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# SOME MARKOVIAN QUEUING RETRIAL SYSTEMS UNDER LIGHT-TRAFFIC CONDITIONS

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*The asymptotic behavior of the time of the first loss of a demand in Markovian single-channel and multichannel queuing retrial systems with a finite buffer and an external Markovian environment is analyzed. Two cases are studied: (a) the ratio of the input rate to the service rate tends to zero and (b) the ratio of the input rate to the service rate and the ratio of the input rate to the retrial rate tend to zero simultaneously. The method of so-called S-sets and the concept of a monotone structure introduced by Anisimov are used to prove the exponential approximation of the time of the first call loss and Poisson approximation of a flow of lost demands for both cases.*

**Keywords:** *Markovian queuing systems, retrials, light-traffic conditions, S-sets, Markovian environment.*

## INTRODUCTION

Mathematical models of actual computer systems and communication networks have, as a rule, a complex hierarchical structure and operate at different time scales. For example, real time and processor time run at different rates. Analytical solution for such models can be found only in special cases; therefore, asymptotic and approximating methods are important for analysis and simulation of such systems. Many practically important models have a so-called small parameter, such as the order of failure probabilities or rates or the ratio $\rho$ of the mean service time to the mean time between arrivals of demands. In queuing theory, a system with $\rho \to 0$ is usually called a light traffic system. In reliability theory, a system with a small ratio of the mean recovery time to the mean time between failures is called an element-reliable system or a fast-recovery system. The use of small parameters intensified the analysis of "rare" events in reliability and queuing theories [1, 2]. Note that the term "quick service" was used in [3, 4] to analyze queuing systems with asymptotically small service time. In applications, rare events usually mean various failures, loss of demands, leaving some domain, excess of a level, etc.

An asymptotic analysis is made of the demand loss time in Markovian service retrial systems under light-traffic conditions. Queuing retrial systems are characterized by the following property: a demand arriving at a system with all servicing servers and buffers occupied is put on a special retrial queue and the service demand is repeated after some random time. Such a property is inherent in many models of computer and telecommunication networks. Let us call a retrial queue an orbit. Systems with retrial queues were studied in [5–8] in detail. Note that explicit formulas for stationary and reliability characteristics cannot be obtained even for Markovian models of single-channel systems. The dimension of the problem increases exponentially with the number of channels. Thus, it is of interest to derive simple approximate formulas oriented to various applications.

In studying various classes of computer and telecommunication networks, various processes in the system often develop at different time scales. This means that transition probabilities or intensities have different orders of smallness. To analyze such classes of models, a special concept of a so-called *S*-set (an asymptotically connected set) was introduced in [9, 10]. The method of *S*-sets allows us to study the asymptotical behavior of the time of the first loss of a demand for wide classes of Markovian and semi-Markovian queuing models with a finite set of states and under quick-recovery or light-traffic conditions. Various applications of this method are presented in [11–17].

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Another area in the asymptotic analysis of queuing systems is the diffusion approximation. The asymptotic technique of switching processes was used in [15, 18] to obtain some results for some classes of queuing systems. The diffusion approximation of some classes of retrial queuing systems was studied in [19–22]. An asymptotic analysis of the first-loss time in retrial systems was mainly made for classical single-channel Markovian models [23].

The asymptotical behavior of the first-loss time and a lost-demand flow in Markovian multichannel retrial systems is studied here. We assume that the characteristics of the system depend on some normalizing parameter $n$, $n \to \infty$. It is expedient to use the following terminology. Let the case where the ratio of the input-flow parameter to the service rate tends to zero be called an intensive service. The case where the ratio of the input-flow parameter to the retrial intensity tends to zero is called intensive retrial.

The present paper is organized as follows. In Section 2, the asymptotical behavior of the first-loss time is studied for single-channel Markovian retrial queuing systems under conditions of intensive service and ordinary or intensive retrials. Cases are also considered where the parameters depend on the queue size and the state of the Markovian environment. Multichannel systems are considered in Section 3. The results of the asymptotic analysis of the time of the first exit from the $S$-set and structures of the state space are illustrated in the Appendix.

## 2. SINGLE-CHANNEL RETRIAL SYSTEMS

Let us consider a single-channel retrial system with $m$ waiting places in the orbit (for retrials) and losses. The system input flow is Poisson with a parameter $\lambda$. The input demands are identified as primary. If the server (channel) is free, the arriving demand is immediately serviced and leaves the system once the service is completed. If the server is busy, the demand is put on the retrial queue (in an orbit) if the queue consists of less than $m$ demands. Otherwise, the demand is considered lost. Each demand in the orbit generates, independently of the other demands, a Poisson flow of calls to the server (secondary demands) with intensity $\nu$. If a secondary demand discovers that the server is free, then it is immediately serviced and leaves the system once the service is completed. For the sake of simplicity, we assume that the service time has the same exponential distribution with the parameter $\mu$ for the initial and secondary demands. Let such a model be called $M / M / 1 / m / wr$.

Using the results presented in the Appendix (on the asymptotic exponentiality of the time of the first exit from the subset of states forming a monotone structure), let us study the asymptotic behavior of the first-loss time for systems with intensive service, with intensive service and intensive retrial, and for systems in a Markovian random environment. Assume that the intensities $\mu = \mu_n$ and $\nu = \nu_n$ can depend on some normalizing parameter $n$, $n \to \infty$. Without loss of generality, assume also that the intensity of the input flow $\lambda$ does not depend on $n$. Let us consider the following cases.

**Case 1.** Let $\mu_n = n\mu$ (intensive service), $\nu_n = \nu$ (ordinary retrial), and $n \to \infty$.

**Case 2.** Let $\mu_n = n\mu$ (intensive service), $\nu_n = V_n \nu$ (intensive retrial), $n \to \infty$, and $V_n \to \infty$.

Let $\Omega_n(j, q)$ be the time of the first loss of a demand provided that $Q_n(0) = q$ and $\delta_n(0) = j$, $0 \le q \le m$, $j = 0, 1$, and $Y_n(t)$ be the number of lost demands on the interval $[0, t]$. Denote by $Q_n(t)$, $t \ge 0$, the number of demands in the orbit at the time $t$, and let the component $\delta_n(t)$ be the state of the server at the time $t$ ($\delta_n(t) = 1$ if the server is busy and $\delta_n(t) = 0$ otherwise).

