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# CONVERGENCE OF SWITCHING REWARD PROCESSES

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ABSTRACT. We study the convergence in the Skorokhod $J$-topology of switching reward processes constructed by sums of conditionally independent random variables or by processes with conditionally independent increments on the trajectories of switching processes. In the case where a switching process satisfies conditions of the averaging principle type and is switched by some asymptotically mixing Markov process, the convergence of the switching reward process to a nonhomogeneous process with independent increments is studied. Some applications to the analysis of reward processes in queueing models are considered.

## 1. INTRODUCTION

A fruitful notion of the $J$-topology introduced by A. V. Skorokhod in 1956 [29] created a new direction in studying the functional convergence to discontinuous processes in the Skorokhod space $\mathcal{D}_T$. A large number of papers are devoted to the analysis of the convergence for stepwise processes constructed from sums of conditionally independent random variables or for processes with conditionally independent increments defined on random sequences or on trajectories of stochastic processes satisfying some types of mixing conditions (see [5, 6, 13, 16, 20, 23, 28]).

In this paper we consider the $J$-convergence in the scheme of series of switching reward processes constructed by sums of conditionally independent random variables or by processes with conditionally independent increments on the trajectories of the so-called switching processes.

The main property of switching processes (SP) introduced by the author in [2, 3] is that their behavior may spontaneously change at some moments of time that are random functionals of the past of the trajectory. Formally, switching processes are defined as two-component processes $(x(t), \zeta(t))$, $t \geq 0$, such that there exists a sequence $t_1 < t_2 < \cdots$ for which $x(t) = x(t_k)$ on each interval $[t_k, t_{k+1})$ and the behavior of the process $\zeta(t)$ depends only on the value $(x(t_k), \zeta(t_k))$. Instants $t_1 < t_2 < \cdots$ are called switching times and $x(t)$ is the discrete switching component.

In some applications, the component $x(t)$ can be regarded as a random medium or as an operating regime of a system. We note that switching processes can be described in terms of characteristics evaluated explicitly, and that they form a convenient tool to study the asymptotic behavior of stochastic systems with "fast" and "rare" switches (see [6, 7, 9, 12]).

Switching processes are a natural generalization of well-known classes of processes such as Markov and semi-Markov processes, processes homogeneous in the second component [15], processes with independent increments and semi-Markov switches [1], piecewise

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Markov aggregates [14], Markov processes with a semi-Markov random interference [16], and Markov and semi-Markov evolutions [17, 19, 21, 23–26].

Both the *averaging principle* (the convergence of a trajectory $\zeta(t)$ to a solution of some ordinary differential equation) and the *diffusion approximation* (the convergence of a normalized difference to some diffusion process) are proved by the author [7, 10] for different subclasses of switching processes. These results are closely related to the averaging type results for stochastic differential equations with fast Markov switches [22], for dynamic systems in the fast Markov type medium [30–33] and for semi-Markov evolutions [23] obtained by a different technique.

The analysis of switching reward processes constructed on switching processes is a natural step in the analysis of stochastic evolutionary systems.

We consider the case of a scheme of series where the switched component $\zeta_n(t)$ satisfies conditions of the averaging principle type and is possibly switched by some asymptotically mixing Markov process. We prove the convergence of switching reward processes constructed on switching processes to some nonhomogeneous process with independent increments. Note that the convergence of additive functionals on switching processes is considered in [8] by using a different approach in the case where the first and second moment functions of individual terms exist.

We consider also some applications to the analysis of reward processes in queueing models.

## 2. Switching reward processes

**2.1.** First we give the definition of a switching reward process (SRP) on a recurrent process of the semi-Markov type (RPSM).

Let families of jointly independent random variables $\mathcal{F}_k = \{(\xi_k(\alpha), \tau_k(\alpha)), \alpha \in \mathbf{R}^r\}$, $k \geq 0$, and $\mathcal{G}_j = \{\gamma_j(\alpha), \alpha \in \mathbf{R}^r\}$, $j \geq 0$, be given and assume values in the sets $\mathbf{R}^r \times [0, \infty)$ and $\mathbf{R}^d$, respectively. Let a random variable $S_0 \in \mathbf{R}^r$ do not depend on $\mathcal{F}_k$ and $\mathcal{G}_k$, $k \geq 0$. We assume that the random variables are measurable with respect to the $\sigma$-algebra $\mathcal{B}_{\mathbf{R}^r}$. Put

$$(2.1)\qquad \begin{gathered} t_0 = 0, \qquad t_{k+1} = t_k + \tau_k(S_k), \qquad S_{k+1} = S_k + \xi_k(S_k), \qquad k \geq 0, \\ S(t) = S_k \quad \text{for } t_k \leq t < t_{k+1},\ t \geq 0. \end{gathered}$$

Then the process $S(t)$, $t \geq 0$, is called a recurrent process of the semi-Markov type (see [7, 10]).

