A CONJECTURE ON SQUARE-ZERO UPPER TRIANGULAR MATRICES AND CARLSSON'S RANK CONJECTURE

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Berrin Şentürk

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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September 2018

A well-known conjecture states that if an elementary abelian p-group acts freely on a product of spheres, then the rank of the group is at most the number of spheres in the product. Carlsson gives an algebraic version of this conjecture by considering a differential graded module M over the polynomial ring A in r variables: If the homology of M is nontrivial and finite dimensional over the ground field, then $N := \dim_A M$ is at least 2^r.

In this thesis, we state a stronger conjecture concerning varieties of square-zero upper triangular $N \times N$ matrices with entries in A. By stratifying these varieties via Borel orbits, we show that the stronger conjecture holds when $N < 8$ or $r < 3$. As a consequence, we obtain a new proof for many of the known cases of Carlsson's conjecture as well as novel results for $N > 4$ and $r = 2$.

Keywords: Rank conjecture, Projective variety, Borel orbit.

ÖZET

KARESİ SIFIR ÜST ÜÇGENSEL MATRİSLER ÜZERİNDE BİR SANI VE CARLSSON'IN MERTEBE SANISI

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Klasik bir sanı eğer bir elementer abelyen p-grup küreler çarpımının üzerine serbest etki ediyorsa, o grubun rankının en fazla çarpımdaki kürelerin sayısı kadar olacağını belirtir. Carlsson r değişkenli polinom halkası A üzerindeki diferansiyel dereceli M modülünü düşünerek bu sanının cebirsel versiyonunu verir: Eğer M'in homolojisi aşikar değil ve ortamdaki cisim üzerinde sonlu boyutluysa, M'in A üzerindeki boyutu en az 2^r'dir.

Bu tezde üst üçgensel karesi sıfır olan $N \times N$ ve girdileri A'dan matrislerin çeşitlemeleriyle ilgilenerek daha güçlü bir sanı belirtiyoruz. Bu çeşitlemeleri Borel orbitler vasıtasıyla katmanlaştırarak daha güçlü olan sanının $N < 8$ ya da $r < 3$ olduğunda tuttuğunu gösteriyoruz. Sonuç olarak Carlsson sanısının bilinen birçok durumuna yeni bir ispat veriyoruz ve $N > 4$ ve $r = 2$ için yeni sonuçlar elde ediyoruz.

Anahtar sözcükler: Rank sanısı, İzdüşel çeşitleme, Borel orbiti.

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Chapter 1

Introduction

A well-known result of Smith [1] states that if a finite group G acts freely on sphere, then all of its abelian subgroups are cyclic. Motivated by this, there is a long-standing conjecture in algebraic topology that if $(\mathbb{Z}/p\mathbb{Z})^r$ acts freely and cellularly on a finite CW-complex homotopy equivalent to $S^{n_1} \times \ldots \times S^{n_m}$, then r is less than or equal to m. This conjecture is known to be true in several cases. In the equidimensional case $n_1 = \ldots = n_m =: n$, Carlsson [2], Browder [3], and Benson-Carlson [4] gave a proof under the assumption that the induced action on homology is trivial. Without the homology assumption, the equidimensional conjecture was proved by Conner [5] for $m = 2$, Adem-Browder [6] for $p \neq 2$ or $n \neq 1, 3, 7$, and Yalçın [7] for $p = 2$, $n = 1$. In the non-equidimensional case, the conjecture was verified by Heller [8] for $m = 2$, Carlsson [9] for $p = 2$ and $m = 3$, Refai [10] for $p = 2$ and $m = 4$, Hanke [11] for p large relative to the dimension of the product of spheres, and Okutan-Yalçın [12] for products in which the average of the dimensions is sufficiently large compared to the differences between them. The general case for $m \geq 5$ is still open.

