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SOME QUESTIONS CONCERNING THE COFINALITY OF Sym(κ)

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§1. **Introduction.** Suppose that G is a group that is not finitely generated. Then the cofinality of G, written $c(G)$, is defined to be the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups. If κ is an infinite cardinal, then $\mathrm{Sym}(\kappa)$ denotes the group of all permutations of the set $\kappa = \{\alpha \mid \alpha < \kappa\}$. In [1], Macpherson and Neumann proved that $c(\mathrm{Sym}(\kappa)) > \kappa$ for all infinite cardinals κ. In [4], we proved that it is consistent that $c(\mathrm{Sym}(\omega))$ and 2^ω can be any two prescribed regular cardinals, subject only to the obvious requirement that $c(\mathrm{Sym}(\omega)) \leq 2^\omega$. Our first result in this paper is the analogous result for regular uncountable cardinals κ.

THEOREM 1.1. *Let $V \models GCH$. Let κ, θ, $\lambda \in V$ be cardinals such that*
(i) *κ and θ are regular uncountable, and*
(ii) *$\kappa < \theta \leq \mathrm{cf}(\lambda)$.*

Then there exists a notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that if G is $\mathbb{P}$-generic then $V[G] \models c(\mathrm{Sym}(\kappa)) = \theta \leq \lambda = 2^\kappa$.

Theorem 1.1 will be proved in §2. Our proof is based on a very powerful uniformization principle, which was shown to be consistent for regular uncountable cardinals in [2]. This approach does not seem suitable for proving an analogous result for singular cardinals. (The particular uniformization principle which we use is easily seen to be false for singular cardinals of countable cofinality. See Proposition 2.6.)

Question 1.2. Let κ be a singular cardinal. Is it consistent that $c(\mathrm{Sym}(\kappa)) > \kappa^+$?

After proving Theorem 1.1., we had hoped to prove an Easton-type theorem. This would say that the function $\kappa \mapsto c(\mathrm{Sym}(\kappa))$, κ regular, can be any function which satisfies certain "obvious constraints". Macpherson and Neumann [1] found the first such constraint; namely

$$\kappa < c(\mathrm{Sym}(\kappa)) \leq \mathrm{cf}(2^\kappa). \tag{1.3}$$

It is quite difficult to find any other constraints. For example, the following result shows that there are no monotonicity constraints.

THEOREM 1.4. *Let $V \models GCH$. Let $\kappa, \lambda \in V$ be regular cardinals such that $\kappa < \lambda$. Then there exists a notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that if G is $\mathbb{P}$-generic then $V[G] \models c(Sym(\kappa)) > c(Sym(\lambda))$.*

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For some time, we suspected that (1.3) was the only constraint on the function $\kappa \mapsto c(\mathrm{Sym}(\kappa))$. Then we were surprised to find that the following result holds.

THEOREM 1.5. *Let κ be an infinite cardinal. If $c(Sym(\kappa)) > \kappa^+$, then*

$$c(Sym(\kappa^+)) \leq c(Sym(\kappa)).$$

Theorem 1.5 is an easy consequence of a more general result.

DEFINITION 1.6. Let $\kappa \leq \lambda$ be infinite cardinals.

(i) $[\lambda]^\kappa = \{S \mid S \subseteq \lambda, |S| = \kappa\}$.

(ii) $c_\omega(\kappa, \lambda)$ is the least cardinality $|C|$ of an ω-closed unbounded subset C of $[\lambda]^\kappa$.

THEOREM 1.7. *Let $\kappa < \lambda$ be infinite cardinals. If $c(Sym(\kappa)) > c_\omega(\kappa, \lambda)$, then $c(Sym(\lambda)) \leq c(Sym(\kappa))$.*

Notice that Theorem 1.5 follows immediately from Theorem 1.7. Theorems 1.4 and 1.7 will be proved in §3. It is conceivable that a result even stronger than Theorem 1.7 holds. For example, the following problem remains open.

Question 1.8. Is it consistent that $2^\omega > \omega_1$ and $c(\mathrm{Sym}(\omega_1)) > \omega_2$?

Proposition 2.6 shows that such a consistency result cannot be achieved using the approach of §2.

Question 1.9. What is the Easton-type theorem for the function $\kappa \mapsto c(\mathrm{Sym}(\kappa))$?

Notation 1.10. Let κ be an infinite cardinal and let $A \in [\kappa]^\kappa$. Let $\{\alpha_i \mid i < \kappa\}$ be the increasing enumeration of A.

