COMMUNICATIONS IN ALGEBRA®
2018, VOL. 46, NO. 6, 2561–2573
https://doi.org/10.1080/00927872.2017.1392532

On pseudo symmetric monomial curves

Mesut Şahin[a] and Nil Şahin[b]

[a]Department of Mathematics, Hacettepe University, Ankara, Turkey; [b]Department of Industrial Engineering, Bilkent University, Ankara, Turkey

ABSTRACT
We study monomial curves, toric ideals and monomial algebras associated to 4-generated pseudo symmetric numerical semigroups. Namely, we determine indispensable binomials of these toric ideals, give a characterization for these monomial algebras to have strongly indispensable minimal graded free resolutions. We also characterize when the tangent cones of these monomial curves at the origin are Cohen–Macaulay.

ARTICLE HISTORY
Received 17 June 2016
Revised 18 June 2017
Communicated by S. Goto

KEYWORDS
Free resolution; Hilbert function; indispensable binomial; monomial curve; numerical semigroup; Rossi's conjecture; tangent cone

2000 MATHEMATICS SUBJECT CLASSIFICATION
Primary: 13H10, 14H20; Secondary: 13P10

1. Introduction

Characterising numerical functions that may be Hilbert functions of one dimensional Cohen–Macaulay local rings is a hard and still open question of local algebra, see [32]. A necessary condition for the characterization is provided by Sally's conjecture that *the Hilbert function of a one dimensional Cohen–Macaulay local ring with small enough embedding dimension is non-decreasing.* This conjecture is obvious in embedding dimension 1, proved in embedding dimensions 2 by Matlis [26] and 3 by Elias [14]. For embedding dimension 4, Gupta and Roberts gave counterexamples in [20], and for each embedding dimension greater than 4, Orecchia gave counterexamples in [29]. Local rings of monomial curves provided many affirmative answers, see e.g. [10, 12, 23, 30] and references therein. On the other hand, counterexamples were given only in affine 10-space by Herzog and Waldi [22] and in affine 12-space by Eakin and Sathaye [13], and most recently, Oneto et al. [27] and Oneto and Tamone [28] announced some methods for producing Gorenstein monomial curves whose tangent cones have decreasing Hilbert functions. However, the problem is still open for monomial curves in n-space, where $3 < n < 10$. As the original conjecture predicts that the embedding dimension n should be small and 4 is the first case, it is natural to focus on monomial curves in 4-space. Arslan and Mete gave an affirmative answer to the conjecture for local rings corresponding to 4-generated symmetric semigroups in [3] under a numerical condition by proving that the tangent cone is Cohen–Macaulay. Taking the novel approach to use indispensable binomials in the toric ideal, Arslan et al. refined in [2] this by characterising Cohen–Macaulayness of the tangent cone completely. As symmetric and pseudo symmetric semigroups are maximal with respect to inclusion with fixed genus, see [5], the second interesting case is the class of 4-generated pseudo symmetric semigroups which is the content of the present paper. We give characterizations under which the tangent cone is Cohen–Macaulay. This reveals how nice the singularity at the origin is and verifies Sally's conjecture by Garcia [15]. It also reduces the computation of the Hilbert function to that of its Artinian reduction which have only a finite number of nonzero

CONTACT Mesut Şahin mesut.sahin@hacettepe.edu.tr Department of Mathematics, Hacettepe University, Ankara 06800, Turkey.

values, see [33]. Our criteria for the Cohen–Macaulayness is in terms of the five integers determining the semigroup, so they can be used in principal to construct counterexamples if there are any. In order to get these conditions we use indispensable binomials in the toric ideal. Motivated originally from its applications in algebraic statistics many authors have studied the concept of indispensability, see e.g. [7, 16, 24, 36] and later strong indispensability, see [6, 8, 9]. In order to state our results more precisely we introduce some notations.

Let $n_1, \dots, n_4$ be positive integers with $\gcd(n_1, \dots, n_4) = 1$. Then the numerical semigroup $S = \langle n_1, \dots, n_4 \rangle$ is defined to be the set $\{u_1 n_1 + \cdots + u_4 n_4 \mid u_i \in \mathbb{N}\}$. Let K be a field and $K[S] = K[t^{n_1}, \dots, t^{n_4}]$ be the semigroup ring of S, then $K[S] \simeq A/I_S$ where, $A = K[X_1, \dots, X_4]$ and I_S is the kernel of the surjection $A \xrightarrow{\phi_0} K[S]$, where $X_i \mapsto t^{n_i}$.

Pseudo Frobenious numbers of S are defined to be the elements of the set $PF(S) = \{n \in \mathbb{Z} - S \mid n + s \in S \text{ for all } s \in S - \{0\}\}$. The largest pseudo Frobenious number not belonging to S is called the Frobenius number and is denoted by $g(S)$. S is called pseudo symmetric if $PF(S) = \{g(S)/2, g(S)\}$, see [5] or [31, Chapter 3]. By Komeda [25, Theorems 6.4 and 6.5], the semigroup S is pseudo symmetric if and only if there are integers $\alpha_i > 1$, $1 \le i \le 4$, and $\alpha_{21} > 0$, with $\alpha_{21} < \alpha_1$, such that $n_1 = \alpha_2\alpha_3(\alpha_4-1)+1$, $n_2 = \alpha_{21}\alpha_3\alpha_4+(\alpha_1-\alpha_{21}-1)(\alpha_3-1)+\alpha_3$, $n_3 = \alpha_1\alpha_4+(\alpha_1-\alpha_{21}-1)(\alpha_2-1)(\alpha_4-1)-\alpha_4+1$, $n_4 = \alpha_1\alpha_2(\alpha_3-1)+\alpha_{21}(\alpha_2-1)+\alpha_2$.

From now on, S is assumed to be a pseudo symmetric numerical semigroup. Then, by Komeda [25], $K[S] = A/(f_1, f_2, f_3, f_4, f_5)$, where

$$f_1 = X_1^{\alpha_1} - X_3 X_4^{\alpha_4 - 1}, \qquad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}} X_4, \qquad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1 - \alpha_{21} - 1} X_2,$$
$$f_4 = X_4^{\alpha_4} - X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3 - 1}, \qquad f_5 = X_3^{\alpha_3 - 1} X_1^{\alpha_{21} + 1} - X_2 X_4^{\alpha_4 - 1}.$$

In Section 2, we determine indispensable binomials of I_S and prove that $K[S]$ has a strongly indispensable minimal S-graded free resolution if and only if $\alpha_4 > 2$ and $\alpha_1 - \alpha_{21} > 2$, see Theorem 2.6, filling a missing case in [6].

In Section 3, we consider the affine curve C_S with parametrization

$$X_1 = t^{n_1},\ X_2 = t^{n_2},\ X_3 = t^{n_3},\ X_4 = t^{n_4}$$

corresponding to S. Recall that the local ring corresponding to the monomial curve C_S is $R_S = K[[t^{n_1}, \dots, t^{n_4}]]$ and its Hilbert function is defined as the Hilbert function of its associated graded ring, $gr_m(K[[t^{n_1}, \dots, t^{n_4}]])$, which is isomorphic to the ring $K[S]/I_{S*}$. Here, I_{S*} is the defining ideal of the tangent cone of C_S at the origin and is generated by the homogeneous summands f_* of the elements $f \in I_S$. We characterize when the tangent cone of C_S is Cohen–Macaulay in terms of the defining integers α_i and α_{21}. As a byproduct of our proofs, we provide explicit generating sets for Cohen–Macaulay tangent cones.