### 2.1. Case 1: Intensive Service

**THEOREM 2.1.** For the model described above under intensive service, if $\lambda > 0$, $\mu > 0$, and $\nu > 0$, then the distribution of $n^{-m-1}\Omega_n(j, q)$, irrespective of the initial state $(j, q)$, $0 \le q \le m$, $j = 0, 1$, weakly converges to the exponential distribution

$$\lim_{n \to \infty} \mathbf{P}\{n^{-m-1}\Omega_n(j, q) \ge t\} = \exp\{-\Lambda t\}, \quad t > 0,$$

where

$$\Lambda = \frac{\lambda \rho^{m+1}}{m!\,\nu^m} \prod_{k=1}^{m} (\lambda + k\nu), \quad \rho = \lambda / \mu. \tag{2.1}$$

**Proof.** Let us consider a multidimensional process $z_n(t) = (\delta_n(t), Q_n(t))$, $t \ge 0$. It is a homogeneous Markovian process (MP) with continuous time and a set of states $Z = \{(j, q),\ j = 0, 1;\ q = 0, 1, \dots, m\}$. The process $z_n(t)$ describes the dynamics of the system. Denote by $\hat{Q}_n(t)$ the number of demands in the orbit for an auxiliary system with an infinite number

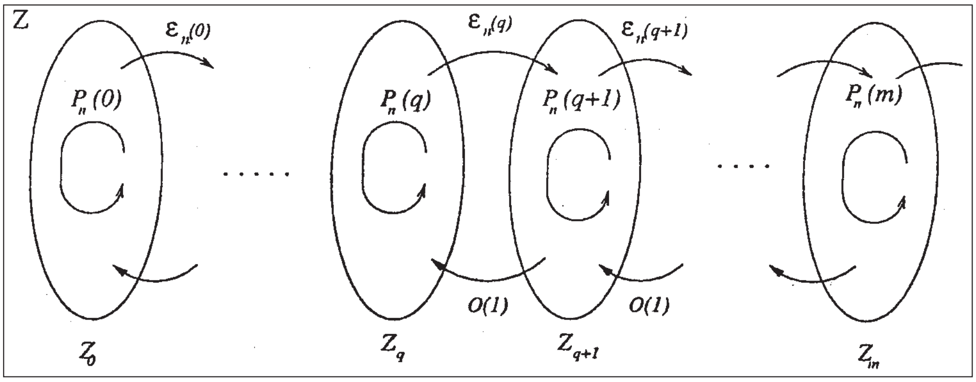


Fig. 1. Monotone structure.

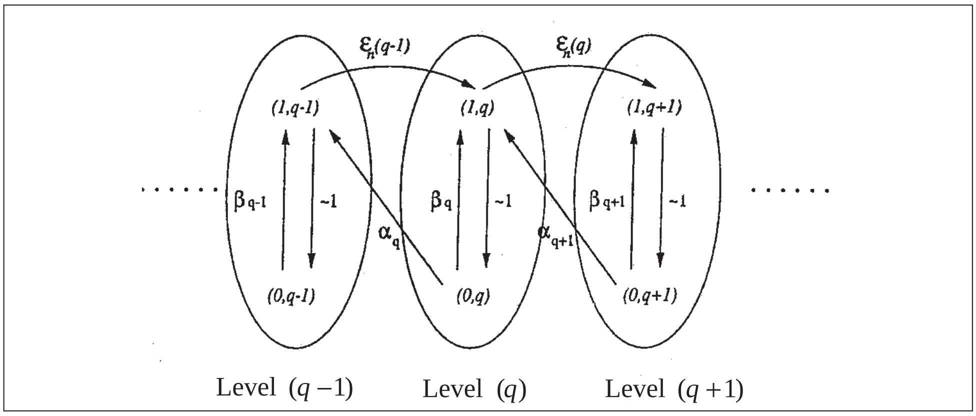


Fig. 2. Monotone structure for a single-channel quick-service system.

of waiting places in the orbit and put $\hat{z}_n(t) = (\delta_n(t), \hat{Q}_n(t))$. Note that $\hat{z}_n(t)$ is an MP in $\{0,1\} \times \{0,1,\dots\}$. Then $\Omega_n(j,q)$ is equivalent to the time of the first exit of $\hat{z}_n(t)$ from the subset $Z$:

$$\Omega_n(j,q) = \min\{t : t > 0,\ \hat{Q}_n(t) > m \text{ provided that } \hat{Q}_n(0) = q,\ \delta_n(0) = j\}.$$

One can easily calculate the intensities of transitions for $z_n(t)$ and see that $Z$ forms a monotone structure (see Definition A2). Here, the set $Z_q = \{(j,q),\ j = 0,1\}$ forms a $q$-level for each $q = 0,1,\dots,m$ (see Fig. 1).

A monotone structure for the model is presented in Fig. 2, where $\alpha_q = q\nu(\lambda + q\nu)^{-1}$, $\beta_q = \lambda(\lambda + q\nu)^{-1}$, and $\varepsilon_n(q) = \lambda / (n\mu)$.

In the state $(j,q)$, the process $\hat{z}_n(t)$ spends exponential time with the parameter $\Lambda_n(j,q) = \lambda + n\mu$ if $j = 1$ and $\Lambda_n(j,q) = \lambda + q\nu$ if $j = 0$. Transition probabilities are calculated by the formulas

$$p_n((1,q),(1,q+1)) \sim n^{-1}\rho \to 0, \quad p_n((0,q),(1,q-1)) = q\nu(\lambda + q\nu)^{-1},$$

$$p_n((0,q),(1,q)) = \lambda(\lambda + q\nu)^{-1}, \quad p_n((1,q),(0,q)) = n\mu(\lambda + n\mu)^{-1} \to 1.$$

Denote by $\overline{\pi}_n(q) = (\pi_n(0,q), \pi_n(1,q))$ the stationary distribution of the embedded Markov chain for $z_n(t)$. Let $\pi_i = \pi(i,0)$, $i = 0,1$, $(\overline{\pi} = (\pi_0, \pi_1))$ be a stationary distribution of states at the level $Z_0 = \{(0,0),(1,0)\}$ (0-level) in the limit. Since $Z_0$ forms one significant class, the limit exists and it is easy to calculate that $\pi_0 = \pi_1 = 1/2$.