(i) If $\pi \in \mathrm{Sym}(\kappa)$, then $\pi^A \in \mathrm{Sym}(A)$ is defined by $\pi^A(\alpha_i) = \alpha_{\pi(i)}$ for all $i < \kappa$.

(ii) $A \in [\kappa]^\kappa$ is a *moiety* if $|\kappa \setminus A| = \kappa$.

§**2. Uniformization principles.** In this section, we shall prove Theorem 1.1.

DEFINITION 2.1. Let κ be a regular uncountable cardinal and let $\mathscr{A} = \langle A_i \mid i < \lambda \rangle$ be a sequence of elements of $[\kappa]^\kappa$.

(i) A *colouring* of $\mathscr{A}$ is a sequence $\langle c_i \mid i < \lambda \rangle$ such that $c_i : A_i \to \kappa$ for each $i < \lambda$.

(ii) The function $g : \bigcup_{i<\lambda} A_i \to \kappa$ *uniformizes* $\langle c_i \mid i < \lambda \rangle$ if for each $i < \lambda$ there exists $\beta_i < \kappa$ such that $g(\alpha) = c_i(\alpha)$ for all $\beta_i \leq \alpha \in A_i$.

(iii) $\mathscr{A}$ has the *uniformization property* if every colouring of $\mathscr{A}$ can be uniformized.

LEMMA 2.2. *Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Let χ be a regular cardinal such that $\chi > \kappa$, and suppose that $\mathscr{A} = \langle A_i \mid i < \chi \rangle$ is a sequence of elements of $[\kappa]^\kappa$ with the uniformization property. Let $\pi_i \in Sym(A_i)$ for each $i < \chi$. Then there exist $I \in [\chi]^\chi$ and $g \in Sym(\kappa)$ such that $g \restriction A_i = \pi_i$ for all $i \in I$.*

PROOF. For each $i < \chi$, define $c_i : A_i \to \kappa \times \kappa$ by $c_i(\alpha) = \langle \pi_i(\alpha), \pi_i^{-1}(\alpha) \rangle$. Since $\mathscr{A}$ has the uniformization property, there exists a function $h : \bigcup_{i<\chi} A_i \to \kappa \times \kappa$ such that for all $i < \chi$ there exists $\beta_i < \kappa$ such that $h(\alpha) = c_i(\alpha)$ for all $\beta_i \leq \alpha \in A_i$. Since $\kappa^{<\kappa} = \kappa$, there exist $\beta < \kappa$ and $I \in [\chi]^\chi$ such that

(i) $\beta_i = \beta$ for all $i \in I$, and

(ii) $A_i \cap \beta = A_j \cap \beta$ and $c_i \restriction A_i \cap \beta = c_j \restriction A_j \cap \beta$ for all $i, j \in I$.

Thus $k = \bigcup_{i \in I} \pi_i$ is a function from $\bigcup_{i \in I} A_i$ into κ such that $k \upharpoonright A_i = \pi_i$ for all $i \in I$. It is clear that range$(k) = \bigcup_{i \in I} A_i$. We claim that k is an injection. For suppose that $k(\gamma) = k(\delta)$, where $\gamma \in A_i$ and $\delta \in A_j$ for some $i, j \in I$. Then $\alpha = k(\gamma) = k(\delta) \in A_i \cap A_j$, and so $c_i(\alpha) = c_j(\alpha)$. Hence $\gamma = \pi_i^{-1}(\alpha) = \pi_j^{-1}(\alpha) = \delta$, as required. Thus $k \in \mathrm{Sym}(\bigcup_{i \in I} A_i)$. Let $k \subseteq g \in \mathrm{Sym}(\kappa)$. Then g satisfies our requirements. □

DEFINITION 2.3 [2]. A notion of forcing $\mathbb{P}$ is *κ-strategically complete* if for all $\alpha < \kappa$, Player II has a winning strategy in the following game of length α. Players I and II alternately choose a decreasing sequence p_β, $\beta < \alpha$, of elements of $\mathbb{P}$, where Player I chooses at the even ordinals and Player II at the odd ordinals. Player I wins either if for some $\beta < \alpha$ there is no legal move or if the sequence p_β, $\beta < \alpha$, has no lower bound.

In [2], it is noted that if $\mathbb{P}$ is κ-strategically complete, then $\mathbb{P}$ does not adjoin any new sets S of ordinals such that $|S| < \kappa$. Also if we iterate κ-strategically closed notions of forcing with supports closed under the union of fewer than κ sets, then the resulting notion of forcing is also κ-strategically complete.

