Probabilités/*Probability Theory*

Analytic and asymptotic properties of Linnik's probability densities

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Abstract – The analytic and asymptotic properties of the probability density $p_\alpha(x)$ introduced in 1953 by Ju. V. Linnik and defined by the characteristic function $1/(1+|t|^\alpha)$, $0<\alpha<2$, are studied. Expansions of $p_\alpha(x)$ into convergent and asymptotic series in terms of $\log|x|$, $|x|^{k\alpha}$, $|x|^k$ $(k=0, 1, 2, \ldots)$ are obtained. It turns out that the analytic structure of $p_\alpha(x)$ depends substantially on the arithmetical nature of the parameter α.

Les propriétés analytiques et asymptotiques des densités de probabilités de Linnik

Résumé – *Nous étudions les propriétés analytiques et asymptotiques des densités de probabilités* $p_\alpha(x)$ *introduites par Ju. V. Linnik en 1953 et définies à l'aide de leur fonction caractéristique,* $1/(1+|t|^\alpha)$, $0<\alpha<2$. *Nous obtenons des développements en séries convergentes et asymptotiques de termes* $\log|x|$, $|x|^{k\alpha}$, $|x|^k$, $k=0, 1, 2, \ldots$ *La structure analytique des* $p_\alpha(x)$ *dépend de façon substantielle de la nature arithmétique du paramètre* α.

Version française abrégée – En 1953, Ju. V. Linnik [1] a démontré que la fonction $\varphi_\alpha(t)=1/(1+|t|^\alpha)$, $0<\alpha<2$, est la fonction caractéristique d'une densité de probabilité. Cette densité, notée $p_\alpha(x)$, peut être vue comme une généralisation de la densité de Laplace $p_2(x)=e^{-|x|}/2$ qui correspond au cas $\alpha=2$. Récemment, les densités de Linnik $p_\alpha(x)$ ont connu un regain d'intérêt et des travaux ([2]-[7]) ont exploré leurs propriétés probabilistes.

Comme dans le cas des densités du type stable, la simplicité analytique des fonctions caractéristiques ne se transmet pas aux densités et nous ne connaissons pas de formules explicites pour les densités de Linnik. Le comportement asymptotique à l'infini des $p_\alpha(x)$ et des densités stables présente des similitudes. Cependant la structure analytique des $p_\alpha(x)$ diffère de celle des densités stables. Elles dépendent de façon substantielle de la nature arithmétique du paramètre α.

Le théorème suivant décrit le comportement asymptotique à l'infini des $p_\alpha(x)$.

THÉORÈME 1. – *Pour tout* $\alpha\in(0, 2)$, *le comportement asymptotique à l'infini de* $p_\alpha(x)$ *est donné par la série* (2). *De façon plus précise, pour tout* $N=1, 2, \ldots$, *la formule* (3) *est vraie où* $R_{N,\alpha}(x)$ *vérifie la condition* (4).

Il s'ensuit donc que $p_\alpha(x)$ a un taux de décroissance polynomiale à l'infini et de façon plus précise, la formule (5) est vraie.

Nous formulons maintenant des théorèmes liés à la structure analytique des $p_\alpha(x)$.

THÉORÈME 2. – *Pour tout* $\alpha\in(0, 2)$, *la densité* $p_\alpha(x)$ *est une fonction complètement monotone de* $x>0$.

Puisque $p_\alpha(x)$ est une fonction paire, elle est donc absolument monotone pour $x<0$.

Notons par Q^{odd} l'ensemble des α rationnels représentables sous la forme $\alpha=m/n$ où m et n sont des entiers avec de plus m impair. Notons L l'ensemble des nombres transcendants de Liouville. Le théorème suivant décrit la structure analytique des $p_\alpha(x)$ pour presque tous

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les α (pour la mesure de Lebesgue), bien que l'ensemble exceptionnel soit un ensemble partout dense dans (0, 2) qui de plus a la puissance du continu.

