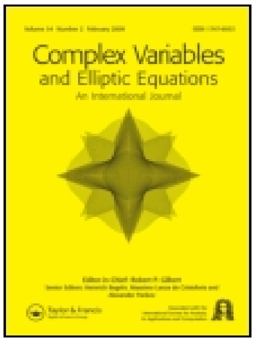

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Parametric Representation of a Class of Multiply Positive Sequences

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1. INTRODUCTION

Multiply positive sequences were introduced by Fekete [1] in 1912 for study of zeros of real polynomials and entire functions. Since then these sequences have been studied by several mathematicians and have found several applications in Analysis (see, e.g. [2,3]).

Recall that the sequence

$$\{a_n\}_{n=0}^{\infty}, \quad a_0 > 0, \quad a_n \geq 0 \; (n = 1, 2, \ldots), \tag{1.1}$$

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is said to be *m-times positive* for $m \in N \cup \{\infty\}$, if all minors of orders $< m+1$ of the infinite matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ . & . & . & . & \cdots \end{pmatrix}$$

are non-negative. The class of all m-times positive sequences is denoted by PF_m.

Evidently, the class PF_1 consists of all sequences (1.1). The class PF_2 consists of all sequences of the form

$$a_n = \exp\{-\psi(n)\}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $\psi : N \cup \{0\} \to (-\infty, +\infty]$, $\psi(0) < \infty$, is a convex function. In 1953, in joint works by Aissen, Edrei, Schoenberg, Whitney (see [2], p. 412) the complete solution of the problem of description of the class PF_∞ was found:

THEOREM AESW. *The function $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ is a generating function of a sequence of PF_∞ if and only if*

$$f(z) = a_0 \exp\{\gamma z\} \frac{\prod_{i=1}^{\infty}(1 + p_i z)}{\prod_{i=1}^{\infty}(1 - q_i z)}, \tag{1.3}$$

where

$$a_0 > 0, \quad \gamma \geq 0, \quad p_i \geq 0, \quad q_i \geq 0 \quad (i = 1, 2, 3, \ldots),$$
$$\sum_{i=1}^{\infty}(p_i + q_i) < \infty. \tag{1.4}$$

Note that (1.3) gives the description of PF_∞ in terms of *independent* parameters a_0, γ, p_i, q_i $(i = 1, 2, 3, \ldots)$. This means: (i) arbitrary values of the parameters can be chosen under the only conditions (1.4); (ii) there is a one-to-one correspondance between collections of these parameters and sequences of PF_∞.

To the best of our knowledges, the problem of the description of the classes PF_m, $3 \leq m < \infty$, in terms of independent parameters has not been solved until now. It is not clear even which kind of parameters could play the corresponding role (similar to that of a_0, p_i, q_i and γ in Theorem AESW).

This paper is devoted to description of a *subclass* $Q_3 \subset PF_3$ in terms of independent parameters. The subclass was for the first time considered by the authors [5] in connection with a generalization of Pólya's theorem on sections of power series.

DEFINITION We say that a sequence (1.1) of PF_3 belongs to Q_3 if all truncated sequences $\{a_n\}_{n=0}^k = \{a_0, a_1, a_2, \ldots, a_k, 0, 0, \ldots\}$, $k \in N$, belong to PF_3. (Note that in [5] we denoted by Q_3 the class of all corresponding generating functions.)

It turns out that the role of independent parameters describing Q_3 play points of the set

$$(0, \infty) \times [0, \infty) \times \mathcal{U}, \tag{1.5}$$

where $\mathcal{U}$ is the set of all sequences $\{\alpha_k\}_{k=2}^{\infty}$, $0 \leq \alpha_k \leq 1$, $k = 2, 3, \ldots$, such that if $\alpha_j = 0$ for some j, then $\alpha_k = 0$ for any $k \geq j$. To give a precise statement of our result, we define the numbers

$$\begin{gathered}[\alpha_2] = 1 + \alpha_2, \quad [\alpha_2\alpha_3] = 1 + \alpha_3\sqrt{[\alpha_2]}, \\ [\alpha_2\alpha_3\alpha_4] = 1 + \alpha_4\sqrt{[\alpha_2\alpha_3]}, \cdots \\ [\alpha_2\alpha_3\cdots\alpha_n] = 1 + \alpha_n\sqrt{[\alpha_2\alpha_3\cdots\alpha_{n-1}]}, \ldots \end{gathered} \tag{1.6}$$

Our main result is the following theorem.

