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# Analytic Properties of Besov Spaces via Bergman Projections

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Abstract. We consider two-parameter Besov spaces of holomorphic functions on the unit ball of $\mathbb{C}^N$. We obtain various exclusions between Besov spaces of different parameters using gap series. We estimate the growth near the boundary and the growth of Taylor coefficients of functions in these spaces. We find the unique function with maximum value at each point of the ball in each Besov space. We base our proofs on Bergman projections and imbeddings between Lebesgue classes and Besov spaces. Special cases apply to the Hardy space $H^2$, the Arveson space, the Dirichlet space, and the Bloch space.

## 1. Besov Spaces

Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^N$ and $H(\mathbb{B})$ the space of holomorphic functions on $\mathbb{B}$. When $N = 1$, $\mathbb{B}$ is the unit disc $\mathbb{D}$. Unless otherwise specified, our main parameters and their range of values are

$$q \in \mathbb{R}, \qquad 0 < p \leq \infty, \qquad s \in \mathbb{R}, \qquad t \in \mathbb{R};$$

given $q$ and $p$, we often choose $t$ to satisfy

$$q + pt > -1. \tag{1}$$

Let $\nu$ be the volume measure on $\mathbb{B}$ normalized with $\nu(\mathbb{B}) = 1$. We also consider on $\mathbb{B}$ the measures

$$d\nu_q(z) = (1 - |z|^2)^q \, d\nu(z),$$

which are finite only for $q > -1$; here $|z|^2 = \langle z, z\rangle$ and $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$. The corresponding Lebesgue classes are $L^p_q$.

Consider the linear transformation $I^t_s$ defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) = (1 - |z|^2)^t \, D^t_s f(z),$$

where $D^t_s$ is a radial differential operator on $H(\mathbb{B})$ of order $t$ for any $s$ and $I^0_s$ is the identity (or inclusion) $I$.

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Definition 1.1. The *Besov space* $B_q^p$ consists of all $f \in H(\mathbb{B})$ for which the function $I_s^t f$ belongs to $L_q^p$ for some $s, t$ satisfying (1).

This particular parametrization of Besov spaces was introduced in [**K1**]. Note that $s$ or $t$ are not among the parameters of the space $B_q^p$, and that $s$ does not appear in (1) at all. It is one of the aims of this paper in Section 2 to emphasize that Definition 1.1 is independent of $s$ and $t$ as long as (1) is satisfied. It follows that the $L_q^p$ norms of various $I_s^t f$ under (1) are all equivalent, where the term "norm" is used even for $0 < p < 1$. We call any one of them the $B_q^p$ norm of $f$ and denote it by $\|f\|_{B_q^p}$. So any $I_s^t$ with (1) is an isometric imbedding of $B_q^p$ into $L_q^p$ when the norm used on $B_q^p$ is $\|I_s^t f\|_{L_q^p}$.

It is clear from Definition 1.1 that each $B_q^p$ space contains all constants and all polynomials, and thus is nonempty. In fact, the ball algebra is dense in any $B_q^p$ with $p < \infty$ by [**BB**, Lemma 5.2]. From a different point of view, condition (1) is required in order that constants and polynomials have finite $B_q^p$ norm.

It is also known that Definition 1.1 is independent of the particular form of the differential operators $D_s^t$ used. The differential operators are often defined as coefficient multipliers on the homogeneous expansion (Taylor expansion for $N = 1$) of $f \in H(\mathbb{B})$. To make this more precise, let $H_k$ denote the space of holomorphic *homogeneous polynomials* in the variables $z_1, \dots, z_N$ of total degree $k = 0, 1, 2, \dots$. An $f \in H(\mathbb{B})$ determines a unique expansion $f = \sum_{k=0}^{\infty} f_k$ with $f_k \in H_k$, where $f_k(z) = c_k z^k$ for $N = 1$. Then $D_s^t f = \sum_{k=0}^{\infty} d_k f_k$, where the dependence of $d_k$ on $s, t$ is not explicitly shown.

In this work, we use the $D_s^t$ and $d_k$ given in [**K3**, Definition 3.1], which we do not repeat here. The coefficients satisfy

$$d_k \neq 0 \quad (k = 0, 1, 2, \dots) \qquad \text{and} \qquad d_k \sim k^t \quad (k \to \infty) \tag{2}$$

for any $s, t$, where $x \sim y$ means that $|x/y|$ is bounded above and below by two positive constants that are independent of the parameters in question ($k$ here). It turns out that each $D_s^t$ is a continuous differential operator of order $t$ on $H(\mathbb{B})$, which is actually integral for $t < 0$. Further, $D_s^0 = I$ for any $s$; $D_{-N}^1 = I + \mathcal{R}$ where $\mathcal{R}$ is the usual radial derivative, $D_s^t(1) = d_0 \neq 0$, and $D_{s+t}^u D_s^t = D_s^{u+t}$. Thus, each $D_s^t$ is invertible on $H(\mathbb{B})$ with two-sided inverse

$$(D_s^t)^{-1} = D_{s+t}^{-t}.$$

After a short explanation of the role of the parameters $q$ and $p$ in Section 2, we relate Besov spaces to well-known spaces in Section 3. There it becomes clear that Besov spaces are a natural continuation of Bergman spaces, and that Hardy spaces are just at the edge. We obtain some inclusion and exclusion relations between Besov spaces with different parameters in Section 4. In Section 5, we introduce the extended Bergman projections and investigate their orthogonality. Then come the applications of Bergman projections. We estimate the growth of Besov functions near the boundary of the ball in Section 6, and the growth of their Taylor coefficients in Section 7. The determination of the unique function yielding the maximality of point evaluations is also in Section 6.

We often use the *Pochhammer symbol*

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

when $a$ and $a+b$ lie off the pole set $-\mathbb{N}$ of the gamma function $\Gamma$. For fixed $a, b$, Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b} \qquad \text{and} \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty). \tag{3}$$

We occasionally use multi-index notation in which $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is an $N$-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, and $0^0 = 1$.

An unadorned $C$ represents a constant whose value varies from one formula to another but never depends on the functions involved or the parameters being investigated.