Further, the matrices $A(j)$ and $P(j)$ in the matrix relations of Theorem A2 have the form

$$A(j) = \begin{bmatrix} 0 & 0 \\ 0 & \lambda/\mu \end{bmatrix}, \quad P(j) = \begin{bmatrix} 0 & \dfrac{\lambda}{\lambda + j\nu} \\ 1 & 0 \end{bmatrix}.$$

Therefore, the stationary distribution for the embedded Markov chain of the process $z_n(t)$ has the form

$$\bar{\pi}_n(q) = \bar{\pi} \prod_{j=0}^{q-1} \left( \begin{bmatrix} 0 & 0 \\ 0 & \lambda/\mu \end{bmatrix} \left( I - \begin{bmatrix} 0 & \dfrac{\lambda}{\lambda + (j+1)\nu} \\ 1 & 0 \end{bmatrix} \right)^{-1} \frac{1}{n} \right) (1 + o(1)), \quad q \geq 1.$$

Transformations yield

$$\bar{\pi}_n(q) = \frac{1}{n^q} \pi_1 \frac{\rho^q}{q!\,\nu^q} \prod_{j=1}^{q} (\lambda + j\nu)\bar{e}(1 + o(1)), \quad q = 1, 2, \dots,$$

where $\rho = \lambda/\mu$ and $\bar{e}$ is a unit vector. Note that $\bar{\pi}_n(q) = O\left( \prod_{s=0}^{q-1} \varepsilon_n(s) \right)$. Similarly, we obtain

$$g_n(Z) = \frac{1}{n^{m+1}} \pi_1 \frac{\rho^{m+1}}{m!\,\nu^m} \prod_{j=1}^{m} (\lambda + j\nu)(1 + o(1)) = \frac{1}{n^{m+1}} G(1 + o(1)).$$

If in (A3) we put $\beta_n = n^{-m-1}$, then $a_{(j,q)}(\theta) = G^{-1}\lambda^{-1}\theta$ for $q = 0$, $j = 0$ and $a_{(j,q)}(\theta) = 0$ otherwise. Thus, for $A(\theta)$ in (A4) we have $A(\theta) = \pi_0 (G\lambda)^{-1}\theta$.

Finally, performing transformations we obtain that the parameter of the limit exponential distribution has the form $\Lambda = \pi_0 (G\lambda)^{-1}$, which corresponds to (2.1).

**2.1.1. Dependence of the Parameters on the System State**. This result may be extended to the case where the intensities depend on the queue size. Let $\lambda(k)$, $\mu(k)$, $k = \overline{0, m}$, and $\nu(k)$, $k = \overline{1, m}$, be given. Then when $Q_n(t) = k$, the instantaneous intensity of the input flow is $\lambda(k)$, the service intensity is $n\mu(k)$, $k = \overline{0, m}$, and the retrial intensity is $\nu(k)$, $k = \overline{1, m}$. Using the same technique, we obtain the following statement.

**Statement 2.1.** Let $\prod_{k=0}^{m} (\lambda(k)\mu(k)) \prod_{j=1}^{m} \nu_j > 0$. Then, under the above assumptions, the distribution $n^{-m-1}\Omega_n(j, q)$ weakly converges to an exponential distribution with the parameter

$$\Lambda = \lambda(0) \frac{1}{m!} \prod_{k=0}^{m} \frac{\lambda(k)}{\mu(k)} \prod_{k=0}^{m-1} \frac{\lambda(k) + (k+1)\nu(k+1)}{\nu(k+1)}.$$

**Remark 2.1.** In both cases (homogeneity or dependence on the state), the process $Y(n^{m+1}t)$ on any interval $[0, T]$ weakly converges (in the sense of weak convergence of finite-dimensional distributions and measures in the Skorokhod space $D_T$) to the ordinary Poisson process with the same parameter $\Lambda$.

### 2.2. Intensive Service and Intensive Retrial

**THEOREM 2.2.** Under conditions of intensive service and intensive retrial for $\lambda > 0$, $\mu > 0$, and $\nu > 0$, the distribution $n^{-m-1}\Omega_n(j, q)$, irrespective of the initial state, weakly converges to an exponential distribution with the parameter $\Lambda$, where $\Lambda = \lambda\rho^{m+1}$.

**Proof.** It is similar to that of Theorem 2.1. Let us consider an auxiliary MP $\hat{z}_n(t) = (\delta_n(t), \hat{Q}_n(t))$, $t \geq 0$. It is easy to verify that the set $Z$ forms a monotone structure and the subset $Z_q = \{(j, q),\ j = 0\}$ is a $q$-level, $q = 0, 1, \dots, m$. The process $\hat{z}_n(t)$ is in the state $(j, q)$ during an exponential time with the parameter $\Lambda_n(j, q) = \lambda + n\mu$ if $j = 1$ and $\Lambda_n(j, q) = \lambda + qV_n\nu$ if $j = 0$. Put $\rho = \lambda/\mu$. The transition probabilities for the embedded MP have the form

$$p_n((1, q), (1, q+1)) \sim n^{-1}\rho \to 0,$$

$$p_n((0, q), (1, q-1)) = qV_n\nu(\lambda + qV_n\nu)^{-1} \to 1,$$

$$p_n((0, q), (1, q)) = \lambda(\lambda + qV_n\nu)^{-1}, \quad p_n((1, q), (0, q)) = n\mu(\lambda + n\mu)^{-1} \to 1.$$

Denote by $\bar{\pi}_n(q) = (\pi_n(0,q), \pi_n(1,q))$, $q \geq 0$, the stationary distribution for $\hat{z}_n(t)$ and let $\pi_j = \pi_0(j,0)$, $j = 0, 1$, $(\bar{\pi} = (\pi_0, \pi_1))$ be a stationary distribution for the level $Z_0$ in the limit. It is easy to see that $\pi_0 = \pi_1 = 1/2$. Further, in the relations of Theorem A2, the matrices $A(j)$ and $P(j+1)$ have the form

$$A(j) = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}, \; P(j+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Performing transformations, we have

$$\bar{\pi}_n(q) = \pi_1 \frac{1}{n^q} \rho^q \, \bar{e}\,(1 + o(1)), \; g_n(Z) = \frac{1}{n^{m+1}} \pi_1 \rho^{m+1}(1 + o(1)).$$

Substituting $\beta_n = n^{-m-1}$, we finally obtain $\Lambda = \lambda\rho^{m+1}$. Note that it is only this case where the result does not depend on the values of $\nu > 0$. However if $\nu = 0$ or $\nu = \nu_n \to 0$, then the result will be different.

