Very Cleanness of Generalized Matrices

Yosum Kurtulmaz

Department of Mathematics, Bilkent University, Ankara, Turkey

yosum@fen.bilkent.edu.tr

Abstract

An element a in a ring R is very clean in case there exists an idempotent $e \in R$ such that $ae = ea$ and either $a - e$ or $a + e$ is invertible. An element a in a ring R is very J-clean provided that there exists an idempotent $e \in R$ such that $ae = ea$ and either $a - e \in J(R)$ or $a + e \in J(R)$. Let R be a local ring, and let $s \in C(R)$. We prove that $A \in K_s(R)$ is very clean if and only if $A \in U(K_s(R))$; $I \pm A \in U(K_s(R))$ or $A \in K_s(R)$ is very J-clean.

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1 Introduction

Throughout this paper all rings are associative with identity. Let R be a ring. Let $C(R)$ be the center of R and $s \in C(R)$. The set containing all 2×2 matrices $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with usual matrix addition and multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

This ring is denoted by $K_s(R)$ and the element s is called the *multiplier* of $K_s(R)$ [3].

Let A, B be rings, ${}_AM_B$ and ${}_BN_A$ be bimodules. A *Morita context* is a 4-tuple $A = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w, z) \to wz$ and $(z, w) \to zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix* ring over R. Thus the ring $K_s(R)$ can be viewed as a special kind of Morita context. It was observed by Krylov [3] that the generalized matrix rings over R are precisely these rings $K_s(R)$ with $s \in C(R)$. When $s = 1, K_1(R)$ is just the matrix ring $M_2(R)$, but $K_s(R)$ can be different from $M_2(R)$. In fact, for a local ring R and $s \in C(R)$, $K_s(R) \cong K_1(R)$ if and only if s is a unit see ([3], Lemma 3 and Corollary 2) and ([4], Corollary 4.10).
In [5], it is said that that an element $a \in R$ is *strongly clean* provided that there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $eu = ue$ and a ring R is called *strongly clean* in case every element in R is strongly clean. In [1], very clean rings are introduced. An element $a \in R$ is *very clean* provided that either a or $-a$ is strongly clean. A ring R is *very clean* in case every element in R is very clean. It is explored the necessary and sufficient conditions under which a triangular 2×2 matrix ring over local rings is very clean. The very clean 2×2 matrices over commutative local rings are completely determined. Motivated by this general setting, the aim of this paper is to investigate the very cleanness of 2×2 generalized matrix rings.
For elements $a, b \in R$, we say that a is equivalent to b if there exist units u, v such that $b = uav$; we use the notation $a \sim b$ to mean that a is similar to b, that is, $b = u^{-1}au$ for some unit u.

Throughout this paper, $M_n(R)$ and $T_n(R)$ denote the ring of all $n\times n$ matrices and the ring of all $n \times n$ upper triangular matrices over R, respectively. We write $R[[x]], U(R)$ and $J(R)$ for the power series ring R, group of units and the Jacobson radical of R, respectively. For $A \in M_n(R), \chi(A)$ stands for the characteristic polynomial $det(tI_n - A)$. Let $\mathbb{Z}(p)$ be the localization of $\mathbb{Z}$

at the prime ideal generated by the prime p.

2 Very Clean Elements

A ring R is *local* if it has only one maximal ideal. It is well known that, a ring R is *local* if and only if $a+b=1$ in R implies that either a or b is invertible. The aim of this section is to investigate elementary properties of very clean matrices over local rings.

Lemma 2.1 ([7], Lemma 1) *Let R be a ring and let $s \in C(R)$. Then* $\begin{pmatrix} a & x \\ y & b \end{pmatrix} \rightarrow \begin{pmatrix} b & y \\ x & a \end{pmatrix}$ *is an automorphism of $K_s(R)$.*

Lemma 2.2 ([7], Lemma 2) *Let R be a ring and $s \in C(R)$. Then the following hold*

(1) $J(K_s(R)) = \begin{pmatrix} J(R) & (s:J(R)) \\ (s:J(R)) & J(R) \end{pmatrix}$, *where* $(s:J(R)) = \{r \in R | rs \in J(R)\}$.

