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A Visual Goodness-of-Fit Test for Econometric Models

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A Visual Goodness-of-Fit Test for Econometric Models

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Abstract. *This paper designs a visual goodness-of-fit test based on the probability integral transformation of the residuals of an estimated model. We illustrate the method with histograms and correlograms of transformed series for different distributions of disturbances in simulated models. An application of the proposed test to the modeling of daily stock-market returns is also presented.*

Keywords. normality, hypothesis testing, probability integral transform, goodness of fit, econometric modeling, visual tests

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1 Introduction

In real-time model evaluation, graphical tools are widely used, as it is easier for traders and other professionals to assess the performance of the underlying model to be used for forecasting purposes. This paper utilizes a visual test of goodness-of-fit for the residuals of an econometric model. The proposed test is based on the probability integral transformation of the residuals of an estimated model with the density of the error distribution. The test implies that if the econometric model is correctly specified under the null hypothesis, then the probability integral transformation of the residuals is identically and independently distributed with $U(0, 1)$. In the recent literature, probability integral transformation has been used within the context of forecasting. Diebold, Gunther, and Tay (1998) used probability integral transformation to evaluate the accuracy of density forecasts. Diebold, Tay, and Wallis (1998) evaluated the density forecasts of inflation.

Consider the following nonlinear econometric model:

$$y_t = x_t(\beta) + \epsilon_t \qquad t = 1, \ldots, n, \tag{1}$$

where y_t is the t^{th} observation on the regressand, which is a scalar variable, and β is a k-vector of unknown parameters. The scalar function $x_t(\beta)$ is a nonlinear function which determines the mean value of y_t

conditional on the regressors, and ϵ_t is a disturbance term. The density of ϵ_t conditionally on the regressors is denoted by $f(\epsilon_t)$.

Our test differs from the others in the literature with its simple visual element of an identical and independent $U(0, 1)$ distribution. The test relies on histogram and correlogram plots, which are easy to implement in practice. The results indicate that this visual test is most suitable for financial time series where the number of observations is large. In Section 2, the test is presented. The simulation results are reported in Section 3. An application of the proposed method to a modeling of daily stock-market returns in Brazil is presented in the last section.

2 The Test

Let $\epsilon_t t = 1, \ldots, n$ be a random variable with distribution function $f(\epsilon_t)$. Let z_t be the probability integral transformation of ϵ_t such that

$$z_t = \int_{-\infty}^{\epsilon_t} f(u)\, du. \quad (2)$$

Rosenblatt (1952) has shown that z_t is distributed with identically and independently $U(0, 1)$. Let the probability density of z_t be denoted by $q(z_t)$ for $t = 1, \ldots, n$. The proposed test of this paper relies on the probability integral transformation of the residuals of the model in Equation 1 with respect to the error density

$$\hat{z}_t = \int_{-\infty}^{\hat{\epsilon}_t} f(u)\, du. \quad (3)$$

If the model in Equation 1 is correctly specified under the null hypothesis, then $f(\hat{\epsilon}_t) = f(\epsilon_t)$, so that $\hat{z}_t$ is identically and independently distributed with $q(\hat{z}_t) \sim U(0, 1)$.

Simple tests of identically and independently distributed $U(0, 1)$ behavior can easily be implemented by Kolmogorov-Smirnov tests or run tests which are joint tests of uniformity and identical and independent distribution. As Diebold, Gunther, and Tay (1998) point out, such tests may not be valuable in practical applications, because the tests provide no guidance as to whether the violation is due to unconditional uniformity, violation of identical and independent behavior, or both if a rejection occurs. In this context, a simple histogram with its confidence intervals is the most informative method to illustrate the unconditional uniformity of z_t. Regarding the identical and independent behavior, we use correlograms of z_t with its Bartlett confidence intervals to examine the behavior of conditional mean, conditional variance, conditional skewness, and conditional kurtosis.

3 Simulations

Simulations are conducted for a linear regression model. The number of observations is set to $n = 5{,}000$. The model is

$$Y = X\beta + \epsilon, \quad (4)$$

where Y is an $n \times 1$ vector of observations on the dependent variable, X is a fixed $n \times k$ matrix of full-column rank k, β is a $k \times 1$ vector of unknown parameters, and ϵ is an $n \times 1$ vector of unobservable error terms. Simulations are conducted for seven distributions: standard normal distribution $N(0, 1)$ as the benchmark, and six alternative distributions. The alternative distributions are student's t distribution with 5 degrees of freedom, t_5; symmetric beta distribution, $\beta(2, 2)$; ordinary gamma distribution with $\mu = \sigma^2 = 10$, $\Gamma(10)$; exponential distribution; chi-squared distribution with 6 degrees of freedom, χ^2_6; and the log-normal distribution. The sample skewness and kurtosis of each distribution are reported in Table 1. In each case, the errors are standardized to have expectation zero and variance 25.