2. Indispensability

In this section, we determine the indispensable binomials in I_S and characterize the conditions under which $K[S]$ has a strongly indispensable minimal S-graded free resolution. First, recall some notions from [7]. The S-degree of a monomial is defined to be $\deg_S(X_1^{u_1} X_2^{u_2} X_3^{u_3} X_4^{u_4}) = \sum_{i=1}^{4} u_i n_i \in S$. Let $V(d)$ be the set of monomials of S-degree d. Denote by $G(d)$ the graph with vertices the elements of $V(d)$ and edges $\{m, n\} \subset V(d)$ such that the binomial $m - n$ is generated by binomials in I_S of S-degree strictly smaller than d. In particular, when $\gcd(m, n) \neq 1$, $\{m, n\}$ is an edge of $G(d)$. $d \in S$ is called a Betti S-degree if there is a minimal generator of I_S of S-degree d and β_d is the number of times d occurs as a Betti S-degree. Both the set B_S of Betti S-degrees and β_d are invariants of I_S. S-degrees of binomials in I_S which are not comparable with respect to $<_S$ constitute a subset denoted by M_S whose elements are called minimal binomial S-degrees, where $s_1 <_S s_2$ if $s_2 - s_1 \in S$. In general, $M_S \subseteq B_S$. By Komeda's result, $B_S = \{d_1, d_2, d_3, d_4, d_5\}$ if d_i's are all distinct, where d_i is the S-degree of f_i, for $i = 1, \dots, 5$. A binomial is called indispensable if it appears in every minimal generating set of I_S. The following useful observation to detect indispensable binomials is not explicitly stated in [7].

Lemma 2.1. *A binomial of S-degree d is indispensable if and only if* $\beta_d = 1$ *and* $d \in M_S$.

Proof. A binomial of S-degree d is indispensable if and only if $G(d)$ has two connected components which are singletons, by Charalambous et al. [7, Corollary 2.10]. From the paragraph just after [7, Corollary 2.8], the condition that $G(d)$ has two connected components is equivalent to $\beta_d = 1$. Finally, Charalambous et al. [7, Proposition 2.4] completes the proof, since the connected components of $G(d)$ are singletons if and only if $d \in M_S$. □

We use the following many times in the sequel.

Lemma 2.2. *If* $0 < v_k < \alpha_k$ *and* $0 < v_l < \alpha_l$, *for* $k \neq l \in \{1, 2, 3, 4\}$, *then* $v_k n_k - v_l n_l \notin S$.

Proof. Assume to the contrary that $v_k n_k - v_l n_l \in S$. Then

$$v_k n_k - v_l n_l = \sum_{i=1}^{4} u_i n_i = u_1 n_1 + u_2 n_2 + u_3 n_3 + u_4 n_4$$

for some non-negative u_k's.

Hence, $(v_k - u_k)n_k = (v_l + u_l)n_l + u_s n_s + u_r n_r \in \langle n_l, n_s, n_r\rangle$. If $v_k - u_k < 0$ then $(v_k - u_k)n_k \in S \cap (-S)$ but this is a contradiction as $S \cap (-S) = \{0\}$. If $v_k - u_k = 0$, then $(v_l + u_l)n_l + u_s n_s + u_r n_r = 0$ and this is impossible as v_l is positive. That is, $v_k - u_k > 0$. This contradicts with the fact that α_i is the smallest positive number with this property as $0 < v_i - u_i \leq v_i < \alpha_i$. □

Now, we determine the minimal binomial S-degrees.

Proposition 2.3. $M_S = \{d_1, d_2, d_3, d_4, d_5\}$ *if* $\alpha_1 - \alpha_{21} > 2$ *and* $M_S = \{d_1, d_2, d_3, d_5\}$ *if* $\alpha_1 - \alpha_{21} = 2$.

Proof. Notice first that

$$\begin{aligned}
d_1 &= \alpha_1 n_1 = n_3 + (\alpha_4 - 1)n_4,\\
d_2 &= \alpha_2 n_2 = \alpha_{21} n_1 + n_4,\\
d_3 &= \alpha_3 n_3 = (\alpha_1 - \alpha_{21} - 1)n_1 + n_2,\\
d_4 &= \alpha_4 n_4 = n_1 + (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3,\\
d_5 &= (\alpha_{21} + 1)n_1 + (\alpha_3 - 1)n_3 = n_2 + (\alpha_4 - 1)n_4.
\end{aligned}$$

Thus, we observe that

$$\begin{aligned}
d_1 - d_2 &= (\alpha_1 - \alpha_{21})n_1 - n_4\\
d_1 - d_3 &= (\alpha_{21} + 1)n_1 - n_2\\
d_1 - d_4 &= n_3 - n_4\\
d_1 - d_5 &= (\alpha_1 - \alpha_{21} - 1)n_1 - (\alpha_3 - 1)n_3\\
d_2 - d_3 &= (\alpha_2 - 1)n_2 - (\alpha_1 - \alpha_{21} - 1)n_1\\
d_2 - d_4 &= n_3 - (\alpha_1 - \alpha_{21})n_1\\
d_2 - d_5 &= (\alpha_2 - 1)n_2 - (\alpha_4 - 1)n_4
\end{aligned}$$

$$d_3 - d_4 = n_3 - n_1 - (\alpha_2 - 1)n_2$$
$$d_3 - d_5 = n_3 - (\alpha_{21} + 1)n_1$$
$$d_4 - d_5 = (\alpha_2 - 1)n_2 - \alpha_{21}n_1.$$

Then, $d_i - d_j = v_k n_k - u_l n_l$ for some $k \neq l \in \{1,2,3,4\}$ with $0 < v_k < \alpha_k$ and $0 < v_l < \alpha_l$ except for $d_3 - d_4$ and $d_4 - d_3$. Hence, we can say $d_i - d_j \notin S$ from Lemma 2.2 for all i, j except 3 and 4.

Assume $d_3 - d_4 \in S$. Then $n_3 - n_1 - (\alpha_2 - 1)n_2 = u_1n_1 + u_2n_2 + u_3n_3 + u_4n_4$ for some non-negative u_i's. So, $(1 - u_3)n_3 = (1 + u_1)n_1 + (\alpha_2 - 1 + u_2)n_2 + u_4n_4 > 0$. This contradicts to α_3 being the minimal number with the property $\alpha_3 n_3 \in \langle n_1, n_2, n_4\rangle$, as $0 < 1 - u_3 < \alpha_3$. Hence $d_3 - d_4$ can not be in S.

There are two possibilities for $d_4 - d_3$. If $\alpha_1 - \alpha_{21} = 2$, then we have $d_4 - d_3 = (\alpha_2 - 2)n_2 + (\alpha_3 - 1)n_3 - (\alpha_1 - \alpha_{21} - 2)n_1 = (\alpha_2 - 2)n_2 + (\alpha_3 - 1)n_3 \in S$.

If $\alpha_1 - \alpha_{21} > 2$, we show that $d_4 - d_3 \notin S$. Assume contrary that $d_4 - d_3 = n_1 + (\alpha_2 - 1)n_2 - n_3 = u_1n_1 + u_2n_2 + u_3n_3 + u_4n_4$. Then, $(\alpha_2 - 1 - u_2)n_2 = (u_1 - 1)n_1 + (u_3 + 1)n_3 + u_4n_4$. If $u_1 > 0$, then $0 < \alpha_2 - 1 - u_2 < \alpha_2$, since $u_3 + 1 > 0$. But this contradicts to the minimality of α_2. Hence $u_1 = 0$ and $n_1 + (\alpha_2 - 1 - u_2)n_2 = (u_3 + 1)n_3 + u_4n_4$ with $\alpha_2 - 1 - u_2 > 0$. (If $\alpha_2 - 1 - u_2 \leq 0$, then $n_1 = (u_2 + 1 - \alpha_2)n_2 + (u_3 + 1)n_3 + u_4n_4$ and this implies $n_1 \in \langle n_2, n_3, n_4\rangle$ which can not happen). Then if $u_4 = 0$, we have $(u_3 + 1)n_3 = n_1 + (\alpha_2 - 1 - u_2)n_2$. As $u_3 + 1 < \alpha_3$ gives a contradiction with the minimality of α_3, we assume $u_3 + 1 = \alpha \geq \alpha_3$. Then $\alpha_3 n_3 + (\alpha - \alpha_3)n_3 = n_1 + (\alpha_2 - 1 - u_2)n_2$ $\Rightarrow (\alpha_1 - \alpha_{21} - 1)n_1 + n_2 + (\alpha - \alpha_3)n_3 = n_1 + (\alpha_2 - 1 - u_2)n_2 \Rightarrow (\alpha_1 - \alpha_{21} - 2)n_1 + (\alpha - \alpha_3)n_3 = (\alpha_2 - 2 - u_2)n_2 \Rightarrow 0 < \alpha_2 - 2 - u_2 < \alpha_2$ and this gives a contradiction with the minimality of α_2. On the other hand, if $u_4 > 0$, then $n_1 + \alpha_2 n_2 = (1 + u_2)n_2 + (u_3 + 1)n_3 + u_4n_4$, and as $\alpha_2 n_2 = 1 + \alpha_{21}n_1 + n_4$, we have $(1 + \alpha_{21})n_1 = (1 + u_2)n_2 + (u_3 + 1)n_3 + (u_4 - 1)n_4$. As $0 < 1 + \alpha_{21} < \alpha_1$, this contradicts with the minimality of α_1. Hence, $d_4 - d_3$ can not be an element of S. □

As a consequence, we determine the indispensable binomials in I_S. Part of this result is remarked at the end of [24].

