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ASYMPTOTIC ANALYSIS OF NONHOMOGENEOUS HIERARCHICAL MARKOV PROCESSES AND APPLICATIONS IN MODELS OF THE CONSOLIDATION OF QUEUEING SYSTEMS

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ABSTRACT. Limit theorems for nonhomogeneous in time hierarchical Markov processes admitting the consolidation of states are studied. Notion of quasi-Markov processes is introduced. The theorem on asymptotic consolidation in the class of switching processes is used in the proof. As an application, asymptotic properties of some models of nonhomogeneous in time queueing systems with local characteristics depending on the current state and a random Markov environment are investigated.

1. INTRODUCTION

A good part of the papers by B. V. Gnedenko is devoted to the development of the theory of limit theorems for sums of random variables and to its applications in queueing and reliability theories [13]–[15]. These papers basically deal with homogeneous in time models.

The asymptotic behavior of nonhomogeneous in time Markov processes (MP) with characteristics slowly varying in some scale of time and with the state space admitting the asymptotic consolidation of states is studied in this paper. The class of Markov systems with two levels of hierarchy is considered. It is convenient to study this class by using the class of the so-called switching processes (SP).

In Section 2 we consider the general construction of switching processes and give a limit theorem on the asymptotic consolidation in the class SP. In Section 3 the asymptotic behavior of additive functionals on quasi-ergodic MP is studied and a model of the asymptotic consolidation of states of nonhomogeneous MP is investigated. Section 4 is devoted to the application of the preceding results to the investigation of asymptotic properties of queueing systems with local characteristics depending on a current state in a random Markov environment.

2. SWITCHING PROCESSES

The main property of switching processes is that their development may spontaneously change at some epochs of time that are random functionals of the preceding trajectory. Formally, SP's are two-component processes $(x(t), \zeta(t))$, $t \geq 0$, assuming values in the space $(X, \mathbf{R}^r)$ for which there exists a sequence of time moments, $t_1 < t_2 < \cdots$ such that $x(t) = x(t_k)$ on every interval $[t_k, t_{k+1})$ and the behavior of the process $\zeta(t)$ on this interval depends only on the values $(x(t_k), \zeta(t_k))$. The moments t_k are called switching moments and $x(t)$ is called the discrete switching component.

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In specific applications, the component $x(t)$ may be viewed as a random environment or as an operating regime of a system. Such processes may be described in terms of constructive characteristics [3–4] and they are a convenient tool to study the asymptotic behavior of stochastic systems with "fast" and "rare" switches ([4–6], [8], [9]).

SP's are a natural generalization of well known classes of processes such as Markov processes, processes homogeneous in the second component [16], processes with independent increments and semi-Markov switches [2], Markov aggregates [12], MP with a semi-Markov interference of a chance [17], and Markov and semi-Markov evolutions [18, 21, 23–25].

2.1. Switching processes. Let jointly independent families

$$F_k = \{(\zeta_k(t,x,\alpha), \tau_k(x,\alpha), \beta_k(x,\alpha)), t \geq 0, x \in X, \alpha \in \mathbf{R}^r\}, \qquad k \geq 0,$$

be given, where $\zeta_k(t,x,\alpha)$, for all fixed k, x, and α, is a stochastic process in the Skorokhod space D^r_∞ and $\tau_k(x,\alpha)$, $\beta_k(x,\alpha)$ are random variables that may depend on $\zeta_k(\cdot,x,\alpha)$, $\tau_k(\cdot) > 0$, $\beta_k(\cdot) \in X$. Denote by (x_0, S_0) the initial value. Put

$$\begin{aligned} &t_0 = 0, \qquad t_{k+1} = t_k + \tau_k(x_k, S_k), \qquad S_{k+1} = S_k + \xi_k(x_k, S_k), \\ &\qquad\qquad x_{k+1} = \beta_k(x_k, S_k), \qquad k \geq 0, \end{aligned} \tag{2.1}$$

where $\xi_k(x,\alpha) = \zeta_k(\tau_k(x,\alpha), x, \alpha)$, and set

$$\zeta(t) = S_k + \zeta_k(t - t_k, x_k, S_k), \qquad x(t) = x_k, \quad \text{for } t_k \leq t < t_{k+1},\ t \geq 0. \tag{2.2}$$

The two-component process $(x(t), \zeta(t))$, $t \geq 0$, is a switching process. We say that a SP is regular if its component $x(t)$ has (with probability one) only a finite number of jumps on any finite interval.

If the distributions of random variables $\{\tau_k(x,\alpha), \beta_k(x,\alpha)\}$ do not depend on k and α, then the sequence x_k is a homogeneous MP and $x(\cdot)$ is a semi-Markov process (SMP). In this case $\zeta(\cdot)$ is a process with semi-Markov switches. If additionally $\zeta_k(t,x,\alpha)$ are processes with independent increments (PII), then $\zeta(\cdot)$ is a PII process with semi-Markov switches (PII SMS) (see [2]). If $x(\cdot)$ is a Markov process, then $(x(t), \zeta(t))$, $t \geq 0$, is a Markov process, homogeneous in the second component [16].

In a particular case, where a nonnegative function $a(x)$, $x \in X$, is given and a SMP $x(t)$ assumes values in X, we denote by $\Pi_{\lambda(t)}(t)$ a Poisson process with parameter $\lambda(t) = a(x(t))$, $t \geq 0$, and call it a process of Poisson type with a parameter switched by the process $x(t)$.