Carlsson proposed several algebraic versions of the conjecture. Let $G = (\mathbb{Z}/p\mathbb{Z})^r$ and k be an algebraically closed field of characteristic p. Assume that G acts freely and cellularly on a finite CW-complex X homotopy equivalent to a product of m spheres. One can consider the cellular chain complex $C_*(X; k)$

as a finite chain complex of free kG-modules whose homology $H_*(X;k)$ is a 2^m-dimensional k-vector space. Hence, a stronger conjecture can be stated as follows: If C_* is a finite chain complex of free kG-modules with nonzero homology then $\dim_k H_*(C_*) \geq 2^r$. Iyengar-Walker in [13] disproved this algebraic conjecture when $p \neq 2$ and $r \geq 8$, but the algebraic version for $p = 2$ remains open, so we can still hope to prove the topological conjecture when $p = 2$ by algebraic means.

Let R be a graded ring. A pair (M, ∂) is a *differential graded* R-module if M is a graded R-module and ∂ is an R-linear endomorphism of M of degree -1 that satisfies $\partial^2 = 0$. A *DG-R-module* is *free* if the underlying R-module is free.

Let $A = k[y_1, \ldots, y_r]$ be the polynomial algebra in r variables of degree -1. Using a functor from the category of chain complexes of kG-modules to the category of differential graded A-modules, Carlsson showed in [14] and [15] that the above algebraic conjecture is equivalent to the following when the characteristic of k is 2:

Conjecture 1. *Let k be an algebraically closed field, $A = k[y_1, \ldots, y_r]$, and N a positive integer. If (M, ∂) is a free DG-A-module of rank N whose homology is nonzero and finite dimensional as a k-vector space, then $N \geq 2^r$.*

When the characteristic of k is 2, Conjecture 1 was proved by Carlsson [9] for $r \leq 3$ and Refai [10] for $N \leq 8$. Avramov, Buchweitz, and Iyengar in [16] dealt with regular rings and in particular they proved Conjecture 1 for $r \leq 3$ without any restriction on the characteristic of k. See also Proposition 1.1 and Corollary 1.2 in [17], and Theorem 5.3 in [18] for results in characteristics not equal to 2.

This thesis is motivated by Conjecture 1, which we consider from the viewpoint of algebraic geometry. We show that Conjecture 1 is implied by the following in Section 3.2:

Conjecture 2. *Let k be an algebraically closed field, r a positive integer, and $N = 2n$ an even positive integer. Assume that there exists a nonconstant morphism ψ from the projective variety $\mathbb{P}_k^{r-1}$ to the weighted quasi-projective* *variety of rank n square-zero upper triangular $N \times N$ matrices (x_{ij}) with $\deg(x_{ij}) = d_i - d_j + 1$ for some N-tuple of nonincreasing integers $(d_1, d_2, \ldots, d_N)$. Then $N \geq 2^r$.*

We will give a more precise statement of Conjecture 2 in Section 3.2.1 after discussing the necessary definitions and notation. We then propose the following:

Conjecture 3. *Let k, r, N, n and ψ be as in Conjecture 2. For $\mathcal{R}$ and $\mathcal{C}$ integers between 1 and N, assume that the value of x_{ij} at every point in the image of ψ is 0 whenever $i \geq N - \mathcal{R} + 1$ or $j \leq \mathcal{C}$. Then $N \geq 2^{r-1}(\mathcal{R} + \mathcal{C})$.*

Note that in this situation we have $2^{r-1}(\mathcal{R} + \mathcal{C}) \geq 2^r$ because $\mathcal{R} \geq 1$ and $\mathcal{C} \geq 1$, so that Conjecture 3 implies Conjecture 2. The main results of this paper are:

Theorem A (Theorem 4.2.1)**.** *Conjecture 3 holds for $N < 8$.*

Theorem B (Theorem 5.1.7)**.** *Conjecture 3 holds for $r \leq 2$.*

As Conjecture 3 is the strongest conjecture we have encountered, we obtain proofs of all the conjectures in this introduction under the same conditions, including the main result of Carlsson in [15]. Also note that for $r = 2$, taking $N > 4$ gives novel results not covered in the literature.

In Chapter 2, we give all the background information on varieties that is necessary for this thesis. We introduce definitions and notation related to the varieties of upper triangular square zero matrices and describe the stratification of these varieties by Borel orbits given by Rothbach [19]. The combinatorial nature of this stratification allows us to represent the irreducible components of these varieties by certain permutations of finite sets.