THEOREM 1. *A sequence* (1.1) *belongs to* Q_3 *if and only if*

$$a_1 = a_0\alpha, \quad a_n = \frac{a_0\alpha^n\alpha_2^{n-1}\alpha_3^{n-2}\cdots\alpha_{n-1}^2\alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2}\cdots[\alpha_2\alpha_3\cdots\alpha_{n-1}]^{3/2}[\alpha_2\alpha_3\cdots\alpha_n]}, \tag{1.7}$$

where

$$a_0 > 0, \quad \alpha \geq 0, \quad \{\alpha_n\}_{n=2}^{\infty} \in \mathcal{U}. \tag{1.8}$$

Thus, the independent parameters are $a_0, \alpha, \alpha_2, \alpha_3, \ldots$. The only restrictions on them are (1.8) i.e. belonging of the point $(a_0, \alpha, \alpha_2, \ldots)$ to the set (1.5).

Since Q_3 is a subclass of PF_2, any sequence $\{a_k\}_{k=2}^{\infty}$ of Q_3 admits the representation (1.2). The question arises what additional conditions on the convex function ψ imply that $\{a_k\}_{k=0}^{\infty}$ belongs to Q_3.

Denote

$$N = \min\{n : \psi(n) = +\infty\} \quad (N = +\infty \text{ if } \psi(n) < +\infty, \forall n \in N). \tag{1.9}$$

Set

$$\Delta_2\psi(n) := \begin{cases} \psi(n) - 2\psi(n-1) + \psi(n-2), & 2 \le n < N, \\ +\infty, & n \ge N \text{ (if } N < +\infty). \end{cases}$$

Define a sequence $\{w_n\}_{n=2}^{\infty}$, putting

$$w_2 = 1, w_n = [\alpha_2\alpha_3 \cdots, \alpha_{n-1}]_{\alpha_2=\alpha_3=\cdots=\alpha_{n-1}=1}, \ n \ge 3.$$

Evidently, the sequence $\{w_n\}_{n=2}^{\infty}$ is increasing and, for any $\{\alpha_n\}_{n=2}^{\infty} \in \mathcal{U}$, we have

$$1 \le [\alpha_2\alpha_3 \cdots \alpha_{n-1}] \le w_n < \lim_{n\to\infty} w_n = 1 + c,$$
$$c := \frac{1+\sqrt{5}}{2} = 1.613\cdots \tag{1.10}$$

THEOREM 2. *For a sequence* (1.2) *to belong to* Q_3 *it is necessary and sufficient that*

$$\Delta_2\psi(n) \ge \log\left(1 + \frac{1}{\sqrt{w_n}}\right), \quad n \ge 2. \tag{1.11}$$

The equality holds in (1.11) *for every* $n \ge 2$ *for the sequence* (1.2) *given by* (1.7) *with* $\alpha_2 = \alpha_3 = \cdots = 1$.

Evidently, the bound in the right hand side of (1.11) has the following properties: (i) it is contained in the half-closed interval $(\log c, \log 2]$, (ii) it is equal to $\log 2$ for $n = 2$, (iii) it tends to $\log c$ as $n \to \infty$. Therefore, using the second assertion of Theorem 2, we obtain the following corollary.

COROLLARY 1. *For a sequence* (1.2) *to belong to* Q_3 *it is necessary to have*

$$\Delta_2\psi(n) > \log c, \quad n \ge 2,$$

and it is sufficient to have

$$\Delta_2\psi(n) \geq \log 2, \quad n \geq 2. \tag{1.12}$$

Both above conditions are unimprovable in the following sense: the first one ceases to be necessary if one replaces $\log c$ *by a larger constant; the second one ceases to be sufficient if one replaces* $\log 2$ *by a smaller constant.*

Note the following immediate corollary of Theorem 2.

COROLLARY 2. *Let* $\psi : [0, +\infty) \to (-\infty, +\infty)$ *be a convex function of* $C^2[0, \infty)$.

(i) If for all sufficiently large x *we have* $\psi''(x) < \log c$, *then the sequence* (1.2) *does not belong to* Q_3.