Let us now consider the case of dependence on the state. Let for $Q_n(t) = q$ the intensity of the input flow be $\lambda(q)$, the service intensity be $n\mu(q)$, $q = \overline{0, m}$, and the retrial intensity be $V_n \nu(q)$, $q = \overline{1, m}$. Here $n \to \infty$ and $V_n \to \infty$.

**Remark 2.2.** If $\prod_{k=0}^{m}(\lambda(k)\mu(k)) \prod_{j=1}^{m} \nu(j) > 0$, then the statement of Theorem 2.2 is true, where

$$\Lambda = \lambda(0) \prod_{k=0}^{m} \frac{\lambda(k)}{\mu(k)}.$$

Here, the process $Y(n^{m+1}t)$ weakly converges to the ordinary Poisson process with the parameter $\Lambda$. In this case, the result also does not depend on the values of $\nu(j) > 0$, $j = \overline{1, m}$.

### 2.3. Dependence on the System State and Markovian Environment

Let us consider a Markovian queuing system with retrials of the type $M_M / M_M / 1 / m / wr$. The system has one server and $m$ places for waiting in the orbit. In addition to the above description, we assume that the system operates in a Markovian environment $x(t)$, $t \geq 0$, which is an ergodic Markovian process with a finite set of states $X = \{1, 2, \ldots, r\}$ determined by the initial state $x(0)$ and transition intensity $a_{ij}$, $i, j \in X$, $i \neq j$. Denote by $\pi_i$, $i = \overline{1, r}$, a stationary distribution $x(t)$.

Let non-negative functions $\lambda(i,q)$, $\nu(i,q)$, and $\mu(i,q)$, be given, $i \in X$, $q = \overline{0, m}$. We will study the case of intensive service. Then for $x(t) = i$, $Q_n(t) = q$, $\lambda(i,q)$ is the instantaneous intensity of the input flow, $n\mu(i,q)$ is the service intensity, and $\nu(i,q)$ is the retrial intensity.

Denote by $\Omega_n(i, j, q)$ the time of the first loss of a demand if $x(0) = i$, $Q_n(0) = q$, and $\delta_n(0) = j$.

**THEOREM 2.3.** Let

$$\max_{i=\overline{1,r}} \lambda(i,q) > 0, \quad q = \overline{0, m}, \quad \max_{i=\overline{1,r}} \nu(i,q) > 0, \quad q = \overline{1, m}; \;,$$

$$\min_{i=\overline{1,r},\, q=\overline{0,m}} \mu(i,q) > 0. \tag{2.2}$$

Then under intensive service in any initial state $(i, j, q) \in Z$, the distribution $n^{-m-1}\Omega_n(j,q)$ weakly converges to an exponential distribution with the parameter $\Lambda$, where

$$\Lambda = \bar{\pi}\, \Lambda G(0)(I - B(1) - \Lambda(1))^{-1}(I - B(1))\, G(1) \ldots$$

$$\ldots G(m-1)(I - B(m) - \Lambda(m))^{-1}(I - B(m))\, G(m)\bar{e}. \tag{2.3}$$

Here $\bar{\pi} = (\pi_1, \ldots, \pi_r)$ (a row vector), $\Lambda G(q)$, $q = \overline{0, m}$, and $\Lambda(q)$, $q = \overline{1, m}$, are diagonal matrices with the diagonal elements $\lambda(i, 0)$, $\lambda(i,q) / \mu(i,q)$, and $\lambda(i,q)(\lambda(i,q) + a_{ii} + q\,\nu(i,q))^{-1}$, respectively,

$$B(q) = \left\| \frac{a_{ij}(1 - \delta_{ij})}{\lambda(i,q) + a_{ii} + q\,\nu(i,q)} \right\|, \quad i, j = \overline{1, r}, \quad q = \overline{1, m},$$

where $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).

**Proof.** Let us consider a MP $z_n(t) = (x(t), \delta_n(t), Q_n(t))$, $t \ge 0$. It is a process with continuous time and a finite set of states $Z = \{(i, j, q),\ i \in X;\ j = 0, 1;\ q = \overline{0, m}\}$.

Let us introduce again an auxiliary MP $\hat{z}_n(t) = (x(t), \delta_n(t), \hat{Q}_n(t))$, $t \ge 0$, where $\hat{Q}_n(t)$ designates the number of demands in the orbit in a system with an infinite number of waiting places. Then $\Omega_n(i, j, q)$ is equivalent to the time of exit of $\hat{z}_n(t)$ from $Z$.

It is easy to verify that $Z$ forms a monotone structure, where for any $q = 0, 1, \dots, m$ the subset $Z_q = \{(i, j, q),\ i = \overline{1, r},\ j = 0, j\}$ forms a $q$-level.

The process is in the state $(i, j, q)$ during exponential time with the parameter $\Lambda_n(i, j, q) = \lambda(i, q) + n\,\mu(i, q) + a_{ii}$ if $j = 1$ and $\Lambda_n(i, j, q) = \lambda(i, q) + q\,\nu(i, q) + a_{ii}$ if $j = 0$, where $a_{ii} = \Sigma_{k \ne i} a_{ik}$. Put $a_n(i, q) = (\lambda(i, q) + a_{ii} + n\,\mu(i, q))^{-1}$, $b(i, q) = (\lambda(i, q) + a_{ii} + q\,\nu(i, q))^{-1}$. Then the transition probabilities of the embedded chain are

$$p_n((i, 1, q)(i, 1, q+1)) = \lambda(i, q) a_n(i, q) \sim n^{-1} \lambda(i, q) / \mu(i, q), \quad i = \overline{1, r},$$

$$p_n((i, 0, q)(i, 1, q-1)) = q\,\nu(i, q) b(i, q), \quad p_n((i, 0, q)(i, 1, q)) = \lambda(i, q) b(i, q),$$

$$p_n((i, 0, q)(k, 0, q)) = a_{ik} b(i, q), \quad i \ne k,$$

$$p_n((i, 1, q)(i, 0, q)) = n\,\mu(i, q)\, a_n(i, q) \to 1,$$

$$p_n((i, 1, q)(k, 1, q)) = a_{ik} a_n(i, q) \to 0, \quad i \ne k.$$

Denote by $\overline{\pi}_n(j, q) = (\pi_n(i, j, q),\ \ i \in X,\ \ j = 0, 1,\ \ q = 0, 1, \dots, m)$ the stationary distribution of the embedded chain for the MP $z_n(t)$. In the relations of Theorem A2, the matrices $A(j)$ and $P(j)$ have the form

$$A(j) = \begin{bmatrix} 0 & 0 \\ 0 & G(j) \end{bmatrix}, \qquad P(j) = \begin{bmatrix} B(j) & \Lambda(j) \\ I & 0 \end{bmatrix}.$$