The regressor matrix (X) is constructed by first obtaining three $5{,}000 \times 1$ vectors of uniformly distributed pseudorandom variables. These uniform pseudorandom numbers are then transformed to have mean zero and variance 25. By adjoining a $5{,}000 \times 1$ vector of ones, the basic $5{,}000 \times 4$ regressor matrix is formed. Since $\hat{\epsilon} = (I - X(X'X)^{-1}X')\epsilon$ regardless of the value taken by β, there is no need to specify parameter values.[1]

[1] This is a similar setting to that used by White and MacDonald (1980).

Table 1
Sample statistics.

Distribution[a]	Skewness	Kurtosis
$N(0, 1)$	0.0049	2.9774
$\beta(2, 2)$	0.0084	2.1778
$\Gamma(10)$	0.6499	3.5970
t_5	0.0419	3.7616
χ^2	1.1205	4.7498
Exponential	2.1844	1.2116
Log-normal	4.8643	39.4950

[a] $N(0, 1)$, $\beta(2, 2)$, $\Gamma(10)$, t_5, and χ^2 refer to the standard normal, beta, Gamma, student's t, and chi-squared distributions.

Table 2
Confidence intervals for a histogram of uniform distribution.[b]

	Sample					
Bins	40	50	100	500	1,000	5,000
5	0.38,1.63	0.50,1.60	0.60,1.40	0.83,1.18	0.88,1.13	0.90,1.10
10	0.25,2.00	0.20,1.80	0.50,1.60	0.74,1.26	0.82,1.19	0.88,1.12
20	0.00,2.50	0.00,2.40	0.20,2.00	0.64,1.40	0.74,1.28	0.83,1.18
40		0.00,3.20	0.00,2.40	0.48,1.60	0.64,1.40	0.83,1.18
50			0.00,2.50	0.40,1.70	0.60,1.45	0.81,1.20
100				0.20,2.00	0.40,1.70	0.74,1.27

[b] The confidence interval for each bin is standardized with the expected value of that particular bin. For example, in a 10-bin histogram with 100 numbers from a uniform distribution, calculated 95% confidence levels for a bin are 5 and 16. Dividing these numbers by the expected value of the bin (10), we find upper (1.6) and lower (0.50) confidence levels.

The linear model is estimated with the $N(0, 1)$ disturbances, and the probability integral transformation of the residuals is evaluated under the $N(0, 1)$ density. Figure 1a represents the time-series behavior of the residuals. The histogram for $\hat{z}_t$ and the correlograms[2] for $(\hat{z}_t - \bar{z})$, $(\hat{z}_t - \bar{z})^2$, $(\hat{z}_t - \bar{z})^3$, and $(\hat{z}_t - \bar{z})^4$ are presented in Figures 1b–1f. The histogram measures the unconditional uniformity, whereas the correlograms measure the autocorrelation, conditional variance, skewness, and kurtosis, respectively. The histogram plot of the $\hat{z}$ series in Figure 1b indicates that the residuals are normal. The probability integral transformation of the residual is within the confidence intervals, and does not violate the $U(0, 1)$ distribution. The correlograms in Figures 1c–1f are calculated up to 100 lags. As expected, correlograms indicate no evidence of persistence.

The 95% confidence intervals for the correlograms are based on the Bartlett standard errors. For the histogram, the 95% confidence interval for each bin is calculated by simulation under the assumption that $\hat{z}$ is iid $U(0, 1)$. Note that for a 50-bin histogram formed from 5,000 observations, the number of $\hat{z}$ values falling in any bin is distributed binomially (5,000, 0.01). We made 1,000 replications of 400 numbers distributed binomially (5,000, 0.01). The resulting 400,000 numbers are sorted in a vector. The numbers corresponding to the 2.5 and 97.5 percentiles of the vector are taken as lower and upper intervals. Confidence intervals for various sample sizes and probabilities of a binomial distribution are reported in Table 2.

For each of the remaining distributions, there are four figures (labeled a–d). The first figure (a) presents the behavior of the disturbances, whereas the second figure (b) describes the behavior of the residuals across the number of observations. The histogram of the residuals is presented in the third figure (c). The histogram of the $\hat{z}$ transformation is presented in the fourth figure (d). We did not include the correlograms of $(\hat{z}_t - \bar{z})^i$, $i = 1, \ldots, 4$, as all distributions studied here are identically and independently distributed.