Note that SP's are mathematical models of the processes of service in queueing systems operating in a random environment. A broad class of systems and networks with parameters depending on a current state of a system (size of a queue etc.) and on an external Markov or semi-Markov environment, may be described in terms of SP's. To this class belong $SM_Q/M_Q/1/\infty$, $M_{SM,Q}/M_{SM,Q}/l/k$, $(M_{SM,Q}/M_{SM,Q}/l_i/k_i)^r$, processes with batch Markov or semi-Markov input flow and service, various types of calls of possibly random size (a volume of information, value of necessary job), etc.

2.2. Convergence of switching processes. In [4–6] limit theorems on the convergence of an SP to another SP (in the class of SP) for the case of "rare" switches were proved and these results form the basis of the consolidation theory of states of nonhomogeneous Markov and semi-Markov processes. We quote a theorem from [5–6] that we use in what follows. This theorem concerns the general models of the consolidation of states in the class of SP's.

The asymptotic consolidation of states means that the parameter space of the initial SP may be split up into subsets in such a way that the characteristics of the limit SP depend on a lesser number of parameters and each parameter corresponds to a concrete subset of the initial space. Assume that, for all $n > 0$, the following families:

$$F_{nk} = \{(\zeta_{nk}(t, x, \alpha), \tau_{nk}(x, \alpha), \beta_{nk}(x, \alpha)),\ x \in X, \alpha \in \mathbf{R}^r\}, \qquad k \geq 0,$$

and the initial value (x_{n0}, S_{n0}), $n = 1, 2, \dots$, are given. This collection, for all $n > 0$, defines, according to relations (2.1) and (2.2), an SP $(x_n(t), \zeta_n(t))$, $t \geq 0$.

Let $K(\cdot): X \to Y$ be a measurable mapping, where Y is a metric space, and let the families

$$\tilde{F}_k = \left\{ \left(\tilde{\zeta}_k(t, y, \alpha), \tilde{\tau}_k(y, \alpha), \tilde{\beta}_k(y, \alpha) \right), t \geq 0, y \in Y, \alpha \in \mathbf{R}^r \right\}, \qquad k \geq 0,$$

and the initial value (y_0, S_0) be given. Using this collection we construct the SP $(y(\cdot), \zeta(\cdot))$ and suppose it to be regular.

Let us give general conditions for the convergence of the processes $(K(x_n(\cdot)), \zeta_n(\cdot))$, $n = 1, 2, \dots$, consolidated with respect to the first component, to a limit SP $(y(\cdot), \zeta(\cdot))$.

Definition 2.1. We say that a sequence of processes $\xi_n(\cdot)$ J-converges as $n \to \infty$ to a process $\xi(\cdot)$ on some interval $[0, T]$ if the sequence of measures generated by the sequence of processes $\xi_n(\cdot)$ weakly converges in the Skorokhod space $D^r_{[0,T]}$ to the corresponding measure generated by $\xi(\cdot)$.

Set

$$\begin{aligned} &\psi_{nk}(\lambda_0, \dots, \lambda_j, t_1, \dots, t_j, \theta, f(\cdot), x, a) \\ &\quad = \mathsf{E} \exp\left\{ i \sum_{l=1}^{j} (\lambda_l, \zeta_{nk}(t_l, x, a)) + i(\lambda_0, \xi_{nk}(x, a)) - \theta \tau_{nk}(x, a) \right\} f(\beta_{nk}(x, a)), \end{aligned} \tag{2.3}$$

where $\xi_{nk}(x, \alpha) = \zeta_{nk}(\tau_{nk}(x, \alpha), x, \alpha)$, $\lambda_l \in \mathbf{R}^r$, $l = 0, \dots, j$, $0 \leq t_1 \leq \dots \leq t_k$, $\theta \geq 0$, $f(\cdot)$ is a continuous function on X. We assume that X is a metric space. Further, let the function $\hat{\psi}_k(\lambda_0, \dots, \lambda_j, t_1, \dots, t_j, \theta, f(\cdot), y, a)$ be defined by (2.3) for the family $\hat{F}_k$, $k \geq 0$.

Theorem 2.1. *Let the following conditions be satisfied:*

1. $(K(x_{n0}), S_{n0}) \overset{\mathrm{w}}{\Rightarrow} (y_0, S_0)$ ($\overset{\mathrm{w}}{\Rightarrow}$ *stands for the weak convergence of distributions*).
2. *There are sequences of sets* $B_m \in \mathcal{B}_Y$ *and* $D_m \in \mathcal{B}_{\mathbf{R}^r}$, $m \geq 0$, *such that for all* $m \geq 0$, $u_0 \in D_m$, $g_0 \in B_m$, *for all* $u_n \to u_0$ *and* $v_n \in X$ *with* $K(v_n) \to g_0$, *and for all* $k \geq 0$, $j \geq 0$, $\lambda_0, \dots, \lambda_j$, $t_1, \dots, t_j$, θ, $f(\cdot)$,
$$\begin{aligned} &\lim_{n \to \infty} \psi_{nk}(\lambda_0, \dots, \lambda_j, t_1, \dots, t_j, \theta, f(K(\cdot)), v_n, u_n) \\ &\qquad = \hat{\psi}_k(\lambda_0, \dots, \lambda_j, t_1, \dots, t_j, \theta, f(\cdot), g_0, u_0), \end{aligned}$$
and the sequence of measures generated by the processes $\zeta_{nk}(t, v_n, u_n)$ *is weakly compact in the Skorokhod space* $D^r_{[0,T_\ell]}$ *on some sequence of intervals* $[0, T_\ell]$, *where* $T_\ell \to \infty$ *as* $\ell \to \infty$;
3. $\mathsf{P}\{\tilde{\tau}_k(y, \alpha) > 0\} = 1$, $y \in Y$, $\alpha \in \mathbf{R}^r$, $k \geq 0$;
4. $\mathsf{P}\{y_k \in B_k\} = 1$, $\mathsf{P}\{S_k \in D_k\} = 1$, $k \geq 0$, *where the sequences* y_k *and* S_k, $k \geq 0$, *are constructed by the process* ζ *according to* (2.1), (2.2).