THÉORÈME 3. – *Pour tout* $\alpha \in (0, 2)\backslash\{Q^{\text{odd}} \cup L\}$, *la représentation* (6) *est vraie, où* $\Lambda_\alpha(z)$ *et* $N_\alpha(z)$ *sont des fonctions entières d'ordre fini respectivement* $1/\alpha$ *et* $1/2$ *données par les formules* (7) *et* (8).

Quand $\alpha \in Q^{\text{odd}}$ la structure analytique des $p_\alpha(x)$ est plus compliquée. Notons Q^1 le sous-ensemble de Q^{odd} formé des $\alpha = 1/n$, $n = 1, 2, \ldots$

THÉORÈME 4. – *Si* $\alpha \in Q^{\text{odd}}\backslash Q^1$, *la représentation* (9) *est vraie, où* $M_\alpha(z)$ *et* $N_\alpha(z)$ *sont des fonctions entières d'ordre fini respectif* $1/\alpha$, 1 *et* $1/2$. *De plus si* $\alpha = m/n$ *où* m *et* n *sont premiers entre eux, ces fonctions peuvent être décrites à l'aide des formules* (10)-(13). *Si* $\alpha = 1/n \in Q^1$, *alors la représentation* (14) *est vraie.*

Les cas particuliers sont donnés par (15) et (16).

Pour $\alpha \in L$ nous n'avons pu trouver une représentation de $p_\alpha(x)$ similaire à celle des théorèmes 3 et 4. Cependant, nous sommes en mesure de démontrer qu'il existe des valeurs de α pour lesquelles les théorèmes 3 et 4 ne sont pas vrais. De plus, l'ensemble de tels α est partout dense et a la puissance du continu. Néanmoins, le comportement asymptotique de $p_\alpha(x)$ à l'origine peut être décrit à l'aide de la formule (18) pour tout $\alpha \notin Q^{\text{odd}}$ et $N = 1, 2, \ldots$ (la première somme est nulle si $\alpha > 1$). Des exemples de formules asymptotiques pour $p_\alpha(x)$, $\alpha \in Q^{\text{odd}}$, $x \to 0$, sont donnés par (19)-(21) où $\gamma = 0,5772157\ldots$ est la constante d'Euler.

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1. INTRODUCTION. – In 1953, Ju. V. Linnik proved [1] that the function

$$\varphi_\alpha(t) = 1/(1 + |t|^\alpha), \qquad 0 < \alpha < 2, \tag{1}$$

is the characteristic function of a probability density. We shall denote this density by $p_\alpha(x)$. It can be viewed as a generalization of the well-known Laplace density $p_2(x) = e^{-|x|}/2$, which corresponds to the case $\alpha = 2$. Recently, Linnik's probability densities $p_\alpha(x)$ attracted attention of a number of researchers who discovered some interesting probabilistic properties and applications (*see,* e.g. [2]-[7]). In this connection, the study of analytic and asymptotic properties of $p_\alpha(x)$ seems to be of interest. Moreover, the study of the structure of these densities is of importance in connection with classification of continuous long-tailed distributions. This Note is devoted to such a study.

As for stable densities, analytic simplicity of characteristic functions (1) is not inherited by the corresponding densities and these densities elude closed form representations. The asymptotic behaviour of $p_\alpha(x)$ at ∞ is similar to some extent to those of stable densities. However, the analytic structure of $p_\alpha(x)$ is quite different from those of stable ones. It depends substantially on the arithmetic nature of the parameter α.