## 2. Two Parameters Suffice

The purpose of this section is to explain that the order $t$ of the derivative $D_s^t$ used in defining $B_q^p$ is not a property of the space as long as (1) holds.

Required reading on holomorphic Besov spaces on $\mathbb{B}$ is [**BB**]. The space $B_q^p$ is called $A_{Q,t}^p$ there ([**BB**, p. 36]), where $Q = 1+q+pt$, and $Q > 0$ is always assumed, which corresponds to our (1). Also $0 < p \leq \infty$ and $t \in \mathbb{R}$. (As a matter of fact, $Q = 0$ is a possibility there too, but this case pertains to Hardy Sobolev spaces, on which [**BB**] is also required reading.) We caution that the measure $v_q$ has power $q-1$ on $1-|z|^2$ there ([**BB**, §0.3]), while our $\nu_q$ has $q$. Conversely, we call the space $A_{Q,t}^p$ there the space $B_{Q-1-pt}^p$ here.

As far as $D_s^t$ or Definition 1.1 are concerned, the value of $s$ makes no difference; see [**BB**, p. 41]. We need it in such exact formulas as Theorem 5.2 (9). Its value is irrelevant in most other results in this paper.

The following result is the content of [**BB**, Theorem 5.12 (i)].

Theorem 2.1. *Suppose*

$$\frac{Q_1 - Q_2}{p} = t_1 - t_2. \tag{4}$$

*Then* $A_{Q_1,t_1}^p = A_{Q_2,t_2}^p$.

If we substitute $1 + q_j + pt_j > 0$ for $Q_j$ in (4), $j = 1, 2$, then (4) takes the form $q_1 = q_2 =: q$, and the conclusion of Theorem 2.1 becomes $A_{Q_1,t_1}^p = A_{Q_2,t_2}^p = B_q^p$. In other words, the space $B_q^p$ remains unchanged by switching from $t_1$ to $t_2$ the order of the derivative in Definition 1.1 as long as both satisfy (1) with the same $q$ and $p$.

This result is proved in a different way in [**K3**, Corollary 4.2]. Yet another proof for $1 \leq p \leq \infty$ is given in Remark 5.4. Thus $B_q^p$ spaces are *all* the Besov spaces that are defined by a norm of pure (Bergman) type. The modifier "diagonal" is used for these spaces in [**AFJP**, Remark 5.2] to distinguish them from the Besov spaces defined by a norm of mixed (Hardy and Bergman) type.

The spaces $B_q^p$ are sometimes called Sobolev spaces, and generalize Bergman spaces to $q \leq -1$. However, the name "Besov space" suits them better, because for $q \leq -1$, their definition requires differentiation, and this differentiation is of fractional order; moreover, there is the connection with Besov spaces with mixed-type norms. Passing from Theorem 2.1 to Definition 1.1 brings out another interesting point; in the usual definition of Sobolev spaces, the maximum order of derivatives is specified; in $B_q^p$ spaces, (1) specifies a "minimum" order of derivatives.

## 3. Special and Extremal Cases

Besov spaces include many known spaces as special cases. Here we fix the exact location of the special cases within the Besov-space family and the location of Besov spaces in reference to yet other known spaces. To begin with, each point in the right half plane of the $pq$-plane can be matched to exactly one $B^p_q$ with $p \neq \infty$. This point of view is illustrated in [**K1**, (2)].

Clearly, (1) is satisfied with $t = 0$ if $q > -1$. Then for $q > -1$, Besov spaces $B^p_q$ are precisely the weighted Bergman spaces with the same parameters. So that part of the half plane above the line $q = -1$ is the Bergman region, and the part on or below it is the proper Besov region. Another critical line is $q = -(N + 1)$. The spaces below it consist only of bounded functions; see Section 6. Several of our results below exhibit a change of behavior either at $q = -1$ or at $q = -(N + 1)$.

When $p = \infty$, (1) takes the form

$$t > 0$$

independently of $q$. Combined with the fact that $L^\infty_q = L^\infty$ for any $q$ (see [**K3**, Proposition 2.3]), we see that all $B^\infty_q$ spaces are one and the same; and taking $t = 1$ shows that they coincide with the Bloch space $\mathcal{B}$. This is the only Besov space not in the right half plane. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ for which some $I^t_s f$ with $t > 0$ vanishes on $\partial\mathbb{B}$.

We can pass between Besov spaces with the same $p$ by differentiation or integration. The following result, inherent in the definition of the spaces $A^p_{Q,t}$ in [**BB**], was rediscovered in [**P**, Corollary 3.9] for $s > N$ and $s + t > N$.

Proposition 3.1. *For any $s, t, q$ and $0 < p < \infty$, $D^t_s(B^p_q) = B^p_{q+pt}$ is an isometric isomorphism when appropriate norms are used in the two spaces.*

Proof. Let $f \in B^p_q$ and put $g = D^t_s f$. Take $u$ so large that $q + p(t + u) > -1$. Then $D^u_{s+t} g = D^u_{s+t} D^t_s f = D^{t+u}_s f$ and $I^{t+u}_s f$ lies in $L^p_q$. This is equivalent to saying that $I^u_{s+t} g$ lies in $L^p_{q+pt}$. Hence $g \in B^p_{q+pt}$, and the norms $\|f\|_{B^p_q} = \|I^{t+u}_s f\|_{L^p_q}$ and $\|g\|_{B^p_{q+pt}} = \|I^u_{s+t} g\|_{L^p_{q+pt}}$ are equal. We conclude by showing $D^{-t}_s(B^p_{q+pt}) = B^p_q$ in the same way. □

Each $B^2_q$ space is a reproducing kernel Hilbert space with reproducing kernel

$$K_q(z, w) = \begin{cases} \dfrac{1}{(1 - \langle z, w\rangle)^{N+1+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(N + 1 + q)_k}{k!} \langle z, w\rangle^k, & \text{if } q > -(N + 1), \\ \dfrac{{}_2F_1(1, 1; 1 - N - q; \langle z, w\rangle)}{-N - q} = \displaystyle\sum_{k=0}^{\infty} \frac{k!\, \langle z, w\rangle^k}{(-(N + q))_{k+1}}, & \text{if } q \le -(N + 1), \end{cases}$$

where ${}_2F_1$ is the hypergeometric function; see [**BB**, p. 13]. Let $b^q_k$ be the coefficient of $\langle z, w\rangle^k$ in the series expansion of $K_q$. To see that ${}_2F_1$ is the right choice to use in $K_q$ for lower values of $q$, we can check by (3) that

$$b^q_k \sim k^{N+q} \qquad (k \to \infty)$$

irrespectively of the value of $q$. Thus $f(z) = \sum_\alpha c_\alpha z^\alpha \in B^2_q$ if and only if

$$\sum_{\alpha \neq 0} \frac{1}{|\alpha|^{N+q}} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 < \infty. \tag{5}$$

This also shows that $K_q$ is bounded if and only if $q < -(N+1)$. We further have

$$D_q^t K_q(z,w) = K_{q+t}(z,w) \tag{6}$$

similar to Proposition 3.1, where differentiation is performed on the holomorphic variable $z$.