The following result was proved in §2 of [2].

THEOREM 2.4. *Let $M \models ZFC$. Suppose that κ, $\mu \in M$ are cardinals such that $\kappa^{<\kappa} = \kappa$ and $\mu > \kappa$. Then there exists a κ-strategically complete notion of forcing $\mathbb{P}_{\kappa,\mu}$ with the κ^+-c.c. such that if G is $\mathbb{P}_{\kappa,\mu}$-generic, then in $M[G]$ there exists a sequence $\mathscr{A} = \langle A_i \mid i < \mu \rangle$ of elements of $[\kappa]^\kappa$ with the uniformization property. Furthermore, $|\mathbb{P}_{\kappa,\mu}| = 2^\mu$.* □

For the rest of this section, $\mathbb{P}_{\kappa,\mu}$ denotes the actual notion of forcing which is defined in §2 of [2].

PROOF OF THEOREM 1.1. Let $V \models GCH$. Let $\kappa, \theta, \lambda \in V$ be cardinals such that κ and θ are regular uncountable, and $\kappa < \theta \leq \mathrm{cf}(\lambda)$. Let $\mathbb{R}$ be the notion of forcing consisting of all partial functions $p : \lambda \to 2$ such that $|p| < \kappa$. Let H be $\mathbb{R}$-generic and let $V_1 = V[H]$. From now on, we will work in V_1. In particular, we have that $\kappa^{<\kappa} = \kappa$ and $\lambda^\mu = \lambda$ for all $\kappa \leq \mu < \mathrm{cf}(\lambda)$. Define a sequence $\langle \mu_i \mid i < \theta \rangle$ of cardinals as follows. If θ is a limit cardinal, let $\langle \mu_i \mid i < \theta \rangle$ be an increasing sequence of cardinals such that $\kappa < \mu_i < \theta$ and $\sup_{i<\theta} \mu_i = \theta$. If $\theta = \mu^+$ is a successor cardinal, define $\mu_i = \mu$ for all $i < \theta$. Now define a $<\kappa$-support iteration $\langle \mathbb{P}_i, \widetilde{\mathbb{Q}}_i \mid i < \theta \rangle$ as follows. Assume that $\mathbb{P}_i$ has been defined. Then, working inside $V^{\mathbb{P}_i}$, set $\widetilde{\mathbb{Q}}_i = \widetilde{\mathbb{P}}_{\kappa,\mu_i}$. Then $\mathbb{P}_i$ is κ-strategically complete for each $i \leq \theta$. This implies that $\mathbb{P}_i$ does not adjoin any new sets S of ordinals such that $|S| < \kappa$. Hence, arguing as in the proofs Lemmas 1.1 and 1.2 of [2], it can be shown that $\mathbb{P}_i$ is κ^+-c.c. for each $i \leq \theta$. Let G be $\mathbb{P}_\theta$-generic, and let $G_i = G \cap \mathbb{P}_i$ for each $i < \theta$. From now on, we will work inside $V_1[G]$. (Note that $V_1[G] \models \kappa^{<\kappa} = \kappa$. This will enable us to apply Lemma 2.2 later in the proof.)

For each $i < \theta$, let $\Gamma_i = \mathrm{Sym}(\kappa) \cap V_1[G_i]$. Then each Γ_i is a proper subgroup of $\mathrm{Sym}(\kappa)$, and $\mathrm{Sym}(\kappa) = \bigcup_{i<\theta} \Gamma_i$. Thus $c(\mathrm{Sym}(\kappa)) \leq \theta$. Suppose that $c(\mathrm{Sym}(\kappa)) = \chi < \theta$. Then we can express $\mathrm{Sym}(\kappa) = \bigcup_{i<\chi} H_i$ as the union of a chain of χ proper subgroups. Fix a moiety A of κ. By Lemma 2.4 of [1], for each $i < \chi$ there exists $\pi_i \in \mathrm{Sym}(\kappa)$ such that $\varphi \upharpoonright A \neq \pi_i^A$ for all $\varphi \in H_i$. Let $\Pi = \langle \pi_i \mid i < \chi \rangle$.