We denote by $\nu(t)$ the total number of switching points on the interval $[0, t]$, that is,

$$(2.2)\qquad \nu(t) = \min\{k\colon k \geq 0, t_{k+1} \geq t\}, \qquad t \geq 0.$$

Let

$$(2.3)\qquad Z(t) = \sum_{k=0}^{\nu(t)} \gamma_k(S_k), \qquad t \geq 0.$$

The process $Z(t)$, $t \geq 0$, is called a switching reward process (SRP).

If the distributions of the random variables $\tau_k(\alpha)$ and $\gamma_k(\alpha)$ do not depend on $k$ and parameter $\alpha$, then $Z(t)$ is a renewal reward process [27].

Further let families of jointly independent random variables $\mathcal{F}_k = \{(\xi_k(x, \alpha), \tau_k(x, \alpha)),$ $x \in X, \alpha \in \mathbf{R}^r\}$, $k \geq 0$, and $\mathcal{G}_j = \{\gamma_j(x, \alpha), x \in X, \alpha \in \mathbf{R}^r\}$, $j \geq 0$, be given and assume values in $\mathbf{R}^r \times [0, \infty)$ and $\mathbf{R}^d$, respectively. Also let $x_l$, $l \geq 0$, be a Markov process assuming values in $X$ and independent of $\mathcal{F}_k$ and $\mathcal{G}_k$, $k \geq 0$. The initial value is $(x_0, S_0)$. Put

$$\begin{aligned} t_0 = 0, \qquad t_{k+1} = t_k + \tau_k(x_k, S_k), \qquad S_{k+1} = S_k + \xi_k(x_k, S_k), \qquad k \geq 0, \\ S(t) = S_k, \qquad x(t) = x_k \quad \text{for } t_k \leq t < t_{k+1},\ t \geq 0. \end{aligned} \tag{2.4}$$

Then $(x(t), S(t))$ is an RPSM with an additional Markov switching. Note that if the distributions of the random variables $\tau_k(\cdot)$ depend on a parameter $\alpha$, then the process $x(t)$ is not, in general, a semi-Markov process (SMP). Put

$$Z(t) = \sum_{k=0}^{\nu(t)} \gamma_k(x_k, S_k), \qquad t \geq 0,$$

where $\nu(t)$ is defined by (2.2). Then the process $Z(t)$, $t \geq 0$, is also an SRP constructed on the RPSM $(x(t), S(t))$.

In a particular case where the distributions of the random variables $(\tau_k(\cdot), \gamma_k(\cdot))$ do not depend on parameters $\alpha$ and $k$, the process $x(t)$ is an SMP and $Z(t)$ is a stepwise process of sums of conditionally independent random variables on $x(t)$.

**2.2.** Consider the definition of SRP on the trajectory of a general SP.

Consider families of jointly independent random variables

$$\mathcal{F}_k = \{(\zeta_k(t, x, \alpha), \tau_k(x, \alpha), \beta_k(x, \alpha)), t \geq 0, x \in X, \alpha \in \mathbf{R}^r\}, \qquad k \geq 0,$$

where, for all fixed $k$, $x$, and $\alpha$, $\zeta_k(t, x, \alpha)$ is a stochastic process in the Skorokhod space $\mathcal{D}_\infty^r$. The random variables $\tau_k(x, \alpha)$ and $\beta_k(x, \alpha)$ are possibly dependent on $\zeta_k(\cdot, x, \alpha)$ and $\tau_k(\cdot) > 0$, $\beta_k(\cdot) \in X$. Also suppose that the initial values $(x_0, S_0)$ do not depend on the above random variables. We assume that the random variables introduced above are measurable in $(x, \alpha)$ with respect to the $\sigma$-algebra $\mathcal{B}_X \times \mathcal{B}_{\mathbf{R}^r}$. For any $k \geq 0$, we put $\xi_k(x, \alpha) = \zeta_k(\tau_k(x, \alpha), x, \alpha)$ and $t_0 = 0$,

$$\begin{aligned} t_{k+1} = t_k + \tau_k(x_k, S_k), \qquad S_{k+1} = S_k + \xi_k(x_k, S_k), \qquad x_{k+1} = \beta_k(x_k, S_k), \\ \zeta(t) = S_k + \zeta_k(t - t_k, x_k, S_k), \qquad x(t) = x_k \quad \text{for } t_k \leq t < t_{k+1},\ t \geq 0. \end{aligned} \tag{2.5}$$

Then $(x(t), \zeta(t))$, $t \geq 0$, is called a switching process (SP); see [2, 3, 6].