(ii) If for all $x \geq 0$ *we have* $\psi''(x) \geq \log 2$, *then the sequence* (1.2) *belongs to* Q_3.

Theorem 2 and Corollaries 1, 2 can be used to determine whether a given sequence belongs to Q_3.

Example Consider the sequence

$$A(\beta, d) := \{\exp(-dn^\beta)\}_{n=0}^{\infty}, \quad \beta > 0, \ d > 0.$$

If $\beta \geq 2$, then the function $\psi(x) := dx^\beta$ satisfies $\Delta_2\psi(n) \geq \Delta_2\psi(2) = d(2^\beta - 2)$ for $n \geq 2$. By Corollary 1, (1.12), we have $A(\beta, d) \in Q_3$ for $d \geq (\log 2)/(2^\beta - 2)$. By Theorem 2, (1.11) with $n = 2$, we have $A(\beta, d) \notin Q_3$ for $d < (\log 2)/(2^\beta - 2)$. If $\beta < 2$, then $\psi''(x) \to 0$, as $x \to \infty$. By Corollary 2 we have $A(\beta, d) \notin Q_3$.

Theorem 2 allows us to describe the possible character of the growth of ψ for sequences (1.2) belonging to Q_3.

Define the *lower order* $\lambda[\psi]$ and the *order* $\rho[\psi]$ of ψ as

$$\lambda[\psi] = \liminf_{n\to\infty} \frac{\log \psi(n)}{\log n}, \quad \rho[\psi] = \limsup_{n\to\infty} \frac{\log \psi(n)}{\log n}.$$

Evidently, $\lambda[\psi] \leq \rho[\psi]$. Since, by (1.12),

$$\psi(n) \geq \frac{n^2}{2}\log c + O(n), \quad n \to \infty,$$

we have $\lambda[\psi] \geq 2$. The question arises about description of the pairs $(\lambda[\psi]; \rho[\psi])$, for which the sequence (1.2) belongs to Q_3. Using Theorem 2, we prove that $2 \leq \lambda[\psi] \leq \rho[\psi] \leq \infty$ is the only restriction.

THEOREM 3. *For any pair of numbers (a, b), $2 \leq a \leq b \leq \infty$, there is a sequence* (1.2) *of Q_3 such that for the corresponding function ψ we have $\lambda[\psi] = a$, $\rho[\psi] = b$.*

2. PROOF OF THEOREM 1

Recall two lemmas and some notation from [5].

LEMMA 1. [5] *Let* (1.1) *be a sequence of PF_2. If $N < \infty$, where N is defined by* (1.9), *then $a_n = 0$ for any $n \geq N$.*

For sequences (1.1) belonging to PF_2, Lemma 1 allows us to introduce the numbers

$$\rho_n = \frac{a_{n-1}}{a_n}, \quad 1 \leq n < N; \qquad \delta_n = \frac{\rho_n}{\rho_{n-1}}, \quad 1 \leq n < N.$$

Evidently,

$$a_n = \frac{a_0}{\prod_{j=1}^{n} \rho_j},$$

and

$$a_n = \frac{a_0 \alpha^n}{\delta_2^{n-1} \delta_3^{n-2} \cdots \delta_n}, \quad 1 \leq n < N, \tag{2.1}$$

where $\alpha = (1/\rho_1) = (a_1/a_0)$.

LEMMA 2. [5] *Let $\{a_k\}_{k=0}^{n} = \{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}$, $a_0 > 0$, $a_n > 0$, $n \geq 2$ be a sequence of PF_3. Then*
1) for $n = 2$, we have $\delta_2 \geq 2$,
2) for $n \geq 3$, we have $\delta_n > 1$ and

$$(\delta_n - 1)^2 \geq 1 - \frac{1}{\delta_{n-1}}.$$

We start the proof of Theorem 1.