Denote $\overline{\pi}_n(q) = (\overline{\pi}_n(0, q), \overline{\pi}_n(1, q))$, where $\overline{\pi}_n(0, q) = (\pi(i, 0, q),\ i = \overline{1, r})$ and $\overline{\pi}_n(1, q) = (\pi(i, 1, q),\ i = \overline{1, r})$, $q = \overline{0, m}$ are row vectors. Using the formula from [24], we obtain

$$(I - P(q))^{-1} = \begin{bmatrix} (I - B(q) - \Lambda(q))^{-1} & (I - B(q) - \Lambda(q))^{-1} \Lambda(q) \\ (I - B(q) - \Lambda(q))^{-1} & (I - B(q) - \Lambda(q))^{-1} (I - B(q)) \end{bmatrix}.$$

Further, using Theorem A2 and performing transformations, we obtain

$$g_n(Z) = \frac{1}{n^{m+1}}\, \overline{\pi}(1, 0) \left( \prod_{j=0}^{m-1} G(j) K(j+1)(I - B(j+1)) \right) G(m)(1 + o(1)),$$

where $K(j) = (1 - B(j) - \Lambda(j))^{-1}$ and $\overline{\pi}(1, 0) = \lim_{n \to \infty} \overline{\pi}_n(1, 0)$.

Since the level $Z_0$ forms in the limit one essential class, the stationary distribution $\pi(i, j, 0)$ of the state $(i, j, 0) \in Z_0$ exists and satisfies the system of equations

$$\pi(i, 0, 0) = \sum_{k \ne i} \pi(k, 0, 0)\, a_{ki} (\lambda(k, 0) + a_{kk})^{-1} + \pi(i, 1, 0),$$

$$\pi(i, 1, 0) = \pi(i, 0, 0)\, \lambda(i, 0) (\lambda(i, 0) + a_{ii})^{-1}, \quad i = \overline{1, r}.$$

Put $B = \sum_{k=1}^{r} \pi_k (2\lambda(k, 0) + a_{kk})$. It is easy to verify that

$$\pi(i, 0, 0) = B^{-1} (\lambda(i, 0) + a_{ii}) \pi_i, \quad \pi(i, 1, 0) = B^{-1} \lambda(i, 0) \pi_i, \quad i = \overline{1, r}.$$

Finally putting $\beta_n = n^{-m-1}$, we obtain the parameter of the exponential distribution in the form (2.3).

Note that the process $Y(n^{m+1}t)$ weakly converges to the ordinary Poisson process with the parameter $\Lambda$. Note also that if $\nu(i,q)=0$, $i=\overline{1,r}$, at some $q$-level, then $Z$ forms a monotone structure and the result will be different.

The case of intensive service and intensive retrials can be studied similarly. By this is meant that for $x(t)=i$, $Q_n(t)=q$, the instantaneous intensity of retrials is $V_n\nu(i,q)$, $i=\overline{1,r}$, $q=0,1,\ldots,$ where $V_n\to\infty$.

**Statement 2.2.** Assume that $\max\limits_{i=\overline{1,r}} \lambda(i,q)>0, q=\overline{0,m}$, and $\nu(i,q)>0, q=\overline{1,m}$, and $\mu(i,q)>0, q=\overline{0,m}$, for all $i=\overline{1,r}$.

Then if the service and retrials are intensive, the statement of Theorem 2.3 holds, where $\Lambda=\bar{\pi}\,\Lambda G(0)G(1)\ldots G(m)\bar{e}$. Note that here the result does not depend on the values of $\nu(i,q)>0$ too.

We may study the mixed case where some values of $\nu(i,q)$ are zero. Let again the retrial intensity have the form $V_n\nu(i,q)$, $i=\overline{1,r}$, $q=0,1,\ldots$

**Statement 2.3.** Let (2.2) hold. Then the statement of Theorem 2.3 is true, where $\Lambda$ is determined by formula (2.3), in which the matrices are calculated as follows: $\Lambda$ and $G(q)$ are the diagonal matrices considered above, $B(q)=||b_{ij}(q)||$, where $b_{ij}(q)=a_{ij}(1-\delta_{ij})(\lambda(i,q)+a_{ii})^{-1}$ if $\nu(i,q)=0$, $b_{ij}(q)=0$ if $\nu(i,q)>0$, and $\Lambda(q)$ is a diagonal matrix with the diagonal elements $\lambda_{ii}(q)=\lambda(i,q)(\lambda(i,q)+a_{ii})^{-1}$ if $\nu(i,q)=0$, and $\lambda_{ij}=0$ if $\nu(i,q)>0$.

In this case, the result does not depend on the values of $\nu(i,q)$ too but depends on the set structure, where $\nu(i,q)>0$.

## 3. MULTICHANNEL RETRIAL SYSTEMS

Let us consider a model that has $s$ identical servers and $m$ waiting places in the orbit of the type $M/M/s/m/wr$. The model is described similarly. The input flow is Poisson with a parameter $\lambda$. If the initial demand finds that one of the servers is free, it is serviced and leaves the system once the service is completed. If all the servers are busy, then the demand goes into the orbit if there are vacant waiting places there; otherwise the demand is lost. In the orbit, demands behave as in a single-channel system. Again assume that the service intensity is identical for initial demands and retrials.

The operation of the system can be described by a two-component Markovian process $(N_n(t), Q_n(t))$, where $N_n(t)$ is the number of busy servers, and $Q_n(t)$ is the number of orbiting demands.

Let us consider two cases.

**Case 1.** Let $\mu_n=n\mu$ (intensive service), $\nu_n=\nu$ (ordinary retrial intensity), and $n\to\infty$.

**Case 2.** Let $\mu_n=n\mu$ (intensive service), $\nu_n=V_n\nu$ (intensive retrial), $n\to\infty$, and $V_n\to\infty$.

Denote by $\Omega_n(j,q)$ the time of the first loss of a demand provided that $Q_n(0)=q$ and $N_n(0)=j$.

**THEOREM 3.1.** In Case 1, if $\lambda>0$, $\mu>0$, and $\nu>0$, the distribution $n^{-s-m}\Omega_n(j,q)$, irrespective of the initial state, weakly converges to an exponential distribution with the parameter $\Lambda=\lambda\rho^{s+m}(s!s^m)^{-1}$, where $\rho=\lambda/\mu$.

