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BRIEF COMMUNICATIONS

Overlapping of Functions

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The *overlapping* of a function means that the graph of its convexification is contained in the convex hull of the graph of the function. Here the *convexification* of a given function is the greatest lower semicontinuous function that is not greater than the given function. Probably, it was Shoples and Shubik [1] who first discussed overlapping, at least in the context of mathematical economics (kernels and competitive balance). Shoples and Shubik pointed out that the overlapping condition can be used to regenerate a certain regularity lost when there are no concavity assumptions (the concavity of the utility function or the convexity of the preference pattern). Regularity is potentially useful in situations other than the direct context of kernels and competitive balances and seems to be essential for any analysis of the relaxation of convexity conditions.

Some sufficient conditions for the overlapping of functions are contained in [1, Theorem 3]. Another sufficient condition was found in connection with the extension of variational problems [2, p. 278]. The author is not aware of other results concerning the overlapping of functions.

The present note is devoted to the study of the overlapping of functions. We obtain a necessary and sufficient overlapping condition for a lower semicontinuous function whose graph contains no lines (Theorem 1). Both results mentioned above are corollaries of Theorem 1. We also study the case of a function whose graph contains lines (Theorem 2).

As usual, by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ we denote the extended real axis, by $\mathbb{R}^n$ real n-dimensional Euclidean space, by $\operatorname{epi} f$, respectively $\operatorname{gr} f$, the epigraph, respectively, the graph, of a function f, and by f^{**} the greatest lower semicontinuous function that is not greater than f. The often used notation f^{**} is suggested by the well known fact that f^{**} is the second Legendre adjoint function of f. By $\operatorname{int} A$ and ∂A we denote the interior and the boundary of a subset $A \subset \mathbb{R}^n$. A point a of a convex set A is said to be an *extreme point* if it can not be represented as a convex combination of two points of A different from a. The definition of an *extreme ray* of a set A is similar. A *forward point* is an extreme point a such that there exists a plane of support passing through a and containing no points of A that differ from a. Also recall that a convex subset $A' \subset A$ is said to be a *face* if the endpoints of any segment of A belong to A' whenever some relatively interior point of this segment belongs to A'.

For a subspace $L \subset \mathbb{R}^n$, by $L^{\perp}$ and P_L we denote its orthogonal complement and the orthogonal projector onto L. A subset M_0 of an affine subspace $M \subset \mathbb{R}^n$ is called an *enveloping subset* if its convex hull is the whole set M.

Definition. A function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *overlapped* if for an arbitrary point $x \in \mathbb{R}^n$ there exist points $x_1, \dots, x_m \in \mathbb{R}^n$ and positive numbers $\lambda_1, \dots, \lambda_m$ (where $\lambda_1 + \dots + \lambda_m = 1$) such that

$$x = \sum_{i=1}^{m} \lambda_i x_i \qquad \text{and} \qquad f^{**}(x) = \sum_{i=1}^{m} \lambda_i f(x_i), \tag{1}$$

i.e., the graph of f^{**} is contained in the convex hull of the graph of f.

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It is easy to show that the integer m here can be bounded by $n+1$.

Theorem 1. *Suppose that a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is lower semicontinuous and linearly bounded below, and $\operatorname{epi} f^{**}$ does not contain lines. Then the function f is overlapped if and only if its graph contains an unbounded subset of each nonvertical extreme ray of the epigraph of f^{**}.*

The condition on the function f that is formulated in the statement of Theorem 1 will be called the *Condition.*

Proof. *Necessity.* Assume that the condition of Theorem 1 does not hold, i.e., there exists a nonvertical extreme ray $\overline{e}$ of an epigraph $\operatorname{epi} f^{**}$ such that the set $\overline{e} \cap \operatorname{gr} f$ is bounded. By e denote the unit vector codirected with the ray $\overline{e}$. Let $\overline{z}$ be the vertex of $\overline{e}$ and z_0 a point of $\overline{e}$ such that the ray $\{(z_0 - e) + \lambda e : \alpha \geq 0\}$ is contained in $\overline{e}$ and does not intersect the graph of f:

$$\{(z_0 - e) + \alpha e : \alpha \geq 0\} \cap \operatorname{gr} f = \varnothing. \tag{2}$$

Then the point $z_0 = (x_0, f^{**}(x_0))$ has no representation of type (1). In fact, if such a representation exists, then z_0 is a relative interior point of the polyhedron[1]

$$P = \operatorname{conv}\{(x_k, f(x_k)) : k = 1, \dots, m\}$$

because λ_k, $k = 1, \dots, m$, are strictly positive numbers. Since $\overline{e}$ is a face of $\operatorname{epi} f^{**}$, Theorem 18.1 [3] implies that the polyhedron P is contained in $\overline{e}$. Combining this with (2), we obtain

$$z_k := (x_k, f(x_k)) \in [\overline{z}, z_0 - e], \qquad \text{where} \quad k = 1, \dots, m.$$

Then the polyhedron P is also contained in the convex set $[\overline{z}, z_0 - e]$, and thus $z_0 \notin P$. This contradicts the inclusion $z_0 \in \operatorname{ri} P$.

Sufficiency. The following statement is an essential part of the proof of sufficiency.

Proposition 1. *Suppose that a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfies the assumptions of Theorem 1. Then $\operatorname{gr} f \supset \operatorname{ex}(\operatorname{epi} f)$, i.e., f and f^{**} have the same restrictions to the projection of the set of extreme points of $\operatorname{epi} f^{**}$ on $\mathbb{R}^n$.*

Further, the proof of Proposition 1 is based on the following statement (which is of interest in itself).