The form of $K_q$ clearly shows that $B^2_{-(N+1)}$ is the Dirichlet space of the ball since $K_{-(N+1)}(z,w) = -\langle z,w\rangle^{-1}\log(1-\langle z,w\rangle)$; $B^2_{-N}$ is the Arveson space, and $B^2_{-1}$ is the Hardy space $H^2$.

For $p \neq 2$ and $p \neq \infty$, we can compare the norms of a monomial $z^\alpha$ in the two spaces to see how close the space $B^p_{-1}$ comes to the space $H^p$. With the aid of [**K3**, Proposition 2.1], one sees easily that

$$\begin{aligned}\|z^\alpha\|^p_{B^p_q} &= d^p_{|\alpha|}\int_{\mathbb{B}} |z^\alpha|^p\,(1-|z|^2)^{q+pt}\,d\nu(z)\\ &= d^p_{|\alpha|}\,\frac{N!\,\Gamma(1+q+pt)\prod_{j=1}^N\Gamma(1+p\alpha_j/2)}{\Gamma(N+1+q+pt+p|\alpha|/2)},\end{aligned}$$

where (1) holds. On the other hand, by [**R**, Definition 5.6.1] and [**K3**, Proposition 2.1],

$$\|z^\alpha\|^p_{H^p} = \int_{\partial\mathbb{B}} |\zeta^\alpha|^p\,d\Sigma(\zeta) = \frac{(N-1)!\prod_{j=1}^N\Gamma(1+p\alpha_j/2)}{\Gamma(N+p|\alpha|/2)}, \tag{7}$$

where $\Sigma$ is the surface measure on the boundary $\partial\mathbb{B}$ of the ball normalized with $\Sigma(\partial\mathbb{B}) = 1$. Now applying (2) and (3), we have

$$\frac{\|z^\alpha\|^p_{B^p_{-1}}}{\|z^\alpha\|^p_{H^p}} \sim |\alpha|^{tp}\,\frac{\Gamma(N+p|\alpha|/2)}{\Gamma(N+pt+p|\alpha|/2)} \sim |\alpha|^0 = 1 \qquad (|\alpha|\to\infty).$$

REMARK 3.2. According to [**BB**, Theorem 5.12 (ii) and (iii)], $B^p_{-1} \subset H^p$ if $0 < p \le 2$ and $H^p \subset B^p_{-1}$ if $2 \le p \le \infty$ with continuous inclusions. In [**BGP**, p. 840], it is shown, using lacunary series in $N = 1$, that these inclusions are proper except for $p = 2$. This reference makes further comparison of the two families.

The Hardy spaces $H^p$ are recovered as a limiting case of $B^p_{-1}$ as the order $t$ of the derivative $D^t_s$ used in the definition of $B^p_{-1}$ tends to 0. This is seen easily by considering the equivalent norm

$$|||\cdot|||^p_{B^p_{-1}} = \frac{(pt)_N}{N!}\,\|\cdot\|^p_{B^p_{-1}}$$

for $B^p_{-1}$, where $t$ satisfies (1) with $q = -1$. If $f \in B^p_q$, then $D^t_s f$ belongs to the Bergman space $B^p_{-1+pt}$ and

$$|||f|||^p_{B^p_{-1}} = \frac{(pt)_N}{N!}\int_{\mathbb{B}} |D^t_s f(z)|^p\,d\nu_{-1+pt}(z),$$

which is the $p$th power of the norm of $D^t_s f$ in $B^p_{-1+pt}$ with respect to the *normalized* version of the measure $\nu_{-1+pt}$. It is known that these normalized weighted volume measures converge weak-$*$ to $\Sigma$ as $t \to 0^+$ as noted in [**BB**, §0.3]. Here is a proof. Switching to polar coordinates and letting

$$d\mu_t(\rho) = \frac{2N\,(pt)_N}{N!}\,\rho^{2N-1}\,(1-\rho^2)^{-1+pt}\,d\rho$$

for $0 \le \rho \le 1$, we have

$$|||f|||^p_{B^p_{-1}} = \int_0^1 d\mu_t(\rho) \int_{\partial\mathbb{B}} |D^t_s f(\rho\zeta)|^p \, d\Sigma(\zeta).$$

To compute the *distribution function* $F_t(x) = \mu_t([0,x])$ of $\mu_t$, we let $y = 1-\rho^2$ and obtain

$$\begin{aligned} F_t(x) &= \frac{2N\,(pt)_N}{N!} \int_0^x \rho^{2N-1}\,(1-\rho^2)^{-1+pt}\,d\rho \\ &= \frac{(pt)_N}{(N-1)!} \int_{1-x^2}^1 y^{-1+pt}\,(1-y)^{N-1}\,dy \\ &= (pt)_N \sum_{j=0}^{N-1} \frac{(-1)^j}{j!\,(N-1-j)!} \int_{1-x^2}^1 y^{-1+pt+j}\,dy \\ &= (pt)_N \sum_{j=0}^{N-1} \frac{(-1)^j}{j!\,(N-1-j)!\,(pt+j)} [1-(1-x^2)^{pt+j}] \end{aligned}$$

for $0 \le x < 1$. A computation with the beta function shows that $F_t(1) = 1$. But $\lim_{t\to0^+} F_t(x) = 0$ if $0 \le x < 1$ and $\lim_{t\to0^+} F_t(1) = 1$. This means that weak-$*$ limit of $\mu_t$ as $t \to 0^+$ is the unit point mass at $\rho = 1$. Consequently,

$$\lim_{t\to0^+} |||f|||^p_{B^p_{-1}} = \int_{\partial\mathbb{B}} |f(\zeta)|^p \, d\Sigma(\zeta) = \|f\|^p_{H^2} \qquad (f \in B^p_{-1})$$

since $D^0_s = I$.