Then the sequence of consolidated processes $(K(x_n(\cdot)), \zeta_n(\cdot))$ J*-converges as* $n \to \infty$ *to the SP* $(y(\cdot), \tilde{\zeta}(\cdot))$ *on any interval* $[0, T_\ell]$.

The proof of this theorem is given in [5, 6]. A limit SP is of a sufficiently simple structure in various applications, for instance, it could be a PII with Markov switches or PII SMS. Therefore Theorem 2.1 provides the basis of a new approach to asymptotic problems, decreasing the dimension for compound stochastic systems.

3. Asymptotic consolidation of states in nonhomogeneous hierarchical Markov systems

If the state space of an MP may be split up into asymptotically connected subsets such that the transient probabilities are asymptotically negligible, then we prove, under rather general conditions, that the accumulation processes on SP's weakly converge to a SMP with Markov switches and with the state space corresponding to the total number of subsets of the initial process.

First we introduce the notion of a class of nonhomogeneous MP's with characteristics slowly varying in some scale of time.

3.1. Quasi-ergodic Markov processes. For the sake of simplicity we consider the case of a finite number of states. Let $x_n(t)$, $t \geq 0$, be an MP with the state space $X = \{1, 2, \ldots, r\}$ generated by the family of instantaneous intensities $\{a_n(i,j,t), i,j \in X, i \neq j, t \geq 0\}$ of the transient probabilities. Denote by $\varphi_n(u,T)$ the uniformly strong mixing coefficient (USMC) for a process $x_n(t)$ on an interval $[0, k_n T]$:

$$\begin{aligned}\varphi_n(u,T) = \sup_{0 \leq t \leq k_n T} \max_{i,j \in X, A \subset X} \big| \mathbf{P}\{x_n(t+u) \in A / x_n(t) = i\} \\ - \mathbf{P}\{x_n(t+u) \in A / x_n(t) = j\} \big|.\end{aligned} \tag{3.1}$$

Assume that there exists a family of continuous functions $\{a_0(i,j,v), i,j \in X, i \neq j, v \geq 0\}$ and a sequence of natural numbers $k_n \to \infty$ such that

$$\lim_{n \to \infty} \sup_{v \leq T} |a_n(i,j,k_n v) - a_0(i,j,v)| = 0 \tag{3.2}$$

for all $i,j \in X$, $i \neq j$, and $T \geq 0$. For any fixed $v \geq 0$ we denote by $x_0^{(v)}(\cdot)$ an auxiliary MP with the state space X generated by the family of intensities $\{a_0(i,j,v), i,j \in X, i \neq j\}$. Similarly to (3.1) we introduce the USMC $\varphi^{(v)}(u)$ for this process. Suppose that there exists q, $0 \leq q < 1$, and that, for all $T > 0$, there exists $r(T) > 0$ such that

$$\varphi^{(v)}(r(T)) \leq q \tag{3.3}$$

for all $v \leq T$.

Assertion 3.1. *Let conditions* (3.2) *and* (3.3) *be satisfied. Then for all* $v > 0$

$$\mathsf{P}\{x_n(k_n v) = j\} \to \pi^{(v)}(j), \qquad j \in X, \tag{3.4}$$

as $n \to \infty$, *where* $\pi^{(v)}(j)$, $j \in X$, *is the stationary distribution* (*existing if* (3.3) *is satisfied*) *for the MP* $x_0^{(v)}(\cdot)$. *Also there exists* q, $0 \leq q < 1$, *and for all* $T > 0$ *there exists* $r(T) > 0$ *such that*

$$\varphi_n(r(T), T) \leq q. \tag{3.5}$$

An MP satisfying (3.4) is called a quasi-ergodic MP.

Proof. Denote by $\tilde{x}_n(t)$, $t \geq 0$, an MP generated at time t by the instantaneous transient intensities $\{a_0(i,j,t/k_n), i,j \in X, i \neq j, u \geq 0\}$ and such that $\tilde{x}_n(0) = x_n(0)$. Let

$$p_n(i,j,u) = \mathsf{P}\{x_n(k_n u) = j / x_n(0) = i\}, \qquad \tilde{p}_n(i,j,u) = \mathsf{P}\{\tilde{x}_n(k_n u) = j / x_n(0) = i\}.$$

By conditions (3.2) and (3.3), for all T we have

$$\sup_{u \leq T} \max_{i,j} |\tilde{p}_n(i,j,u) - p_n(i,j,u)| \to 0$$

as $n \to \infty$ according to results of [6, Chapter 2, Section 1] (see also [7]) and relation (3.5) is satisfied for the process $\tilde{x}_n(t)$. Further, since functions $a_0(i,j,u)$ are continuous, analogously to [6] and [7] we obtain for all $u > 0$ that $|\tilde{p}_n(i,j,u) - \pi^{(u)}(j)| \to 0$ in view of condition (3.3). This proves the statement of Assertion 3.1. □

3.2. Asymptotic behavior of the first exit time from a subset of states. Let $x_n(t)$, $t \ge 0$, be a nonhomogeneous MP with a finite state space $X = \{0,1,\dots,d\}$ generated by a family of instantaneous transient intensities $\{a_n(i,l,t), i,l \in X, i \ne l, t \ge 0\}$. Denote by

$$\Omega_n(i_0) = \inf\{\, t : t > 0,\ x_n(t) = 0,\ \text{given } x_n(0) = i_0 \,\}, \qquad i_0 = 1,\dots,d, \tag{3.6}$$

the first exit time from the subset $X_0 = \{1,2,\dots,d\}$. Let us investigate the behavior of $\Omega_n(i_0)$ if the set $\{1,2,\dots,d\}$ forms a single quasi-ergodic class as $n \to \infty$.