Claim 2.5. *Suppose that B is any moiety of κ. Then there exists $j_B < \chi$ such that if $j_B \leq i < \chi$, then $\varphi \restriction B \neq \pi_i^B$ for all $\varphi \in H_i$.*

Proof of Claim 2.5. There exists $\psi \in \mathrm{Sym}(\kappa)$ such that $\psi \restriction A$ is an order-preserving bijection between A and B. Clearly $j_B = \min\{j \mid \psi \in H_j\}$ satisfies our requirements. □

Let $\alpha < \theta$ be a successor ordinal such that $\Pi \in V_1[G_\alpha]$ and $\mu_\alpha \geq \chi$. There exists a sequence $\mathscr{A} = \langle A_i \mid i < \mu_\alpha \rangle \in V_1[G_\alpha]$ of elements of $[\kappa]^\kappa$ with the uniformization property. Note that each A_i must be a moiety of κ. (For suppose that $|\kappa \backslash A_j| < \kappa$ for some $j < \mu_\alpha$. Then $|A_j \cap A_i| = \kappa$ for all $i < \mu_\alpha$. Define $c_j : A_j \to \kappa$ by $c_j(\alpha) = 1$ for all $\alpha \in A_j$. For each $i < \mu_\alpha$ such that $i \neq j$, define $c_i : A_i \to \kappa$ by $c_i(\alpha) = 0$ for all $\alpha \in A_i$. Then clearly $\langle c_i \mid i < \mu_\alpha \rangle$ cannot be uniformized.) Let $\langle j_{A_i} \mid i < \chi \rangle \in V_1[G]$ be the sequence of ordinals $j_{A_i} < \chi$ given by Claim 2.5. Since $\mathbb{P}_\theta$ is κ^+-c.c., there exists a sequence $\langle F(i) \mid i < \chi \rangle \in V_1$ such that $F(i) \in [\chi]^\kappa$ and $j_{A_i} \in F(i)$ for all $i < \chi$. For each $i < \chi$, let $f(i) = \sup F(i) \geq j_{A_i}$. Then $f \in {}^\chi\chi \cap V_1$. Thus we can define the sequence $\langle \varphi_i^{A_i} \mid i < \chi \rangle \in V_1[G_\alpha]$ by $\varphi_i^{A_i} = \pi_{k_i}^{A_i}$, where $k_i = \max\{i, f(i)\}$. Note that for all $i < \chi$, $\varphi \restriction A_i \neq \varphi_i^{A_i}$ for all $\varphi \in H_i$. By Lemma 2.2, there exist $g \in \mathrm{Sym}(\kappa) \cap V_1[G_\alpha]$ and $I \in [\chi]^\chi$ such that $g \restriction A_i = \varphi_i^{A_i}$ for all $i \in I$. By considering an $i \in I$ such that $g \in H_i$, we obtain a contradiction. Thus $c(\mathrm{Sym}(\kappa)) = \theta$. □

The following observation shows that a different approach is needed to answer Question 1.2 for cardinals κ such that $\mathrm{cf}(\kappa) = \omega$, and to answer Question 1.8.

Proposition 2.6. *Suppose that κ is an uncountable cardinal such that $\kappa^\omega > \kappa$. If $\mathscr{A} = \langle A_i \mid i < \kappa^+ \rangle$ is a sequence of elements of $[\kappa]^\kappa$, then $\mathscr{A}$ does not have the uniformization property.*

Proof. Suppose that $\mathscr{A}$ has the uniformization property. Let $\langle f_i \mid i < \kappa^+ \rangle$ be a sequence of κ^+ distinct elements of ${}^\omega\kappa$. For each $i < \kappa^+$, let $\{\alpha_\zeta^i \mid \zeta < \kappa\}$ be the increasing enumeration of A_i. For each $i < \kappa^+$, define $c_i : A_i \to \kappa \times \kappa$ by $c_i(\alpha_\zeta^i) = \langle \alpha_{\zeta+1}^i, f_i(n) \rangle$, where $\zeta = \lambda + n$ for some limit ordinal λ. Then there exists a function $h : \bigcup_{i<\kappa^+} A_i \to \kappa \times \kappa$ such that for all $i < \kappa^+$ there exists $\beta_i < \kappa$ such that $h(\alpha) = c_i(\alpha)$ for all $\beta_i \leq \alpha \in A_i$. There exist $\beta < \kappa$ and $I \in [\kappa^+]^{\kappa^+}$ such that $\beta_i = \beta$ for all $i \in I$. Let λ be a limit ordinal such that $\beta < \lambda < \kappa$. Then there exist distinct ordinals $i, j \in I$ such that $\alpha_\lambda^i = \alpha_\lambda^j$. This implies that

$$\langle \alpha_{\lambda+1}^i, f_i(0) \rangle = c_i(\alpha_\lambda^i) = c_j(\alpha_\lambda^j) = \langle \alpha_{\lambda+1}^j, f_j(0) \rangle.$$

Continuing in this fashion, we obtain that $f_i = f_j$, which is a contradiction. □

§**3. In search of an Easton-type theorem.** In this section, we shall prove Theorems 1.4 and 1.7.