(2) If R is a local ring with $s \in J(R)$, then $J(K_s(R)) = \begin{pmatrix} J(R) & R \\ R & J(R) \end{pmatrix}$ *and moreover* $\begin{pmatrix} a & x \\ y & b \end{pmatrix} \in U(K_s(R))$ *if and only if $a, b \in U(R)$.*

Lemma 2.3 ([7], Lemma 3) *Let $E^2 = E \in K_s(R)$. If E is equivalent to a diagonal matrix in $K_s(R)$, then E is similar to a diagonal matrix in $K_s(R)$.*

Lemma 2.4 *Let R be a local ring with $s \in C(R)$ and let E be a non-trivial idempotent of $K_s(R)$. Then we have the following.*

(1) If $s \in U(R)$, then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(2) If $s \in J(R)$, then either $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Let $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in R$. Since $E^2 = E$, we have

$$a^2 + sbc = a, \ scb + d^2 = d, \ ab + bd = b, \ ca + dc = c \tag{1}$$

If $a, d \in J(R)$, then $b, c \in J(R)$ and so $E \in J\big(M_2(R; s)\big)$. Hence $E = 0$, a contradiction. Since R is local, we have $a \in U(R)$ or $d \in U(R)$.
Assume that $a \in U(R)$. Then

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^{-1} & a^{-1}b \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & sca^{-1} - d \end{pmatrix} \tag{2}$$

Hence E is equivalent to a diagonal matrix.
Now suppose that $d \in U(R)$. Then

$$\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d^{-1}c & d^{-1} \end{pmatrix} = \begin{pmatrix} a - sbd^{-1}c & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

Hence E is equivalent to a diagonal matrix. According to Lemma 2.3, there exist $P \in U\big(K_s(R)\big)$ and idempotents $f, g \in R$ such that

$$PEP^{-1} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \tag{4}$$

To complete the proof we shall discuss four cases $f = 1$ and $g = 0$ or $f = 0$ and $g = 1$ or $f = 1$ and $g = 1$ or $f = 0$ and $g = 0$. However, E is a non-trivial idempotent matrix, we may discard the latter two cases. Since R is local, $s \in U(R)$ or $s \in J(R)$. We divide the proof into some cases:
(A) Assume that $s \in U(R)$.
Case (i). $f = 1$ and $g = 0$. Then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
Case (ii). $f = 0$ and $g = 1$. Then $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. But since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & s^{-1} \\ s^{-1} & 0 \end{pmatrix}$, we have that $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This proves (1).
(B) Assume that $s \in J(R)$.
Case (iii). $f = 1$ and $g = 0$. Then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
Case (iv). $f = 0$ and $g = 1$. Then $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
To complete the proof of (B), we prove that only one of $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is valid. Indeed, if otherwise, $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. That is, there exists $P = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in U\big(K_s(R)\big)$ such that $P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}P$. By direct calculation one sees that $x = t = 0$. But since $P \in U\big(K_s(R)\big)$ and $s \in J(R)$, we get $x, t \in U(R)$ by Lemma 2.2, a contradiction. This holds (2). $\square$

Lemma 2.5 *Let R be a ring and $s \in C(R)$. Then $A \in K_s(R)$ is very clean if and only if for each invertible $P \in K_s(R)$, $PAP^{-1} \in K_s(R)$ is very clean.*

Proof. If PAP^{-1} is very clean in $K_s(R)$, then either PAP^{-1} or $-PAP^{-1}$is strongly clean for some $P \in U(K_s(R))$. Suppose that $PAP^{-1}$is strongly clean in $K_s(R)$. Then there exist $E^2 = E$, $U \in U\big(K_s(R)\big)$ such that $PAP^{-1} = E + U$ and $EU = UE$. Then $A = P^{-1}EP + P^{-1}UP$, $(P^{-1}EP)^2 = P^{-1}EP$, $P^{-1}UP \in U\big(K_s(R)\big)$, $P^{-1}EP$ and $P^{-1}UP$ commute; $\big(P^{-1}EP\big)\big(P^{-1}UP\big) = P^{-1}EUP = P^{-1}UEP = \big(P^{-1}UP\big)\big(P^{-1}EP\big)$. So A is strongly clean. If $-PAP^{-1}$ is very clean in $K_s(R)$, then $-A$ is strongly clean by using the similar argument. Hence A is very clean. Conversely assume that $A \in K_s(R)$ is very clean i.e. either A or $-A$ is strongly clean. Suppose that $-A$ is strongly clean. There exist $F^2 = F \in K_s(R)$ and $W \in U\big(K_s(R)\big)$ such that $-A = F + W$ with $FW = WF$. Let $P \in K_s(R)$ be an invertible matrix. $P^{-1}(-A)P = P^{-1}FP + P^{-1}WP$ is strongly clean since $P^{-1}FP$ is an idempotent, $P^{-1}WP \in U(K_s(R))$, $P^{-1}FP$ and $P^{-1}WP$ commute.