Chapter 3 provides a brief summary of Carlsson's work on Conjecture 1. We transfer the conjecture from algebraic topology to algebraic geometry by giving an explicit proof that Conjecture 2 implies Conjecture 1, see Theorem 3.2.3. Then we rephrase Conjectures 2 and 3 by using the new terminology given in the previous chapter. Using ideas of Iyengar-Walker in [13], we give a counterexample to all these conjectures when $r = 8$ and the characteristic of the field k is not 2. The case of characteristic 2 is still open.

In Chapter 4, we introduce operations on the morphism ψ of Conjectures 2 and 3 in order to give our first main result, Theorem A. We also restrict our attention to certain permutations of maximal rank as it is enough to consider those in the proof of the theorem.

Chapter 5 contains technical results. By using earlier results on the dimension formula for Borel orbits associated to certain varieties, we obtain our second main result, Theorem B.

In Chapter 6, we conclude with some examples and problems.

Chapter 2

Varieties of square-zero matrices

In this chapter, we recall the basic algebraic geometry that will be used in this thesis. We mention some basic results on algebraic varieties in affine or projective space over any algebraically closed field k, see, e.g., [20], [21] and [22]. We also give definitions and facts on varieties that will be of interest to this thesis.

2.1 Projective varieties

Let $\mathbb{A}_k^r$ denote the affine r-space over k. An *affine variety* is the set of common zeroes in $\mathbb{A}_k^r$ of a collection of polynomials. Let S be any set of polynomials in $k[x_1, \ldots, x_r]$. Then the affine variety of S is

$$V(S) := \{a \in \mathbb{A}_k^r \mid f(a) = 0 \ \forall f \in S\}.$$

A variety $V \subset \mathbb{A}_k^r$ is *irreducible* if V is nonempty and cannot be written as the union of two proper subvarieties

$$V = V_1 \cup V_2.$$

Otherwise, V is called *reducible.* For $f \in k[x_1, \ldots, x_r]$, $V(f)$ is an irreducible variety when f is an irreducible polynomial. The *dimension* of an irreducible variety $V \subset \mathbb{A}_k^r$ is defined by

$$\dim V = \min\{\dim T_P V\},$$

where $T_P V$ is the tangent space to V at a point $P \in V$. More generally, if V is an affine variety with irreducible components

$$V = V_1 \cup \ldots \cup V_m,$$

then $\dim V = \max\{\dim V_i\}$.

Proposition 2.1.1. *[20, Proposition 1.13] The dimension of an irreducible variety $Y \in \mathbb{A}_k^r$ is $r-1$ if and only if $Y = Z(f)$, where f is a nonconstant irreducible polynomial in $k[x_1, \ldots, x_r]$.*

The *projective r-space* over k, denoted $\mathbb{P}_k^r$, is the quotient

$$\mathbb{A}_k^{r+1} - \{0\}/\sim,$$

where $\sim$ is the equivalence relation that identifies all points of the same line through the origin. A *projective variety* is a subset in $\mathbb{P}_k^r$ such that there is a set of homogeneous polynomials $H \subset k[x_1, \ldots, x_r]$ with

$$V = \{p \in \mathbb{P}_k^r \mid f(p) = 0 \ \forall f \in H\}.$$

We denote the homogeneous coordinate of the point $p \in \mathbb{P}_k^r$ by $(x_0 : \ldots : x_r)$. An open subset of a $\mathbb{P}_k^r$ is a *quasi-projective variety.* Let $V \subseteq \mathbb{P}_k^r$ be an irreducible projective variety with affine covering

$$V = V_0 \cup \ldots \cup V_r,$$

that is, $V_i = V \cap U_i$, where $U_i = \{(x_0 : \ldots : x_r) \in \mathbb{P}_k^r \mid x_i \neq 0\}$ is identified with $\mathbb{A}_k^r$ via the bijection $(x_0 : \ldots : x_i : \ldots : x_r) \to (\frac{x_0}{x_i}, \ldots, \frac{x_i-1}{x_i}, \frac{x_i+1}{x_i}, \ldots, \frac{x_r}{xi})$, so that V_i may be considered as an affine variety. If V is not contained in $\mathbb{P}_k^r \setminus U_i$ for any i, each V_i has the same dimension. Then the *dimension* of V is given by the common dimension of the V_i.