(i) Let (1.1) be a sequence of Q_3. Let us show that the representation (1.7) is valid. By Lemma 2 the numbers

$$y_n := \frac{1}{\delta_n - 1}, \quad 2 \le n < N, \tag{2.2}$$

are well-defined and the following inequalities hold:

$$\begin{cases} y_n^2 \le y_{n-1} + 1, & 3 \le n < N; \\ y_2 \le 1. \end{cases} \tag{2.3}$$

Define the parameters

$$\alpha := \frac{a_1}{a_0}, \quad \alpha_n := \begin{cases} y_2, & n = 2; \\ \dfrac{y_n}{\sqrt{1 + y_{n-1}}}, & 3 \le n < N; \\ 0, & n \ge N, \end{cases} \tag{2.4}$$

which will take part in (1.7). From (2.3), (2.4) it follows $\{\alpha_n\}_{n=2}^{\infty} \in \mathcal{U}$. From (2.4) we obtain

$$y_n = \alpha_n \sqrt{1 + \alpha_{n-1}\sqrt{1 + \cdots + \alpha_3\sqrt{1 + \alpha_2}}}, \quad 2 \le n < N.$$

Hence, by (2.2),

$$\delta_n = 1 + \frac{1}{\alpha_n \sqrt{1 + \alpha_{n-1}\sqrt{1 + \cdots + \alpha_3\sqrt{1 + \alpha_2}}}}, \quad 2 \le n < N.$$

Using (2.1), we have for $2 \le n < N$

$$a_n = \frac{a_0 \alpha^n}{\left(1 + \frac{1}{\alpha_2}\right)^{n-1} \left(1 + \frac{1}{\alpha_3\sqrt{1+\alpha_2}}\right)^{n-2} \cdots \left(1 + \frac{1}{\alpha_n \sqrt{1 + \alpha_{n-1}\sqrt{1 + \cdots + \alpha_3\sqrt{1+\alpha_2}}}}\right)}$$

$$= \frac{a_0 \alpha^n \alpha_2^{n-1} \alpha_3^{n-2} \cdots \alpha_n}{(1+\alpha_2)^{n/2}(1 + \alpha_3\sqrt{1+\alpha_2})^{(n-1)/2} \cdots (1 + \alpha_n\sqrt{1 + \cdots + \alpha_3\sqrt{1+\alpha_2}})}$$

and $a_n = 0$ for $n \ge N$. Taking into account notation (1.6), we obtain (1.7).

(ii) Take any sequence $\{\alpha_n\}_{n=2}^{\infty} \in \mathcal{U}$ and any numbers $a_0 > 0$, $\alpha \geq 0$, and form the sequence $\{a_k\}_{k=0}^{\infty}$ according to (1.7). Let us prove that $\{a_k\}_{k=0}^{\infty} \in Q_3$.

Let $i := \min\{n : \alpha_n = 0\}$ ($i := \infty$ if $\alpha_n \neq 0$, $\forall n \geq 2$). Then we have $a_n > 0$ for $0 \leq n < i$, and $a_n = 0$ for $n \geq i$. Form the sequence

$$y_n := \begin{cases} \alpha_n\sqrt{1+y_{n-1}}, & 3 \leq n < i, \\ \alpha_2, & n = 2. \end{cases} \tag{2.5}$$

Since $0 \leq \alpha_n \leq 1$, $n \geq 2$, the following inequalities hold:

$$\begin{cases} y_n \leq \sqrt{1+y_{n-1}}, & 3 \leq n < i, \\ y_2 \leq 1. \end{cases} \tag{2.6}$$

Denote

$$\begin{cases} \delta_n = 1 + 1/y_n, & 2 \leq n < i, \\ \delta_n = \infty, & n \geq i. \end{cases} \tag{2.7}$$

Then we have

$$a_n = \frac{a_0\alpha^n}{\delta_2^{n-1}\delta_3^{n-2}\cdots\delta_n}, \quad n \geq 2.$$

The proof of $\{a_n\}_{n=0}^{\infty} \in Q_3$ is very close to that of Lemma 5 of [5]. Nevertheless, we will present the proof for the reader's convenience.

As in [5], we use the following theorem of Schoenberg.

THEOREM ([4]) *Let $\{b_n\}_{n=0}^{k}$ be a finite sequence of numbers. Consider m matrices*

$$B_n = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_k & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{k-1} & b_k & 0 & \cdots & 0 \\ 0 & 0 & b_0 & \cdots & b_{k-2} & b_{k-1} & b_k & \cdots & 0 \\ . & . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & . & . & . & \cdots & b_k \end{pmatrix}, \quad n = 1, 2, \ldots, m,$$

where B_n consists of n rows and $k+n$ columns. Assume the following condition is satisfied for $n = 1, 2, \ldots, m$: all $n \times n$-minors of B_n, consisting of consecutive columns are strictly positive. Then the sequence $\{b_0, b_1, \ldots, b_k, 0, 0 \ldots\}$ is m-times positive.