Lemma 3.1. *Let $V \models GCH$. Let $\kappa, \lambda \in V$ be regular cardinals such that $\kappa < \lambda$. Then there exists a κ^+-c.c. notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that if G is $\mathbb{P}$-generic then $V[G] \models c(\mathit{Sym}(\kappa)) = \lambda^{++}$.*

Proof. For $\kappa > \omega$, this was shown in the proof of Theorem 1.1. So suppose that $\kappa = \omega$. There exists a c.c.c. notion of forcing $\mathbb{P}$ such that if G is $\mathbb{P}$-generic, then $V[G] \models \mathrm{MA} + 2^\omega = \lambda^{++}$. By [4], MA implies that $c(\mathrm{Sym}(\omega)) = 2^\omega$. □

DEFINITION 3.2. Let λ be an infinite cardinal.

(i) If $f, g \in {}^{\lambda}\lambda$, then $f \leq^* g$ iff there exists $\alpha_o \in \lambda$ such that $f(\beta) \leq g(\beta)$ for all $\alpha_o \leq \beta < \lambda$.

(ii) A family $F \subseteq {}^{\lambda}\lambda$ is *dominating* if for every $g \in {}^{\lambda}\lambda$, there exists $f \in F$ such that $g \leq^* f$.

(iii) d_λ is the minimal cardinality of a dominating family F of ${}^{\lambda}\lambda$.

LEMMA 3.3. $c(Sym(\lambda)) \leq d_\lambda$.

PROOF. Let $\mu = d_\lambda$ and let $F = \{f_i \mid i < \mu\}$ be a dominating family. We may assume that each f_i is strictly increasing. For each $\theta < \mu$, define

$$G_\theta = \langle g \in \mathrm{Sym}(\lambda) \big| \text{ There exist } i, j < \theta \text{ such that } g \leq^* f_i \text{ and } g^{-1} \leq^* f_j \rangle.$$

Then $\mathrm{Sym}(\lambda) = \bigcup_{\theta<\mu} G_\theta$. Arguing as in the proof of Proposition 1.4 of [3], we can easily see that each G_θ is a proper subgroup. □

PROOF OF THEOREM 1.4. Let $\mathbb{P}$ be the notion of forcing given by Lemma 3.1. Let $F = {}^{\lambda}\lambda \cap V$. Since $V \vDash \mathrm{GCH}$, $|F| = \lambda^+$. Since $\mathbb{P}$ is κ^+-c.c., for each $h \in {}^{\lambda}\lambda \cap V[G]$ there exists a sequence $\langle H(i) \mid i < \lambda \rangle$ such that $H(i) \in [\lambda]^\kappa$ and $h(i) \in H(i)$ for all $i < \lambda$. It follows that $V[G] \vDash F$ is a dominating family in ${}^{\lambda}\lambda$. By Lemma 3.3, $V[G] \vDash c(\mathrm{Sym}(\lambda)) \leq |F| < \lambda^{++} = c(\mathrm{Sym}(\kappa))$. □

The rest of this section will be devoted to the proof of Theorem 1.7. Let $\kappa < \lambda$ be infinite cardinals, and suppose that $c(\mathrm{Sym}(\kappa)) = \theta > c_\omega(\kappa, \lambda)$. Let $\mathrm{Sym}(\kappa) = \bigcup_{i<\theta} G_i$, where each G_i is a proper subgroup. From now on, let C be a fixed ω-closed unbounded subset of $[\lambda]^\kappa$ such that $|C| = c_\omega(\kappa, \lambda)$. Also, for each $T \in C$, fix a bijection $f_T : T \to \kappa$.

Convention 3.4. If $\Omega \subseteq \lambda$, then we identify $\mathrm{Sym}(\Omega)$ with the subgroup $\{g \in \mathrm{Sym}(\lambda) | g(\alpha) = \alpha \text{ for all } \alpha \in \lambda \backslash \Omega\}$ of $\mathrm{Sym}(\lambda)$. In particular, we regard $\mathrm{Sym}(\kappa)$ as a subgroup of $\mathrm{Sym}(\lambda)$.