Corollary 2.4. *Indispensable binomials of I_S are $\{f_1, f_2, f_3, f_4, f_5\}$ if $\alpha_1 - \alpha_{21} > 2$ and are $\{f_1, f_2, f_3, f_5\}$ if $\alpha_1 - \alpha_{21} = 2$.*

Proof. This follows from Lemma 2.1 and Proposition 2.3, since $\beta_{d_i} = 1$, for all $i = 1, \ldots, 5$. □

A minimal graded free resolution of $K[S]$ is given in [6, Theorem 6] as follows:

Theorem 2.5. *If S is a 4-generated pseudosymmetric semigroup, then the following is a minimal graded free A-resolution of $K[S]$:*

$$(\mathbf{F}, \phi) : 0 \longrightarrow \bigoplus_{j=1}^{2} A[-c_j] \xrightarrow{\phi_3} \bigoplus_{j=1}^{6} A[-b_j] \xrightarrow{\phi_2} \bigoplus_{j=1}^{5} A[-d_j] \xrightarrow{\phi_1} A \longrightarrow 0$$

where $\phi_1 = (f_1, f_2, f_3, f_4, f_5)$,

$$\phi_2 = \begin{pmatrix} X_2 & 0 & X_3^{\alpha_3-1} & 0 & X_4 & 0 \\ 0 & f_3 & 0 & X_1X_3^{\alpha_3-1} & X_1^{\alpha_1-\alpha_{21}} & X_4^{\alpha_4-1} \\ X_1^{\alpha_{21}+1} & -f_2 & X_4^{\alpha_4-1} & 0 & X_1X_2^{\alpha_2-1} & 0 \\ 0 & 0 & 0 & X_2 & X_3 & X_1^{\alpha_{21}} \\ -X_3 & 0 & -X_1^{\alpha_1-\alpha_{21}-1} & X_4 & 0 & X_2^{\alpha_2-1} \end{pmatrix},$$

and $$\phi_3 = \begin{pmatrix} X_4 & -X_1 & 0 & X_3 & -X_2 & 0 \\ -X_2^{\alpha_2-1}X_3^{\alpha_3-1} & X_4^{\alpha_4-1} & f_2 & -X_1^{\alpha_1-1} & X_1^{\alpha_{21}}X_3^{\alpha_3-1} & -f_3 \end{pmatrix}^T.$$

The numbers b_j and c_j above can be obtained from the maps ϕ_2 and ϕ_3 as in [6, Corollary 16]. For instance, the S-degrees of the non-zero entries in the first column of ϕ_2 gives us $b_1 = d_1 + n_2 = d_3 + (\alpha_{21} + 1)n_1 = d_5 + n_3$. Similarly we get:

$$\begin{aligned}
b_2 &= d_2 + d_3\\
b_3 &= d_1 + (\alpha_3 - 1)n_3 = d_3 + (\alpha_4 - 1)n_4 = d_5 + (\alpha_1 - \alpha_{21} - 1)n_1\\
b_4 &= d_4 + n_2 = d_2 + n_1 + (\alpha_3 - 1)n_3 = d_5 + n_4\\
b_5 &= d_1 + n_4 = d_2 + (\alpha_1 - \alpha_{21})n_1 = d_3 + n_1 + (\alpha_2 - 1)n_2 = d_4 + n_3\\
b_6 &= d_2 + (\alpha_4 - 1)n_4 = d_4 + \alpha_{21}n_1 = d_5 + (\alpha_2 - 1)n_2
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= b_1 + n_4 = b_2 + n_1 = b_4 + n_3 = b_5 + n_2\\
c_2 &= b_1 + (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3\\
&= b_2 + (\alpha_4 - 1)n_4\\
&= b_3 + d_2 = b_3 + \alpha_2 n_2 = b_3 + \alpha_{21}n_1 + n_4\\
&= b_4 + (\alpha_1 - 1)n_1\\
&= b_5 + \alpha_{21}n_1 + (\alpha_3 - 1)n_3\\
&= b_6 + d_3 = b_6 + \alpha_3 n_3 = b_6 + (\alpha_1 - \alpha_{21} - 1)n_1 + n_2.
\end{aligned}$$

Note that the resolution $(\mathbf{F}, \phi)$ is called strongly indispensable if for any graded minimal resolution $(\mathbf{G}, \theta)$, we have an injective complex map $i\colon (\mathbf{F}, \phi) \longrightarrow (\mathbf{G}, \theta)$. We finish this section with its main result to characterize when $K[S]$ has a strongly indispensable minimal graded free resolution.

Theorem 2.6. *Let S be a 4-generated pseudo-symmetric semigroup. Then $K[S]$ has a strongly indispensable minimal graded free resolution if and only if $\alpha_4 > 2$ and $\alpha_1 - \alpha_{21} > 2$.*

Proof. According to Barucci et al. [6, Proposition 29], $K[S]$ has a strongly indispensable minimal graded free resolution if and only if the differences $d_i - d_j$ and $b_i - b_j$ do not belong to S, for any i and j. Indeed, $d_i - d_j \notin S$ if and only if $\alpha_1 - \alpha_{21} > 2$ from the proof of Proposition 2.3. For the other differences, we use the identities in c_1 and c_2. As a result, from c_1, we get the differences

$$\begin{aligned}
b_1 - b_2 &= n_1 - n_4,\\
b_1 - b_4 &= n_3 - n_4,\\
b_1 - b_5 &= n_2 - n_4,\\
b_2 - b_4 &= n_3 - n_1,\\
b_2 - b_5 &= n_2 - n_1,\\
b_4 - b_5 &= n_2 - n_3.
\end{aligned}$$

Similarly, from c_2, we get the differences

$$\begin{aligned}
b_1 - b_3 &= n_2 - (\alpha_3 - 1)n_3,\\
b_1 - b_6 &= n_3 - (\alpha_2 - 1)n_2,\\
b_3 - b_4 &= (\alpha_1 - \alpha_{21} - 1)n_1 - n_4,\\
b_3 - b_5 &= (\alpha_3 - 1)n_3 - n_4,\\
b_3 - b_6 &= (\alpha_1 - \alpha_{21} - 1)n_1 - (\alpha_2 - 1)n_2\\
b_4 - b_6 &= n_2 - \alpha_{21}n_1,\\
b_5 - b_6 &= n_3 - \alpha_{21}n_1.
\end{aligned}$$

Observe that $b_i - b_j = v_k n_k - v_l n_l$ for any $i < j$ and for some $k \neq l \in \{1, 2, 3, 4\}$ with $0 < v_k < \alpha_k$ and $0 < v_l < \alpha_l$. By Lemma 2.2, we have $\mp(b_i - b_j) \notin S$, for any $i < j$, except for $i = 2$ and $j = 3, 6$.

Furthermore, $b_2 - b_3 = \alpha_2 n_2 - (\alpha_4 - 1)n_4 = \alpha_{21} n_1 - (\alpha_4 - 2)n_4$. Again by Lemma 2.2, we have $\mp(b_2 - b_3) \notin S$ when $\alpha_4 > 2$. On the other hand, if $\alpha_4 = 2$, then $b_2 - b_3 = \alpha_{21} n_1 \in S$.

