1}/\mathbb{Z}/n$ of the free linear action of $\mathbb{Z}/n$ on S^{2k+1}. If $(q, n) = 1$, then $L^k(n)$ is a $\mathbb{Z}/q$-homology sphere with the covering projection being the $\mathbb{Z}/q$-homology isomorphism. Let $L^k(n, \mu)$ denote the orbit space of a free action μ of $\mathbb{Z}/n$ on S^{2k+1} where μ acts by homeomorphisms. Such $L^k(n, \mu)$ are often called fake lens spaces (see [24] and [25], [26] for more details on topological and [27], [28] for smooth fake lens spaces). For any such action μ, one can always find a lens space $L^k(n)$ homotopy equivalent to $L^k(n, \mu)$ (see [26] P. 456). If p is a prime and k is an integer with $k \leq 2p - 3$, Theorem 1.1 applies to any fake lens space $L^k(p, \mu)$. If $k < p - 1$, then $S = \emptyset$ and if $p - 1 \leq k \leq 2p - 3$, then $S = \{p\}$ (note that dimension of $L^k(p, \mu)$ is $2k+1$). In this case, if $T = \{q \text{ prime} : q \leq k+1\}\backslash\{p\}$, then for any prime $q \in T$, $L^k(p, \mu)$ is a $\mathbb{Z}/q$ homology sphere and $K(L^k(p, \mu))$ does not have any element of order q for q odd. In general, if n is a natural number not divisible by primes less than or equal to $\frac{k+2}{2}$, μ is an action of $\mathbb{Z}/n$ on S^{2k+1}, $S = \{p \text{ prime} : p|n\}$ and $T = \{q \text{ prime} : q \leq k+1\}\backslash S$, then Theorem 1.1 applies to $L^k(n, \mu)$ where the image of Ψ consists of orbits in $\mathfrak{K}(L^k(n, \mu))_{(S, \mathfrak{q}_1)}$ that are represented by elements in $\ker(f^*)$, where f is the covering projection. Again, for $q \in T$ odd prime, the group $K(L^k(n, \mu))$ does not have any q-torsion. We refer to Theorem 2 in [29] for the K-theory of a lens space and to [30] for calculation of the group of homotopy classes of self homotopy equivalences of a lens space. For the particular cases when $k = p - 3$ and $k = 2p - 4$, we can get from Theorem 2.A in [30] that $\mathrm{Aut}_s(K(L^k(p, \mu)))$ has only 2-elements, namely the identity and the automorphism mapping an element to its algebraic inverse. Hence, in these cases each orbit (except the orbit of 0) in $\mathfrak{K}(L^k(p, \mu))$ has exactly two elements.

Another class of examples can be obtained from spherical space forms. There is a vast literature on classification of spherical space forms, see for example [31], [32] and [33]. Let Σ be a homotopy m-sphere with $m \geq 5$. Let π be a group that can act freely and smoothly on Σ and let $X = \Sigma/\pi$, so that $f : \Sigma \to X$ is a principal π-bundle. Let $p \geq 3$ be the smallest prime dividing the order of π. Then there is a map $\varphi : X \to B\pi$ that classifies f. The group of self equivalences $\mathrm{Aut}(X)$ of X contains a normal subgroup isomorphic to all inner automorphism $\mathrm{Inn}(\pi)$ of π (see [30] Corollary 1 and Theorem 1.4). Note that, an inner automorphism induces the identity on all (generalized) cohomology groups of $B\pi$, due to the commutativity in cohomology. Let $\alpha : X \to X$ in $\mathrm{Aut}(X)$ be a self equivalence. Consider the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & B\pi \\
\alpha \downarrow & & \downarrow \alpha_* \\
X & \xrightarrow{\varphi} & B\pi
\end{array}
$$

so that α and α_* induce the same map on $\pi_1(X) = \pi$. By an argument as in [34] Theorem 7.26, the diagram commutes up to homotopy. It is well-known that φ induces a surjection on K (see for example [23]). Thereby, inner automorphisms of π induce identity on $K(X)$ as well. Denote by $\mathrm{Out}(X) = \mathrm{Aut}(X)/\mathrm{Inn}(\pi)$ *the group of outer self-equivalences of X*. Then, in order to determine $\overline{\mathrm{Aut}_s}(K(X))$, we only need to consider automorphisms induced by self equivalences belonging to a fixed set of representatives of cosets in $\mathrm{Out}(X)$.