Suppose the families of jointly independent random variables $\mathcal{G}_j = \{\gamma_j(x, \alpha), x \in X, \alpha \in \mathbf{R}^r\}$, $j \geq 0$, are independent of $\mathcal{F}_k$, $k \geq 0$, and assume values in $\mathbf{R}^d$. Also let a parametric family of functions $\{\psi(\theta, x, \alpha), x \in X, \alpha \in \mathbf{R}^r, \theta \in \mathbf{R}^d\}$ be given such that, for all fixed $x$ and $\alpha$, the function $\psi(\theta, x, \alpha)$ is the cumulant of an infinitely divisible law. This means that $\exp\{\psi(\theta, x, \alpha)\}$ is the characteristic function of an infinitely divisible random variable with values in $\mathbf{R}^d$. Assume that, for all fixed $\theta$ and $x$, the function $\psi(\theta, x, \alpha)$ is continuous with respect to $\alpha$. Further we put

$$\phi_k(\theta, x, \alpha) = \mathsf{E} \exp\{i(\theta, \gamma_k(x, \alpha))\}, \qquad x \in X,\ \alpha \in \mathbf{R}^r,\ \theta \in \mathbf{R}^d,\ i = +\sqrt{-1}.$$

Now we construct the process $Z(t)$, $t \geq 0$, on the trajectory $(x(\cdot), \zeta(\cdot))$ as follows. Given a fixed trajectory $(x(\cdot), \zeta(\cdot))$, the conditional characteristic function of $Z(t)$ is of the form

$$\Psi\bigl(\theta, t \mid (x(u), \zeta(u)), 0 \leq u \leq t\bigr) = \prod_{k=0}^{\nu(t)} \phi_k(\theta, x_k, S_k) \exp\left\{\int_0^t \psi(\theta, x(u), \zeta(u))\, du\right\} \tag{2.6}$$

on the interval $[0, t]$. This means that the process $Z(t)$, $t \geq 0$, has conditionally independent increments.

In a particular case where the above families do not depend on the parameter $\alpha$, $x(t)$, $t \geq 0$, is a Markov process, while the pair $(x(t), Z(t))$, $t \geq 0$, is a Markov process homogeneous with respect to the second component [15].

## 3. Limit theorems for SRP

Consider limit theorems for SRP in the case of fast switches. Let a sequence of processes $(x_n(t), \zeta_n(t), Z_n(t))$, $t \geq 0$, be given on the interval $[0, nT]$. Assume that the switching process $(x_n(\cdot), \zeta_n(\cdot))$ depends on a scaling parameter $n$ $(n \to \infty)$ in such a way that the total number of switches on every interval $[na, nb]$, $0 < a < b < T$, tends to infinity in probability. Under some natural assumptions, the normalized trajectory of the process $\zeta_n(nt)$ uniformly converges in probability to some nonrandom function $s(t)$ which is a solution of an ordinary differential equation (this is proved in the papers [7, 10]). This assertion is called the averaging principle. In this case one can prove that the sequence of SRP $Z_n(nt)$ normalized in an appropriate way $J$-converges to a process with independent increments whose cumulant is constructed on $s(t)$ and averaged with respect to some quasi-stationary measure corresponding to the component $x(t)$. First we prove this result for SRP constructed on RPSM.

**3.1. A simple RPSM.** For any $n = 1, 2, \dots$, let families of jointly independent random variables

$$\mathcal{F}_{nk} = \{(\xi_{nk}(\alpha), \tau_{nk}(\alpha)), \alpha \in \mathbf{R}^r\}, \qquad k \geq 0,$$

and $\mathcal{G}_{nj} = \{\gamma_{nj}(\alpha), \alpha \in \mathbf{R}^r\}$, $j \geq 0$, be given, with values in the sets $\mathbf{R}^r \times [0, \infty)$ and $\mathbf{R}^d$, respectively. For simplicity we assume that their distributions do not depend on indices $k$ and $j$. Also let the initial values $S_{n0} \in \mathbf{R}^r$ do not depend on $\mathcal{F}_{nk}$ and $\mathcal{G}_{nk}$, $k \geq 0$. For every $n$, we construct a sequence of processes $(S_n(t), Z_n(t))$, $t \geq 0$, according to relations (2.1)–(2.3). Under the conditions introduced above the trajectory $S_n(nt)$ is of order $O(n)$, and therefore we can assume that the above variables depend on the normalized variable $n^{-1}S_n(\cdot)$. Put

$$\phi_n(\theta, \alpha) = \mathsf{E} \exp\{i(\theta, \gamma_{n1}(n\alpha))\}, \qquad \theta \in \mathbf{R}^d, \ \alpha \in \mathbf{R}^r, \ i = +\sqrt{-1}.$$

Assume that there exist a normalizing factor $\rho_n$ and a function $\Psi(\theta, \alpha)$, continuous in $\alpha$ for any $\theta$, such that $\Psi(0, \alpha) \equiv 0$ and

$$n(\phi_n(\rho_n\theta, \alpha) - 1) = \Psi(\theta, \alpha) + o_n(\theta, \alpha) \tag{3.1}$$

for all $\alpha$ and $\theta$, where

$$\sup_{|\alpha| \leq N} |o_n(\theta, \alpha)| \to 0$$

as $n \to \infty$ for all $N > 0$.