Theorem 3.1. *Let there exist a sequence $k_n \to \infty$ satisfying condition (3.2) for all $i,l \in X_0$, $i \ne l$. Further, let the auxiliary homogeneous MP $\tilde{x}_0^{(v)}(\cdot)$ generated by the intensities $\{a_0(i,l,v), i,l \in X_0, i \ne l\}$ satisfy relation (3.3), and for all $T > 0$*

$$\limsup_{n\to\infty} \max_{i\in X_0} \sup_{u<T} k_n a_n(i,0,k_n u) < C_T < \infty. \tag{3.7}$$

Then for all $i_0 \in X_0$

$$\lim_{n\to\infty} \sup_{u\ge 0} |\mathsf{P}\{\Omega_n(i_0) > k_n u\} - \exp\{-\Lambda_n(u)\}| = 0, \tag{3.8}$$

where

$$\Lambda_n(u) = k_n \int_0^u \Big(\sum_{i\in X_0} \pi^{(v)}(i) a_n(i,0,k_n v)\Big)\, dv,$$

and $\pi^{(v)}(i)$, $i \in X_0$, is the stationary distribution for the MP $\tilde{x}_0^{(v)}(\cdot)$.

Remark. Under the same conditions we obtain in the homogeneous case (i.e. $a_n(i,l,t) \equiv a_n(i,l)$) that $\Lambda_n(u) = u k_n \sum_{i\in X_0} \pi(i) a_n(i,0)$, which means the exponential approximation for $\Omega_n(i_0)$.

Proof. Denote by $\tilde{x}_n(t)$ the auxiliary nonhomogeneous MP with the state space $X_0 = \{1,2,\dots,d\}$ and intensities of transitions $\{a_n(i,l,t),\ i,l \in X_0, i \ne l, t \ge 0\}$. Further, we denote by $(\tilde{x}_n(t), \Pi_n(t))$, $t \ge 0$, a two-component MP such that $\Pi_n(t)$ is a Poisson process switched by $\tilde{x}_n(t)$ and having instantaneous intensity of a jump $a_n(x_n(t),0,t)$ at the moment t. Put

$$\tilde{\Omega}_n(i_0) = \inf\Big\{\, t : t > 0,\ \tilde{\Pi}_n(t) \ge 1,\ \text{given } \tilde{x}_n(0) = i_0 \,\Big\}, \qquad i_0 \in X_0.$$

It is not hard to prove (see [6]) that for all $i_0 \in X_0$ random variables $\Omega_n(i_0)$ (see (3.6)) and $\tilde{\Omega}_n(i_0)$ have the same distribution. According to relations (3.2) and (3.3), $\tilde{x}_n(\cdot)$ is a quasi-ergodic MP and Assertion 3.1 holds.

Now we use the representation

$$\mathsf{P}\{\Omega_n(i_0) > k_n u\} = \mathsf{E} \exp\Big\{ - \int_0^{k_n u} a_n(\tilde{x}_n(t),0,t)\, dt \Big\}. \tag{3.9}$$

Put $\tilde{\Lambda}_n(u) = \mathsf{E} \int_0^{k_n u} a_n(\tilde{x}_n(t),0,t)\, dt$. Using the inequality $|e^\alpha - e^\beta - e^\beta(\alpha-\beta)| \le \frac{1}{2}|\alpha-\beta|^2$, valid for $\alpha,\beta \le 0$, we obtain via (3.9) that

$$\Big|\mathsf{P}\{\Omega_n(i_0) > k_n u\} - \exp\big\{-\tilde{\Lambda}_n(u)\big\}\Big| \le \frac{1}{2} \mathsf{E} \Big| \int_0^{k_n u} a_n(\tilde{x}_n(t),0,t)\, dt - \tilde{\Lambda}_n(u)\Big|^2. \tag{3.10}$$

Denote by $\varphi_n(u,T)$ the USMC for the process $\tilde{x}_n(t)$ (defined similarly to (3.1)) on the interval $[0,k_nT]$. Conditions (3.2) and (3.3) imply that for all $T>0$ there exist q_1, $q<q_1<1$, and $r(T)$ such that

$$\varphi_n(r(T),T)\le q_1. \tag{3.11}$$

Taking into account the well-known inequality

$$\begin{aligned}\mathsf{E}\big|a_n(x_n(u),0,u)a_n(x_n(v),0,v)-\mathsf{E}\,a_n(x_n(u),0,u)\,\mathsf{E}\,a_n(x_n(v),0,v)\big|\\ \le \sup_{x,s} a_n(x,0,s)\varphi_n(v-u),\end{aligned}$$

valid for all $u<v$, it is not hard to prove that

$$\mathsf{E}\left|\int_0^{nu} a_n(x_n(t),t)\,dt-\tilde{\Lambda}_n(u)\right|^2\to 0.$$

Relation (3.4) together with (3.7) implies that $\tilde{\Lambda}_n(u)-\Lambda_n(u)\to 0$. This completes the proof of the statement of Theorem 3.1. □

Note that Theorem 3.1 generalizes results on exponential approximation for the first exit time from a subset of states independently proved for homogeneous MP and SMP by different methods in [1, 2] and [20], respectively.