In Figure 2, the linear model is estimated with the $\beta(2, 2)$ disturbances. The disturbance and the residual plots indicate that the variation in the residuals is tighter than in the disturbances. The distribution of the residuals indicates a fat symmetric shape and is centered around zero. The probability integral transformation

[2] $\bar{z}$ is the sample mean value of $\hat{z}_t$ for $t = 1, \ldots, n$.

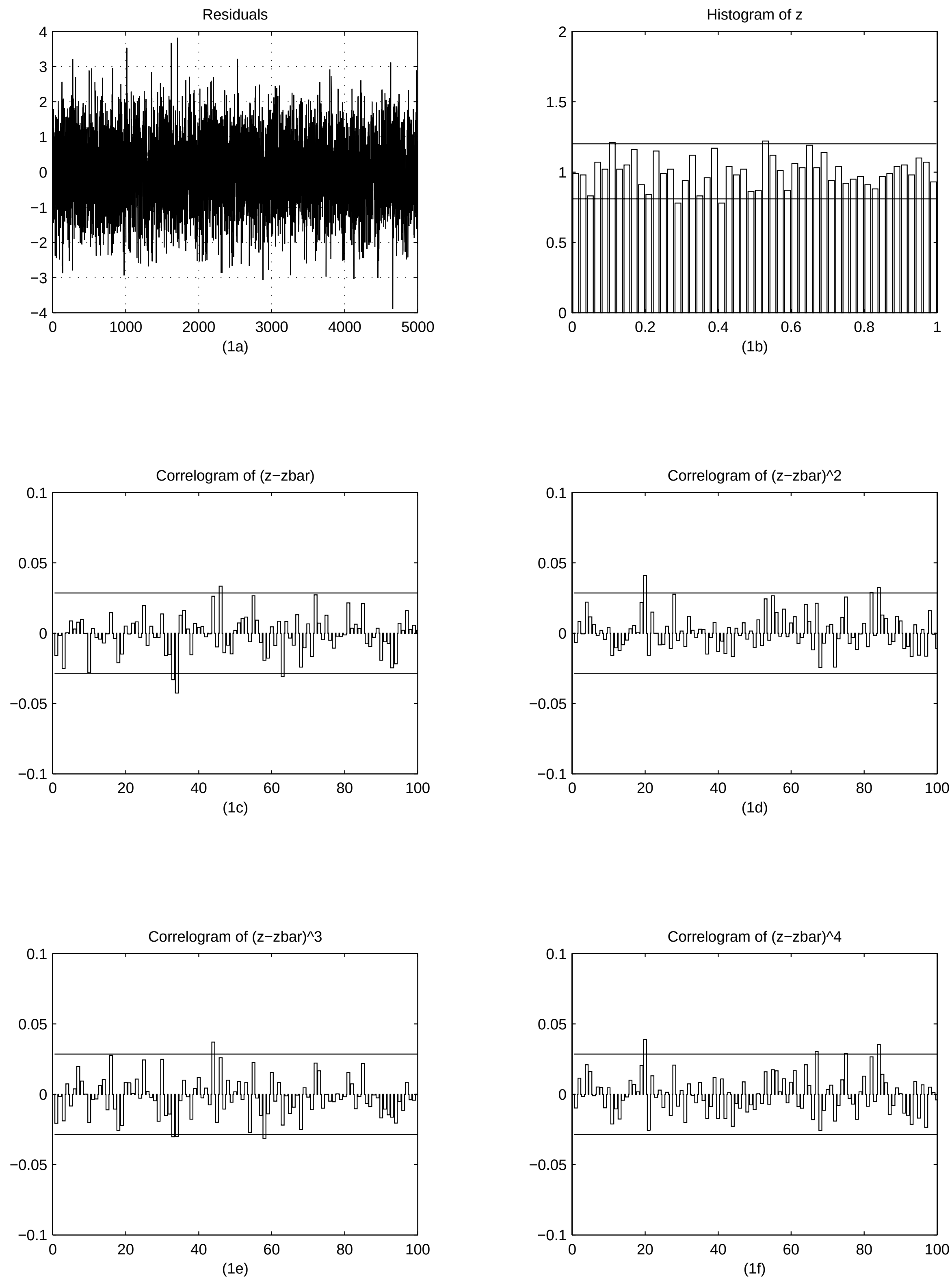

Figure 1
Linear model with $N(0, 1)$ disturbances. See text for discussion in greater detail.