Put $m_n(\alpha) = \mathsf{E}\, \tau_{n1}(n\alpha)$ and $b_n(\alpha) = \mathsf{E}\, \xi_{n1}(n\alpha)$ for $\alpha \in \mathbf{R}^r$.

**Theorem 3.1.** *Assume that condition* (3.1) *holds and, for all* $N > 0$,

$$\lim_{L \to \infty} \limsup_{n \to \infty} \sup_{|\alpha| < N} \{\mathsf{E}\, \tau_{n1}(n\alpha)\chi(\tau_{n1}(n\alpha) > L) + \mathsf{E}\, |\xi_{n1}(n\alpha)|\chi(|\xi_{n1}(n\alpha)| > L)\} = 0$$

*and*

$$|m_n(\alpha_1) - m_n(\alpha_2)| + |b_n(\alpha_1) - b_n(\alpha_2)| \leq C_N|\alpha_1 - \alpha_2| + \alpha_n(N)$$

*if* $\max(|\alpha_1|, |\alpha_2|) < N$, *where* $C_N$ *are some bounded constants, and* $\alpha_n(N) \to 0$ *uniformly in* $|\alpha_1| < N$ *and* $|\alpha_2| < N$. *We also assume that there are functions* $m(a) > 0$ *and* $b(a)$ *and a nonrandom number* $s_0$ *such that* $n^{-1}S_{n0} \xrightarrow{\mathsf{P}} s_0$ *as* $n \to \infty$ *and*

$$m_n(\alpha) \to m(\alpha), \qquad b_n(\alpha) \to b(\alpha)$$

*for all* $\alpha \in \mathbf{R}^r$. *Further we assume that a solution of the equation*

$$\eta(0) = s_0, \qquad d\eta(u) = b(\eta(u))\,du \tag{3.2}$$

*exists and is unique on every finite interval and that* $T$ *is a positive number such that* $y(+\infty) > T$, *where* $y(t) = \int_0^t m(\eta(u))\,du$.

*Then the sequence of processes* $\rho_n Z_n(nt)$ $J$*-converges*[1] *on the interval* $[0,T]$ *to a nonhomogeneous process with independent increments* $z_0(t)$ *whose characteristic function is of the form*

$$\mathsf{E}\exp\{i(\theta, z_0(t))\} = \exp\left\{\int_0^t m(s(u))^{-1}\Psi(\theta, s(u))\,du\right\}, \tag{3.3}$$

*where the function* $s(t)$ *satisfies the equation*

$$s(0) = s_0, \qquad ds(t) = m(s(t))^{-1} b(s(t))\,dt \tag{3.4}$$

*whose solution exists and is unique on the interval* $[0,T]$.

*Proof.* We introduce the sequences $\eta_{nk} = n^{-1}S_{nk}$ and $y_{nk} = n^{-1}t_{nk}$, $k \ge 0$, and put

$$\eta_n(u) = \eta_{nk}, \qquad y_n(u) = y_{nk} \quad \text{for } n^{-1}k \le u < n^{-1}(k+1),\ u \ge 0.$$

Let $R_n(u) = \sum_{i=0}^{[nu]} \gamma_{ni}(\eta_{ni})$, where the symbol $[a]$ denotes the integer part of $a$. As before, let $\nu_n(t) = \min\{k\colon k > 0, t_{nk+1} > t\}$ and $\mu_n(t) = \inf\{u\colon u > 0, y_n(u) > t\}$. Then $\nu_n(nt) = n\mu_n(t) - 1$, $S_n(nt) = S_{n\nu_n(nt)}$, and the following representations hold:

$$\begin{aligned} n^{-1}S_n(nt) &= \eta_n\left(n^{-1}\nu_n(nt)\right) = \eta_n(\mu_n(t) - 1/n), \\ \rho_n Z_n(nt) &= \rho_n R_n(\mu_n(t) - 1/n). \end{aligned} \tag{3.5}$$