Finally, $b_2 - b_6 = \alpha_3 n_3 - (\alpha_4 - 1)n_4$. Using the identity

$$d_5 = (\alpha_{21} + 1)n_1 + (\alpha_3 - 1)n_3 = n_2 + (\alpha_4 - 1)n_4,$$

we obtain $b_2 - b_6 = n_2 + n_3 - (\alpha_{21} + 1)n_1$. If $b_2 - b_6 \in S$, then there are non-negative u_i such that

$$n_2 + n_3 - (\alpha_{21} + 1)n_1 = b_2 - b_6 = u_1 n_1 + u_2 n_2 + u_3 n_3 + u_4 n_4.$$

Then $(1 - u_2)n_2 + (1 - u_3)n_3 = (\alpha_{21} + 1 + u_1)n_1 + u_4 n_4 > 0$. It follows that $u_2 = u_3 = 0$. Thus, $n_2 + n_3 = (\alpha_{21} + 1 + u_1)n_1 + u_4 n_4$.

If $u_4 = 0$ then $\alpha n_1 \in \langle n_2, n_3 \rangle$ with $\alpha < \alpha_1$ because if $\alpha \geq \alpha_1$, then $n_2 + n_3 = (\alpha - \alpha_1)n_1 + \alpha_1 n_1 = (\alpha - \alpha_1)n_1 + n_3 + (\alpha_4 - 1)n_4$. This leads to a contradiction as $n_2 = (\alpha - \alpha_1)n_1 + n_4(\alpha_4 - 1) \in \langle n_1, n_4 \rangle$. So $u_4 > 0$ in which case, $n_2 + n_3 = \alpha_{21} n_1 + (1 + u_1)n_1 + n_4 + (u_4 - 1)n_4 = (1 + u_1)n_1 + \alpha_2 n_2 + (u_4 - 1)n_4$ $\Rightarrow n_3 = (u_1 + 1)n_1 + (\alpha_2 - 1)n_2 + (u_4 - 1)n_4 \in \langle n_1, n_2, n_4 \rangle$, another contradiction. Hence, $b_2 - b_6 \notin S$.

If $b_6 - b_2 = (\alpha_{21} + 1)n_1 - n_2 - n_3 = u_1 n_1 + u_2 n_2 + u_3 n_3 + u_4 n_4$, for some non-negative u_i, then $(\alpha_{21} + 1 - u_1)n_1 = (u_2 + 1)n_2 + (u_3 + 1)n_3 + u_4 n_4 > 0$. Then $0 < \alpha_{21} + 1 - u_1 < \alpha_1$, a contradiction with the minimality of α_1. Hence, $b_6 - b_2$ can not be an element of S either, completing the proof. □

3. Cohen–Macaulayness of the tangent cone

In this section, we give conditions for the Cohen–Macaulayness of the tangent cone. For some recent and past activity about the tangent cone of C_S, see [1, 2, 11, 21, 34, 35].

Recall that for an ideal I with a fixed monomial ordering "$<$", a finite set $G \subset I$ is called a standard basis of I if the leading monomials of the elements of G generate the leading ideal of I that is, if for any $f \in I - \{0\}$, there exits $g \in G$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(f)$. Note that a standard basis is also a basis for the ideal and when the ordering "$<$" is global, standard basis is actually a Gröbner basis [18].

Remark 3.1. Depending on the ordering among n_1, n_2, n_3 and n_4 there are 24 possible cases. We illustrate in Table 1 that there are pseudo symmetric monomial curves with Cohen–Macaulay tangent cones in all of these cases. We will determine standard bases and characterize Cohen–Macaulayness completely in the first 12 cases in terms of the defining integers. For the remaining 12 cases, finding a general form for the standard basis is not possible, and instead of giving a characterization as in [2], we give some partial results involving the defining integers α_i and α_{21}.

3.1. Cohen–Macaulayness of the tangent cone when n_1 is smallest

In this section, we assume that n_1 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. Using the indispensable binomials of I_S, we characterize the Cohen–Macaulayness of the tangent cone of C_S. First, we get the necessary conditions.

Lemma 3.2. *If the tangent cone of the monomial curve C_S is Cohen–Macaulay, then the following must hold*
(C1.1) $\alpha_2 \leq \alpha_{21} + 1$,
(C1.2) $\alpha_{21} + \alpha_3 \leq \alpha_1$,
(C1.3) $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$.

Proof. Corollary 2.4 implies that f_2 and f_3 are indispensable binomials of I_S, which means that they appear in every standard basis. To prove C(1.1), assume contrary that $\alpha_2 > \alpha_{21} + 1$. Then,

Table 1. Cases.

	α_{21}	α_1	α_2	α_3	α_4	n_1	n_2	n_3	n_4
$n_1 < n_2 < n_3 < n_4$	2	5	3	2	2	7	12	13	22
$n_1 < n_2 < n_4 < n_3$	2	5	3	2	4	19	20	29	22
$n_1 < n_3 < n_2 < n_4$	3	5	4	2	3	17	21	19	33
$n_1 < n_3 < n_4 < n_2$	3	6	3	3	5	37	52	42	45
$n_1 < n_4 < n_2 < n_3$	3	6	3	2	4	19	28	33	27
$n_1 < n_4 < n_3 < n_2$	3	8	3	4	6	61	88	83	81
$n_2 < n_1 < n_3 < n_4$	2	6	6	3	5	73	39	86	88
$n_2 < n_1 < n_4 < n_3$	2	5	4	2	4	25	20	35	30
$n_2 < n_3 < n_1 < n_4$	2	4	4	2	4	25	19	22	26
$n_2 < n_3 < n_4 < n_1$	3	5	6	2	6	61	39	50	51
$n_2 < n_4 < n_1 < n_3$	2	5	4	2	5	33	24	45	30
$n_2 < n_4 < n_3 < n_1$	2	4	4	2	5	33	23	28	26
$n_3 < n_1 < n_2 < n_4$	1	3	2	3	3	13	14	9	15
$n_3 < n_1 < n_4 < n_2$	3	6	3	4	6	61	82	51	63
$n_3 < n_2 < n_1 < n_4$	2	4	4	5	4	61	49	22	74
$n_3 < n_2 < n_4 < n_1$	2	4	5	4	5	81	59	28	74
$n_3 < n_4 < n_1 < n_2$	2	4	2	4	6	41	55	24	28
$n_3 < n_4 < n_2 < n_1$	2	4	3	4	5	49	47	24	43
$n_4 < n_1 < n_2 < n_3$	2	5	2	2	5	17	24	29	14
$n_4 < n_1 < n_3 < n_2$	2	4	2	2	4	13	19	16	12
$n_4 < n_2 < n_1 < n_3$	1	4	2	2	4	13	12	19	11
$n_4 < n_2 < n_3 < n_1$	1	3	2	2	4	13	11	12	9
$n_4 < n_3 < n_1 < n_2$	2	5	2	4	6	41	58	35	34
$n_4 < n_3 < n_2 < n_1$	1	4	2	4	6	41	34	29	27

$\mathrm{LM}(f_2) = X_1^{\alpha_{21}} X_4$ is divisible by X_1. This leads to a contradiction as [4, Lemma 2.7] implies that the tangent cone is not Cohen–Macaulay. Similarly, when $\alpha_{21} + \alpha_3 > \alpha_1$, $\mathrm{LM}(f_3) = X_1^{\alpha_1 - \alpha_{21} - 1} X_2$ is divisible by X_1. So, if the tangent cone is Cohen–Macaulay, then C(1.1) and C(1.2) must hold.

To show the last inequality holds, assume not: $\alpha_4 > \alpha_2 + \alpha_3 - 1$. Then $\mathrm{LM}(f_4) = X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3 - 1}$ is divisible by X_1. If $\alpha_1 > \alpha_{21} + 2$, f_4 is indispensable by Corollary 2.4 again. As before, the tangent cone is not Cohen–Macaulay, a contradiction. So, we must have $\alpha_1 = \alpha_{21} + 2$. In this case, there exists a binomial g in a minimal standard basis of I_S such that $\mathrm{LM}(g) \mid \mathrm{LM}(f_4)$ and $X_1 \nmid \mathrm{LM}(g)$. Hence $\mathrm{LM}(g) = X_2^a X_3^b$ with $0 < a \le \alpha_2 - 1$ and $0 < b \le \alpha_3 - 1$ since the case $a = 0$ contradicts with the minimality of d_2 and the case $b = 0$ contradicts with the minimality of d_3. By Proposition 2.3 and its proof, $M_S = \{d_1, d_2, d_3, d_5\}$ are the minimal degrees and the only degree that is smaller than d_4 is d_3. Since $\deg(g) < d_4$, we must have $d_3 < \deg(g) < d_4$. Hence, $\deg(g) - d_3 = a n_2 - (\alpha_3 - b) n_3 \in S$ with $0 < a < \alpha_2$ and $0 < \alpha_3 - b < \alpha_3$ but this contradicts to Lemma 2.2. So, C(1.3) must hold as well. □

Before we check if the conditions C(1.1), C(1.2) and C(1.3) are sufficient, we note the following.