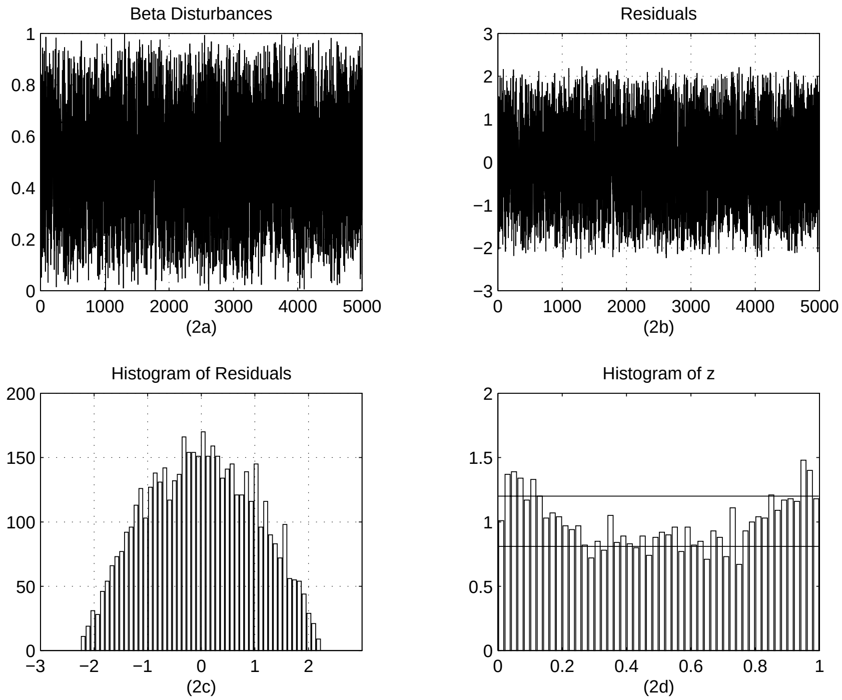

Figure 2
Linear model with $\beta(2, 2)$ disturbances.

of the residuals under the $N(0, 1)$ distribution indicates two peaks at the far end corners of the histogram, and there is evidence of the violation of the $U(0, 1)$-distribution assumption.

In Figure 3, the linear model with gamma distribution disturbances is presented. The comparison of the disturbance and the residual plots indicates that the variation of the residuals is a lot tighter than the variations in the disturbances. The distribution of the residuals is more skewed to the right, and the probability integral transformation under the $N(0, 1)$ distribution indicates extreme outliers at the far-right end of the z-histogram. In the far-left corner of the z-histogram, there are fewer observations that also lie outside of the confidence intervals. Here, there is evidence that the $U(0, 1)$-distribution assumption is violated.

In Figure 4, the t distribution with 5 degrees of freedom is presented. This is the closest case to the $N(0, 1)$ disturbances, and the probability integral transformation does not indicate significant violations of the normality assumption except for four cases which lie outside of the lower and upper confidence intervals. In Figures 5, 6, and 7, the linear model with χ^2_6, exponential, and log-normal disturbances is presented. In all three cases, the probability integral transformations of the residuals under $N(0, 1)$ indicate the skewness and the violation of the $U(0, 1)$ distribution. Overall, the simulation experiments indicate that the probability integral transformation test is an easy, yet efficient visual method to test the goodness-of-fit of the residuals of an econometric model.

4 An Application to the Brazilian Stock Market

In this section, Sao Paulo's 51-share Bovespa index daily returns is studied. The sample is from February 2, 1996 to February 2, 1998, a total of 497 observations.[3] First, we assume that the log-difference of the Bovespa index is $N(\mu, \sigma^2)$, despite the fact that financial returns (especially in emerging markets) are not normally distributed. We plot the data in Figure 8a. The histogram plot of the $\hat{z}$ series in Figure 8b indicates that the series under consideration is not normally distributed. Too many observations lie around the mean, relative to a normal density. As several observations are outside the 95% confidence interval, we reject the normality

[3]Data set is obtained from Reuters data base.