Fix any k, $2 \leq k < i$, and consider the sequence

$$\{a_0, a_1, \ldots, a_{k-1}, a_k - \varepsilon, 0, 0, \ldots\}, \tag{2.8}$$

where $\varepsilon > 0$ will be chosen sufficiently small later. Form three matrices

$$A_1 = (a_0 \quad a_1 \quad \cdots \quad a_{k-1} \quad a_k - \varepsilon),$$

$$A_2 = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} & a_k - \varepsilon & 0 \\ 0 & a_0 & a_1 & \cdots & a_{k-2} & a_{k-1} & a_k - \varepsilon \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} & a_k - \varepsilon & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_{k-2} & a_{k-1} & a_k - \varepsilon & 0 \\ 0 & 0 & a_0 & \cdots & a_{k-3} & a_{k-2} & a_{k-1} & a_k - \varepsilon \end{pmatrix}.$$

All minors of A_1 are evidently positive for $0 < \varepsilon < a_k$. Since

$$\frac{a_n}{a_{n+1}} = \alpha\delta_2\delta_3\cdots\delta_{n+1}, \quad 1 \le n < i-1, \quad \delta_n > 1, \tag{2.9}$$

we have

$$\frac{a_{n-1}}{a_n} < \frac{a_n}{a_{n+1}}, \quad n = 1, 2, \ldots, k-2; \quad \frac{a_{k-2}}{a_{k-1}} < \frac{a_{k-1}}{a_k - \varepsilon}. \tag{2.10}$$

for sufficiently small $\varepsilon > 0$. Therefore all minors of A_2 are positive for such ε.

Further, consider the determinants

$$N_n = \begin{vmatrix} a_n & a_{n+1} & 0 \\ a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & a_{n-1} & a_n \end{vmatrix}, \quad 2 \le n < i.$$

In virtue (2.9), (2.7) and (2.6), we have

$$\begin{aligned} N_n &= a_n^3 + a_{n+1}^2 a_{n-2} - 2a_{n+1}a_n a_{n-1} \\ &= a_{n+1}^2 a_{n-2}(\delta_{n+1}^2\delta_n + 1 - 2\delta_{n+1}\delta_n) \\ &= a_{n+1}^2 a_{n-2}\delta_n\left((\delta_{n+1} - 1)^2 + \frac{1}{\delta_n} - 1\right) \\ &= a_{n+1}^2 a_{n-2}\delta_n\left(\frac{1}{y_{n+1}^2} - \frac{1}{y_n + 1}\right) \ge 0, \end{aligned}$$

where $2 \le n < i-1$. Moreover, (2.7) and (2.6) yield non-negativity of the minor

$$N_1 = \begin{vmatrix} a_1 & a_2 & 0 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} = a_0a_1a_2\delta_2\left(1 - \frac{2}{\delta_2}\right) = a_0a_1a_2\delta_2(1 - y_2).$$

Now, consider 3×3-minors of A_3 consisting of consecutive columns:

$$M_0 = \begin{vmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{vmatrix}, \quad M_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix},$$

$$M_n = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} \\ a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & a_{n-1} & a_n \end{vmatrix}, \quad n = 2, 3, \ldots, k-3;$$

$$M_{k-2}(\varepsilon) = \begin{vmatrix} a_{k-2} & a_{k-1} & a_k - \varepsilon \\ a_{k-3} & a_{k-2} & a_{k-1} \\ a_{k-4} & a_{k-3} & a_{k-2} \end{vmatrix},$$

$$M_{k-1}(\varepsilon) = \begin{vmatrix} a_{k-1} & a_k - \varepsilon & 0 \\ a_{k-2} & a_{k-1} & a_k - \varepsilon \\ a_{k-3} & a_{k-2} & a_{k-1} \end{vmatrix},$$

$$M_k(\varepsilon) = \begin{vmatrix} a_k - \varepsilon & 0 & 0 \\ a_{k-1} & a_k - \varepsilon & 0 \\ a_{k-2} & a_{k-1} & a_k - \varepsilon \end{vmatrix}.$$