## 4. Inclusions and Exclusions

Various inclusions among Besov spaces follow directly from Definition 1.1. For example, $B^p_{q_1} \subset B^p_{q_2}$ if $q_1 < q_2$ for fixed $p$. For fixed $q > -1$ (in the Bergman region), $B^{p_1}_q \supset B^{p_2}_q$ if $p_1 < p_2$. These are shown graphically in [**K1**, (2)]. Many others can be derived from [**BB**, Theorem 5.13]. This result is very general, and several special cases of it have been rediscovered in later papers. For example, the inclusion $B^{p_1}_{-(N+1)} \subset B^{p_2}_{-(N+1)}$ for $p_1 < p_2$ follows immediately from it, but was also obtained later for $p_1 > 1$ using the Möbius invariance of these particular Besov spaces. For connections between Besov, Bergman, Hardy, and Hardy Sobolev spaces, one should first look at again [**BB**, Theorem 5.13] and [**BB**, Theorem 5.12 (ii) and (iii)]. Note that Hardy spaces are labeled $A^p_{0,0}$ and Hardy Sobolev spaces $A^p_{0,t}$ in this source.

It is also interesting to show that Besov spaces with different parameters are different from each other. For the subfamily of spaces $B^2_q$, we can use the orthogonality of the monomials to construct explicit series that lie in $B^2_{q_2}$ but not in $B^2_{q_1}$ for $q_1 < q_2$. This was done in [**AK**, Example 2.4] for $q \ge -(N+1)$; as mentioned in the note added in proof at the end of that paper, it works for all $q$.

To construct further examples for exclusions, we set $N = 1$ in the remaining part of this section and consider lacunary series, mentioned earlier in Remark 3.2.

Definition 4.1. A power series $f(z) = \sum_k c_k z^{n_k} \in H(\mathbb{D})$ is said to have *Hadamard gaps* ($f \in \mathrm{HG}$) if $n_{k+1}/n_k \ge \lambda$ for all $k$ for some $\lambda > 1$.

The following result appears for all $q$ in [**K2**]; for particular values of $q$, it had been noted earlier by various authors.

THEOREM 4.2. *A gap series* $f(z) = \sum_k c_k z^{n_k} \in \mathrm{HG}$ *belongs to* $B^p_q$ *if and only if* $\sum_k n_k^{-(1+q)} |c_k|^p$ *converges.*

PROOF. This is done by adapting [**M**, Theorem 1] to our case. □

Similar to the inclusion relation for the spaces $B^p_{-(N+1)}$ noted above, we have the following, where we concentrate on the proper Besov region for $N = 1$.

COROLLARY 4.3. *Let* $q \leq -1$ *and* $p_1 < p_2$*. Then* $B^{p_1}_q \cap \mathrm{HG} \subset B^{p_2}_q \cap \mathrm{HG}$.

EXAMPLE 4.4. Let $q \leq -1$ and $p_1 < p_2$. Choose $c_k = k^{-1/p_1}\, 2^{k(1+q)/p_2}$ and $n_k = 2^k$, and consider the power series $f$ formed with them. Then $f \in B^{p_2}_q \backslash B^{p_1}_q$ by Theorem 4.2.

EXAMPLE 4.5. Let $q_1 < q_2$. Choose $c_k = k^{-(1+1/p)}\, 2^{k(1+q_2)/p}$ and $n_k = 2^k$, and consider the power series $f$ formed with them. Then $f \in B^p_{q_2} \backslash B^p_{q_1}$ by Theorem 4.2 again. With appropriate modifications, this example works also for $N > 1$.

## 5. Bergman Projections and Orthogonality

In the remaining part of the paper, we consider only $1 \leq p \leq \infty$, as our proofs depend on Definition 5.1 and Theorem 5.2, the latter of which is not available for smaller $p$.

DEFINITION 5.1. (Extended) *Bergman projections* are the linear transformations $P_s$ defined by

$$P_s f(z) = \int_{\mathbb{B}} K_s(z, w)\, f(w)\, d\nu_s(w) \qquad (z \in \mathbb{B})$$

for suitable $f$.

THEOREM 5.2. [**K3**] *The operator* $P_s : L^p_q \to B^p_q$ *is bounded if and only if*

$$q + 1 < p\,(s + 1). \tag{8}$$

*Given an* $s$ *satisfying* (8)*, if* $t$ *satisfies* (1)*, then*

$$P_s I^t_s f = \frac{N!}{(1 + s + t)_N}\, f =: \frac{1}{C_{s+t}}\, f \qquad (f \in B^p_q). \tag{9}$$

Thus $P_s : L^p_q \to B^p_q$ is onto and $C_{s+t} I^t_s : B^p_q \to L^p_q$ is a right inverse for it. A rudimentary case of (9) for $p = \infty$ is in [**C**, Corollary 13]. In the Bergman region, it is possible to take $s = q$ except when $p = 1$. In the proper Besov region, $s$ must be strictly greater than $q$ for any value of $p$; however, $s = -q$ works for any $p$, and $s = 0$ works for any $p$ and $q$ in this region.

When written explicitly, (9) is an integral representation for $f \in B^p_q$. In a sense, this paper is about showing how powerful such a formula can be.

When $p = \infty$, Theorem 5.2 is about operators from $L^\infty$ onto $\mathcal{B}$, and (8) has the form

$$s > -1.$$

There is a similar result for the little Bloch space, where $\mathcal{C}$ denotes continuous functions on $\mathbb{B}$ and $\mathcal{C}_0$ the subspace of functions in $\mathcal{C}$ whose restriction to $\partial\mathbb{B}$ is 0.