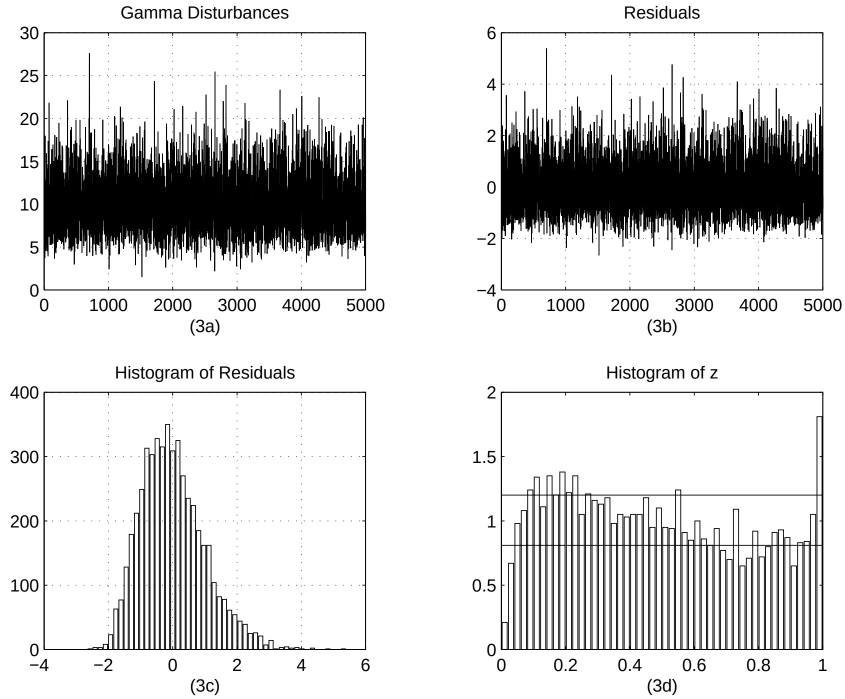

Figure 3
Linear model with $\Gamma(10)$ disturbances.

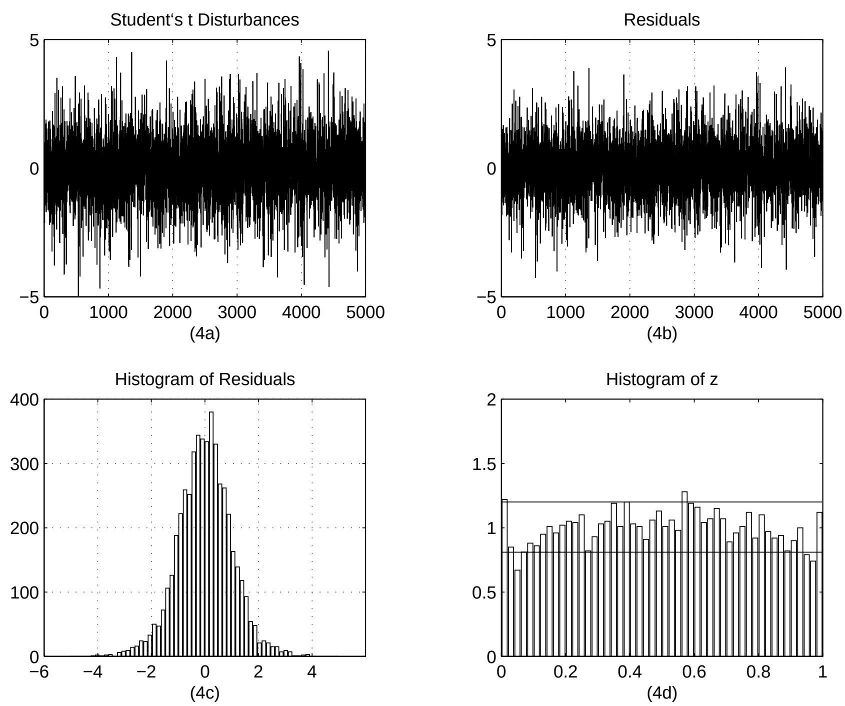

Figure 4
Linear model with t_5 disturbances.

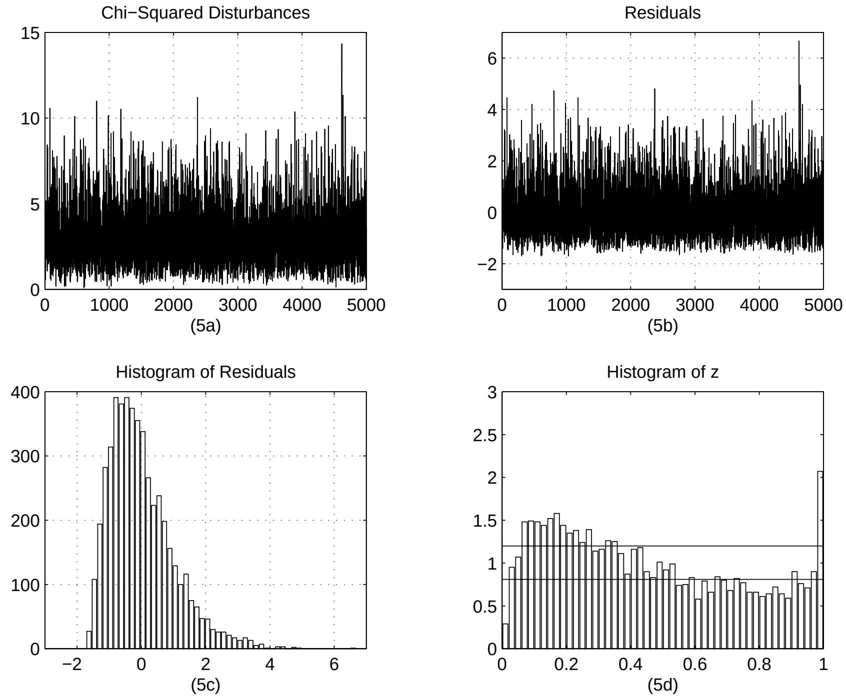

Figure 5
Linear model with $\chi^2(6)$ disturbances.