3.3. Asymptotic consolidation of nonhomogeneous MP. Consider applications of Theorems 2.1 and 3.1 in models of the asymptotic consolidation for nonhomogeneous MP. Let $x_n(t)$, $t\ge 0$, for $n>0$ be a nonhomogeneous MP assuming values in $X=\{1,2,\dots,d\}$ and generated by a family of instantaneous intensities of transient probabilities $a_n(i,l,t)$, $i,l=1,\dots,d$, $i\ne l$. Assume that the state space X may be represented in the form

$$X=\bigcup_{j\in Y} X_j, \tag{3.12}$$

where $X_{j_1}\cap X_{j_2}=\varnothing$ for $j_1\ne j_2$. Introduce a map $K(\cdot)$ from X to Y such that $K(i)=j$ for all $i\in X_j$ and consider the consolidated process $K(x_n(t))=j$ for $x_n(t)\in X_j$, $t\ge 0$. Assume that the following representation is valid:

$$a_n(i,l,t)=a_n^{(0)}(i,l,t)+\frac{1}{n}b_n(i,l,t),\qquad i,l=1,\dots,d, \tag{3.13}$$

where for all $T>0$

$$\limsup_{n\to\infty}\max_{i,l}\sup_{t\le nT}|b_n(i,l,t)|<C_T<\infty, \tag{3.14}$$

and for all $j\in Y$, $t>0$

$$a_n^{(0)}(i,l,t)\equiv 0\qquad i\in X_j,\quad l\notin X_j. \tag{3.15}$$

We assume that the functions $a_n^{(0)}(i,l,t)$ regularly depend on the parameter t in the following way: there exists a family of continuous functions $\{a_0(i,l,u),\ i,l=1,\dots,d,\ i\ne l,\ u\ge 0\}$ such that for all $j\in Y$ and $T>0$

$$\lim_{n\to\infty}\sup_{u\le T}\left|a_n^{(0)}(i,l,nu)-a_0(i,l,u)\right|=0,\qquad i,l\in X_j. \tag{3.16}$$

For all $j \in Y$ and a fixed $v \geq 0$ we denote by $x_0^{(j)}(t,v)$, $t \geq 0$, the auxiliary homogeneous MP with the state space X_j and transient intensities $a_0(i,l,v)$, $i,l \in X_j$, $i \neq l$. Introduce the USMC

$$\varphi_0^{(j)}(u,v) = \max_{i_1,i_2 \in X_j, A \subset X_j} \Big| \mathsf{P}\left\{x_0^{(j)}(u,v) \in A \,/\, x_0^{(j)}(0,v) = i_1\right\} - \mathsf{P}\left\{x_0^{(j)}(u,v) \in A \,/\, x_0^{(j)}(0,v) = i_2\right\}\Big|. \tag{3.17}$$

Also assume that there exists q, $0 \leq q < 1$, and for all $T > 0$ there exists $r(T)$ such that for all $j \in Y$, $v \leq T$

$$\varphi_0^{(j)}(r(T),v) \leq q. \tag{3.18}$$

Note that conditions (3.16)–(3.18) mean that every X_j is a quasi-ergodic subset. Further, for all $v \geq 0$ denote by $\pi_0^{(j)}(i,v)$, $i \in X_j$, the stationary distribution of the MP $x_0^{(j)}(t,v)$ (that exists under condition (3.18)). For all $j \in Y$, $m \in Y$, $j \neq m$ we put

$$\hat{a}_n(j,m,v) = \sum_{i \in X_j} \pi_0^{(j)}(i,v) \sum_{l \in X_m} b_n(i,l,nv).$$

Assume that for all $j,m \in Y$, $j \neq m$ and for all $u > 0$ there exist the following limits:

$$\Lambda(j,m,u) = \lim_{n \to \infty} \int_0^u \hat{a}_n(j,m,v)\,dv, \tag{3.19}$$

where the functions $\Lambda(j,m,u)$ may be written in the form:

$$\Lambda(j,m,u) = \int_0^u \hat{\lambda}_0(j,m,v)\,dv, \tag{3.20}$$

and $\hat{\lambda}_0(j,m,v)$ are continuous functions.

Theorem 3.2. *Let conditions* (3.12)–(3.20) *be satisfied and*

$$K(x_n(0)) \xrightarrow{\mathsf{P}} j_0.$$

Then the sequence of consolidated processes $K(x_n(nu))$ *J-converges on any interval* $[0,T]$ *to a nonhomogeneous MP* $y(u)$ *having the state space* Y *and initial value* j_0 *and generated by the family of transient intensities* $\hat{\lambda}_0(j,m,u)$, $j,m \in Y$, $j \neq m$. *Moreover, for all* $i \in X_j$ *and* $u > 0$,

$$\lim_{n \to \infty} \mathsf{P}\{x_n(nu) = i\} = \pi_0^{(j)}(i,u)\mathsf{P}\{y(u) = j\}.$$

Further we consider the convergence of integral functionals. Let a family of continuous functions $\{f(i,v),\ i \in X, v \geq 0\}$ be given. Put

$$S_n(u) = \int_0^u f(x_n(nv),v)\,dv.$$

Set $\hat{f}(j,v) = \sum_{i \in X_j} \pi_0^{(j)}(i,v) f(i,v)$, $j \in Y$, $v \geq 0$.