THEOREM 5.3. *The operator* $P_s$ *maps either of* $\mathcal{C}$ *or* $\mathcal{C}_0$ *boundedly onto* $\mathcal{B}_0$ *if and only if* $s > -1$*. Given such an* $s$*, if also* $t > 0$*, then there is a constant* $C$ *such that* $P_s I^t_s f = C f$ *for* $f \in \mathcal{B}_0$.

Proof. The if part and the case $t = 1$ are proved in [**C**, Theorem 2]. The equality for other $t$ follow from the $p = \infty$ case of Theorem 5.2. Now consider $f(w) = -w_1/\log(1-|w|^2) \in \mathcal{C}_0$. It is easily shown that $P_s f$ fails to exist if $s \le -1$, and this proves the only-if part. □

Remark 5.4. Let us show once again that Besov spaces are fully described by the parameters $q$ and $p$, for $1 \le p \le \infty$ now. Suppose $t_1, t_2$ satisfy (1) with the same $q, p$, and that $g = I_s^{t_1} f \in L_q^p$, where $s$ satisfies (8). By Theorem 5.2, $P_s I_s^{t_1} f = Cf$. Apply $I_s^{t_2}$ to both sides to get $V_s^{t_2} g = C I_s^{t_2} f$. The operator $V_s^{t_2} := I_s^{t_2} P_s$ is bounded on $L_q^p$ by [**K3**, Remark 5.2 and Theorem 2.4] because of (1) and (8). Then also $I_s^{t_2} f \in L_q^p$.

The operator $P_s$ is not a projection on a subspace in the true sense of the word, because $B_q^p$ need not be subspace of $L_q^p$. However, for any $t$ satisfying (1), $M_{qs}^{pt} := C_{s+t} I_s^t(B_q^p)$ is an isometric copy (up to a constant multiple) of $B_q^p$ in $L_q^p$ and thus a closed subspace of $L_q^p$ by Definition 1.1 and the discussion on norms following it. Then (9) shows that $V_s^t = C_{s+t} I_s^t P_s$ is a projection indeed on $M_{qs}^{pt}$ for any $s$ satisfying (8). To determine whether or not this projection is orthogonal when $p = 2$, we proceed by computing the exact operator norm of $P_s$.

Proposition 5.5. *If $P_s : L_q^2 \to B_q^2$ is bounded and the norm on $B_q^2$ is $\|I_s^t(\cdot)\|_{L_q^2}$, then*

$$\|P_s\| = \frac{N!\,\sqrt{\Gamma(1-q+2s)\,\Gamma(1+q+2t)}}{\Gamma(N+1+s+t)}.$$

Proof. Let $f \in L_q^2$. First,

$$P_s f(z) = \sum_{k=0}^{\infty} b_k^s \int_{\mathbb{B}} f(w)\,(1-|w|^2)^{-q+s}\,\langle z, w\rangle^k\,d\nu_q(w). \tag{10}$$

The spaces $Y_k = (1-|z|^2)^{-q+s} H_k$ lie in $L_q^2$ by (8); and they, as well as $H_k$, are pairwise orthogonal by [**FR**, Proposition 2.4 (23)]. Let $Y$ be the closure of the span of $Y_k$; $P_s$ annihilates the orthogonal complement of $Y$ in $L_q^2$ by (10). Denote by $P$ the orthogonal projection from $L_q^2$ to $Y$. Then $Pf(z) = \sum_{k=0}^{\infty}(1-|z|^2)^{-q+s} f_k(z)$ for some $f_k \in H_k$, and $P_s f = P_s P f$. By replacing $f$ by $Pf$ in the integral in (10), we obtain

$$P_s f(z) = \sum_{k=0}^{\infty} b_k^s \int_{\mathbb{B}} f_k(w)\,(1-|w|^2)^{-q+2s}\,\langle z, w\rangle^k\,d\nu(w)$$

by the orthogonality of $Y_k$. Hence, by [**FR**, Proposition 2.4 (26)],

$$P_s f(z) = \frac{N!}{(1-q+2s)_N} \sum_{k=0}^{\infty} b_k^s \frac{k!}{(N+1-q+2s)_k}\, f_k(z).$$

Note that (8) now has the form $-q + 2s > -1$. Pick $t$ so that (1) is satisfied, that is, $q + 2t > -1$. Together $s + t > -1$. Then, using the orthogonality of $H_k$

and [**FR**, Proposition 2.4 (25)] twice, we compute that

$$\begin{aligned}\|P_s f\|^2_{B^2_q} &= \|I^t_s P_s f\|^2_{L^2_q} \\ &= \frac{(N!)^2}{(1-q+2s)^2_N} \sum_{k=0}^{\infty} \frac{d_k^2\,(b_k^s)^2\,(k!)^2}{(N+1-q+2s)^2_k} \int_{\mathbb{B}} (1-|z|^2)^{q+2t}\,|f_k(z)|^2\,d\nu(z) \\ &= \frac{(N!)^2}{(1-q+2s)_N\,(1+q+2t)_N} \sum_{k=0}^{\infty} \frac{d_k^2\,(b_k^s)^2\,(k!)^2}{(N+1-q+2s)_k\,(N+1+q+2t)_k} \\ &\quad \cdot \int_{\mathbb{B}} (1-|z|^2)^{-q+2s}\,|f_k(z)|^2\,d\nu(z).\end{aligned}$$

The values of the coefficients $d_k$ and $b_k^s$ both depend on whether $s > -(N+1)$ or not. Yet it turns out that the end result is the same in either case. We show the details for $s > -(N+1)$ only. Then $d_k b_k^s = (N+1+s+t)_k/k!$, and

$$\begin{aligned}\|P_s f\|^2_{B^2_q} &= \frac{(N!)^2\,\Gamma(1-q+2s)\,\Gamma(1+q+2t)}{\Gamma(N+1+s+t)^2} \sum_{k=0}^{\infty} \frac{(N+1-q+2s+k)_{q-s+t}}{(N+1+s+t+k)_{q-s+t}} \\ &\quad \cdot \int_{\mathbb{B}} (1-|z|^2)^{-q+2s}\,|f_k(z)|^2\,d\nu(z) \qquad (11) \\ &\le \frac{(N!)^2\,\Gamma(1-q+2s)\,\Gamma(1+q+2t)}{\Gamma(N+1+s+t)^2} \sum_{k=0}^{\infty} \int_{\mathbb{B}} (1-|z|^2)^{-q+2s}\,|f_k(z)|^2\,d\nu(z) \\ &= \frac{(N!)^2\,\Gamma(1-q+2s)\,\Gamma(1+q+2t)}{\Gamma(N+1+s+t)^2}\,\|f\|^2_{L^2_q},\end{aligned}$$

where the inequality follows from [**FR**, Proposition 2.6 (33)], which is also valid for $x = 0$ when $y = 0$.