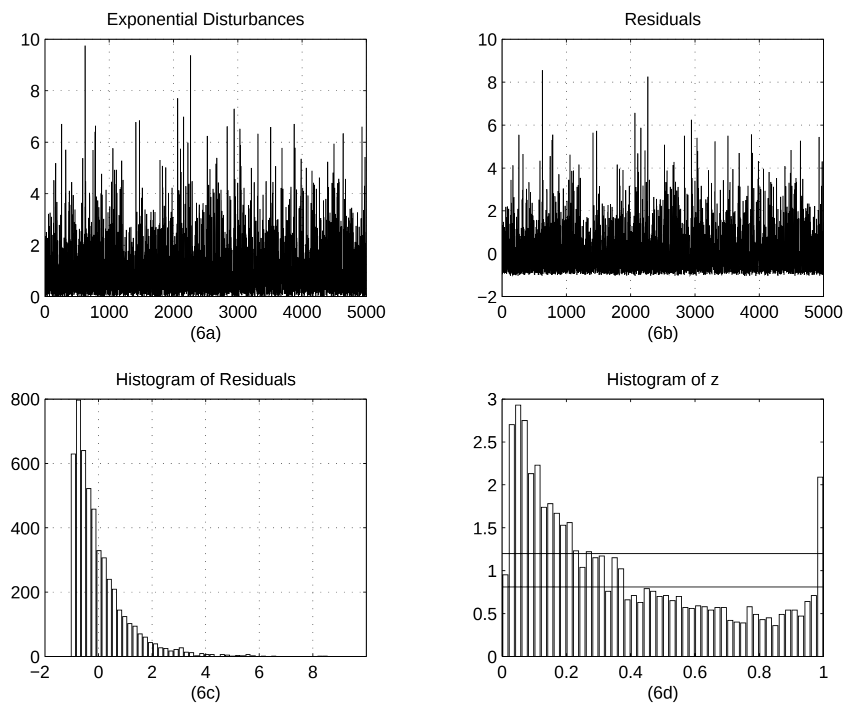

Figure 6
Linear model with exponential disturbances.

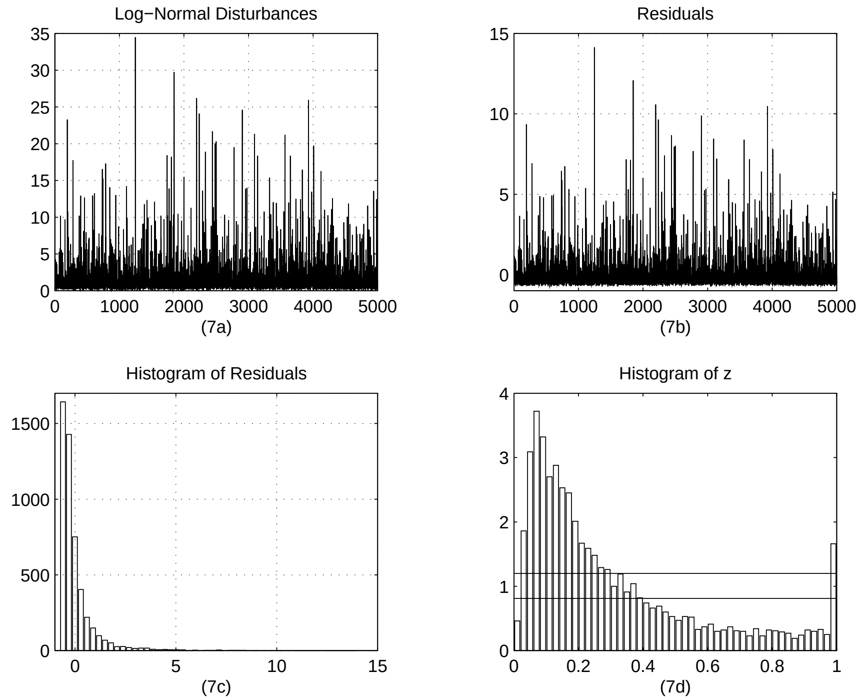

Figure 7
Linear model with log-normal disturbances.

assumption. Correlograms in Figures 8c and 8d indicate that the variance and the kurtosis dynamics are not captured by a simple $N(\mu, \sigma^2)$ model. The strong autocorrelations of $(\hat{z}_t - \bar{z})^2$ and $(\hat{z}_t - \bar{z})^4$ point to a model that should handle persistence in the second and fourth conditional moments. Accordingly, we estimate a t-GARCH (1,1) model for the same sample,

$$y_t = \mu + \epsilon_t, \qquad \epsilon_t = v_t h_t, \qquad \epsilon_t \mid \Omega_{t-1} \sim (0, h_t^2), \tag{5}$$

and

$$h_t^2 = \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 h_{t-1}^2, \tag{6}$$

where $v_t \sim (0, 1)$ and $\Omega_{t-1} = \{\epsilon_{t-1}, \epsilon_{t-2}, \ldots\}$.