**Proof.** Let us introduce an auxiliary MP $(N_n(t), \hat{Q}_n(t))$, where $\hat{Q}_n(t)$ is the number of orbiting demands in a system with infinite number of waiting places, and note that $\Omega_n(j,q)$ is equivalent to the time of exit from the subset $Z$. The process is in the state $(i,q)$ with the parameter $\Lambda_n(j,q)=\lambda+sn\mu$ during exponential time if $j=s$ and $\Lambda_n(j,q)=\lambda+jn\mu+q\nu$ if $j\in\{0,1,2,\ldots,s-1\}$.

It is easy to determine the transition probabilities for the embedded Markov chain and to verify that the set $Z$ forms a monotone structure with the following levels: $Z_0=\{(0,0),(1,0)\}$, $Z_1=\{(2,0)\},\ldots,$ $Z_{s-1}=\{(s,0)\}$, $Z_{s-1+q}=\{(j,q),\ j=\overline{0,s}\}$, $q=1,\ldots,m$.

Denote by $\pi_n(i,q)$, $i\in\{0,\ldots,s\}$, $q\in\{0,\ldots,m\}$, the stationary distribution of the embedded Markov chain for $(N_n(t),\hat{Q}_n(t))$. Let us first consider the 0-level $Z_0$. Since it forms in the limit one essential class, we denote $\pi_i=\lim\limits_{n\to\infty}\pi_n(i,0)$, $i=0,1$. It is easy to verify that $\pi_0=\pi_1=1/2$. Further, note that the set $\{(i,0),\ i=0,1,\ldots,s\}$ also forms a monotone structure, and, according to relation (A6), we obtain

$$\pi_n(i,0)=\frac{1}{2}\frac{1}{n^{i-1}}\frac{\rho^{i-1}}{(i-1)!}(1+o(1)),\ i=2,3,\ldots,s.$$

Denote $\overline{\pi}_n(q) = (\pi_n(i, q),\ i = 0, 1, \ldots, s)$ for $q = s, s+1, \ldots, s+m$. Taking into consideration the structure of the transition probabilities and relation (A7), for any $q = 1, 2, \ldots, m$ we recursively obtain

$$\pi_n(s, q) \sim \pi_n(s, q-1)\, \rho\, (ns)^{-1},$$

$$\pi_n(s, q) \sim \pi_n(s-1, q) \sim \ldots \sim \pi_n(2, q),$$

$$\pi_n(1, q) \sim \pi_n(2, q) + \pi_n(0, q)\, \lambda\, (\lambda + q\, \nu)^{-1}, \quad \pi_n(0, q) \sim \pi_n(1, q). \tag{3.1}$$

Finally, for any $q = 1, 2, \ldots, m$

$$\pi_n(s, q) \sim \pi_n(s, 0) \left( \frac{\rho}{ns} \right)^q = \frac{1}{2} \frac{1}{n^{s+q-1}} \frac{\rho^{s+q-1}}{s!\, s^{q-1}} (1 + o(1)),$$

$$\pi_n(i, q) \sim \pi_n(s, q), \quad i = 2, \ldots, s-1,$$

$$\pi_n(0, q) \sim \pi_n(1, q) \sim \pi_n(s, q)(\lambda + q\, \nu)(q\nu)^{-1}. \tag{3.2}$$

Since $g_n(Z) \sim \pi_n(s, m)\, \rho\, (ns)^{-1}$, we have

$$g_n(Z) = \frac{1}{2} \frac{1}{n^{s+m}} \frac{\rho^{s+m}}{s!\, s^m} (1 + o(1)). \tag{3.3}$$

Putting $\beta_n = n^{-m-s}$, we obtain $\Lambda = \lambda \rho^{s+m} (s!\, s^m)^{-1}$, similarly to Theorem 2.1.

Let us now study Case 2 ($\mu_n = n\mu$ and $\nu_n = V_n \nu$ ).

**THEOREM 3.2.** In Case 2, for $\lambda > 0$, $\mu > 0$, and $\nu > 0$, the statement of Theorem 3.1 with the same parameter $\Lambda$ holds, irrespective of the initial state.

**Proof.** It is similar to that of Theorem 3.1. The set $Z$ is again a monotone structure with the same levels as in Case 1. The process during exponential time is in the state $(j, q)$ with the parameter $\Lambda(j, q) = \lambda + sn\mu$ if $j = s$ and $\Lambda(j, q) = \lambda + jn\mu + qn\nu$ if $j \in (0, 1, \ldots, s-1)$.

Relations (3.1) and (3.2) can be proved similarly and it is possible to show that

$$\pi_n(s, q) \sim \pi_n(s-1, q) \sim \ldots \sim \pi_n(1, q) \sim \pi_n(0, q).$$

Finally, we have expression (3.3) and prove the statement of Theorem 3.2.

Note that in both cases the result does not depend on the value of $\nu_n$ if $\nu_n \not\to 0$. The results may similarly be extended to multichannel systems in a Markov environment.

## 4. CONCLUSIONS

Here, we have studied the asymptotical behavior of the time of the first loss of a demand in Markov single- and multi-channel queuing retrial systems and a finite buffer and in the presence of a Markov environment.

Two cases have been considered: the ratio of the input-flow parameter to the service intensity tends to zero and the ratios of the input-flow parameter to the service intensity and the retrial intensity tend to zero. The method of $S$-sets and the concept of a monotone structure have been used to prove the exponential approximation of the first-loss time and the Poisson approximation of the lost-demand flow.

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## APPENDIX

## ASYMPTOTICAL BEHAVIOR OF THE TIME OF THE FIRST EXIT FROM THE *S*-SET

Let us consider the important concept of an $S$-set (asymptotically connected set) and the exponential approximation of the time of the first exit from this set. Let us introduce a special class of hierarchical $S$-sets (a monotone structure). The results presented above give an analytical means for asymptotic analysis and simulation of the reliability characteristics of hierarchical Markovian and semi-Markovian models and, in particular, retrial queuing systems.

Let for each $n$ a Markovian process (MP) $x_{nk}$, $k \geq 0$, with a finite set of states $X = \{1, 2, \dots, r\}$ and a matrix of transition probabilities $P_n = || p_n(i, j)||$, $i, j = \overline{1, r}$, be given. We will fix some subset $X_0 \subset X$. Denote by

$$\nu_n(i) = \min\{k : k > 0, \quad x_{nk} \notin X_0 \text{ provided that } x_{n0} = i\}, \; i \in X_0, \tag{A1}$$

the number of steps to the first exit from $X_0$, beginning from the state $i \in X_0$.