Therefore RPSM $n^{-1}S_n(nt)$ and SRP $\rho_n Z_n(nt)$ are represented in the form of a superposition of two processes: $\eta_n(u)$ and $\mu_n(t)$ and $\rho_n R_n(u)$ and $\mu_n(t)$, respectively. First we study the limit behavior of the processes $\eta_n(u)$, $y_n(u)$, and $\rho_n R_n(u)$, $u \ge 0$. Then we consider $\mu_n(t)$ and their superpositions. Using the averaging principle for stochastic recurrent sequences, it is proved in [7] and [10] that

$$\sup_{0\le u\le t} |\eta_n(u) - \eta(u)| \xrightarrow{\mathsf{P}} 0, \qquad \sup_{0\le u\le t} |y_n(u) - y(u)| \xrightarrow{\mathsf{P}} 0 \tag{3.6}$$

for all $t \ge 0$ (see relation (3.2)). Since $m(a) > 0$, the process $y(t)$ strictly increases. This means that the inverse function $y^{-1}(t) = \mu(t)$ exists for all $t < y(+\infty)$, is continuous, and

$$\sup_{u\le t} |\mu_n(u) - \mu(u)| \xrightarrow{\mathsf{P}} 0. \tag{3.7}$$

Using a result in [13] on the uniform convergence of the superposition of random functions, representation (3.5), and relations (3.6) and (3.7) we obtain

$$\sup_{0\le t\le T} \left|n^{-1}S_n(nt) - s(t)\right| \xrightarrow{\mathsf{P}} 0, \tag{3.8}$$

where the function $s(t)$ satisfies equation (3.4).

Now we study the convergence of $\rho_n R_n(u)$. By relation (3.6), the probability of the event $\{|\eta_n(t)| \le N, t \le T\}$ is close to one for any fixed $T > 0$ and large $N$. This

[1]The $J$-convergence means that the measures generated by $\rho_n Z_n(nt)$ in the Skorokhod space $\mathcal{D}_T$ are weakly convergent.

means that in what follows one can assume that $\{|\eta_n(t)| \le N, t \le T\}$. Put $\varphi_n(\theta,\alpha) = \ln \phi_n(\rho_n\theta,\alpha)$. One has the following representation:

$$\mathsf{E}\exp\{i(\theta,\rho_n R_n(u))\} = \mathsf{E}\exp\left\{\sum_{k=0}^{[nu]} \varphi_n(\theta,\eta_{nk})\right\}, \qquad u \ge 0. \tag{3.9}$$

According to (3.1), the right-hand side in (3.9) is equivalent to

$$G_n(\theta,u) = \mathsf{E}\exp\left\{\frac{1}{n}\sum_{k=0}^{[nu]} \Psi(\theta,\eta_{nk})\right\}. \tag{3.10}$$

Using relation (3.6) we get

$$\frac{1}{n}\sum_{k=0}^{[nu]} \Psi(\theta,\eta_{nk}) \xrightarrow{\mathsf{P}} \int_0^u \Psi(\theta,\eta(v))\,dv \tag{3.11}$$

for $u \ge 0$. It follows from condition (3.1) that $\exp\{\Psi(\theta,\alpha)\}$ is the characteristic function of an infinitely divisible law. This means that $\operatorname{Re}\Psi(\theta,\alpha) \le 0$. Therefore the absolute value of the expression under the expectation sign on the right-hand side of (3.10) does not exceed 1 and relation (3.11) implies that

$$G_n(\theta,u) \to G_0(\theta,u) = \exp\left\{\int_0^u \Psi(\theta,\eta(v))\,dv\right\}$$

for all $u \ge 0$. This relation implies the convergence of one-dimensional distributions of the sequence of processes $\rho_n R_n(u)$ to a process with independent increments $R_0(u)$ whose characteristic function is $G_0(\theta,u)$. The convergence of finite-dimensional distributions follows in a similar way.

To prove the weak compactness of the corresponding measures generated by the sequence of processes $\rho_n R_n(u)$ in the Skorokhod space $\mathcal{D}_T$ we use the criteria given in [18]. In the case under consideration one needs to check that for all fixed $\theta$ and all $T > 0$ and $L > 0$,

$$\lim_{c\to+0}\limsup_{n\to\infty}\sup_{\substack{t\le T, v\le c\\ |\alpha|\le L}} \left|\mathsf{E}\left[\exp\left\{i\rho_n\sum_{k=[nt]}^{[n(t+v)]}(\theta,\gamma_{nk}(\eta_{nk}))\right\} \,\Big|\, \eta_{n,[nt]} = \alpha\right] - 1\right| = 0. \tag{3.12}$$

Using representation (3.9) and relation (3.1) we obtain that

$$\begin{aligned} &\mathsf{E}\left[\exp\left\{i\rho_n\sum_{k=[nt]}^{[n(t+v)]}(\theta,\gamma_{nk}(\eta_{nk}))\right\} \,\Big|\, \eta_{n,[nt]} = \alpha\right] \\ &\quad\asymp \mathsf{E}\left[\exp\left\{\frac{1}{n}\sum_{k=[nt]}^{[n(t+v)]}\Psi(\theta,\eta_{nk})\right\} \,\Big|\, \eta_{n,[nt]} = \alpha\right] \end{aligned} \tag{3.13}$$

uniformly in $|\alpha| \le L$, $t \le T$, and $v \le c$. Taking (3.6) into account we prove that, uniformly in $|\alpha| \le L$, $t \le T$, and $v \le c$, the right-hand side of (3.13) is equivalent to

$$\delta(t,v,\alpha) = \exp\left\{\int_t^{t+v}\Psi(\theta,\eta(v))\,dv\right\},$$

where $\eta(t) = \alpha$. Applying the continuity of $\Psi(\theta,\alpha)$ in $\alpha$ we obtain $\delta(t,v,\alpha) \to 1$ uniformly in $|\alpha| \le L$, $t \le T$, and $v \le c$ as $c \to +0$. This yields relation (3.12) and proves the $J$-convergence of the sequence of processes $\rho_n R_n(u)$ to $R_0(u)$ on every interval $[0,T]$.