2. ASYMPTOTIC BEHAVIOUR OF $p_\alpha(x)$ AT ∞.

THEOREM 2.1. – *For any* $\alpha \in (0, 2)$, *the asymptotic behaviour of* $p_\alpha(x)$ *at* ∞ *can be described by the following asymptotic (divergent) series*

$$p_\alpha(x) \sim \frac{1}{\pi} \sum_{k=1}^{\infty} \left\{(-1)^{k-1} \Gamma(1 + \alpha k) \sin \frac{\pi \alpha k}{2}\right\} |x|^{-1-\alpha k}, \qquad x \to \infty. \tag{2}$$

More precisely, for any $\alpha \in (0,\ 2)$ and $N = 1,\ 2,\ 3, \ldots$, the following formula is valid

(3) $$p_\alpha(x) = \frac{1}{\pi} \sum_{k=1}^{N} \left\{ (-1)^{k-1} \Gamma(1 + \alpha k) \sin \frac{\pi \alpha k}{2} \right\} |x|^{-1-\alpha k} + R_{N,\alpha}(x),$$

where

(4) $$|R_{N,\alpha}(x)| \leqq \frac{\Gamma(1 + \alpha(N+1))}{\pi \sin(\pi\alpha/2)} |x|^{-1-\alpha(N+1)}.$$

COROLLARY. – *For any $\alpha \in (0,\ 2)$, the density $p_\alpha(x)$ decreases at ∞ at the rate of a power function, more precisely*

(5) $$p_\alpha(x) \sim \frac{1}{\pi} \left\{ \Gamma(1 + \alpha) \sin \frac{\pi\alpha}{2} \right\} |x|^{-1-\alpha}, \qquad x \to \infty.$$

3. ANALYTIC PROPERTIES OF $p_\alpha(x)$.

THEOREM 3.1. – *For any $\alpha \in (0,\ 2)$, the density $p_\alpha(x)$ is a completely monotonic function of x on the positive ray.*

COROLLARY. – *For any $\alpha \in (0,\ 2)$, the density $p_\alpha(x)$ is a restriction to the positive ray of a function analytic in the right half-plane.*

It is evident that $p_\alpha(x)$ is an even function of x, therefore the analogous properties are valid on the negative ray.

Now, we shall consider representations of $p_\alpha(x)$ by convergent series in terms of $|x|^{k\alpha}$, $|x|^k$ $(k = 0,\ 1,\ 2, \ldots)$ and $\log 1/|x|$. Denote by Q^{odd} the set of all rational numbers $\alpha \in (0, 2)$ representable in the form $\alpha = m/n$, where m and n are integers and, moreover, m is odd. Denote by L the set of all Liouville transcendental numbers $\alpha \in (0,\ 2)$. It is well-known that the set L is of zero Lebesgue measure.

The following theorem describes the analytic structure of $p_\alpha(x)$ for almost all $\alpha \in (0,\ 2)$ in the sense of Lebesgue measure, though exceptional values of α form an everywhere dense in $(0,\ 2)$ set of the power of the continuum.

THEOREM 3.2. – *For any $\alpha \in (0,\ 2) \backslash \{Q^{\text{odd}} \cup L\}$ the following representation is valid*

(6) $$p_\alpha(x) = \frac{1}{|x|} \Lambda_\alpha(|x|^\alpha) + N_\alpha(x^2)$$

where $\Lambda_\alpha(z)$ and $N_\alpha(z)$ are entire functions of the finite orders $1/\alpha$ and $1/2$ correspondingly. These functions admit the following explicit power series representations

(7) $$\Lambda_\alpha(z) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{\Gamma(\alpha k) \cos(\pi \alpha k/2)},$$

(8) $$N_\alpha(z) = \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(2k+1) \sin(x(2k+1)/\alpha)}.$$

In the case $\alpha \in Q^{\text{odd}}$, the analytic structure of $p_\alpha(x)$ is more complicated. Denote by Q^1 the subset of Q^{odd} consisting of the numbers $\alpha = 1/n$, $n = 1,\ 2, \ldots$