Remark 3.3. $\alpha_1 \ge \alpha_4$ holds. Indeed, as f_1 is S-homogeneous and $n_1 < n_4$, we have $(\alpha_4 - 1) n_4 < n_3 + (\alpha_4 - 1) n_4 = \alpha_1 n_1 < \alpha_1 n_4$ implying $\alpha_1 > \alpha_4 - 1$.

Next, we compute a standard basis for I_S, when C(1.1), C(1.2) and C(1.3) hold.

Lemma 3.4. *If* C(1.1), C(1.2) *and* C(1.3) *hold, the set* $G = \{f_1, f_2, f_3, f_4, f_5\}$ *is a minimal standard basis for* I_S *with respect to a negative degree reverse lexicographical ordering making* X_1 *the smallest variable.*

Proof. We will apply standard basis algorithm to the set $G = \{f_1, f_2, f_3, f_4, f_5\}$ with the normal form algorithm NFMORA, see [18] for details. We need to show $NF(\mathrm{spoly}(f_i, f_j) | G) = 0$ for any $i \neq j$ with $1 \le i, j \le 5$. Observe that the conditions (C1.1) and (C1.3) imply that $\alpha_4 \le \alpha_{21} + \alpha_3 (*)$ and hence,

- $\mathrm{LM}(f_1) = \mathrm{LM}(X_1^{\alpha_1} - X_3 X_4^{\alpha_4 - 1}) = X_3 X_4^{\alpha_4 - 1}$, by Remark 3.3
- $\mathrm{LM}(f_2) = \mathrm{LM}(X_2^{\alpha_2} - X_1^{\alpha_{21}} X_4) = X_2^{\alpha_2}$, by (C1.1).

- $\mathrm{LM}(f_3) = \mathrm{LM}(X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2) = X_3^{\alpha_3}$, by (C1.2)
- $\mathrm{LM}(f_4) = \mathrm{LM}(X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}) = X_4^{\alpha_4}$, by (C1.3)
- $\mathrm{LM}(f_5) = \mathrm{LM}(X_1^{\alpha_{21}+1}X_3^{\alpha_3-1} - X_2X_4^{\alpha_4-1}) = X_2X_4^{\alpha_4-1}$, by (*).

Then we conclude the following:

- $NF(\mathrm{spoly}(f_i,f_j)|G) = 0$ as $\mathrm{LM}(f_i)$ and $\mathrm{LM}(f_j)$ are relatively prime, for $(i,j) \in \{(1,2),(2,3),(2,4),(3,4),(3,5)\}$.
- $\mathrm{spoly}(f_1,f_3) = X_1^{\alpha_1}X_3^{\alpha_3-1} - X_1^{\alpha_1-\alpha_{21}-1}X_2X_4^{\alpha_4-1}$ and by (*) its leading monomial is $X_1^{\alpha_1-\alpha_{21}-1}X_2X_4^{\alpha_4-1}$, which is divisible only by $\mathrm{LM}(f_5)$. As $\mathrm{ecart}(f_5) = \mathrm{ecart}(\mathrm{spoly}(f_1,f_3))$ and $\mathrm{spoly}(f_5,\mathrm{spoly}(f_1,f_3)) = 0$, we have

$$NF(\mathrm{spoly}(f_1,f_3)|G) = 0.$$

- $\mathrm{spoly}(f_1,f_4) = X_1^{\alpha_1}X_4 - X_1X_2^{\alpha_2-1}X_3^{\alpha_3}$.

$$\alpha_2 \leq \alpha_{21}+1 \text{ from (C1.1). Then,}$$
$$\alpha_2+\alpha_3 \leq \alpha_3+\alpha_{21}+1 \text{ then as } \alpha_3 \leq \alpha_1-\alpha_{21} \text{ from (C1.2)}$$
$$\alpha_2+\alpha_3 \leq \alpha_1+1.$$

As a result, $\mathrm{LM}(\mathrm{spoly}(f_1,f_4)) = X_1X_2^{\alpha_2-1}X_3^{\alpha_3}$. Only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1,f_4))$ and $\mathrm{ecart}(\mathrm{spoly}(f_1,f_4)) \geq \mathrm{ecart}(f_3)$. Then, $\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_4)) = X_1^{\alpha_1}X_4 - X_1^{\alpha_1-\alpha_{21}}X_2^{\alpha_2}$. As $\alpha_2 \leq \alpha_{21}+1$ from (C1.1), $\alpha_1-\alpha_{21}+\alpha_2 \leq \alpha_1+1$ and hence $\mathrm{LM}(\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_4))) = X_1^{\alpha_1-\alpha_{21}}X_2^{\alpha_2}$. Among the leading monomials of elements of G, only $\mathrm{LM}(f_2)$ divides this with $\mathrm{ecart}(f_2) = \alpha_{21}+1-\alpha_2 = \mathrm{ecart}(\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_4)))$. Then $\mathrm{spoly}(f_2,\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_4))) = 0$ implying

$$NF(\mathrm{spoly}(f_1,f_4)|G) = 0.$$

- $\mathrm{spoly}(f_1,f_5) = X_1^{\alpha_{21}+1}X_3^{\alpha_3} - X_1^{\alpha_1}X_2$ with $\mathrm{LM}(\mathrm{spoly}(f_1,f_5)) = X_1^{\alpha_{21}+1}X_3^{\alpha_3}$ by (C1.2). Only $\mathrm{LM}(f_3)$ divides this. As $\mathrm{ecart}(\mathrm{spoly}(f_1,f_5)) = \alpha_1-\alpha_{21}+\alpha_3 = \mathrm{ecart}(f_3)$ and $\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_5)) = 0$, $NF(\mathrm{spoly}(f_1,f_5)|G) = 0$.
- $\mathrm{spoly}(f_2,f_5) = X_1^{\alpha_{21}+1}X_2^{\alpha_2-1}X_3^{\alpha_3-1} - X_1^{\alpha_{21}}X_4^{\alpha_4}$. As (C1.3) implies $\alpha_{21}+\alpha_4 \leq \alpha_{21}+\alpha_2+\alpha_3-1$, $\mathrm{LM}(\mathrm{spoly}(f_2,f_5)) = X_1^{\alpha_{21}}X_4^{\alpha_4}$. Only $\mathrm{LM}(f_4)$ divides this. As $\mathrm{ecart}(\mathrm{spoly}(f_2,f_5)) = \alpha_2+\alpha_3-1-\alpha_4 = \mathrm{ecart}(f_4)$ and $\mathrm{spoly}(f_4,\mathrm{spoly}(f_2,f_5)) = 0$, $NF(\mathrm{spoly}(f_2,f_5)|G) = 0$. Finally,
- $\mathrm{spoly}(f_4,f_5) = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1}X_4 - X_1X_2^{\alpha_2}X_3^{\alpha_3-1}$. Then $\alpha_2 \leq \alpha_{21}+1$ implies $\alpha_2+\alpha_3 \leq \alpha_{21}+1+\alpha_3$ and hence $\mathrm{LM}(\mathrm{spoly}(f_4,f_5)) = X_1X_2^{\alpha_2}X_3^{\alpha_3-1}$. Only $\mathrm{LM}(f_2)$ divides this. Since $\mathrm{ecart}(\mathrm{spoly}(f_4,f_5)) = \alpha_{21}+1-\alpha_2 = \mathrm{ecart}(f_2)$ and $\mathrm{spoly}(f_2,\mathrm{spoly}(f_4,f_5)) = 0$, $NF(\mathrm{spoly}(f_4,f_5)|G) = 0$.

It is not hard to see that this standard basis is minimal, so we are done. □

We are now ready to give the complete characterization of the Cohen–Macaulayness of the tangent cone.

