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WAVELET BASIS IN THE SPACE $C^\infty[-1,1]$

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Abstract. We show that the polynomial wavelets suggested by T. Kilgor and J. Prestin in [12] form a topological basis in the space $C^\infty[-1,1]$.

During the last twenty years wavelets have found a lot of applications in mathematics, physics and engineering. Our interest in wavelets is related to their ability to represent a function, not only in the corresponding Hilbert space, but also in other function spaces with perhaps quiet different topology. Wavelets form unconditional Schauder bases in Lebesgue spaces ([16], [8], see also [3] and [11]) and in the Hardy space ([23], [16]). Weighted spaces $L^p(w)$, $H^p(w)$ were considered in [4], [5]. For the multidimensional case, see also [19]. Wavelet topological bases were found in Sobolev spaces ([9], [2]) and in their generalizations, as in Besov ([1], [10]) and Triebel-Lizorkin ([14]) spaces. The list is far from being complete. Using “multiresolution analysis” of the space of continuous functions, Girgensohn and Prestin constructed in [6] (see also [18], [15] and [13]) a polynomial Schauder basis of optimal degree in the space $C[-1,1]$. Here we show that the polynomial wavelets suggested in [12] form a topological basis in the space $C^\infty[-1,1]$. As far as we know this is the first (but we are sure not the last!) example when wavelets form a topological basis in non-normed Fréchet space. Since the space is nuclear, the basis is absolute.

1. Polynomial wavelets.

T. Kilgor and J. Prestin suggested in [12] the following wavelets constructed from the Chebysev polynomials. Let $\Pi_n$ denote the set of all polynomials of degree at most $n$. For $n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$ and $|x| \leq 1$ let $T_n(x) = \cos(n \cdot \arccos x)$ be the Chebyshev polynomial of the first kind. Let $\omega_0(x) = 1 - x^2$ and for $n \in \mathbb{N}_0$ let

$$\omega_{n+1}(x) = 2^{n+1}(1 - x^2)T_1(x)T_2(x)T_4(x) \cdots T_{2^n}(x).$$

The scaling functions are given by the condition

$$\varphi_{j,k}(x) = \frac{\omega_j(x)}{\omega_j(x) \left( x - \cos \frac{k\pi}{2^j} \right)} \quad k = 0, 1, \cdots, 2^j, \quad j \in \mathbb{N}_0.$$
Now the Kilgor-Prestin wavelets are defined as

$$
\psi_{j,k}(x) = \frac{T_{2j}(x)}{2^j (x - x_{j,k})} [2\omega_j(x) - \omega_j(x_{j,k})], \quad k = 0, 1, \ldots, 2^j - 1, \quad j \in \mathbb{N}_0
$$

with $x_{j,k} = \cos \left( \frac{(2k+1)\pi}{2^{j+1}} \right)$.

Then (see [12] for more details) the subspaces $W_{-1} := \Pi_1$ and $W_j := \text{span}\{\psi_{j,k}, k = 0, 1, \ldots, 2^j - 1\} = \text{span}\{T_{2j+1}, T_{2j+2}, \ldots, T_{2^{j+1}}\}, \quad j \in \mathbb{N}_0$ give the decomposition

$$
\Pi_{2^{j+1}} = W_{-1} \oplus W_0 \oplus \cdots \oplus W_j
$$

which is orthogonal with respect to the inner product

$$
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2} \, dx.
$$

By $H$ we denote the corresponding Hilbert space. Let $\varepsilon_{j,n}$ take the value $1$ for $1 \leq n \leq 2^j - 1$ and $\varepsilon_{j,0} = \varepsilon_{j,2^j} = 1/2$.

**Lemma 1.** *(Lemma 2.2 in [12])* The wavelets can be written as

$$
\psi_{j,k}(x) = 2^{1-j} \sum_{n=2^j+1}^{2^{j+1}} T_n(x)T_n(x_{j,k})\varepsilon_{j+1,n}.
$$

Let us express the Chebyshev polynomials in terms of the system $\{\psi_{j,k}\}$.