To show that equality holds, we take $f(z) = (1-|z|^2)^{-q+s}\,f_k(z)$ with $f_k \in H_k$. By (11), we have

$$\|P_s f\|^2_{B^2_q} = \frac{(N!)^2\,\Gamma(1-q+2s)\,\Gamma(1+q+2t)}{\Gamma(N+1+s+t)^2}\,\frac{(N+1-q+2s+k)_{q-s+t}}{(N+1+s+t+k)_{q-s+t}}\,\|f\|^2_{L^2_q}.$$

The second fraction has limit 1 as $k \to \infty$ by [**FR**, Proposition 2.6]. This yields the desired result. □

Corollary 5.6. *If $V^t_s : L^2_q \to M^{2t}_{qs}$ is bounded, then*

$$\|V^t_s\| = \frac{\sqrt{\Gamma(1-q+2s)\,\Gamma(1+q+2t)}}{\Gamma(1+s+t)}.$$

Given an $s$ satisfying (8), taking $t = -q+s$ always satisfies (1), and this makes $\|V_s^{-q+s}\| = 1$. Therefore $V_s^{-q+s} : L^2_q \to M^{2(-q+s)}_{qs}$ is an orthogonal projection. For the classical Bergman space $B^2_0$, we need $s > -1/2$ and $t = 0$ is customary. Then Corollary 5.6 reduces to [**FR**, Theorem (9)]. However, we can also take $t = s$ with $B^2_0$, say $s = t = 1$. Then $V^1_1 : L^2 \longrightarrow \frac{1}{2}(N+1)(N+2)(1-|z|^2)D^1_1(B^2_0)$ is also an orthogonal projection, although $P_1 : L^2 \to B^2_0$ is not orthogonal. So by adjusting the range space by $I^t_s$, we can generate orthogonal projections from previously nonorthogonal ones.

Adjusting by $I^t_s$ is useful in other formulas, too, when $s, t$ satisfy (8) and (1). For example, we have $P_s C_{s+t} I^t_s \overline{f(z)} = \overline{f(0)}$, where $D^t_s \overline{f}$ is defined as $\overline{D^t_s f}$. Also, $\|I^t_s f\|_{L^p_q} \le C\,\|I^t_s(\operatorname{Re} f)\|_{L^p_q}$ for some $C$.

## 6. Boundary Growth and Maximal Point Evaluations

We now deduce several properties of Besov spaces as consequences of Bergman projections. Recall that our standing hypothesis is $1 \le p \le \infty$.

THEOREM 6.1. *Given $q$, $1 \le p < \infty$, and $s, t$ to be used in $\|\cdot\|_{B^p_q}$, there is a constant $C$ such that for all $f \in B^p_q$ and $z \in \mathbb{B}$,*

$$|f(z)| \le C\,\|f\|_{B^p_q} \begin{cases} (1-|z|^2)^{-(N+1+q)/p}, & \text{if } q > -(N+1); \\ \log(1-|z|^2)^{-1}, & \text{if } q = -(N+1); \\ 1, & \text{if } q < -(N+1). \end{cases}$$

This theorem, as well as the next two corollaries, can be found in a form that also covers the case $0 < p < 1$ in [**BB**, Lemma 5.6]. The novelty here is that we have quick proofs based on Theorem 5.2.

PROOF. First let $1 < p < \infty$. We begin by applying Theorem 5.2 (9) to $f \in B^p_q$ with $s, t$ satisfying (8), (1), and also $s > -(N+1)$ for convenience. This gives

$$f(z) = C \int_{\mathbb{B}} \frac{I^t_s f(w)\,(1-|w|^2)^{-q+s}}{(1-\langle z,w\rangle)^{N+1+s}}\,d\nu_q(w) \qquad (z \in \mathbb{B}).$$

Applying Hölder's inequality with $p' = p/(p-1)$, we obtain

$$\begin{aligned} |f(z)| &\le C\,\|f\|_{B^p_q} \left( \int_{\mathbb{B}} \frac{(1-|w|^2)^{(-q+s)p'}}{|1-\langle z,w\rangle|^{(N+1+s)p'}}\,d\nu_q(w) \right)^{1/p'} \\ &= C\,\|f\|_{B^p_q} \left( \int_{\mathbb{B}} \frac{(1-|w|^2)^{a}}{|1-\langle z,w\rangle|^{N+1+a+c}}\,d\nu(w) \right)^{1/p'}, \end{aligned}$$

where $a = (-q+ps)/(p-1) > -1$ by (8) and $c/p' = (N+1+q)/p$. A glance at [**R**, Proposition 1.4.10] yields all three inequalities for $p \ne 1$. Since the Lebesgue norms are continuous functions of $p$, the proof is completed by letting $p \to 1^+$. □

When $q = -1$, we see that the spaces $B^p_{-1}$ have the same growth rate near $\partial\mathbb{B}$ as those of Hardy spaces $H^p$ given in [**R**, Theorem 7.2.5 (a)]. Another similarity between the two families has already been noted in Section 3.

COROLLARY 6.2. *Given $q, r, u$, $1 \le p < \infty$, and $s, t$ to be used in $\|\cdot\|_{B^p_q}$, there is a constant $C$ such that for all $f \in B^p_q$ and $z \in \mathbb{B}$,*

$$|D^u_r f(z)| \le C\,\|f\|_{B^p_q} \begin{cases} (1-|z|^2)^{-(N+1+q+pu)/p}, & \text{if } q > -(N+1+pu); \\ \log(1-|z|^2)^{-1}, & \text{if } q = -(N+1+pu); \\ 1, & \text{if } q < -(N+1+pu). \end{cases}$$

PROOF. Just combine Theorem 6.1 with Proposition 3.1. Note that $u < 0$ is a possibility. □

When $p = \infty$, these results take the following form. The proof is similar to and easier than that above, or we can just set $p = \infty$.