Theorem 3.3. *Under the conditions of Theorem* 3.2, *the sequence of processes*

$$\big(K(x_n(nu)), S_n(u)\big)$$

J-converges on any interval $[0,T]$ *to a process* $(y(u), S(u))$, *where* $y(u)$ *is defined in Theorem* 3.2 *and*

$$S(u) = \int_0^u \hat{f}(y(v), v)\, dv. \tag{3.21}$$

Further we consider the convergence of Poisson type processes with switches. Let a family of continuous nonnegative functions $\{\mu_n(i,t),\ i \in X, t \geq 0\}$ be given. Construct a process of Poisson type on the trajectory $x_n(\cdot)$ as follows: if $x_n(t) = i$, then the instantaneous intensity of a jump is $\mu_n(i,t)$. Denote by $\Pi_n(t)$ the total number of jumps on the interval $[0,t]$ and put

$$A_n^{(j)}(u) = \int_0^u n \sum_{i \in X_j} \pi_0^{(j)}(i,v)\mu_n(i,nv)\, dv.$$

Theorem 3.4. *Let conditions of Theorem* 3.2 *be satisfied,*

$$\limsup_{n\to\infty} \max_{i\in X} \sup_{u<T} n\mu_n(i,nu) < C_T < \infty \quad \textit{for all } T > 0,$$

and suppose there exists a family of continuous functions $\{\hat{\mu}(j,v), j \in Y, v \geq 0\}$ *such that for all* $u > 0$

$$\lim_{n\to\infty} A_n^{(j)}(u) = \int_0^u \hat{\mu}(j,v)\, dv.$$

Then the sequence of processes $(K(x_n(nu)), \Pi_n(nu))$ *J-converges on any interval* $[0,T]$ *to a process* $(y(u), \Pi(u))$, *where the MP* $y(u)$ *is defined in Theorem* 3.2 *and* $\Pi(u)$ *is a Poisson process on the trajectory* $y(\cdot)$ *generated by the intensities* $\hat{\mu}(j,u)$ (*if* $y(u) = j$ *at a moment* u, *then the local intensity of a jump is* $\hat{\mu}(j,u)$).

Proofs of Theorems 3.2–3.4. First we represent the process $(K(x_n(t)), S_n(t))$ as a SP. In this case, switching moments are the moments of transitions between subsets X_j, and the processes $\zeta_n(t,j,\alpha)$ are the corresponding additive functionals of the auxiliary processes $\tilde{x}_n^{(j)}(\cdot)$ on subsets X_j.

For all $j \in Y$ and $l \in X_j$ we denote by $\tilde{x}_n^{(j)}(t,l)$ an auxiliary MP with the state space X_j and initial state l generated by transient intensities $a_n(i,k,t)$, $i,k \in X_j$, $i \neq k$. Further we denote by $\Pi_n^{(j)}(t)$ a nonhomogeneous compound Poisson process switched by $\tilde{x}_n^{(j)}(t,l)$ and having, at the moment t, the instantaneous intensity $n^{-1}b_n^{(j)}(i,t)$ of a jump from a state i, where $b_n^{(j)}(i,t) = \sum_{l\notin X_j} b_n^{(j)}(i,l,t)$, and the jump size $\varkappa_n^{(j)}(i,t)$ equals s with probability $b_n^{(j)}(i,t)^{-1}b_n^{(j)}(i,s,t)$, $s \notin X_j$. Consider a two-component process $(\tilde{x}_n^{(j)}(t,l), \Pi_n^{(j)}(t))$. Denote by $u + \tau_n^{(j)}(u,l)$ the first moment of a jump of the process $\Pi_n^{(j)}(t)$ on the interval $[u,\infty)$ and by $\beta_n^{(j)}(u,l)$ we denote the jump size. Construct a SP $y_n(t)$ with values in Y using random variables $\{\tau_n^{(j)}(u,l),\ \beta_n^{(j)}(u,l), l \in X_j\}$, $j \in Y$. Let $i_0 \in X_{j_0}$ be the initial value. Put $t_{n0} = 0$, $t_{nk+1} = t_{nk} + \tau_n^{(j_{nk})}(t_{nk}, i_{nk})$, $i_{nk+1} = \beta_n^{(j_{nk})}(t_{nk}, i_{nk})$, $j_{nk} = K(i_{nk})$, $k \geq 0$, and set $y_n(t) = j_{nk}$ for $t_{nk} \leq t < t_{nk+1}$, $t \geq 0$. By construction, the process $y_n(\cdot)$ is equivalent to the process $K(x_n(\cdot))$. Denote by $\Pi^{(j)}(t)$ the compound Poisson process with the instantaneous intensity of a jump $\hat{\lambda}_0(j,v) = \sum_{m\neq j} \hat{\lambda}_0(j,m,v)$ (see (3.19) and (3.20)) and such that the jump size $\varkappa_0^{(j)}(i,v)$ equals m with probability $\lambda_0(j,m,v)\lambda_0(j,v)^{-1}$, $l \notin X_j$. If $n \to \infty$, then, according to Theorem 3.1, the distribution of the random variable $n^{-1}\tau_n^{(j)}(nv,l)$ for all $l \in X_j$, $v \geq 0$, weakly converges to the random variable that may be described as follows: it is the moment of the first jump of the process $\Pi^{(j)}(t)-\Pi^{(j)}(v)$ on the interval $[v,\infty)$ minus v. Respectively, the distribution of the random variable $\beta_n^{(j)}(v,l)$ weakly converges to the distribution of the size of this jump. It is easily seen that the SP constructed in such a way, is equivalent to the MP $y(u)$ introduced in Theorem 3.2. Finally, using Theorem 2.1 we complete the proof of the statement of Theorem 3.2.