**Lemma 2.** If $2^j + 1 \leq n \leq 2^{j+1}$ for $j \in \mathbb{N}_0$ then

$$
T_n = \sum_{k=0}^{2^j-1} T_n(x_{j,k})\psi_{j,k}.
$$

*Proof:*

Since the decomposition (1) is orthogonal, we get $T_n = \sum_{k=0}^{2^j-1} d_k^{(n)} \psi_{j,k}$. To find $d_k^{(n)}$ we can use the following interpolational property of wavelets ([12], (2.4))

$$
\psi_{j,k}(x_{j,m}) = \delta_{m,k} \quad \text{for} \quad m, k = 0, 1, \ldots, 2^j - 1.
$$

Hence, $d_k^{(n)} = T_n(x_{j,k})$. □

**Lemma 3.** Any function $f \in H$ can be represented in the form

$$
f = \frac{1}{\pi} \langle f, T_0 \rangle T_0 + \frac{2}{\pi} \langle f, T_1 \rangle T_1 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{j,k} \psi_{j,k}
$$

where

$$
c_{j,k} = \frac{2}{\pi} \sum_{n=2^j+1}^{2^{j+1}} \langle f, T_n \rangle T_n(x_{j,k})
$$

and convergence is considered with respect to the Hilbert norm.
Proof: The Chebyshev polynomials form a Hilbert basis in the space $H$. Since $\langle T_n, T_n \rangle = \pi/2$ for $n \geq 1$ and $\langle T_0, T_0 \rangle = \pi$, we have $f = \frac{1}{\pi} \langle f, T_0 \rangle T_0 + \frac{2}{n} \langle f, T_1 \rangle T_1 + \frac{2}{\pi} \sum_{j=0}^{\infty} \sum_{n=2j+1}^{2j+1} \langle f, T_n \rangle T_n$. By using Lemma 2 and changing the order of summation we get the desired result. □

Lemma 4. For any $j \in \mathbb{N}_0$ the matrices $X_j = (T_n(x_{j,k}))_{n=2j+1, k=0}^{2j+1, 2j-1}$ and $Y_j = 2^{-j} (T_n(x_{j,k}) \varepsilon_{j+1,n})_{k=0, n=2j+1}^{2j-1, 2j+1}$ are not singular.

Proof: We get the matrix $Y_j$ if we transpose $X_j$, then multiply the last column by $1/2$ and take the common coefficient $2^{1-j}$. Let us multiply the $p$–th row of $Y_j$ by the $q$–th column of $X_j$:

$$2^{1-j} \sum_{n=2j+1}^{2j+1} T_n(x_{j,p})T_n(x_{j,q})\varepsilon_{j+1,n} = \psi_{j,p}(x_{j,q}) = \delta_{p,q}.$$ 

Therefore, $Y_j \cdot X_j = I$ and both matrices are not singular. Since $\det(Y_j) = 2^{-j} \det(X_j)$, we get $\det(X_j) = \pm 2^{j/2}$ and $\det(Y_j) = \pm 2^{-j/2}$. □

Remark. If we multiply the $p$–th row of $X_j$ by the $q$–th column of $Y_j$, then we get the orthogonality property (1.141) from [21].

2. Wavelet Schauder basis in $C^\infty[-1, 1]$.

Topology $\tau$ of the space $C^\infty[-1, 1]$ of all infinitely differentiable functions on $[-1, 1]$ can be given by the system of norms

$$|f|_p = \sup\{|f^{(i)}(x)| : |x| \leq 1, i \leq p\}, \quad p \in \mathbb{N}_0.$$ 