Positivity of M_0 and $M_k(\varepsilon)$ for $0 < \varepsilon < a_k$ is evident. Since

$$M_1 = N_1 + a_3 \begin{vmatrix} a_0 & a_1 \\ 0 & a_0 \end{vmatrix},$$

$$M_n = N_n + a_{n+2} \begin{vmatrix} a_{n-1} & a_n \\ a_{n-2} & a_{n-1} \end{vmatrix}, \quad n = 2, 3, \ldots, k-3;$$

$$M_{k-2}(\varepsilon) = N_{k-2} + (a_k - \varepsilon) \begin{vmatrix} a_{k-3} & a_{k-2} \\ a_{k-4} & a_{k-3} \end{vmatrix},$$

we have $M_n > 0$, $n = 1, 2, \ldots, k-3$, and $M_{k-2}(\varepsilon) > 0$. Further, since

$$M_{k-1}(0) = N_{k-1} \geq 0,$$

$$M'_{k-1}(0) = 2a_{k-1}a_{k-2} - 2a_k a_{k-3} = 2a_k a_{k-2} \left(\frac{a_{k-1}}{a_k} - \frac{a_{k-3}}{a_{k-2}} \right) > 0,$$

we have $M_{k-1} > 0$ for sufficiently small ε.

Applying Schoenberg's theorem, we conclude that the sequence (2.8) belongs to PF_3 for sufficiently small $\varepsilon > 0$. Taking a limit as $\varepsilon \to 0$, we obtain that the truncated sequence $\{a_n\}_{n=0}^{k} = \{a_0, a_1, \ldots, a_k, 0, 0, \ldots\}$ is 3-times positive and hence $\{a_n\}_{n=0}^{\infty} \in Q_3$.

3. PROOF OF THEOREM 2

Assume $\{a_k\}_{k=0}^{\infty} \in Q_3$. Then the representation (1.7) of Theorem 1 yields the following expression of the function ψ connected with $\{a_k\}_{k=0}^{\infty}$ by (1.2):

$$\psi(n) = -\log \frac{a_0 \alpha^n \alpha_2^{n-1} \alpha_3^{n-2} \cdots \alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2} \cdots [\alpha_2\alpha_3 \cdots \alpha_{n-1}]^{3/2}[\alpha_2\alpha_3 \cdots \alpha_n]}$$

$$= -\log \frac{a_0\alpha^n}{\left(1+\frac{1}{\alpha_2}\right)^{n-1}\left(1+\frac{1}{\alpha_3\sqrt{[\alpha_2]}}\right)^{n-2} \cdots \left(1+\frac{1}{\alpha_n\sqrt{[\alpha_2\alpha_3\cdots\alpha_{n-1}]}}\right)}$$

$$= -\log a_0 - n\log\alpha + (n-1)\log\left(1+\frac{1}{\alpha_2}\right) + (n-2)\log\left(1+\frac{1}{\alpha_3\sqrt{[\alpha_2]}}\right) + \cdots + \log\left(1+\frac{1}{\alpha_n\sqrt{[\alpha_2\alpha_3\cdots\alpha_{n-1}]}}\right). \tag{3.1}$$

This yields

$$\Delta_2\psi(2) = \log\left(1+\frac{1}{\alpha_2}\right),$$

$$\Delta_2\psi(n) = \log\left(1+\frac{1}{\alpha_n\sqrt{[\alpha_2\alpha_3\cdots\alpha_{n-1}]}}\right), \quad n \geq 3. \tag{3.2}$$

Using the definition of the numbers w_n and (1.10), we conclude that

$$\Delta_2\psi(n) \geq \log\left(1+\frac{1}{\alpha_n\sqrt{w_n}}\right) \geq \log\left(1+\frac{1}{\sqrt{w_n}}\right),$$

i.e. (1.11) is true.

Now, assume that (1.11) holds. Set

$$a_0 = \exp\{-\psi(0)\}, \quad a_1 = \exp\{-\psi(1)\}, \quad \alpha = \exp\{\psi(1) - \psi(0)\}$$

and show that the numbers $\alpha_2, \alpha_3, \ldots \in [0, 1]$ can be chosen such that the representation (1.7) is true.