Theorem 3.5. *Suppose n_1 is the smallest number in $\{n_1,n_2,n_3,n_4\}$. The tangent cone of C_S is Cohen–Macaulay if and only if*
(C1.1) $\alpha_2 \leq \alpha_{21}+1$,
(C1.2) $\alpha_{21}+\alpha_3 \leq \alpha_1$,
(C1.3) $\alpha_4 \leq \alpha_2+\alpha_3-1$.

Proof. If the tangent cone of C_S is Cohen–Macaulay, then C(1.1), C(1.2) and C(1.3) hold, by Lemma 3.2. If C(1.1), C(1.2) and C(1.3) hold, then from Lemma 3.4, a minimal standard basis for I_S is $G = \{f_1,f_2,f_3,f_4,f_5\}$ and $X_1 \nmid \mathrm{LM}(f_i)$ for $i = 1,2,3,4,5$. Thus, it follows from [4, Lemma 2.7] that the tangent cone is Cohen–Macaulay. □

3.2. Cohen–Macaulayness of the tangent cone when n_2 is smallest

In this section, we deal with the Cohen–Macaulayness of the tangent cone when n_2 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. As before, we get the necessary conditions first.

Lemma 3.6. *Suppose n_2 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. If the tangent cone of the monomial curve C_S is Cohen–Macaulay, then the following must hold*
(C2.1) $\alpha_{21} + \alpha_3 \le \alpha_1$,
(C2.2) $\alpha_{21} + \alpha_3 \le \alpha_4$,
(C2.3) $\alpha_4 \le \alpha_2 + \alpha_3 - 1$,
(C2.4) $\alpha_{21} + \alpha_1 \le \alpha_4 + \alpha_2 - 1$.

Proof. If tangent cone is Cohen–Macaulay then C(2.1) and C(2.2) comes from the indispensability of f_3 and f_5. If $\alpha_1 > \alpha_{21} + 2$, f_4 is indispensable, in which case C(2.3) follows. If $\alpha_1 = \alpha_{21} + 2$, f_4 is not indispensable. To prove C(2.3) in this case, assume contrary that $\alpha_4 > \alpha_2 + \alpha_3 - 1$. Then $\mathrm{LM}(f_4) = X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}$. As $f_4 \in I_S$, there exists a binomial g in a minimal standard basis of I_S such that $\mathrm{LM}(g) \mid \mathrm{LM}(f_4)$ and as the tangent cone is Cohen–Macaulay $X_2 \nmid \mathrm{LM}(g)$. Hence $\mathrm{LM}(g) = X_1^aX_3^b$ with $a \le 1$ and $b \le \alpha_3 - 1$. Then $\deg(f_5) - \deg(g) = (\alpha_{21} + 1 - a)n_1 + (\alpha_3 - 1 - b)n_3 \in S$ but this contradicts with the minimality of $\deg(f_5)$. Hence, C(2.3) must hold.

For the last condition, the result follows immediately if $\alpha_4 \ge \alpha_1$, as in this case, $\alpha_{21}+\alpha_1 \le \alpha_4+\alpha_{21} \le \alpha_4 + \alpha_2 - 1$. When $\alpha_4 < \alpha_1$, assume contrary that $\alpha_{21} + \alpha_1 > \alpha_4 + \alpha_2 - 1$. Then, as $(\alpha_1 + \alpha_{21})n_1 = \alpha_2n_2 + n_3 + (\alpha_4 - 2)n_4$, the binomial $f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2}X_3X_4^{\alpha_4-2} \in I_S$ and $\mathrm{LM}(f_6) = X_2^{\alpha_2}X_3X_4^{\alpha_4-2}$ is divisible by X_2. As the tangent cone is Cohen–Macaulay there exists a nonzero polynomial f in a minimal standard basis of I_S such that $\mathrm{LM}(f) \mid \mathrm{LM}(f_6)$ and $X_2 \nmid \mathrm{LM}(f)$. This implies that $\mathrm{LM}(f) = X_3^aX_4^b$, where $a \le 1$ and $b \le \alpha_4 - 2$, and that $\deg(f_1) - \deg(f) = (1 - a)n_3 + (\alpha_4 - 1 - b)n_4$ is also in S which contradicts with the minimality of $\deg(f_1)$. Hence, C(2.4) must hold. □

Before computing a standard basis, we observe the following.

Remark 3.7. When n_2 is the smallest number in $\{n_1, n_2, n_3, n_4\}$, $\alpha_{21} + 1 \le \alpha_2$ holds automatically. Indeed, as f_2 is S-homogeneous, $\alpha_{21}n_1 < \alpha_{21}n_1 + n_4 = \alpha_2n_2 < \alpha_2n_1$ implying $\alpha_{21} < \alpha_2$.

Now, we compute a standard basis under the conditions C(2.1), C(2.2), C(2.3), and C(2.4).

Lemma 3.8. *Let n_2 be the smallest number in $\{n_1, n_2, n_3, n_4\}$ and*
(C2.1) $\alpha_{21} + \alpha_3 \le \alpha_1$,
(C2.2) $\alpha_{21} + \alpha_3 \le \alpha_4$,
(C2.3) $\alpha_4 \le \alpha_2 + \alpha_3 - 1$,
(C2.4) $\alpha_{21} + \alpha_1 \le \alpha_4 + \alpha_2 - 1$.
then a minimal standard basis for I_S is
(i) $\{f_1, f_2, f_3, f_4, f_5\}$ *if* $\alpha_1 \le \alpha_4$,
(ii) $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2}X_3X_4^{\alpha_4-2}\}$ *if* $\alpha_1 > \alpha_4$, *with respect to negative degree reverse lexicographical ordering with* $X_3, X_4 > X_1 > X_2$.

Proof. Omitted as it can be done similarly. □

We are now ready to give the full characterization.

Theorem 3.9. *Suppose n_2 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. Tangent cone of the monomial curve C_S is Cohen–Macaulay if and only if*
(C2.1) $\alpha_{21} + \alpha_3 \le \alpha_1$,

(C2.2) $\alpha_{21} + \alpha_3 \leq \alpha_4$,
(C2.3) $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$,
(C2.4) $\alpha_{21} + \alpha_1 \leq \alpha_4 + \alpha_2 - 1$.

Proof. If tangent cone is Cohen–Macaulay then C(2.1), C(2.2), C(2.3) and C(2.4) must hold by Lemma 3.6. If C(2.1), C(2.2), C(2.3) and C(2.4) hold, then a minimal standard basis with respect to the negative degree reverse lexicographic ordering making X_2 the smallest variable is $G = \{f_1, f_2, f_3, f_4, f_5\}$ in the case $\alpha_4 \geq \alpha_1$ and $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ in the case $\alpha_4 < \alpha_1$ from Lemma 3.8. X_2 does not divide $\mathrm{LM}(f_i)$ in both cases, so the tangent cone is Cohen–Macaulay by Arslan et al. [4, Lemma 2.7]. □

3.3. Cohen–Macaulayness of the tangent cone when n_3 is smallest

In this section, we deal with the Cohen–Macaulayness of the tangent cone when n_3 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. As before, we get the necessary conditions first.