The first basis in $C^\infty[-1, 1]$, namely the Chebyshev polynomials, was found by Mityagin ([17], L.25). And what is more, by the Dynin-Mityagin theorem ([17], T.9), every topological basis of nuclear Fréchet space is absolute. In our case we see that the series $\frac{1}{\pi} \langle f, T_0 \rangle T_0 + \frac{2}{\pi} \sum_{n=1}^{\infty} \langle f, T_n \rangle T_n$ converges to $f \in C^\infty[-1, 1]$ in the topology $\tau$. The convergence is absolute, that is for any $p \in \mathbb{N}_0$ the series $\sum_{n=0}^{\infty} |\langle f, T_n \rangle| \cdot |T_n|_p$ converges. Furthermore, if $\{e_n, \xi_n\}$ is a biorthogonal system with the total (that is $\xi_n(f) = 0, \forall n \Rightarrow f = 0$) over $C^\infty[-1, 1]$ sequence of functionals and for every $p \in \mathbb{N}_0$ there exist $q \in \mathbb{N}_0$ and $C > 0$ such that

$$|e_n|_p \cdot |\xi_n|_{-q} \leq C \quad \text{for all} \quad n,$$

then $(e_n)$ is a Schauder basis in $C^\infty[-1, 1]$.

Here and subsequently, $|\cdot|_{-q}$ denotes the dual norm: for a bounded linear functional $\xi$ let $|\xi|_{-q} = \sup\{|\xi(f)| : |f|_q \leq 1\}$.

Theorem 1. The system $\{T_0, T_1, (\psi_{j,k})_{j=0, k=0}^{\infty, 2j-1}\}$ is a topological basis in the space $C^\infty[-1, 1]$.

Proof: We suggest two proofs of the theorem. The 1st proof is similar in spirit to the arguments of Mityagin in [17], L.25. Let $\xi_0(f) = \frac{1}{\pi} \langle f, T_0 \rangle$, $\xi_1(f) = \frac{2}{\pi} \langle f, T_1 \rangle$ and for $j \in \mathbb{N}_0, 0 \leq k \leq 2^j - 1$ let
\[ \xi_{j,k}(f) = c_{j,k}, \] where \( c_{j,k} \) are given in Lemma 3. Then \( \xi_{j,k}(\psi_{i,l}) = 0 \) if \( i \neq j \), as is easy to see. For the wavelets and functionals of the same level we get

\[ \xi_{j,k}(\psi_{j,l}) = \frac{2}{\pi} \sum_{n=2^j+1}^{2^{j+1}} T_n(x_{j,k}) \cdot 2^{1-j} \sum_{m=2^j+1}^{2^{j+1}} T_m(x_{j,l}) \varepsilon_{j+1,m} \langle T_m(\cdot), T_n(\cdot) \rangle = \]

\[ = 2^{1-j} \sum_{n=2^j+1}^{2^{j+1}} T_n(x_{j,k}) T_n(x_{j,l}) \varepsilon_{j+1,n} = \psi_{j,l}(x_{j,k}) = \delta_{l,k}. \]

Therefore the functionals \( \{ \xi_0, \xi_1, (\xi_{j,k})_{j=0, k=0}^{\infty, 2^j-1} \} \) are biorthogonal to the system \( \{ T_0, T_1, (\psi_{j,k})_{j=0, k=0}^{\infty, 2^j-1} \} \). Let us check that this sequence of functionals is total over \( C^\infty[-1,1] \). Suppose that \( \xi_{j,k}(f) = 0 \) for all \( j \) and \( k \). For fixed \( j \) we get the system of \( 2^j \) linear equations \( \langle f, T_n \rangle = 0, n = 2^j+1, \ldots, 2^{j+1} \) with unknowns \( \langle f, T_n \rangle \). By Lemma 4 the system has only the trivial solution. Together with \( \xi_0(f) = \xi_1(f) = 0 \) it follows that \( \langle f, T_n \rangle = 0 \) for all \( n \). But the Chebyshev polynomials form a basis in \( C^\infty[-1,1] \) and so \( f = 0 \). Thus it is enough to check the Dynin-Mityagin condition. Let us fix \( p \in \mathbb{N}_0 \). For Chebyshev polynomials we have (see e.g. [21])

\[ |T_n|_m = T_n^{(m)}(1) = \frac{n^2(n^2-1)(n^2-2^2) \cdots (n^2-(m-1)^2)}{1 \cdot 3 \cdot 5 \cdots (2m-1)}. \]