Similarly, strong cleanness of A implies strong cleanness of $P^{-1}AP$. This completes the proof. $\square$

Lemma 2.6 *Let R be a local ring and $s \in C(R)$. Then $A \in K_s(R)$ is very clean if and only if either*

(1) $I \pm A \in U(K_s(R))$, or

(2) $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, where $v \in J(R)$, $w \in \pm 1 + J(R)$ and $s \in U(R)$, or

(3) either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ or $A \sim \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$, where $v \in J(R)$, $w \in \pm 1 + J(R)$ and $s \in J(R)$.

Proof. " $\Leftarrow$: " If $I \pm A \in U(K_s(R))$, then A is obviously very clean. If $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, where $v \in J(R)$, $w \in \pm 1 + J(R)$ and $s \in U(R)$, then $\begin{pmatrix} v-1 & 0 \\ 0 & w \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, $\begin{pmatrix} v-1 & 0 \\ 0 & w \end{pmatrix}$ is invertible and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is idempotent. Then $\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ is strongly clean. Similarly $\begin{pmatrix} -v & 0 \\ 0 & -w \end{pmatrix}$ is strongly clean. Since either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ or $A \sim \begin{pmatrix} -v & 0 \\ 0 & -w \end{pmatrix}$ we have $PAP^{-1} = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ is very clean. By Lemma 2.5, A is very clean.

Similarly, if either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ or $A \sim \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$, where $v \in J(R)$, $w \in \pm 1 + J(R)$ and $s \in J(R)$, then A is very clean.
" $\Rightarrow$: " Assume that A is very clean and $\pm A, I \pm A \notin U(K_s(R))$. Then either $A - E$ or $A + E$ is in $U(K_s(R))$ where $E^2 = E \in K_s(R)$.
Case 1. If $A - E$ is in $U(K_s(R))$, then $A - E = V$ and $EV = VE$, where $V \in U(K_s(R))$. If $s \in U(R)$, then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by Lemma 2.4. Then there exists

$P \in U(K_s(R))$ such that $PEP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. From Lemma 2.5, $PAP^{-1} - PEP^{-1} = PVP^{-1}$ is very clean. Let $W = [w_{ij}] = PVP^{-1}$ and $PEP^{-1} = F$. Since $WF = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = FW$,we find $w_{12} = w_{21} = 0$ and $w_{11}, w_{22} \in U(R)$. Hence $A \sim \begin{pmatrix} w_{11}+1 & 0 \\ 0 & w_{22} \end{pmatrix} = B$. Note that $A \in U(K_s(R))$ if and only if $PAP^{-1} \in U(K_s(R))$. This gives that $B \notin U(K_s(R))$ and $I \pm B \notin U(K_s(R))$. Since R is local, we have $w_{22} \in \pm 1 + J(R)$ and $\pm 1 + w_{11} \in J(R)$. If $s \in J(R)$, then either $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ by Lemma 2.4. Using the previous argument, one can easily show that either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ or $\begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$ where $v \in \pm 1 + J(R)$ and $w \in J(R)$.

Case 2. If $A + E$ is in $U(K_s(R))$, then $A + E = V$ and $EV = VE$, where $V \in U(K_s(R))$.
If $s \in U(R)$, then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by Lemma 2.5. Then there exists $P \in U(K_s(R))$ such that $PAP^{-1} + PEP^{-1} = PVP^{-1}$. Let $W = [w_{ij}] = PVP^{-1}$ and $PEP^{-1} = F$. Since $WF = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = FW$, we find $w_{12} = w_{21} = 0$ and $w_{11}, w_{22} \in U(R)$. Thus $A \sim \begin{pmatrix} w_{11}-1 & 0 \\ 0 & w_{22} \end{pmatrix} = B$. Note that $A \in U(K_s(R))$ if and only if $PAP^{-1} \in U(K_s(R))$. This gives that $B \notin U(K_s(R))$ and $I \pm B \notin U(K_s(R))$. Since R is local, we have $w_{22} \in \pm 1 + J(R)$ and $1 + w_{11} \in J(R)$. If $s \in J(R)$, then either $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ by Lemma 2.5. In this case, using the previous argument, one can easily show that either $A \sim \begin{pmatrix} w_{11}-1 & 0 \\ 0 & w_{22} \end{pmatrix}$ or $A \sim \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22}-1 \end{pmatrix}$. □