Lemma 3.10. *Suppose n_3 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. If the tangent cone of the monomial curve C_S is Cohen–Macaulay, then the following must hold*
(C3.1) $\alpha_1 \leq \alpha_4$,
(C3.2) $\alpha_4 \leq \alpha_{21} + \alpha_3$,
(C3.3i) $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$ *if* $\alpha_1 - \alpha_{21} > 2$,
(C3.3ii) $\alpha_4 \leq \alpha_2 + 2\alpha_3 - 3$ *if* $\alpha_1 - \alpha_{21} = 2$,

Proof. If tangent cone is Cohen–Macaulay then C(3.1) and C(3.2) comes from the indispensability of f_1 and f_5. If $\alpha_1 > \alpha_{21} + 2$, f_4 is indispensable, in which case C(3.3*i*) follows. If $\alpha_1 = \alpha_{21} + 2$, f_4 is not indispensable. To prove C(3.3*ii*) in this case, assume contrary that $\alpha_4 > \alpha_2 + 2\alpha_3 - 3$. Then $\alpha_4 > \alpha_2 + \alpha_3 - 1$ and $\mathrm{LM}(f_4) = X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3 - 1}$. As $\mathrm{LM}(f_3) = X_1 X_2 \mid \mathrm{LM}(f_4)$, f_4 can not be in a minimal standard basis of I_S. It can not be in a minimal generating set since a minimal generating set would lie in a minimal standard basis. Since Betti S-degrees are invariant, there must be a binomial of degree d_4 in a minimal generating set. We prove that $f_4' = X_4^{\alpha_4} - X_2^{\alpha_2 - 2} X_3^{2\alpha_3 - 1}$ must belong to a minimal generating set and so to a minimal standard basis. This will follow from [7] and the claim that

$$\deg_S^{-1}(d_4) = \{X_4^{\alpha_4}\} \cup \{X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3 - 1}, X_2^{\alpha_2 - 2} X_3^{2\alpha_3 - 1}\}.$$

In order to prove the claim above, take $m \in \deg_S^{-1}(d_4)$. Since d_3 is the only S-degree smaller than d_4 and $\deg_S^{-1}(d_3) = \{X_3^{\alpha_3}, X_1 X_2\}$, it follows that $X_3^{\alpha_3} \mid m$ or $X_1 X_2 \mid m$ if $\deg_S(m) = d_4$. If $X_3^{\alpha_3} \mid m$, then $m = X_3^{\alpha_3} m'$. If $m' \neq X_2^{\alpha_2 - 2} X_3^{\alpha_3 - 1}$, then $m' - X_2^{\alpha_2 - 2} X_3^{\alpha_3 - 1} \in I_S$, as this binomial is S-homogeneous of S-degree $d = d_4 - d_3$. As d_3 is the only S-degree smaller than d_4, it follows that $d_3 <_S d <_S d_4$. So, $2d_3 <_S d_4$. On the other hand, by Lemma 2.2, we have

$$d_4 - 2d_3 = n_1 + (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3 - n_1 - n_2 - \alpha_3 n_3 = (\alpha_2 - 2)n_2 - n_3 \notin S.$$

Thus, $m' = X_2^{\alpha_2 - 2} X_3^{\alpha_3 - 1}$ and so $m = X_2^{\alpha_2 - 2} X_3^{2\alpha_3 - 1}$. By the same argument, if $X_1 X_2 \mid m$ then $m = X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3 - 1}$, hence the claim follows.

If $\alpha_4 > \alpha_2 + 2\alpha_3 - 3$, $\mathrm{LM}(f_4') = X_2^{\alpha_2 - 2} X_3^{2\alpha_3 - 1}$ is divisible by X_3, contradicting to the Cohen–Macaulayness of the tangent cone. So, C(3.3*ii*) follows. □

Before computing a standard basis, we observe the following.

Remark 3.11. When n_3 is the smallest number, $\alpha_1 - \alpha_{21} < \alpha_3$ holds automatically. Indeed, as f_3 is S-homogeneous, $(\alpha_1 - \alpha_{21})n_3 < (\alpha_1 - \alpha_{21} - 1)n_1 + n_2 = \alpha_3 n_3$.

Now, we compute a standard basis.

Lemma 3.12. *Let n_3 be the smallest number in $\{n_1, n_2, n_3, n_4\}$ and $\alpha_2 \le \alpha_{21} + 1$, then a minimal standard basis for I_S is*
(i) $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ *if* $C(3.1)$, $C(3.2)$ *and* $C(3.3i)$ *hold*
(ii) $\{f_1, f_2, f_3, f_4', f_5, f_6\}$ *when* $C(3.1)$, $C(3.2)$ *and* $C(3.3ii)$ *hold, with respect to negative degree reverse lexicographical ordering with* $X_2 > X_1, X_4 > X_3$, *where* $f_6 = X_1^{\alpha_1 - 1}X_4 - X_2^{\alpha_2 - 1}X_3^{\alpha_3}$.

Proof. Omitted as it can be done similarly. □

We are now ready to give a list of sufficient conditions.

Corollary 3.13. *Let n_3 is the smallest number and and $\alpha_2 \le \alpha_{21} + 1$.*
(i) *If* $C(3.1)$, $C(3.2)$ *and* $C(3.3i)$ *hold, then the tangent cone of the monomial curve* C_S *is Cohen–Macaulay.*
(ii) *When* $C(3.1)$, $C(3.2)$ *and* $C(3.3ii)$ *hold, the tangent cone of the monomial curve* C_S *is Cohen–Macaulay if and only if* $\alpha_1 \le \alpha_2 + \alpha_3 - 1$.

Proof.
(i) If C(3.1), C(3.2) and C(3.3*i*) hold, then a minimal standard basis with respect to the negative degree reverse lexicographic ordering making X_3 the smallest variable is $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ from Lemma 3.12. X_3 does not divide $\mathrm{LM}(f_i)$, so the tangent cone is Cohen–Macaulay by Arslan et al. [4, Lemma 2.7].

(ii) When C(3.1), C(3.2) and C(3.3*ii*) hold, then a minimal standard basis with respect to the negative degree reverse lexicographic ordering making X_3 the smallest variable is $G = \{f_1, f_2, f_3, f_4', f_5, f_6\}$ from Lemma 3.12. X_3 does not divide $\mathrm{LM}(f_i)$, for $i = 1, \ldots, 5$, so the tangent cone is Cohen–Macaulay by Arslan et al. [4, Lemma 2.7] if and only if X_3 does not divide $\mathrm{LM}(f_6)$ if and only if $\alpha_1 \le \alpha_2 + \alpha_3 - 1$. □

We finish the section by illustrating that $\alpha_2 \le \alpha_{21} + 1$ is not a necessary condition.

Example 3.14. Let $(\alpha_{21}, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2, 4, 5, 4, 5)$. Then $(n_1, n_2, n_3, n_4) = (81, 59, 28, 74)$. SINGULAR computes a minimal standard basis for I_S as $\{X_1X_2 - X_3^5, X_1^2X_4 - X_2^4, X_1^4 - X_3X_4^4, X_2^5 - X_1X_3^5X_4, X_2X_4^4 - X_1^3X_3^4, X_4^5 - X_1X_2^3X_3^4\}$ and thus I_{S^*} is generated by $G_* = \{X_1X_2, X_1^2X_4, X_1^4, X_2^5, X_2X_4^4, X_4^5\}$. As X_3 does not divide these elements, the tangent cone is Cohen–Macaulay from [4, Lemma 2.7].

3.4. Cohen–Macaulayness of the tangent cone when n_4 is smallest

We get some necessary conditions first as before.

Lemma 3.15. *Suppose n_4 is the smallest number in $\{n_1, n_2, n_3, n_4\}$. If the tangent cone of the monomial curve C_S is Cohen–Macaulay then*
(C4.1) $\alpha_1 \le \alpha_4$,
(C4.2) $\alpha_2 \le \alpha_{21} + 1$,
(C4.3) $\alpha_3 + \alpha_{21} \le \alpha_4$.

Proof. The results follow immediately from the indispensabilities of f_1, f_2 and f_5 respectively. □

Remark 3.16. If n_4 is the smallest number in $\{n_1, n_2, n_3, n_4\}$ then $\alpha_4 > \alpha_2 + \alpha_3 - 1$. Indeed, as f_4 is S-homogeneous and n_4 is the smallest number in $\{n_1, n_2, n_3, n_4\}$ $\alpha_4 n_4 = n_1 + (\alpha_2 - 1)n_2 + (\alpha_3 - 1)n_3 > (\alpha_2 + \alpha_3 - 1)n_4$ implying $\alpha_4 > \alpha_2 + \alpha_3 - 1$.

Lemma 3.17. *Let n_4 be the smallest number in $\{n_1, n_2, n_3, n_4\}$ and $\alpha_3 \leq \alpha_1 - \alpha_{21}$. If the conditions $C(4.1)$, $C(4.2)$ and $C(4.3)$ hold, then $\{f_1, f_2, f_3, f_4, f_5\}$ is a minimal standard basis for I_S with respect to negative degree reverse lexiographical ordering with $X_3 > X_1, X_2 > X_4$.*

Proof. Omitted as it can be done similarly. □

Corollary 3.18. *Let n_4 be the smallest number in $\{n_1, n_2, n_3, n_4\}$ and $\alpha_3 \leq \alpha_1 - \alpha_{21}$. If the conditions $C(4.1)$, $C(4.2)$ and $C(4.3)$ hold, then the tangent cone of C_S is Cohen–Macaulay.*

Proof. By hypothesis $\{f_1, f_2, f_3, f_4, f_5\}$ is a minimal standard basis for I_S with respect to negative degree reverse lexiographical ordering with X_4 the smallest variable from Lemma 3.17 and $X_4 \nmid \mathrm{LM}(f_i)$ for $i = 1, 2, 3, 4, 5$. Thus, it follows from [4, Lemma 2.7] that the tangent cone is Cohen–Macaulay. □

However, the tangent cone may be Cohen–Macaulay even if $\alpha_3 > \alpha_1 - \alpha_{21}$.