In the case of Theorem 3.3 we put $\zeta_n(u,j,l)=\int_0^u f(\tilde{x}_n^{(j)}(nv,l),v)\,dv$. Then the process $(K(x_n(nu)),S_n(u))$, as before, may be represented as a SP by using the above notation and the processes $\zeta_n(u,j,l)$. However, the process $\tilde{x}_n^{(j)}(t,l)$ satisfies the uniform strong mixing condition on every subset X_j. This implies that the process $\zeta_n(u,j,l)$ uniformly converges for all $l\in X_j$ to the deterministic function $\int_0^u \hat{f}(j,v)\,dv$ and the limit process constructed by the limit variables corresponds to relation (3.21). The proof of Theorem 3.4 is similar. □

Note that the consolidation of states of homogeneous MP and SMP was studied in [20, 22] using operator methods. A direct constructive method was proposed in [19]. An approach based on the switching processes was developed in [2, 4, 6].

3.4. The asymptotic consolidation in queueing systems. Now we consider applications of Theorems 3.1–3.4 to problems of the asymptotic consolidation of states in queueing systems.

3.4.1. A state-dependent system $M_{M,Q}/M_{M,Q}/s/m$ in a fast varying environment.

1) *Consolidation of states of the environment.* Let families of continuous (in t) nonnegative functions,

$$\{a(i,l,t,q),\lambda(i,t,q),\mu(i,t,q),\ i,l\in X, i\neq l, q\in\{0,1,2,\dots\}\},$$

be given, where $X=\{1,2,\dots,r\}$. The system consists of s servers and m waiting places. Denote by $x_n(t)$ a stochastic process forming the environment of the system. Let us describe the evolution of the system. We suppose that calls enter the system one by one. If, at the moment t, the total number of calls in the system is Q and $x_n(t)=i, i=1,\dots,r$, then the instantaneous intensity of the input flow is $\lambda(i,t,Q)$, the instantaneous intensity of the service for any busy server is $\mu(i,t,Q)$, and the process $x_n(t)$ may jump from a state i to a state l with the intensity $na(i,l,t,Q)$ (n is a scale parameter and $n\to\infty$). After the service is completed, the call leaves the system.

For every fixed (v,q) we consider the auxiliary homogeneous MP $x(u,v,q)$, $u\geq 0$, assuming values in X and generated by the transient intensities $\{a(i,l,v,q),i,l\in X,i\neq l\}$. Let $\varphi(u,v,q)$ be its USMC (see (3.1)). Assume that there exists g, $0\leq g<1$, and for all $T>0$ there exists $r(T)>0$ such that

$$\varphi(r(T),v,q)\leq g \tag{3.22}$$

for all $v\leq T$ and $q\in\{0,1,\dots\}$. Denote by $\{\pi(i,v,q),i\in X\}$ the stationary distribution of the process $x(u,v,q)$, $u\geq 0$, and put

$$\hat{\lambda}(v,q)=\sum_{i\in X}\lambda(i,v,q)\pi(i,v,q),\qquad \hat{\mu}(v,q)=\sum_{i\in X}\mu(i,v,q)\pi(i,v,q). \tag{3.23}$$

By $Q_n(t)$ we denote the total number of calls in the system at the moment t (the size of a queue).

Introduce the system $M_Q/M_Q/s/m$ described as follows: if the size of a queue at the moment t is $Q(t)=Q$, then the instantaneous intensity of the input flow is $\hat{\lambda}(t,Q)$ and the instantaneous intensity of service for any busy server is $\hat{\mu}(t,Q)$ (the environment is absent in such a system). Suppose the process $Q(t)$ is regular.

Assertion 3.2. *Under the above assumptions and condition* (3.22) *the process* $Q_n(t)$ *J-converges on any finite interval to the process* $Q(t)$.

Remark. This means that the size of a queue in the initial system may be approximated by the size of a queue in the limit system with averaged characteristics.

Proof. Consider the MP $(Q_n(t), x_n(t))$ and describe it as a SP. In this case the component $Q_n(\cdot)$ is the environment and $x_n(\cdot)$ is a process of Markov type switched by $Q_n(\cdot)$. Therefore the statement directly follows from Theorem 3.2. □

2) *Consolidation of states of the environment.* Now we consider the preceding system in the case where the process $x_n(\cdot)$ admits the asymptotic consolidation of states. Let families of continuous nonnegative functions $\{\lambda(i,t,q), \mu(i,t,q),\ i,l \in X, i \neq l, q \in \{0,1,2,\dots\}\}$ be given, where $X = \{1,2,\dots,r\}$. We suppose that representation (3.12) is valid (it is possible to consider the case where different q correspond to different partitions) and families of continuous nonnegative functions $\{a^{(j)}(i,l,t,q),\ i,l \in X_j, i \neq l,\ b^{(j)}(i,k,t,q),\ i \in X_j, k \notin X_j,\ j \in Y,\ t \geq 0,\ q \in \{0,1,2,\dots\}\}$ are given.