THEOREM 3.3. – *If $\alpha \in Q^{\mathrm{odd}} \backslash Q^1$, then the following representation is valid*

$$p_\alpha(x) = \frac{1}{|x|}\,\Lambda_\alpha(|x|^\alpha) + \log\frac{1}{|x|}\cdot M_\alpha(|x|) + N_\alpha(x^2) \tag{9}$$

where $\Lambda_\alpha(z)$, $M_\alpha(z)$, $N_\alpha(z)$ are entire functions of the orders $1/\alpha$, 1, $1/2$ correspondingly. Letting $\alpha = m/n$, where m and n are relatively prime integers, we can describe the power series representation of these functions in the following way

$$\Lambda_\alpha(z) = \sum_{k=1}^{\infty} \lambda_k^{(\alpha)}\, z^k, \tag{10}$$

$$\lambda_k^{(\alpha)} = \begin{cases} \dfrac{(-1)^{k+1}}{2\,\Gamma(\alpha k)\cos(\pi\alpha k/2)}, & \text{for non-integer } \dfrac{k}{n} \\[2ex] \dfrac{(-1)^{(m+n)\,s}}{\Gamma(ms)}\left[\dfrac{\Gamma'(ms)}{\Gamma(ms)}\sin\dfrac{\pi\, ms}{2} - \dfrac{1}{2}\cos\dfrac{\pi\, ms}{2}\right], & \\ \qquad \text{for}\quad k = ns,\quad s = 1, 2, \ldots & \end{cases} \tag{11}$$

$$M_\alpha(z) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{(m+n)\,k}\sin(\pi\, mk/2)}{\Gamma(mk)}\, z^{mk-1}, \tag{12}$$

$$N_\alpha(z) = \frac{1}{\alpha}\sum_{k=0,\ (2k+1)/m\notin N}^{\infty} \frac{(-1)^k z^k}{\Gamma(2k+1)\sin(\pi(2k+1)/\alpha)}. \tag{13}$$

If $\alpha = 1/n \in Q^1$, then the following representation is valid

$$p_\alpha(x) = \frac{1}{|x|}\,\Lambda_\alpha(|x|^\alpha) + \frac{(-1)^{n=1}}{\pi}\cdot\cos x\cdot\log\frac{1}{|x|} \tag{14}$$

where the entire function $\Lambda_\alpha(z)$ is defined above.

We mention the following particular cases of representations of $p_\alpha(x)$ by theorem 3.3:

$$p_1(x) = \frac{1}{\pi}\cos x\cdot\log\frac{1}{|x|} + \frac{1}{2}\sin|x| + \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma'(2k+1)}{\Gamma^2(2k+1)}\, x^{2k}; \tag{15}$$

$$\begin{aligned} p_{1/2}(x) = {} & \frac{1}{\sqrt{2|x|}}\sum_{k=0}^{\infty}\frac{(-1)^{[(k+1)/2]}\,|x|^k}{\Gamma(k+(1/2))} - \frac{1}{\pi}\cos x\cdot\log\frac{1}{|x|} + \frac{1}{2}\sin|x| \\ & - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)\frac{\Gamma'(2k+1)}{\Gamma^2(2k+1)}\, x^{2k}. \end{aligned} \tag{16}$$

In the case $\alpha \in L$ we could not find any representation of $p_\alpha(x)$ similar to (6), (9), (14), therefore we have to restrict ourselves to the following less convenient one.

THEOREM 3.4. – *If $\alpha \in (0, 2)\backslash Q^{\mathrm{odd}}$, then the following representation is valid*

$$p_\alpha(x) = \frac{1}{|x|} \lim_{s\to\infty} \left\{ \frac{1}{2} \sum_{k=1}^{s} \frac{(-1)^{k+1} |x|^{k\alpha}}{\Gamma(\alpha k) \cos(\pi\alpha k/2)} + \frac{1}{\alpha} \sum_{1 \leqq 2k+1 < \alpha(s+(1/2))} \frac{(-1)^k |x|^{2k+1}}{\Gamma(2k+1) \sin(\pi(2k+1)/\alpha)} \right\}. \tag{17}$$

The limit is uniform with respect to x on any compact set of $\mathbf{R}$.

Note that the values of α such that the separate limits of the each of the two sums in the right hand side of (17) do not exist form an everywhere dense in (0, 2) set of the power of the continuum. It is possible to give a more detailed description of this set.