**Definition A1.** The set $X_0$ is called an $S$-set if for any $i, j \in X_0$

$$\mathbf{P}\{\text{there exists } k, \quad k < \nu_n(i) \text{ such that } x_{nk} = j \,|\, x_{n0} = i\} \to 1 \text{ for } n \to \infty.$$

Further, let a semi-Markovian process (SMP) $x_n(t)$ with a finite set of states $X = \{1, 2, \dots, r\}$ be given. Denote by $x_{nk}$ an embedded Markov chain and by $\tau_n(j)$ the time of stay in the state $j = \overline{1, r}$. For the sake of simplicity, we assume that the time of stay does not depend on the subsequent transition. Denote by $\Omega_n(i) = \inf\{t : t > 0, \quad x_n(t) \notin X_0 \text{ provided that } x_n(0) = i\}$ the time of the first exit from $X_0$, beginning from the state $i \in X_0$. We will study the asymptotic behavior of $\Omega_n(i)$. Let us construct an auxiliary MP $\tilde{x}_{nk}$ with a set of states $X_0$ and the matrix of transition probabilities $\tilde{P}_n(X_0) = || \tilde{p}_n(i, j)||$, $i, j \in X_0$, where

$$\tilde{p}_n(i, j) = p_n(i, j)\, p_n(i, X_0)^{-1}, \; i, j \in X_0, \; p_n(i, X_0) = \sum_{l \in X_0} p_n(i, l).$$

Assume that $X_0$ forms an $S$-set and denote by $\tilde{\pi}_n(i)$, $i \in X_0$, the stationary distribution $\tilde{x}_{nk}$ (which exists for rather large $n$). Assume also that $g_n(X_0)$ is the stationary (aggregated) probability of the exit from $X_0$,

$$g_n(X_0) = \sum_{i \in X_0} \tilde{\pi}_n(i)(1 - p_n(i, X_0)). \tag{A2}$$

**THEOREM A1.** Let $X_0$ form an $S$-set and there exist a normalizing factor $\beta_n$ and functions $a_i(\theta)$ $(a_i(\pm 0) = 0)$ such that as $n \to \infty$

$$g_n(X_0)^{-1}\big(1 - \mathbf{E}\exp\{-\beta_n \theta\, \tau_n(i)\}\big) \to a_i(\theta), \quad i \in X_0. \tag{A3}$$

Then for any initial state $i_0 \in X_0$

$$\lim_{n \to \infty} \mathbf{E}\exp\{-\beta_n \theta\, \Omega_n(i_0)\} = (1 + A(\theta))^{-1},$$

where

$$A(\theta) = \lim_{n \to \infty} \sum_{i \in X_0} \tilde{\pi}_n(i)\, a_i(\theta). \tag{A4}$$

**The proof** with an algorithm of check on an $S$-set is presented in [9–11]. It is based on an asymptotic analysis of a matrix equation for the characteristic function of the normalized vector $\{\beta_n \Omega_n(i), \; i \in X_0\}$, the technique of recurrent aggregation of states, and the representation

$$(I - \tilde{P}_n(X_0))^{-1} = g_n(X_0)^{-1}\, \tilde{\Pi}_n(X_0),$$

where $I$ is a unit matrix and $\tilde{\Pi}_n(X_0) = ||\tilde{\pi}_n(i)(1 + o_{ij}(1))||$, $i, j \in X_0$.

**COROLLARY A1.** If $X_0$ forms an $S$-set, then for any $i_0 \in X_0$

$$\lim_{n \to \infty} \mathbf{P}\{g_n(X_0)\, \nu_n(i_0) > t\} = \exp\{-t\}, \quad t > 0, \tag{A5}$$

which means exponential approximation of the time of the first exit from $X_0$.

The convergence rate in (A5) is estimated in [25]. Let us consider, in particular, the exponential approximation $\Omega_n(i_0)$.

**COROLLARY A2.** Assume that the times of stay $\tau_n(i)$ have expectations $m_n(i) = \mathbf{E}\,\tau_n(i)$, $m_n(i) \to m_i$, $i = \overline{1, r}$, and $\mathbf{E}\,\tau_n(i)^2 < C < \infty$ for any $i = \overline{1, r}$. Denote

$$M = \lim_{n \to \infty} \sum_{i \in X_0} \tilde{\pi}_n(i)\, m_n(i) < \infty.$$

If $X_0$ forms an $S$-set, then for any $i_0 \in X_0$ the distribution $g_n(X_0)\Omega_n(i_0)$ weakly converges to an exponential distribution with the parameter $M^{-1}$.

**Proof.** It follows from relations (A3) and (A4), where $a_i(\theta) = m_i\theta$, $i \in X_0$ for the case being considered.

Note that a different method is used in [26] to prove the asymptotic exponentiality of the time of exit from the subset forming in the limit one class.

**COROLLARY A3.** Assume that $x_n(t)$, $t \ge 0$, is a MP with continuous time given by an embedded Markov chain with a matrix of transition probabilities $P_n$ and intensities of exit from the states $\lambda_n(i)$, $i = \overline{1, r}$, the set $X_0$ forms an $S$-set, and as $n \to \infty$

$$\min_i \lambda_n(i) \not\to 0, \quad \sum_{i \in X_0} \tilde{\pi}_n(i) / \lambda_n(i) \not\to 0.$$

Put $\beta_n = g_n(X_0)\left(\sum_{i \in X_0} \tilde{\pi}_n(i) / \lambda_n(i)\right)^{-1}$. Then the distribution $\beta_n \Omega_n(i_0)$ weakly converges to an exponential distribution with the parameter 1.

Note that in this case $\beta_n$ is asymptotically equivalent to the quantity

$$\tilde{\Lambda}_n(X_0) = \sum_{i \in X_0} \tilde{\rho}_n(i) \sum_{k \notin X_0} \lambda_n(i, k),$$

where $\lambda_n(i, j)$, $i, j \in X_0$, $i \ne j$, are the intensities of transitions for $x_n(\cdot)$, and $\tilde{\rho}_n(i)$, $i \in X_0$, is the stationary distribution of an auxiliary MP with continuous time, a set of states $X_0$, and intensities of transitions $\lambda_n(i, j)$, $i, j \in X_0$, $i \ne j$. The quantity $\tilde{\Lambda}_n(X_0)$ designates the stationary (aggregated) intensity of exit from $X_0$.