DEFINITION 3.5. For each $\varphi \in \mathrm{Sym}(\kappa)$ and $T \in C$, we define $\varphi_T \in \mathrm{Sym}(T)$ by

$$\varphi_T \restriction T = f_T^{-1} \circ \varphi \circ f_T.$$

DEFINITION 3.6. For each $i < \theta$, H_i is the *set* of all elements $\pi \in \mathrm{Sym}(\lambda)$ such that for some ω-closed unbounded subset $D \subseteq C$ of $[\lambda]^\kappa$, for all $T \in D$,

(i) $\pi[T] = T$, and

(ii) there exists $\varphi \in G_i$ such that $\pi \restriction T = \varphi_T \restriction T$.

LEMMA 3.7. *For each $i < \theta$, H_i is a subgroup of $Sym(\lambda)$.*

PROOF. Left to the reader. □

LEMMA 3.8. *For each $\pi \in Sym(\lambda)$, there exists $i < \theta$ such that $\pi \in H_i$.*

PROOF. There exists an ω-closed unbounded subset $D \subseteq C$ of $[\lambda]^\kappa$ such that $\pi[T] = T$ for all $T \in D$. For each $T \in D$, there exists $i_T < \theta$ such that $\pi \restriction T = \varphi_T \restriction T$ for some $\varphi \in G_{i_T}$. Note that $\theta = c(\mathrm{Sym}(\kappa))$ is regular. Since $|D| = c_\omega(\kappa, \lambda) < \theta$, it follows that $\sup_{T \in D} i_T < \theta$. □

Clearly $H_i \subseteq H_j$ for all $i < j < \theta$. So the following lemma completes the proof of Theorem 1.7.

LEMMA 3.9. *For each $i < \theta$, H_i is a proper subgroup of $Sym(\lambda)$.*

PROOF. Suppose that $H_i = \mathrm{Sym}(\lambda)$ for some $i < \theta$. Let

$$C^* = \{T \in C \mid \kappa \subset T, |T \backslash \kappa| = \kappa\}.$$

For each $T \in C^*$, let $X_T = f_T[\kappa]$. For each pair of elements $S, T \in C^*$, there exists $\Pi_{S,T} \in \mathrm{Sym}(\kappa)$ such that

$$f_T \circ f_S^{-1} \restriction X_S = \Pi_{S,T} \restriction X_S.$$

In particular, $\Pi_{S,T}[X_S] = X_T$. Since $|C^*| < \theta$, we can assume that $\Pi_{S,T} \in G_i$ for all $S, T \in C^*$.

Now fix some $R \in C^*$. Let $\varphi \in \mathrm{Sym}(X_R)$. (Remember that we are using Convention 3.4 during this proof. Thus $\varphi \in \mathrm{Sym}(X_R)$ means that φ is a permutation of λ such that $\varphi(\alpha) = \alpha$ for all $\alpha \in \lambda \setminus X_R$.) We shall show that there exists $\sigma \in G_i$ such that $\sigma \restriction X_R = \varphi \restriction X_R$. Let $\pi = \varphi_R$. Then $\pi \in \mathrm{Sym}(\kappa)$. Since $\pi \in H_i$, there exist $T \in C^*$ and $\psi \in G_i$ such that $\pi \restriction T = \psi_T \restriction T$. Clearly $\psi \in \mathrm{Sym}(X_T)$. Let $\sigma = \Pi_{T,R} \circ \psi \circ \Pi_{T,R}^{-1}$. Then

$$\begin{aligned}\sigma \restriction X_R &= f_R \circ f_T^{-1} \circ \psi \circ f_T \circ f_R^{-1} \restriction X_R = f_R \circ \pi \circ f_R^{-1} \restriction X_R \\ &= f_R \circ f_R^{-1} \circ \varphi \circ f_R \circ f_R^{-1} \restriction X_R = \varphi \restriction X_R.\end{aligned}$$

Thus we have shown that the setwise stabilizer of X_R in G_i induces $\mathrm{Sym}(X_R)$ on X_R. By Lemma 2.4 of [1], there exists $g \in \mathrm{Sym}(\kappa)$ such that $\mathrm{Sym}(\kappa) = \langle G_i, g \rangle$. Let $g \in G_j$, where $i < j < \theta$. Then $G_j = \mathrm{Sym}(\kappa)$, which is a contradiction. □

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