3 Very J-clean element

Let R be a ring. In [2], an element $a \in R$ is said to be *strongly J-clean* provided that there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$. A ring R is *strongly J-clean* in case every element in R is strongly J-clean. We say that an element $a \in R$ is *very J-clean* if there exists an idempotent $e \in R$ such that $ae = ea$ and either $a - e \in J(R)$ or $a + e \in J(R)$. A ring R is *very J-clean* in case every element in R is very J-clean. A very J-clean ring need not be strongly J-clean. For example $\mathbb{Z}_{(3)}$ is very J-clean but not strongly J-clean.

Lemma 3.1 *Every very J-clean element is very clean.*

Proof. Let $e^2 = e \in R$ and $w \in J(R)$. If $x - e = w$,then $x - (1 - e) = 2e - 1 + w \in U(R)$ since $(2e - 1)^2 = 1$. Similarly if $x + e = w$, then $x + (1 - e) = 1 - 2e + w \in U(R)$ since $(1 - 2e)^2 = 1$. □

The converse statement of Lemma 3.1 need not hold in general.

Example 3.2 Let S be a commutative local ring and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ be in $R = M_2(S)$. A is an invertible matrix and it is very clean. Since R is a 2-projective-free ring, by [6, Proposition 2.1], it is easily checked that any idempotent E in R is one of the following :

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $x \in S$. But A is not very J-clean since neither of the above mentioned idempotents E does not satisfy $A - E \notin J(R)$ or $A + E \notin J(R)$.

Lemma 3.3 *Let R be a ring and $s \in C(R)$. Then $A \in K_s(R)$ is very J-clean if and only if $PAP^{-1} \in K_s(R)$ is very J-clean for some $P \in U(K_s(R))$.*

Proof. " $:\Rightarrow$ " Assume that $A \in K_s(R)$ is very J-clean. Then there exists $E^2 = E \in K_s(R)$ such that $A - E = W \in J(K_s(R))$ or $A + E = W \in J(K_s(R))$ and $EW = WE$. Let $F = PEP^{-1}$ and $V = PWP^{-1}$. Then $F^2 = F$, $V \in J(K_s(R))$ and $FV = VF$. If $A - E = W \in J(K_s(R))$, then $PAP^{-1} - F = V \in J(K_s(R))$. Thus PAP^{-1} is very J-clean. The same result is obtained when $A + E \in J(K_s(R))$.
" $\Leftarrow:$ " Assume that PAP^{-1} is very J-clean for some $P \in U(K_s(R))$. Then by using a similar argument, A is very J-clean. $\square$

Lemma 3.4 *Let R be a local ring and $s \in C(R)$. Then $A \in K_s(R)$ is very J-clean if and only if either*

(1) $I \pm A \in J(K_s(R))$, *or*

(2) $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, *where* $v \in \pm 1 + J(R)$, $w \in J(R)$ *and* $s \in U(R)$, *or*

(3) either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ *or* $A \sim \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$, *where* $v \in \pm 1 + J(R)$, $w \in J(R)$ *and* $s \in J(R)$.

Proof. " $\Leftarrow:$ " If either $I \pm A \in J(K_s(R))$, then A is obviously very J-clean. If $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, where $v \in \pm 1 + J(R)$, $w \in J(R)$ and $s \in U(R)$, then $\begin{pmatrix} v+1 & 0 \\ 0 & w \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in J(K_s(R))$. Then by Lemma 3.3, A is very J-clean. Similarly, if either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ or $A \sim \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$, where $v \in \pm 1 + J(R)$, $w \in J(R)$ and $s \in J(R)$, then A is very J-clean.
" $\Rightarrow:$ " Assume that A is very J-clean and $I \pm A \notin J(K_s(R))$. Then either $A - E$ or $A + E$ is in $J(K_s(R))$ where $E^2 = E \in K_s(R)$ is a non-trivial idempotent.
Case 1.If $A - E$ is in $J(K_s(R))$, then $A - E = M$ and $EM = ME$, where $M \in J(K_s(R))$. If $s \in U(R)$, then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by Lemma 2.4. Then there