Now relations (3.5) and (3.7) and a result on the $J$-convergence of the superposition of stochastic processes (see [4]) easily imply that the sequence $\rho_n Z_n(nt)$ is $J$-convergent to the process $R_0(\mu(t))$. Using the differential equations for the functions $\eta(u)$ and $s(t)$ we get after simple calculations that

$$\mathsf{E}\exp\{i(\theta, R_0(\mu(t)))\} = \exp\left\{\int_0^{\mu(t)} \Psi(\theta, \eta(u))\,du\right\} = \exp\left\{\int_0^t m(s(v))^{-1}\Psi(\theta, s(v))\,dv\right\}.$$

This completes the proof of Theorem 3.1. □

*Remark 3.1.* Theorem 3.1 can be extended to the case where the initial value $s_0$ is a proper random variable.

**3.2. SRP on RPSM with additional Markov switches.** We consider limit theorems for SRP constructed on RPSM with additional Markov switches.

Assume that for any $n > 0$, the families of jointly independent random variables $\mathcal{F}_{nk} = \{(\xi_{nk}(x,\alpha), \tau_{nk}(x,\alpha)), x \in X, \alpha \in \mathbf{R}^r\}$, $k \geq 0$, and $\mathcal{G}_{nj} = \{\gamma_{nj}(x,\alpha), x \in X, \alpha \in \mathbf{R}^r\}$, $j \geq 0$, are given and assume values in $\mathbf{R}^r \times [0,\infty)$ and $\mathbf{R}^d$, respectively. Let $x_{nl}$, $l \geq 0$, be a Markov process with values in $X$ and independent of $\mathcal{F}_{nk}$ and $\mathcal{G}_{nk}$, $k \geq 0$, and let $(x_{n0}, S_{n0})$ be the initial value.

We construct the RPSM by relation (2.4), namely we put

$$t_{n0} = 0, \qquad t_{n,k+1} = t_{nk} + \tau_{nk}(x_{nk}, S_{nk}), \qquad S_{n,k+1} = S_{nk} + \xi_{nk}(x_{nk}, S_{nk}), \qquad k \geq 0,$$
$$S_n(t) = S_{nk}, \quad x_n(t) = x_{nk} \quad \text{for} \quad t_{nk} \leq t < t_{n,k+1}, \quad t \geq 0.$$

The process $(x_n(t), S_n(t))$ is an RPSM with additional Markov switches. Put

$$Z_n(t) = \sum_{k=0}^{\nu_n(t)} \gamma_{nk}(x_{nk}, S_{nk}), \qquad t \geq 0,$$

where the random variable $\nu_n(t)$ is defined by (2.2) for the sequence $t_{nk}$.

We study the convergence of the sequence $\rho_n Z_n(nt)$, $t \geq 0$. For the sake of simplicity we consider the homogeneous case (the distributions of random variables $\{\xi_{nk}(\cdot), \tau_{nk}(\cdot), \gamma_{nk}(\cdot)\}$ do not depend on $k \geq 0$). Denote by

$$\varphi_n(r) = \sup_{x,y,A} |\mathsf{P}\{x_{nr} \in A \mid x_{n0} = x\} - \mathsf{P}\{x_{nr} \in A \mid x_{n0} = y\}|, \qquad r > 0,$$

the uniform strong mixing coefficient for the Markov process $x_{nk}$. Assume that there exist a sequence of integers $r_n$ and a number $q$ such that $0 \leq q < 1$ and

$$n^{-1} r_n \to 0, \qquad \varphi_n(r_n) \leq q. \tag{3.14}$$

Under this condition, for every $n > 0$, the Markov process $x_{nk}$, $k \geq 0$, is an ergodic process with the stationary measure $\pi_n(A)$, $A \in \mathcal{B}_X$. Put

$$m_n(x,\alpha) = \mathsf{E}\,\tau_n(x,\alpha), \qquad b_n(x,\alpha) = \mathsf{E}\,\xi_n(x, n\alpha),$$
$$m_n(\alpha) = \int_X m_n(x,\alpha)\pi_n(dx), \qquad b_n(\alpha) = \int_X b_n(x,\alpha)\pi_n(dx)$$