4. ASYMPTOTIC BEHAVIOUR OF $p_\alpha(x)$ AT 0.

THEOREM 4.1. – *If $\alpha \in (0, 2)\backslash Q^{\mathrm{odd}}$, then for any $N = 1, 2, \ldots$ the following asymptotic formula is valid*

$$\begin{aligned} p_\alpha(x) = {} & \frac{1}{2} \sum_{k=1}^{[1/\alpha]} \frac{(-1)^{k+1} |x|^{k\alpha-1}}{\Gamma(\alpha k) \cos(\pi\alpha k/2)} + \frac{1}{\alpha \sin(\pi/\alpha)} \\ & + \sum_{k=[1/\alpha]+1}^{[3/\alpha]} \frac{(-1)^{k+1} |x|^{k\alpha-1}}{\Gamma(\alpha k) \cos(\pi\alpha k/2)} - \frac{x^2}{2\alpha \sin(3\pi/\alpha)} + \cdots \\ & + \frac{1}{2} \sum_{k=[(2N-3)/\alpha]+1}^{[(2N-1)/\alpha]} \frac{(-1)^{k+1} |x|^{k\alpha-1}}{\Gamma(\alpha k) \cos(\pi\alpha k/2)} \\ & + \frac{(-1)^{N-1} x^{2N-2}}{\Gamma(2N-1) \sin((2N-1)\pi/\alpha)} + o(|x|^{2N-2}), \qquad x \to 0. \end{aligned} \tag{18}$$

(The first sum vanishes for $\alpha > 1$.)

Thus, the case $\alpha \in L$ is not considered here separately.

For $\alpha \in Q^{\mathrm{odd}}$, the asymptotic formula of $p_\alpha(x)$ as $x \to 0$ is a corollary of theorem 3.3. However, its general form is rather complicated. Therefore we restrict our attention to a few examples:

$$\begin{cases} p_1(x) = \frac{1}{\pi} \log \frac{1}{|x|} - \frac{\gamma}{\pi} + \frac{1}{2} |x| - \frac{1}{2\pi} x^2 \log \frac{1}{|x|} + O(x^2), \\ \qquad x \to 0, \end{cases} \tag{19}$$

$$\begin{cases} p_{1/n}(x) = \frac{1}{2} \sum_{k=1}^{n-1} \frac{(-1)^{k+1} |x|^{(k/n)-1}}{\Gamma(k/n) \cos(k\pi/2n)} + \frac{(-1)^{n+1}}{\pi} \log \frac{1}{|x|} + (-1)^n \frac{\gamma}{\pi} \\ \qquad + \frac{(-1)^{n+1} n\, |x|^{1/n}}{2\Gamma(1/n) \sin(\pi/2n)} + O(|x|^{2/n}), \\ \qquad x \to 0, \quad n = 2, 3, \ldots, \end{cases} \tag{20}$$

$$\begin{aligned} p_{3/2}(x) = {} & \frac{4}{3\sqrt{3}} - \sqrt{\frac{2}{\pi}} \cdot |x|^{1/2} \\ & + \frac{1}{2\pi} x^2 \log \frac{1}{|x|} + \frac{3-2\gamma}{4\pi} + O(|x|^{7/2}), \qquad x \to 0, \end{aligned} \tag{21}$$

where $\gamma = -\Gamma'(1) = 0.5772157\ldots$ is the Euler constant.

The proofs of the main results of this Note are based on the following representation of $p_\alpha(x)$ in terms of Cauchy type integral

$$x^{1/\alpha} p_\alpha(x^{1/\alpha}) = (1/\alpha)\,\mathrm{Im}\, f_\alpha(-x e^{-i\pi\alpha/2}), \qquad x > 0,$$

where

$$f_\alpha(z) = \frac{1}{\pi}\int_0^\infty \frac{e^{-v^{1/\alpha}}\, v^{1/\alpha}}{v - z}\, dv$$

and on the explicit solution of a Riemann-Hilbert boundary problem.

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