Example 3.19. Let $(\alpha_{21}, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (4, 7, 3, 4, 9)$. Then $(n_1, n_2, n_3, n_4) = (97, 154, 87, 74)$. SINGULAR computes a minimal standard basis for I_S as $\{X_1^2X_2 - X_3^4, X_2^3 - X_1^4X_4, X_1X_2^2X_3^3 - X_4^9, X_2^2X_3^4 - X_1^6X_4, X_1^7 - X_3X_4^8, X_1^5X_3^3 - X_2X_4^8, X_2X_3^7 - X_1X_4^9, X_1^3X_3^7 - X_2^2X_4^8, X_3^{11} - X_1^3X_4^9\}$ and so the ideal I_{S^*} is generated by the set $G_* = \{X_1^2X_2, X_2^3, X_1X_2^2X_3^3, X_2^2X_3^4, X_1^7, X_1^5X_3^3, X_2X_3^7, X_1^3X_3^7 - X_2^2X_4^8, X_3^{11}\}$. As X_4 does not divide elements, the tangent cone is Cohen–Macaulay from [4, Lemma 2.7].

Acknowledgments

We would like to thank F. Arslan, A. Katsampekis and M. Morales for very helpful suggestions. We also thank an anonymous referee for his/her comments that improved the presentation. Most of the examples are computed by using the computer algebra systems Macaulay 2 and singular, see [17, 19].

Funding

The authors were supported by the project 114F094 under the program 1001 of the Scientific and Technological Research Council of Turkey.

ORCID

Mesut Şahin http://orcid.org/0000-0002-0310-2542

References

[1] Arslan, F. (2000). Cohen-Macaulayness of tangent cones. *Proc. Am. Math. Soc.* 128:2243–2251.
[2] Arslan, F., Katsabekis, A., Nalbandiyan, M. On the Cohen-Macaulayness of tangent cones of monomial curves in $\mathbb{A}^4(K)$. arXiv:1512.04204.
[3] Arslan, F., Mete, P. (2007). Hilbert functions of Gorenstein monomial curves. *Proc. Am. Math. Soc.* 135:1993–2002.
[4] Arslan, F., Mete, P., Şahin, M. (2009). Gluing and Hilbert functions of monomial curves. *Proc. Am. Math. Soc.* 137:2225–2232.
[5] Barucci, V., Dobbs, D. E., Fontana, M. (1997). Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains. *Mem. Am. Math. Soc.* 125(598):x+78.
[6] Barucci, V., Fröberg, R., Şahin, M. (2014). On free resolutions of some semigroup rings. *J. Pure Appl. Algebra* 218(6):1107–1116.
[7] Charalambous, H., Katsabekis, A., Thoma, A. (2007). Minimal systems of binomial generators and the indispensable complex of a toric ideal. *Proc. Am. Math. Soc.* 135:3443–3451.
[8] Charalambous, H., Thoma, A. (2009). On simple A-multigraded minimal resolutions. *Contemp. Math.* 502:33–44.
[9] Charalambous, H., Thoma, A. (2010). On the generalized Scarf complex of lattice ideals. *J. Algebra* 323:1197–1211.
[10] Cortadellas, T., Jafari, R., Zarzuela, S. (2013). On the Apéry sets of monomial curves. *Semigroup Forum* 86(2): 289–320.

[11] Cortadellas, T., Zarzuela, S. (2011). Apery and micro-invariants of a one-dimensional Cohen-Macaulay local ring and invariants of its tangent cone. *J. Algebra* 328:94–113.
[12] D'Anna, M., Di Marca, M., Micale, V. (2015). On the Hilbert function of the tangent cone of a monomial curve. *Semigroup Forum* 91(3):718–730.
[13] Eakin, J., Sathaye, A. (1976). Prestable ideals. *J. Algebra* 41:439–454.
[14] Elias, J. (1993). The conjecture of Sally on the Hilbert function for curve singularities. *J. Algebra* 160(1):42–49.
[15] Garcia, A. (1982). Cohen-Macaulayness of the associated graded ring of a semigroup ring. *Commun. Algebra* 10:393–415.
[16] Garcia-Sanchez, P. A., Ojeda, I. (2010). Uniquely presented finitely generated commutative monoids. *Pac. J. Math.* 248:91–105.
[17] Grayson, D., Stillman, M. Macaulay 2–A system for computation in algebraic geometry and commutative algebra. Available at: http://www.math.uiuc.edu/Macaulay2.
[18] Greuel, G.-M., Pfister, G. (2002). *A Singular Introduction to Commutative Algebra*. Berlin: Springer-Verlag.
[19] Greuel, G.-M., Pfister, G., Schönemann, H. (2001). Singular 2.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern. Available at: http://www.singular.uni-kl.de.
[20] Gupta, S. K., Roberts, L. G. (1983). Cartesian squares and ordinary singularities of curves. *Commun. Algebra* 11(2):127–182.
[21] Herzog, J., Stamate, D. I. (2014). On the defining equations of the tangent cone of a numerical semigroup ring. *J. Algebra* 418:8–28.
[22] Herzog, J., Waldi, R. (1975). A note on the Hilbert function of a one-dimensional Cohen-Macaulay ring. *Manuscripta Math.* 16:251–260.
[23] Jafari, R., Zarzuela, S. (2014). On monomial curves obtained by gluing. *Semigroup Forum* 88(2):397–416.
[24] Katsabekis, A., Ojeda, I. (2014). An indispensable classification of monomial curves in $\mathbb{A}^4$. *Pac. J. Math.* 268:95–116.
[25] Komeda, J. (1982). On the existence of Weierstrass points with a certain semigroup. *Tsukuba J. Math.* 6(2):237–270.
[26] Matlis, E. (1973). *One-dimensional Cohen-Macaulay Rings*. Lecture Notes in Mathematics, Vol. 327. Berlin: Springer-Verlag.
[27] Oneto, A., Strazzanti, F., Tamone, G. (2017). One-dimensional Gorenstein local rings with decreasing Hilbert function. *J. Algebra* 489:91–114.
[28] Oneto, A., Tamone, G. (2016). On semigroup rings with decreasing Hilbert function. *J. Algebra* 489:373–398.
[29] Orecchia, O. (1980). One-dimensional local rings with reduced associated graded ring and their Hilbert functions. *Manuscripta Math.* 32:391–405.
[30] Patil, D. P., Tamone, G. (2011). CM defect and Hilbert functions of monomial curves. *J. Pure Appl. Algebra* 215:1539–1551.
[31] Rosales, J. C., Garcia-Sanchez, P. A. (2009). *Numerical Semigroups*. Developments in Mathematics, Vol. 20. New York: Springer.
[32] Rossi, M. E. (2011). Hilbert functions of Cohen-Macaulay rings. In: Alberto, C., Claudia, P., eds. *Commutative Algebra and Its Connections to Geometry - Contemporary Mathematics*, Vol. 555. Rhode Island: AMS, pp. 173–200.
[33] Sally, J. (1980). Good embedding dimensions for Gorenstein singularities. *Math. Ann.* 249:95–106.
[34] Shen, Y.-H. (2011). Tangent cone of numerical semigroup rings of embedding dimension three. *Commun. Algebra* 39(5):1922–1940.
[35] Shibuta, T. (2008). Cohen-Macaulayness of almost complete intersection tangent cones. *J. Algebra* 319:3222–3243.
[36] Takemura, A., Aoki, S. (2004). Some characterizations of minimal Markov basis for sampling from discrete conditional distributions. *Ann. Inst. Stat. Math.* 56(1):1–17.