Proposition 2. *Suppose that a lower semicontinuous nonnegative function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfies the Condition. Then the zero set of the function f^{**} coincides with the convex hull of the zero set of f.*

Now if *Condition* holds, then it is obvious that any nonvertical extreme ray of a convex set $\operatorname{epi} f^{**}$ is contained in $\operatorname{co}(\operatorname{gr} f)$. By Proposition 1, $\operatorname{gr} f$ contains all the extreme points of a convex set $\operatorname{epi} f^{**}$. This implies, in particular, that every vertical extreme ray of $\operatorname{epi} f^{**}$ is contained in $\operatorname{epi} f$. Since $\operatorname{epi} f^{**}$ does not contain lines, Theorem 18.5 from [3] implies that $\operatorname{conv}(\operatorname{gr} f \cup V) \supset \operatorname{epi} f^{**}$, and so $\operatorname{conv}(\operatorname{epi} f \cup V) \supset \operatorname{gr} f^{**}$. Here by V we denote the union of all vertical extreme rays of $\operatorname{epi} f^{**}$.

Therefore, $(x_0, f^{**}(x_0)) \in \operatorname{co}(\operatorname{epi} f)$ whenever $x_0 \in \operatorname{dom} f^{**}$. Then there exist points $x_1, \dots, x_m \in \mathbb{R}^n$ and numbers $\lambda_1, \dots, \lambda_m > 0$ and $\alpha_1, \dots, \alpha_m > 0$ such that $\sum_{k=1}^m \lambda_k = 1$ and

$$(x_0, f^{**}(x_0)) = \sum_{k=1}^m \lambda_k (x_k, f(x_k) + \alpha_k).$$

If some of the numbers α_k, $k = 1, \dots, m$, are positive, then we have

$$x_0 = \sum_{k=1}^m \lambda_k x_k \qquad \text{and} \qquad f^{**}(x_0) > \sum_{k=1}^m \lambda_k f(x_k).$$

This contradicts the definition of the convexification of f^{**}.

Now let us draw a hyperplane of support of the convex set at $(x_0, f^{**}(x_0))$. Using the Carathéodory theorem (see [3]), it can be easily shown that the number m in the representation (1) can be chosen so that $m \leq n+1$. $\square$

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[1] *Editor's note.* The author uses the standard notation of convex analysis (conv , ri , etc., cf. [3]).

Corollary 1. *Suppose that* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *is a lower semicontinuous function linearly bounded below and such that the set* $\operatorname{dom} f \subset \mathbb{R}^n$ *contains no lines. Then the function* f *is overlapped if and only if its graph contains an unbounded subset of each nonvertical extreme ray of* $\operatorname{epi} f^{**}$.

Proof. It is easy to see that the set $\operatorname{epi} f^{**}$ does not contain nonvertical lines. Indeed, otherwise the projection of this line on $\mathbb{R}^n$ is a line contained in $\operatorname{dom} f$. Obviously, $\operatorname{epi} f^{**}$ contains no vertical lines also. The application of Theorem 1 completes the proof. □

To formulate the next corollary we need some definitions.

A function $f\colon \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ is said to be *sublinear* if for any linear function l with nonnegative coefficients the difference $f - l$ is bounded below by a constant. A function f is said to be *strictly decreasing* if $f(x) < f(y)$ whenever $x, y \in \mathbb{R}^n_+$ and $y - x \in \mathbb{R}^n_+ \setminus \{0\}$.

Corollary 2 (Shoples and Shubik [1, Theorem 3]). *If a function* $f\colon \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ *is continuous, sublinear, and strictly decreasing, then* f *is overlapped.*

Corollary 3. *If a lower semicontinuous function* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *satisfies the condition*

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty, \tag{3}$$

then f *is overlapped.*

Proof. It is clear that f is bounded below. It is easy to see that the convexification of f^{**} also satisfies condition (3). Thus, $\operatorname{epi} f^{**}$ does not contain nonvertical extreme rays. Theorem 1 implies that the function f is overlapped. □

Corollary 3 is stronger than the above mentioned Ekeland–Temam lemma [2, p. 278]; in this lemma the following more severe constraint is assumed instead of the growth condition (3):

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|^\alpha} = \infty \qquad (\alpha > 1).$$

The following example shows that the growth condition is essential.

Example. Let

$$f(x) = \begin{cases} 2|x| & \text{for } |x| \le 1, \\ |x| + 1 & \text{for } |x| > 1. \end{cases}$$

Then $f^{**}(x) = |x|$ and f is not overlapped.

Theorem 2. *Let* $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a function linearly bounded below with nontrivial space of linearity of the epigraph* $\operatorname{epi} f^{**}$. *Then* f *is overlapped if and only if the two following conditions hold: for each extreme ray of* $\operatorname{epi} f^{**} \cap L$ *the projection* $P_{L^\perp}(\operatorname{gr} f \cap \operatorname{epi} f^{**})$ *contains an unbounded subset of each extreme ray* $\operatorname{epi} f^{**} \cap L^\perp$ *and for any extreme point* $\overline{z}$ *of the set* $\operatorname{epi} f^{**} \cap L^\perp$ *the graph of the function contains an enveloping subset of* $\overline{z} + L$.

The proof of this theorem is based on Theorem 1 and Theorem 8.1 from [3].

References

1. L. S. Shoples and M. Shubik, *Econometrica*, **34**, 805–827 (1966).
2. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems (Studies in Mathematics and its Applications*, Vol. 1 North-Holland, Amsterdam–Oxford, and American Elsevier Publ. Co., New York (1976).
3. R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, N.J. (1970).

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