All coefficients of the estimated model are statistically significant at less than the 1% significance level. Residuals from the estimated model are plotted in Figure 9a. Figure 9b plots the histogram of $\hat{z}$, which is the probability integral transformation of the residuals of the estimated model in Equations 5 and 6, with respect to the error density (student's t). Compared to the misspecified model, the histogram is closer to unity. Correlograms at Figures 9d and 9f indicate that the estimated model is capable of explaining the variance and kurtosis dynamics as the majority of the autocorrelation coefficients lie within the 95% confidence level.

5 Conclusions

This paper presents a visual goodness-of-fit test based on the probability integral transformation of the residuals of an estimated model with the density of the error distribution. The test implies that if the econometric model is correctly specified under the null hypothesis, then the probability integral transformation of the residuals is identically and independently distributed with $U(0, 1)$. The test relies on histogram and correlogram plots, which are easy to implement in practice.

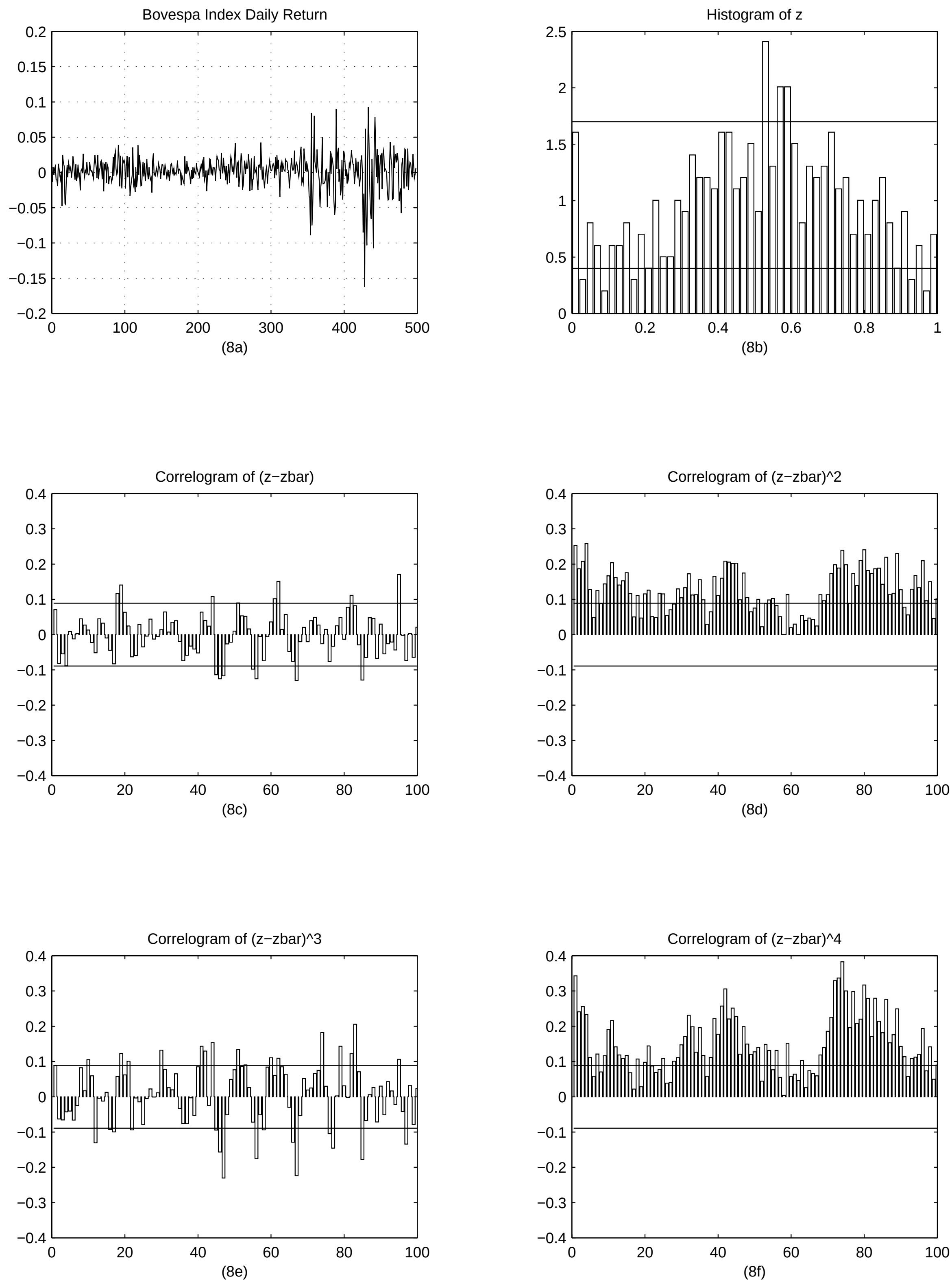

Figure 8
Bovespa index daily return ($N(\mu, \sigma^2)$ model).