COROLLARY 6.3. *Given $r, u$ and $s, t$ to be used in $\|\cdot\|_{\mathcal{B}}$, there is a constant $C$ such that for all $f \in \mathcal{B}$ and $z \in \mathbb{B}$,*

$$|D^u_r f(z)| \le C\,\|f\|_{\mathcal{B}} \begin{cases} (1-|z|^2)^{-u}, & \text{if } u > 0; \\ \log(1-|z|^2)^{-1}, & \text{if } u = 0; \\ 1, & \text{if } u < 0. \end{cases}$$

That the above results do not depend on the particular $s, t$ used in their proofs confirms the work of Section 2.

Corollary 6.4. *If $q < -(N+1)$ and $1 \le p < \infty$, then $B^p_q \subset H^\infty$, the space of bounded holomorphic functions on $\mathbb{B}$, and the inclusion is continuous.*

Corollary 6.5. *Let $q > -(N+1)$ and $1 \le p < \infty$. If $f \in B^p_q$, then*

$$\lim_{|z| \to 1^-} (1 - |z|^2)^{(N+1+q)/p} \, |f(z)| = 0.$$

*Further, the exponent $(N+1+q)/p$ on $(1-|z|^2)^{-1}$ in Theorem 6.1 cannot be replaced by a smaller one.*

Proof. The first claim is proved the same way as [**R**, Theorem 7.2.5] by the properties of the dilates of $f$ given in the proof of [**BB**, Lemma 5.2]. For the second claim, we modify the example given following [**R**, Theorem 7.2.5] that contains the case $q = 0$. Let $c < (N+1+q)/p$ and $g(z) = (1 - z_1)^{-c} = (1 - \langle z, e_1 \rangle)^{-c}$, where $e_1 = (1, 0, \ldots, 0)$. We take a $t$ satisfying (1) and compute $\|g\|_{B^p_q}$ using (6). By [**R**, Proposition 1.4.10], $g \in B^p_q$. So $B^p_q$ has an element whose growth rate has exponent $c$ for any $c$ lower than the one claimed. □

Here is another application. A general result also containing the case $0 < p < 1$ is in [**BB**, Theorem 5.7], but the proof is now trivial, given the above results.

Theorem 6.6. *Given a compact set $E$ in $\mathbb{B}$, $q, p, r, u$, and $s, t$ to be used in $\| \cdot \|_{B^p_q}$, there is a constant $C$ such that*

$$\sup_{z \in E} |D^u_r f(z)| \le C \, \|f\|_{B^p_q}.$$

*Therefore evaluation of $f$ and of any $D^u_r f$ at any point of $\mathbb{B}$ is a bounded linear functional. Consequently, the Besov space $B^p_q$ is a complete space.*

Proof. The first and second statements follow immediately from Theorem 6.1. It is a standard matter to obtain the last statement from these. □

Associated to point evaluations is the extremal problem of finding

$$\sup \big\{ \, f(a) > 0 \, : \, a \in \mathbb{B}, \; f \in B^p_q, \; \|f\|_{B^p_q} = \|I^t_s f\|_{L^p_q} = 1 \, \big\}, \tag{12}$$

which has been solved for Bergman spaces in [**V**]. More generally, we have the following result.

Theorem 6.7. *For $0 < p < \infty$, the extremal function solving (12) exists and is unique.*

Proof. Take $s, t$ satisfying (1) and put $Q = q + pt > -1$. Then $D^t_s f \in B^p_Q$, which is in the Bergman region; we use $\|f\|_{B^p_q} = \|D^t_s f\|_{B^p_Q}$. First take $a = 0$. By the proof of [**V**, Theorem] applied to $D^t_s f$, we have

$$D^t_s f(0) \le \exp\left( C_Q \int_{\mathbb{B}} \log |D^t_s f(z)| \, d\nu_Q(z) \right) \le C_Q^{1/p} \, \|f\|_{B^p_q} = C_Q^{1/p}.$$

The great advantage of using the radial differential operators $D^t_s$ now becomes apparent. We have $D^t_s f(0) = d_0 f(0)$. Hence $f(0) \le C_Q^{1/p}/d_0$, and equality holds if and only if $f$ is constant. Thus the unique extremal function at 0 is $f_0(z) = C_Q^{1/p}/d_0$.

To pass to other $a \in \mathbb{B}$, we use an idea in [**RS**, Theorem 5.2]. By Proposition 3.1, the map $U_a = D_{s+t}^{-t} T_a D_s^t$ is a surjective involutive isometry on $B_q^p$, because the $T_a$ in [**V**, Lemma] has the same properties on $B_Q^p$. Therefore the unique extremal function at $a$ is

$$f_a(z) = U_a f_0(z) = D_{s+t}^{-t} T_a C_Q^{1/p}(z) = C_{q+pt}^{1/p}\, D_{s+t}^{-t} \left( \frac{1-|a|^2}{(1-\langle z, a\rangle)^2} \right)^{(N+1+q+pt)/p}.$$

When $s = (N+1+q+pt)2/p - (N+1) - t$, we can make $f_a$ a little more explicit by using (6). Then

$$f_a(z) = \left( \frac{(1+q+pt)_N}{N!} \right)^{1/p} (1-|a|^2)^{(N+1+q+pt)/p}\, K_{(N+1+q+pt)2/p-(N+1)-t}(z, a).$$

When $q > -1$, we can take $t = 0$, and $f_a$ reduces to the function given in [**V**, Theorem] when we take into account the difference in the normalization of the measures. □

We can also consider the problem of maximizing the values of derivatives of functions in $B_q^p$; by Theorem 3.1, this reduces to problem (12) in another Besov space, so we omit the details. A particular case is in [**RS**, Theorem 5.3].

Another topic closely related to boundary growth is the growth of the *integral means* $M_r(f, R)$ of functions in a $B_q^p$ space; see [**BB**, p. 23] for a definition of these means. We are content with summarizing the situation in this matter.