Let us describe the evolution of the system. We assume that calls enter the system one by one. If the total number of calls in the system at the moment t is Q and $x_n(t) = i \in X_j$, then the instantaneous intensity of the input flow is $\lambda(i,t,Q)$, the instantaneous intensity of service for any busy server is $\mu(i,t,Q)$, and the process $x_n(t)$ may jump from a state i to a state $l \in X_j$ with intensities $na^{(j)}(i,l,t,Q)$, $l \in X_j$, or it may jump to a state $k \in X_m$, $m \neq j$, with intensities $b^{(j)}(i,k,t,Q)$, $k \notin X_j$.

For every fixed (j,v,q) consider the auxiliary homogeneous MP $x(u,j,v,q)$, $u \geq 0$, assuming values in X_j and generated by transient intensities $\{a^{(j)}(i,l,v,q),\ i,l \in X_j, i \neq l\}$. Let $\varphi(u,j,v,q)$ be its USMC (see (3.1)). Assume that the USMC satisfies condition (3.22) for all $j \in Y$. Denote by $\{\pi(i,j,v,q), i \in X_j\}$ the stationary distribution of the process $x(u,j,v,q)$, $u \geq 0$, and put

$$
\begin{aligned}
&\hat{\lambda}(j,v,q) = \sum_{i \in X_j} \lambda(i,v,q)\pi(i,j,v,q), \qquad \hat{\mu}(j,v,q) = \sum_{i \in X_j} \mu(i,v,q)\pi(i,j,v,q), \\
&\hat{b}(j,m,v,q) = \sum_{i \in X_j} \pi(i,j,v,q) \sum_{k \in X_m} b^{(j)}(i,k,v,q).
\end{aligned}
\tag{3.24}
$$

Also denote by $Q_n(t)$ the size of a queue in the system at the moment t and put $y_n(t) = K(x_n(t))$.

Introduce the system $M_{M,Q}/M_{M,Q}/s/m$ switched by the process $y(\cdot)$ and described in the following way: if the size of a queue at the moment t is $Q(t) = Q$ and $y(t) = j$, then the instantaneous intensity of the input flow is $\hat{\lambda}(j,t,Q)$, the instantaneous intensity of the service for any busy server is $\hat{\mu}(j,t,Q)$, and the intensity of the transition of the process $y(\cdot)$ from a state j to a state m is $\hat{b}(j,m,t,Q)$ (note that the process $y(\cdot)$ in the general case is not Markov, since its transient intensities depend also on the current size of a queue). Assume that the process $(y(t), Q(t))$ is regular.

Assertion 3.3. *If under the above conditions* $x_n(0) = i_0 \in X_{j_0}$, *then the process* $(y_n(t), Q_n(t))$ *J-converges on any finite interval to the process* $(y(t), Q(t))$, *where* $y(0) = j_0$.

Remark. In this case the limit system operates in an environment with a consolidated state space and with averaged characteristics in every asymptotically connected subset.

3.4.2. Analysis of losses in the system $M_{M,Q}/M_{M,Q}/s/m$. As another example we consider the same system operating in the same fast time scale as the environment. Let the system be described in the same way as that in Section 3.4.1 2) with the only difference that the instantaneous intensities of the input flow and service at the moment t are $n\lambda(i,t,Q)$ and $n\mu(i,t,Q)$, respectively. We assume that $s<\infty$ and $m<\infty$. Denote by $p_n(t)$ the probability to lose a call entering the system at a moment t. Let $Z_n(t)$ denote the total number of calls lost on an interval $[0,t]$.

For all fixed (j,v), $j\in Y$, $v\geq 0$, we introduce the auxiliary, homogeneous in time system $M_{M,Q}^{(j,v)}/M_{M,Q}/s/m$ described by the two-component MP $(y(t,j,v),Q(t))$ assuming values in $X_j\times\{0,1,\dots,s+m\}$, as follows: if $Q(t)=Q$ and $y(t,j,v)=i$ at a moment t, then the instantaneous intensity of the input flow is $\lambda(i,v,Q)$, the instantaneous intensity of the service for any busy server is $\mu(i,v,Q)$, and the transition probability of the process $y(t,j,v)$ from a state i to a state $l\in X_j$ is $a^{(j)}(i,l,v,Q)$. Let $\pi(i,q,j,v)$, $(i,q)\in X_j\times\{0,1,\dots,s+m\}$ denote the stationary distribution of the process $(y(t,j,v),Q(t))$ and $g(j,v)$ is the stationary probability to lose a call.

Introduce the random variables

$$\hat{b}(j,m,v)=\sum_{i\in X_j,0\leq q\leq s+m}\pi(i,q,j,v)\sum_{k\in X_m}b^{(j)}(i,k,v,q)$$

and denote the MP with transient intensities $\hat{b}(j,m,t)$, $j\neq m$, by $\tilde{y}(t)$.

Assertion 3.4. *If $x_n(0)=i_0\in X_{j_0}$, then under the above assumptions $p_n(t)\to \mathsf{E}\,g(\tilde{y}(t),t)$ for all $t>0$ and the process $n^{-1}Z_n(t)$ J-converges on any fixed interval to the process $\int_0^t g(y(u),u)\,du$, where $y(0)=j_0$.*

We also may consider a system with "fast" service (i.e. $\mu(i)=\mu_n(i)$ and $\mu_n(i)\to\infty$ while λ is fixed). Some applications to the analysis of the behavior of compound renewable systems with "fast" service are obtained in [9]–[11] for homogeneous models.

The above results obtained for systems may be extended to a wide class of nonhomogeneous in time stochastic networks and provide us with a new approach to problems of analytical modelling of compound hierarchical service systems.

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