For $m \neq 1 (\mathrm{mod}\, 8)$ and $m < 2p - 2$, Theorem 1.1 applies to X, so that $\Psi : \mathcal{M}(X) \to \mathfrak{K}(X)$ is surjective. In this case $S = \emptyset$. If $m < 4p - 4$ and no other prime between p and $2p$ divides the order of π, then the image of Ψ is determined by the first mod-p Wu class of the Spivak normal bundle of X. In general, if S is the set of primes between p and $2p$ which divide the order of π, then the image of Ψ is $\mathfrak{K}(X)_{(S, \mathfrak{q}_1)}$. We refer to [23] for the computation of $K(X)$ (for $X = \Sigma / \pi$ as above) and the results given in [35] (and although indirectly, in [36]) for the computation of $\mathrm{Aut}(X)$. Of course we only need these computations when $\pi_1(X)$ is of odd order. The action of $\overline{\mathrm{Aut}_s}(K(X))$ on $K(X)$ is given by the restriction of the usual canonical action of the automorphism group $\mathrm{Aut}(K(X))$ on $K(X)$, which can be understood once $K(X)$ is known.

Let m and S be as in Theorem 1.1. Let X_0 and X_1 with given maps $f_i : S^m \to X_i$ for $i = 0, 1$ be two $\mathbb{Z}/q$-homology m-spheres, so that both f_0 and f_1 are $\mathbb{Z}/q$-homology isomorphisms for $q < (m + 2)/2$ with $q \notin S$, i.e. Theorem 1.1 applies to both X_0 and X_1. Then, $X_0 \# X_1$ is also a $\mathbb{Z}/q$-homology sphere for primes $q < (m + 2)/2$ with $q \notin S$ (which follows easily from Mayer–Vietoris sequence) and the fundamental group of $X_0 \# X_1$ is the free product $\pi_1(X_0) * \pi_1(X_1)$ (which follows from a simple application of Van Kampen's Theorem). Thus, Corollary 3.6 applies to the connected sum $X_0 \# X_1$. If there exists a map $f : S^m \to X_0 \# X_1$ satisfying the conditions of Theorem 1.1, then we can choose g as the identity map. In this case, $\mathrm{im}(\Psi)$ consists of all orbits in $\mathfrak{K}(X_0 \# X_1)_{(S, \mathfrak{q}_1)}$ which are represented by elements in $\ker(f^*)$. If we can not find such a map f, then we can apply Corollary 3.6 by using the zigzags $S^m \xrightarrow{f_i} X_i \xleftarrow{g_i} X_0 \# X_1$, where g_i's are the obvious collapse maps. In this case, $\mathrm{im}(\Psi)$ contains all orbits in $\mathfrak{K}(X_0 \# X_1)_{(S, \mathfrak{q}_1)}$ which are represented by elements in $g^*(\ker(f^*))$.

For a given finite CW-complex X, denote by $\overline{\mathrm{Aut}}(K^i(X))$ the subgroup of $\mathrm{Aut}(K^i(X))$ that consists of automorphisms induced by elements in $\mathrm{Aut}(X)$ (we simply write $\overline{\mathrm{Aut}}(K(X))$ when $i = 0$). Due to the naturality of suspension isomorphism and Bott periodicity, we can identify $\overline{\mathrm{Aut}}(K(X))$ with $\overline{\mathrm{Aut}}(K(\Sigma^{8i} X))$. Hence, the natural map from $\mathrm{Aut}(X)$ to $\overline{\mathrm{Aut}}(K(X))$ factors through the group of stable self equivalences of X, which is equal to $\mathrm{colim}_i \mathrm{Aut}(\Sigma^{8i} X)$ (see for example [37], [38] and [39] for more details about the group of stable self equivalences). If X is a $\mathbb{Z}/q$-homology sphere for a prime q, then all Betti numbers of X are less than or equal to 1. Thus, due to Theorem 1.1-(a) in [39], the group of stable self equivalences of X is finite, which implies that $\overline{\mathrm{Aut}_s}(K(X))$ (which is a subgroup of $\overline{\mathrm{Aut}}(K(X))$) is finite.

Acknowledgements

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