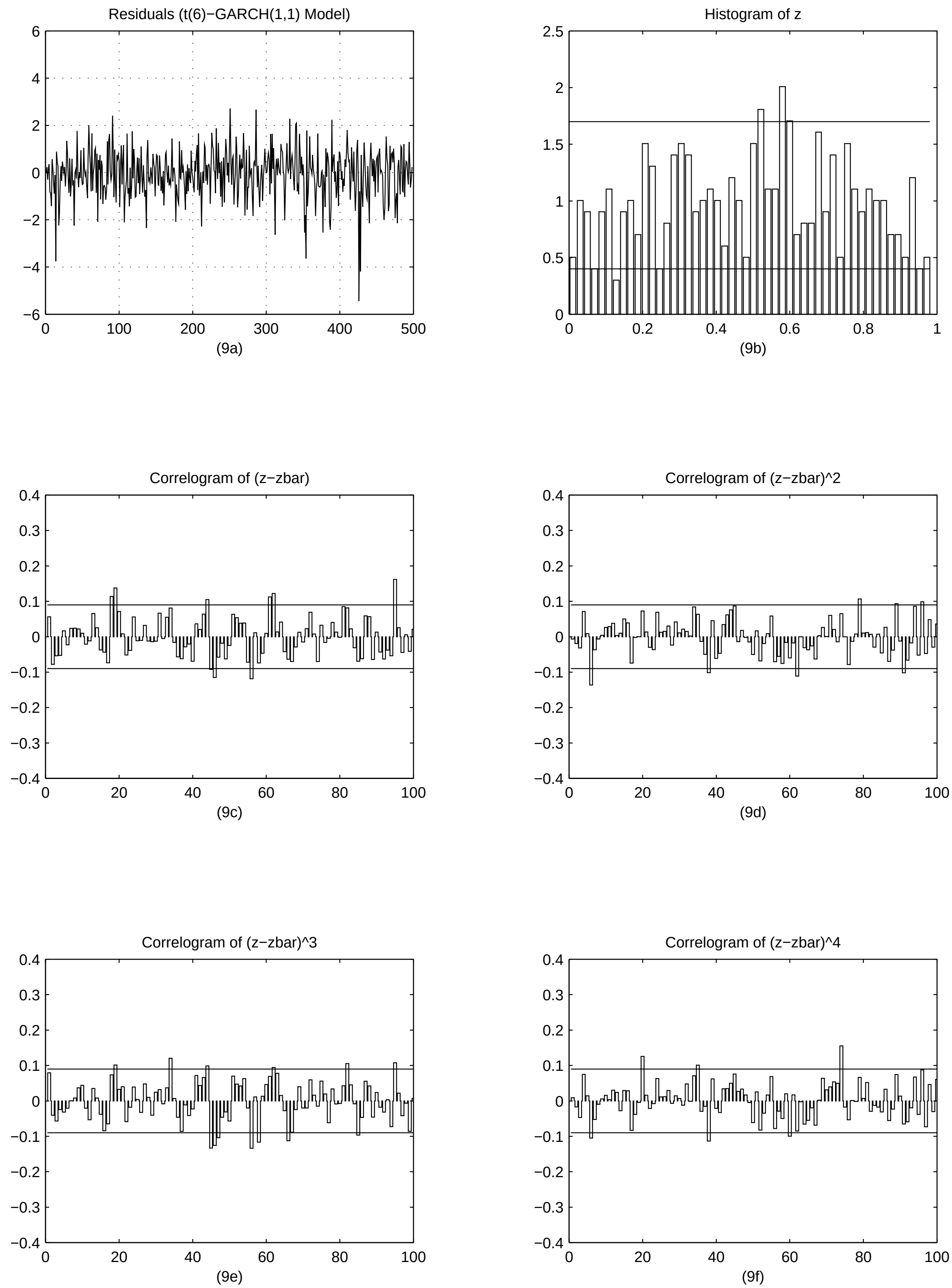

Figure 9
Bovespa index daily return (t_6-GARCH(1, 1) model).

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