(2)

By Lemma 1,

\[ |\psi_{j,k}|_p \leq 2^{1-j} \sup_{m \leq p} \sum_{n=2^j+1}^{2^{j+1}} |T_n|_m \leq 2^{1-j} 2^j |T_{2^{j+1}}|_p \leq 2^{(j+1)2p+1}. \]

(3)

On the other hand, by orthogonality

\[ \langle f, T_n \rangle = \int_0^\pi f(\cos t) \cos nt dt = \int_0^\pi (f(\cos t) - Q(\cos t)) \cos nt dt \]

for any polynomial \( Q \in \Pi_{n-1} \). As in [7] we can take the polynomial \( Q = Q_{n-1} \) of best approximation to \( f \) on \([-1,1]\) in the norm \( |\cdot|_0 \). By the Jackson theorem (see e.g. [20], T.1.5) for any \( q \in \mathbb{N}_0 \) there exists a constant \( C_q \) such that for any \( n > q \)

\[ |f - Q_{n-1}|_0 \leq C_q n^{-q} |f|_q. \]

Therefore, \( |\langle f, T_n \rangle| \leq \pi C_q n^{-q} |f|_q \) and for \( 2^j > q \) we get

\[ |\xi_{j,k}|_{-q} \leq 2 C_q 2^j (2^j)^{-q}. \]

Taking into account (3) we see that the values \( q = 2p + 1 \) and \( C = 4^{p+1} C_q \) will give us the desired conclusion.

In the 2nd proof we introduce the operator \( A \) first on the basis \( (T_n) \) and then by linearity. Let \( AT_0 = T_0, AT_1 = T_1 \) and \( AT_n = \psi_{j,k} \) for \( n = 2^j + k + 1 \), where \( j \in \mathbb{N}_0, k = 0, 1, \ldots, 2^j - 1 \). Let us show that for any \( p \in \mathbb{N}_0 \) there exist \( q \in \mathbb{N}_0 \) and \( C > 0 \) such that

\[ |\psi_{j,k}|_p \leq C |T_{2^j+k+1}|_q \]

for all \( j \) and \( k \).
For the left side we already have the bound (3). Also, from (2) we obtain

$$|T_{2^j+k}|_q \geq |T_{2^j}|_q \geq \frac{1}{1 \cdot 3 \cdot 5 \cdots (2q-1)} (2^{2j} - q^2)^q.$$ 

Clearly, the value $q = p + 1$ provides the inequality (4) for large enough $j$.
Hence there exists $C$ depending only on $p$ that ensures the result for all $j$ and $k$.

From (4) we deduce that the operator

$$A : C^\infty[-1, 1] \longrightarrow C^\infty[-1, 1] : f = \sum_{n=0}^{\infty} \xi_n T_n \mapsto \sum_{n=0}^{\infty} \xi_n A T_n$$

is well defined and continuous. If $Af = 0$, then for any $j \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{2^j-1} \xi_{2^j+k} \psi_{j, k} = 0.$$ 

Lemma 4 implies $\xi_{2^j+k} = 0$. Therefore, $\text{ker} A = 0$. In the same way, one can easily show that $A$ is surjective. Therefore the operator $A$ is a continuous linear bijection. By the open mapping theorem, $A$ is an isomorphism. Thus the system \{\(T_0, T_1, (\psi_{j, k})_{j=0, k=0}^{\infty, 2^j-1}\)\} is a topological basis and what is more, it is equivalent to the classical basis \((T_n)_{n=0}^{\infty}\) (see e.g. [22] for the definition of equivalent bases).

□

Remark. Since \{\(T_0, T_1, (\psi_{j, k})_{j=0, k=0}^{\infty, 2^j-1}\)\} is a block-system with respect to the basis \((T_n)_{n=0}^{\infty}\), one can suggest also a third proof based on a generalization of Corollary 7.3 from [22], Ch.1 for the case of countably normed space.

References


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