and $\phi_n(\theta, x, \alpha) = \mathsf{E}\exp\{i(\theta, \gamma_{n1}(x, n\alpha))\}$, $\theta \in \mathbf{R}^d$, $x \in X$, $\alpha \in \mathbf{R}^r$. Assume that there exist a normalizing factor $\rho_n$ and a function $\Psi(\theta, x, \alpha)$ such that $\Psi(0, x, \alpha) \equiv 0$ and for all $\alpha$, $x$, and $\theta$,

$$n(\phi_n(\rho_n\theta, x, \alpha) - 1) = \Psi(\theta, x, \alpha) + o_n(\theta, x, \alpha), \tag{3.15}$$

where for all $\theta \in \mathbf{R}^d$ and $N > 0$ the function $\Psi(\theta, x, \alpha)$ is bounded in the domain $x \in X$, $|\alpha| \le N$, and $\sup_{|\alpha| \le N, x \in X} |o_n(\theta, x, \alpha)| \to 0$ as $n \to \infty$. Let

$$\Psi_n(\theta, \alpha) = \int_X \Psi(\theta, x, \alpha)\, \pi_n(dx).$$

**Theorem 3.2.** *Let conditions* (3.14) *and* (3.15) *hold. Assume that* $n^{-1} S_{n0} \xrightarrow{\mathsf{P}} s_0$,

$$\lim_{L\to\infty} \limsup_{n\to\infty} \sup_{|\alpha|<N} \sup_{x\in X} \Big\{ \mathsf{E}\, \tau_{n1}(x, n\alpha) \chi(\tau_{n1}(x, n\alpha) > L) + \mathsf{E}\, |\xi_{n1}(x, n\alpha)| \chi(|\xi(x, n\alpha)| > L) \Big\} = 0$$

*for all* $N > 0$, *and*

$$|b_n(x, \alpha_1) - b_n(x, \alpha_2)| + |m_n(x, \alpha_1) - m_n(x, \alpha_2)| \le C_N |\alpha_1 - \alpha_2| + \alpha_n(N)$$

*for all* $x$, *where* $\max(|\alpha_1|, |\alpha_2|) < N$, $C_N$ *are some constants, and* $\alpha_n(N) \to 0$ *uniformly in* $|\alpha_1| < N$, $|\alpha_2| < N$. *Further let there exist functions* $b(\alpha)$ *and* $m(\alpha)$ *and a function* $\Psi(\theta, \alpha)$ *that is continuous for all* $\theta$, *such that* $b_n(\alpha) \to b(\alpha)$ *and* $m_n(\alpha) \to m(\alpha) > 0$ *for all* $\alpha \in \mathbf{R}^r$, *and, for fixed* $\theta$ *and* $N > 0$, $\Psi_n(\theta, \alpha) \to \Psi(\theta, \alpha)$ *uniformly in* $|\alpha| \le N$.

*Also assume that a solution of equation* (3.2) *exists and is unique on every finite interval. Let* $T$ *be a positive number such that* $y(+\infty) > T$, *where* $y(t) = \int_0^t m(\eta(u))\, du$.

*Then the sequence* $\rho_n Z_n(nt)$ *J-converges on the interval* $[0, T]$ *to a nonhomogeneous process with independent increments* $z_0(t)$ *whose characteristic function is given by relations* (3.3) *and* (3.4).

*Remark 3.2.* Condition (3.14) can be satisfied under weaker assumptions than the assumption that the process $x_{nk}$ is asymptotically ergodic. For example, an appropriate condition is that its state space forms an $n$-$S$-set (see [1, 9, 12]).

*Proof.* The proof is analogous to that of Theorem 3.1. In a similar way we introduce the sequences $\eta_{nk}$, $y_{nk}$, $k \ge 0$, and processes $\eta_n(u)$, $y_n(u)$, $u \ge 0$, and put

$$R_n(u) = \sum_{i=0}^{[nu]} \gamma_{ni}(x_{ni}, \eta_{ni}).$$

Now we can write representation (3.5).

It is proved in [7, Theorem 4.1], that relations (3.6), (3.7), and (3.8) hold under the conditions of Theorem 3.2, where the function $s(t)$ satisfies equation (3.4).