exists $P \in U(K_s(R))$ such that $PEP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = F$. From Lemma 3.3, $PAP^{-1} - PEP^{-1} = PMP^{-1}$ is very J-clean. Let $v = [v_{ij}] = PMP^{-1}$. Since $VF = FV$, we find $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in J(R)$. Hence $A \sim \begin{pmatrix} v_{11}+1 & 0 \\ 0 & v_{22} \end{pmatrix}$. If $s \in J(R)$, then either $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ by Lemma 2.4. Using the previous argument, one can easily show that either $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ or $\begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$ where $v \in \pm 1 + J(R)$ and $w \in J(R)$.

Case 2. If $A + E$ is in $J(K_s(R))$, then $A + E = M$ and $EM = ME$, where $M \in J(K_s(R))$.
If $s \in U(R)$, then $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by Lemma 2.4. Then there exists $P \in U(K_s(R))$ such that $PAP^{-1} + PEP^{-1} = PVP^{-1}$. Let $V = [v_{ij}] = PVP^{-1}$ and $PEP^{-1} = F$. Since $VF = FV$, we find $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in J(R)$. Thus $A \sim \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, where $v = v_{11} - 1 \in \pm 1 + J(R), w = v_{22} \in J(R)$. Similarly, if $s \in J(R)$, then either $E \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $E \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ by Lemma 2.4. In this case, using the previous argument, one can easily show that either $A \sim \begin{pmatrix} v_{11}-1 & 0 \\ 0 & v_{22} \end{pmatrix}$ or $A \sim \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22}-1 \end{pmatrix}$. □

Theorem 3.5 *Let R be a local ring, and let $s \in C(R)$. Then $A \in K_s(R)$ is very clean if and only if $A \in U(K_s(R)), I \pm A \in U(K_s(R))$ or $A \in K_s(R)$ is very J-clean.*

Proof. The proof is clear by combining Lemma 2.6 and Lemma 3.4. □

Lemma 3.6 *Let R be a local ring with $s \in C(R) \cap J(R)$, and $A \in K_s(R)$ be very J-clean. Then either $I \pm A \in J(K_s(R))$ or $A \sim \begin{pmatrix} w & 1 \\ v & u \end{pmatrix}$ or $A \sim \begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in \pm 1 + J(R)$, $v \in U(R)$ and $w \in J(R)$.*

Proof. Assume that $I \pm A \notin J(K_s(R))$. By Lemma 2.6 either $A \sim \begin{pmatrix} v_1 \pm 1 & 0 \\ 0 & w_1 \end{pmatrix}$ or $A \sim \begin{pmatrix} v_1 & 0 \\ 0 & w_1 \pm 1 \end{pmatrix}$, where $v_1, w_1 \in J(R)$ and $s \in J(R)$.
Case 1 : Let $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a = v_1 \in J(R)$, $b = w_1 \pm 1 \in \pm 1 + J(R)$. Clearly $b - a \in \pm 1 + J(R) = U(R)$.
$B \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}$
$\sim \begin{pmatrix} 1 & 0 \\ -b & b-a \end{pmatrix} \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (b-a)^{-1}b & (b-a)^{-1} \end{pmatrix}$
$= \begin{pmatrix} a+sb & 1 \\ (b-a)b(b-a)^{-1}b - ba - sb^2 & (b-a)b(b-a)^{-1} - sb \end{pmatrix}$, where $u = a + sb \in J(R), v = (b-a)b(b-a)^{-1}b - ba - sb^2 \in U(R)$ and $w = (b-a)b(b-a)^{-1} - sb \in \pm 1 + J(R)$. Thus $A \sim \begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ where $u \in J(R), v \in U(R)$ and $w \in \pm 1 + J(R)$.
Case 2. Let $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c = 1 + v_1 \in \pm + J(R), d = w_1 \in J(R)$. Similarly, we show that $A \sim \begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ where $u \in \pm 1 + J(R), v \in U(R)$ and $w \in J(R)$. $\square$

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