The above results show that to determine the parameter in the exponential approximation of the time of exit, it is sufficient to estimate the principal order of smallness of the stationary probabilities $\tilde{\pi}_n(i)$, $i \in X_0$, for the auxiliary MP $\tilde{x}_{nk}$. This is a very complicated task for applied problems.

Further, assume that after the system exits from $X_0$, it comes back to $X_0$ in some random time. Denote by $Y_n(t)$ the number of exits on the interval $[0, t]$, which corresponds to the number of lost demands in queuing systems.

Using the asymptotic exponentiality of the time of exit and its independence of the initial state, we can prove the following statement.

**Statement A1.** If the time of return to $X_0$ is stochastically bounded uniform in $n$, then the process $Y_n(\beta_n^{-1}t)$ weakly converges on any finite interval (in the sense of the weak convergence of finite-dimensional distributions and measures in the Skorokhod space $D_T$) to an ordinary Poisson process with the same parameter, as in exponential approximation of the time of exit.

Note that an asymptotic analysis of flows of rare events on trajectories of stochastic systems (or those switched by some random environment) is made in [27], where the approximation by an inhomogeneous Poisson flow is proved for the case where the environment satisfies asymptotic mixing conditions.

Let us now consider a special class of MP with a monotone structure of the state space, investigated in [11, 15, 27]. It is possible to derive explicit formulas for the principal parts of stationary probabilities of this class. Such models usually arise in asymptotic analysis of queuing systems and reliability models with transition probabilities of different orders (quick service, rare failures, etc.).

Let $x_{nk}$, $k \ge 0$, be an MP with a finite set of states $X$, $X = \bigcup_{s=0}^{m+1} (X_s, s)$, where $X_s$ $(s = 0, 1, \ldots, m+1)$ are some sets. Then individual states have the form $\{(l, q)\}$. Let us consider a subset of states $Z = \{(i, s),\ i \in X_s,\ s = \overline{0, m}\}$. Denote by $p_n((i, s), (j, q))$ transition probabilities.

**Definition A2.** The subset $Z = \{(i, s),\ i \in X_s,\ s = \overline{0, m}\}$ is called a monotone structure if the following asymptotic relations hold:

(1) $p_n((i, s), (j, s+1)) = \varepsilon_n(s) a_{ij}(s)(1 + o(1)), i \in X_s, j \in X_{s+1}$, where $\varepsilon_n(s) \to 0$, $s = \overline{0, m}$;

(2) $p_n((i, s), (j, s+k)) = 0$, $i \in X_s$, $j \in X_{s+k}$, $s = \overline{0, m-2}$, $k > 1$;

(3) $p_n((i, s), (j, s)) = p_{ij}(s)(1 + o(1))$, $i, j \in X_s$, $s = \overline{0, m}$,

where the matrix $I - P(s)$ is reversible for each $s = \overline{1, m}$, $P(0)$ is a nonreducible matrix with the stationary distribution $\pi_i$, $i \in X_0$ (here $P(s) = \| p_{ij}(s) \|$, $i, j \in X_s$), $I$ is a unit matrix, and $o(1) \to 0$ for $n \to \infty$.

The subset $Z_q = \{(i,q), i \in X_q\}$ is called a $q$-level (see Fig. 1). Denote $\overline{\pi}_n(s) = (\pi_n(i,s),\ i \in X_s),\ \ s = \overline{0,m}$, $\overline{\pi} = (\pi_i,\ i \in X_0)$ and $\overline{b} = (b_i,\ i \in X_m)$ are row vectors, where $\pi_n(i,s)$ is the stationary probability of the state $(i,s)$ for an MP with a set of states $Z$ and the matrix of transition probabilities

$$\widetilde{P}_n(Z) = \| p_n((i,s),(j,q))\, p_n((i,s),Z)^{-1} \|,\ \ (i,s),(j,q) \in Z,$$

$$p_n((i,s),Z) = \sum_{(l,g)\in Z} p_n((i,s),(l,g)),\ \ b_i = \sum_{k\in X_{m+1}} a_{ik}(m).$$

**THEOREM A2.** If the set $Z = \{(i,s),\ i \in X_s,\ \ s = \overline{0,m}\}$ is a monotone structure, then it also forms an $S$-set and for any $q = \overline{1,m}$,

$$\overline{\pi}_n(q) = \overline{\pi}\left(\prod_{j=0}^{q-1} A(j)(I - P(j+1))^{-1}\varepsilon_n(j)\right)(1 + o(1)),$$

$$g_n(Z) = \overline{\pi}\left(\prod_{j=0}^{m-1} A(j)(I - P(j+1))^{-1}\varepsilon_n(j)\right)\varepsilon_n(m)\,\overline{b}^{*}(1 + o(1)), \tag{A.6}$$

where $A(s) = \|a_{ij}(s)\|,\ i, j \in X_s,\ \overline{b}^{*}$ is a transposed vector, $\prod_{j=k}^{s} C(j) = C(k)C(k+1)\ldots C(s)$ for $k \le s$.

**Proof.** It is performed recursively in the order of the monotone structure. The main task is to estimate the stationary probabilities. It is possible to show that as $n \to \infty$,

$$\pi_n(i,q) = O\left(\prod_{s=0}^{q-1} \varepsilon_n(s)\right),\quad i = \overline{1,r},\quad q > 0.$$

Further, from the matrix relation

$$\overline{\pi}_n(q) = \overline{\pi}_n(q)\,P_n(q) + \overline{\pi}_n(q-1)\,\varepsilon_n(q-1)\,A(q-1) + O\left(\prod_{s=0}^{q} \varepsilon_n(s)\right),$$

where $P_n(q) = \| p_n((i,q),(j,q))\|,\ i, j \in X_q$, we obtain

$$\overline{\pi}_n(q) = \pi_n(q-1)\,A(q-1)(I - P_n(q))^{-1}\varepsilon_n(q-1)(1 + o(1)),$$

whence (A6) follows. The expression for $g_n(Z)$ follows from (A2).

The method of $S$-sets makes it possible to study the asymptotic behavior of the time of the first loss of a demand for wide classes of systems and queuing networks with a finite buffer under small loading [11–15, 28, 23].

Note that the limit distribution of the time of exit from an $S$-set does not depend on the initial state. This allows us to study models of asymptotic integration (aggregation) of the state space [11, 28].

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