Now we study the convergence of $\rho_n R_n(u)$. Put $\varphi_n(\theta, x, \alpha) = \ln \phi_n(\rho_n \theta, x, \alpha)$. Similarly to (3.9), the following representation holds:

$$\mathsf{E} \exp\{i(\theta, \rho_n R_n(u))\} = \mathsf{E} \exp\Big\{ \sum_{k=0}^{[nu]} \varphi_n(\theta, x_{nk}, \eta_{nk}) \Big\}, \qquad u \ge 0. \tag{3.16}$$

The further proof can be carried out under the assumption that $\{|\eta_n(t)| \le N, t \le T\}$. According to condition (3.15), the right-hand side of (3.16) is equivalent to

$$G_n(\theta, u) = \mathsf{E} \exp\Big\{ \frac{1}{n} \sum_{k=0}^{[nu]} \Psi(\theta, x_{nk}, \eta_{nk}) \Big\}.$$

Using condition (3.14), the boundedness of the function $\Psi(\theta, x, \alpha)$ in the domain $\{x \in X, |\alpha| \leq N\}$, where $\theta$ is fixed, and a known inequality [13],

$$\left| \int_X f(x) P(dx) - \int_X f(x) Q(dx) \right| \leq 4 \sup_x |f(x)| \sup_A |P(A) - Q(A)|,$$

which holds for any bounded complex function $f(x)$ and all probability measures $P(\cdot)$ and $Q(\cdot)$, we obtain, in the same way as in [5], that

$$\mathsf{E} \left| \frac{1}{n} \sum_{k=0}^{[nu]} \Psi(\theta, x_{nk}, \eta_{nk}) - \frac{1}{n} \sum_{k=0}^{[nu]} \Psi_n(\theta, \eta_{nk}) \right|^2 \to 0.$$

This relation, (3.6), and the uniform convergence of $\Psi_n(\theta, \alpha)$ to $\Psi(\theta, \alpha)$ in every domain $\{|\alpha| \leq N\}$ imply that

$$\frac{1}{n} \sum_{k=0}^{[nu]} \Psi_n(\theta, \eta_{nk}) \xrightarrow{\mathsf{P}} \int_0^u \Psi(\theta, \eta(v))\, dv$$

for all $u \geq 0$. Similarly to the proof of Theorem 3.1, we obtain the convergence of finite-dimensional distributions of the sequence $\rho_n R_n(u)$ to the process with independent increments $R_0(u)$ whose characteristic function is of the form $\exp\{\int_0^u \Psi(\theta, \eta(v))\, dv\}$. The weak compactness of measures generated by the process $\rho_n R_n(u)$ in the space $\mathcal{D}_T$ can be proved in a similar way. The further proof follows the proof of Theorem 3.1. □

Analogous results can be proved for SRP constructed on trajectories of general switching processes (see (2.5) and (2.6)) as well as for time nonhomogeneous models [11].

**3.3. SRP in queueing systems.** As an example we consider the behavior of an SRP constructed on a trajectory of an overloaded queueing system $G/M/\infty$. Let the arrival process form an ordinary renewal process with the interarrival time $\tau$. Assume that there are infinitely many identical servers with the exponential rate of service $n^{-1}\mu$, where $n \to \infty$.

We denote by $t_{nk}$, $k > 0$, the times when customers arrive at the system and let $Q_n(t)$ be the total number of customers served in the system at time $t$. Assume that if $Q_n(t_{nk} + 0) = nq$, then we get a reward $\gamma_k(q)$, where $\{\gamma_k(q), q \geq 0\}$, $k \geq 0$, is a sequence of independent random variables in $\mathbf{R}$ whose distributions do not depend on $k$. By $Z_n(t)$ we denote the total reward on the interval $[0, t]$.

**Proposition 3.1.** *Let*

$$n^{-1} Q_n(0) \xrightarrow{\mathsf{P}} s_0$$

*as $n \to \infty$, where $s_0 > 0$ is nonrandom. Assume that the first moment functions*

$$m = \mathsf{E}\,\tau, \qquad g(q) = \mathsf{E}\,\gamma(q), \quad q \geq 0,$$

*exist and $g(q)$ is continuous.*

*Then for all $T > 0$,*

$$\sup_{t \leq T} \left| \frac{1}{n} Z_n(nt) - \frac{1}{m} \int_0^t g(s(u))\, du \right| \xrightarrow{\mathsf{P}} 0,$$

*where*

$$s(t) = (m\mu)^{-1} + \left(s_0 - (m\mu)^{-1}\right) e^{-\mu t}. \tag{3.17}$$

*If $g(q) = 0$, $q \geq 0$, and there exists a continuous function $\sigma^2(q) = \operatorname{Var}\gamma(q)$, then the sequence $n^{-1/2}Z_n(nt)$ converges in the $U$-topology (see [29]) on every interval $[0,T]$ to the process*

$$z_0(t) = \frac{1}{\sqrt{m}} \int_0^t \sigma(s(u))\, dw(u), \qquad t \geq 0,$$

*where $w(u)$, $u \geq 0$, is a standard Wiener process in $\mathbf{R}$.*

*Proof.* We use Theorem 7.2 in [7] which is proved for a more general system. As a result we easily obtain that for all $T > 0$,

$$\sup_{t \leq T} \left| \frac{1}{n} Q_n(nt) - s(t) \right| \xrightarrow{\mathsf{P}} 0,$$

where the function $s(t)$ is defined by (3.17). Now Proposition 3.1 follows immediately from Theorem 3.1. □

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