For Besov spaces in the Bergman region and for the Bloch space, the problem has been dealt with in [**BB**, Theorem 3.2] in considerable detail. A Besov space $B_q^p$ in the region $q < -1$ lies in a Hardy space $H^{p_2}$ with $p_2 > p$ by [**BB**, Theorem 5.13] and hence lies in $H^p$. This is true also for the spaces $B_{-1}^p$ for $0 < p \le 2$ by Remark 3.2. More strongly, by Corollary 6.4, any $B_q^p$ space with $q < -(N+1)$ lies in $H^\infty$. Then one can consult the numerous results in [**D**, Chapter 5] on the growth of the integral means of Hardy spaces.

These remarks omit the case of $B_{-1}^p$ with $p > 2$. Let $N = 1$ for now and $f \in B_{-1}^p$. Then $f' \in B_{p-1}^p$, which is a Bergman space. By [**BB**, Theorem 3.2], we have $M_p(f', R) = \mathcal{O}((1-R^2)^{-1})$. Then [**GP**, Theorem 1] yields $M_p(f, R) = \mathcal{O}((\log(1-R^2)^{-1})^b)$ for all $b > 1/2$ and that this result is sharp.

## 7. Taylor Coefficients

In this section, our standing hypotheses are $N = 1$ and $1 \le p \le \infty$.

Theorem 7.1. *Given $q, p$ and $s, t$ to be used in $\|\cdot\|_{B_q^p}$, there is a constant $C$ such that for all $f \in B_q^p$, we have*

$$|c_k| \le C\, \|f\|_{B_q^p}\, k^{(1+q)/p},$$

*where $c_k = f'(0)/k!$.*

Notice the natural similarity with Theorem 4.2.

Proof. As in the proof of Theorem 6.1, we begin by applying Theorem 5.2 (9) to $f \in B_q^p$ with $s, t$ satisfying (8), (1), and also $s > -2$ for convenience. This gives

$$f(z) = C \int_{\mathbb{D}} \frac{I_s^t f(w)}{(1 - z\overline{w})^{2+s}}\, dA_s(w) \qquad (z \in \mathbb{D}),$$

where $A$ is the normalized area measure. Then

$$f^{(k)}(z) = C\,(2+s)_k \int_{\mathbb{D}} \frac{I_s^t f(w)\,\overline{w}^k}{(1-z\overline{w})^{2+s+k}}\,dA_s(w)$$

and

$$c_k = \frac{f^{(k)}(0)}{k!} = C\,\frac{(2+s)_k}{k!} \int_{\mathbb{D}} I_s^t f(w)\,\overline{w}^k\,(1-|w|^2)^{-q+s}\,dA_q(w).$$

Consider first $1 < p < \infty$. Applying Hölder's inequality with $p' = p/(p-1)$, switching to polar coordinates, and letting $y = |w|^2$ yield

$$\begin{aligned}
|c_k| &\le C\,\frac{(2+s)_k}{k!}\,\|f\|_{B_q^p}\left(\int_{\mathbb{D}} |w|^{p'k}\,(1-|w|^2)^{(-q+s)p'+q}\,dA(w)\right)^{1/p'} \\
&= C\,\|f\|_{B_q^p}\,\frac{\Gamma(2+s+k)}{\Gamma(1+k)}\left(\int_0^1 y^{1+p'k/2}\,(1-y)^{-q(p'-1)+p's}\,dy\right)^{1/p'} \\
&= C\,\|f\|_{B_q^p}\,\frac{\Gamma(2+s+k)}{\Gamma(1+k)}\left(\frac{\Gamma(p'k/2+2)}{\Gamma(p'k/2-q(p'-1)+p's+3)}\right)^{1/p'},
\end{aligned}$$

where $-q(p'-1)+p's > -1$ by (8), and factors not involving $k$ are gathered together in $C$. Now we estimate using (3). Thus

$$|c_k| \le C\,\|f\|_{B_q^p}\,k^{1+s}\left(k^{(q-ps)/(p-1)-1}\right)^{(p-1)/p} \le C\,\|f\|_{B_q^p}\,k^{(1+q)/p}.$$

The cases $p=1$ and $p=\infty$ follow as in Section 6, or the case $p=\infty$ can be directly computed. Likewise, the final result does not depend on the particular $s,t$ used in the proof, once again confirming the work of Section 2. □

The case $q=-2$ of Theorem 7.1 has been taken care of in [**Zh**, Theorem 8]. There is a stronger result for $0<p<1$ in [**BK**, Theorem 2.8].

The next step in Taylor coefficients is Hardy-Littlewood-type results, and these have already been derived in [**BK**] for $q > -1$. They extend to all $q$ readily as follows.

THEOREM 7.2. *Let $f(z) = \sum_k c_k z^k$. For $0 < p \le 2$, there is a constant $C$ such that*

$$\sum_k \frac{k^{p-2}}{k^{1+q}}\,|c_k|^p < C\,\|f\|_{B_q^p}.$$

*For $2 \le p < \infty$, there is a constant $C$ such that*

$$\|f\|_{B_q^p} < C \sum_k \frac{k^{p-2}}{k^{1+q}}\,|c_k|^p.$$

*The exponents on $k$ are sharp.*

PROOF. First, $\|f\|_{B_q^p} = \|g\|_{B_Q^p}$, where $g = D_s^t f$ and $Q = q+pt > -1$ for some $s,t$ satisfying (1). Second, $g(z) = D_s^t f(z) \sim \sum_k k^t c_k z^k$. We are done by using [**BK**, Theorems 2.1 and 2.2] on $g$. □

When $p=2$, the exponent on $k$ is the same as that in (5) for $N=1$, which also shows that this exponent is sharp. It is impossible to distinguish between $B_{-1}^p$ and $H^p$ in this respect. See [**D**, Chapter 6] for the results on Hardy spaces. The case $q=-2$ for $p>1$ first appears in [**HW**, Theorem 2]. The very special case $q=-2$ and $p=1$ is also in [**Zh**, Theorem 8 (2)].

All the results in this section generalize to $N > 1$, although they become less pleasing in appearance. We omit them, but refer the reader to [**BK**, Theorem 3.4] for more attractive results lacking $q$ in the exponents and thus more reminiscent of the classical Hardy-Littlewood inequalities.

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