Since by (1.11) we have $\Delta_2\psi(2) \geq \log 2$, the number α_2 defined by the first equality (3.2) belongs to $[0, 1]$. Further, we will define the numbers $\alpha_3, \alpha_4, \ldots$ inductively by means of the second equality (3.2). Since, for fixed $\alpha_2, \alpha_3 \cdots \alpha_{n-1}$ $(n \geq 3)$, the function $\log(1 + (1/\alpha_n\sqrt{[\alpha_2\alpha_3\cdots\alpha_{n-1}]}))$ as a function of $\alpha_n \in [0, 1]$ is decreasing and its range covers the closed half-ray $[\log 2, \infty]$, the number α_n is defined by (3.2) uniquely if $\alpha_2, \alpha_3, \ldots \alpha_{n-1}$ have been defined. Having defined $\{\alpha_k\}_{k=2}^{\infty} \in \mathcal{U}$, we have (3.2) and (3.1) for all $n \geq 2$. Inverting the calculations in (3.1), we see that the representation (1.7) holds. By Theorem 1, we conclude that $\{a_k\}_{k=0}^{\infty} \in Q_3$.

4. PROOF OF THEOREM 3

For $a = b = \infty$, it suffices to consider any sequence (1.8) with $\alpha_n = 0$, $n \geq N$, where N is any integer ≥ 1. For such a sequence, we have $\psi(n) = +\infty$ for $n \geq N$.

Now, let us assume $2 \leq a \leq b < \infty$. Consider an auxiliary function

$$\psi_0(x) = \exp\{\vartheta(x)\},$$

$$\vartheta(x) = \frac{1}{2}\{a + b + (b - a)\sin(\log\log\log x)\}\log x,$$

well-defined for $x > x_0 = e^e$. Evidently, $\vartheta(x) \geq a\log x \geq 2\log x$ for $x > x_0$, therefore $\psi_0(x) \geq x^2$, $x > x_0$. Moreover,

$$\lambda[\psi_0] = \liminf_{x\to\infty}\frac{\vartheta(x)}{\log x} = a, \quad \rho[\psi_0] = \limsup_{x\to\infty}\frac{\vartheta(x)}{\log x} = b.$$

Since

$$\vartheta'(x) = \frac{a + b + (b - a)\sin(\log\log\log x)}{2x} + o\left(\frac{1}{x}\right), \quad x \to +\infty,$$

$$\vartheta''(x) = -\frac{a + b + (b - a)\sin(\log\log\log x)}{2x^2} + o\left(\frac{1}{x^2}\right), \quad x \to +\infty,$$

we have

$$\psi_0'(x) = \vartheta'(x)\psi_0(x) \geq \left(\frac{a}{x} + o\left(\frac{1}{x}\right)\right)\psi_0(x), \quad x \to +\infty,$$

$$\psi_0''(x) = \{\vartheta''(x) + \vartheta'^2(x)\}\psi_0(x)$$

$$= \frac{1}{x^2}\left\{\frac{a+b+(b-a)\sin(\log\log\log x)}{2}\right.$$
$$\left.\times\left(\frac{a+b+(b-a)\sin(\log\log\log x)}{2}-1\right)+o(1)\right\}\psi_0(x)$$
$$\geq \frac{1}{x^2}\{a(a-1)+o(1)\}\psi_0(x).$$

Using the estimate $\psi_0 \geq x^2$, $x > x_0$, we obtain

$$\psi_0''(x) \geq a(a-1)+o(1) \geq 2+o(1) \geq 1$$

for sufficiently large x.

Choose a constant A so large that the function $\psi(x) := \psi_0(x+A)$ satisfies the condition $\psi''(x) \geq 1$ for all $x \geq 0$. Hence,

$$\Delta_2\psi(n) = \int_{n-1}^{n} dx \int_{x-1}^{x} \psi''(u)du \geq 1 > \log 2.$$

Corollary 1 shows that the sequence $a_k = \exp\{-\psi(k)\}$ belongs to Q_3. Evidently,

$$\lambda[\psi] = \lambda[\psi_0] = a, \quad \rho[\psi] = \rho[\psi_0] = b.$$

The case $2 \leq a < b = +\infty$ can be considered analogously by choosing

$$\psi_0(x) = \exp\{\vartheta(x)\},$$
$$\vartheta(x) = \{a + (\log\log\log x)\sin^2(\log\log\log x)\}\log x.$$

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