Let $Y \subseteq \mathbb{P}_k^r$ be an irreducible quasi-projective variety. A function $f : Y \to k$ is called *regular* at a point $p \in Y$ if there exists an open subset $U \subseteq Y$ containing p

and homogeneous polynomials $g, h \in k[x_0, \dots, x_r]$ of the same degree such that h is nonzero on U and $f = g/h$. When f is regular at every point of Y, it is *regular* on Y. Let X be a second irreducible quasi-projective variety. A map $\psi : X \to Y$ is called a *morphism* if it is a continuous map such that for every open set $W \subseteq Y$, and for every regular function $f : W \to k$, the function $f \circ \psi : \psi^{-1}(W) \to k$ is regular (see [20]).

Now we consider the intersection of varieties in affine or projective space.

Proposition 2.1.2. *[20, Proposition 7.1] Let X and Y be irreducible varieties of dimensions s and t in $\mathbb{A}_k^r$. Then every irreducible component of $X \cap Y$ has dimension at least $s + t - r$.*

The proposition also holds for projective varieties. In particular, if $s + t \geq r$, then $X \cap Y$ is nonempty, see [20, Proposition 7.2].

We will use the following result to prove Theorem 4.2.1.

Theorem 2.1.3. *Let $f_1, f_2, \dots, f_l$ be nonconstant homogeneous polynomials in $k[x_1, \dots, x_r]$. If $r > l$, then there exists a nonzero $\gamma \in \mathbb{A}_k^r$ such that $f_1(\gamma) = 0$, $f_2(\gamma) = 0, \dots, f_l(\gamma) = 0$.*

Proof. Suppose first that the f_i's are irreducible nonconstant homogeneous polynomials in $k[x_1, \dots, x_r]$. We have $\dim(V(f_i)) = r - 1$ for $i \in \{1, \dots, l\}$ by Proposition 2.1.1. Now consider the dimension of the intersection of the set of zeroes of the f_i. For instance, $\dim(V(f_1) \cap V(f_2)) \geq (r-1) + (r-1) - r = r - 2$ by Proposition 2.1.2. Inductively we have

$$\dim(V(f_1) \cap V(f_2) \cap \dots \cap V(f_l)) \geq r - l > 0.$$

Since $r > l$, $\dim(V(f_1) \cap V(f_2) \cap \dots \cap V(f_l)) > 0$. This means $V(f_1) \cap \dots \cap V(f_l)$ has infinitely many points, so $V(f_1) \cap \dots \cap V(f_l)$ has point γ other than 0.

For the general case, note that any polynomial f can be written as product of irreducible components

$$f = f_1 \dots f_n$$

for some $n \in \mathbb{N}$, and $f_i = 0$ implies $f = 0$ for $i \in \{1, \ldots, n\}$. The result follows for any homogeneous polynomials in $k[x_1, \ldots, x_r]$; even when they are not assumed irreducible. $\square$

Let k^* denote the unit group of k and $a_0, \ldots, a_r$ be positive integers, consider the action of k^* on $\mathbb{A}_k^{r+1} - \{0\}$ by

$$\lambda.(x_0, \ldots, x_r) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_r} x_r).$$

Given $a_0, \ldots, a_r$, the corresponding *weighted projective space* is defined by

$$\mathbb{A}_k^{r+1} - \{0\} \Big/ k^*,$$

where $a_0, \ldots, a_r$ are called the *weights*. For example, $\mathbb{P}_k^r$ is the weighted projective space corresponding to the weights $1, \ldots, 1$.

Let $k[x_0, \ldots, x_r]$ be the polynomial ring in $r+1$ variables with $\deg(x_i) = a_i$ for some positive integers $a_0, \ldots, a_r$. We call $f \in k[x_0, \ldots, x_r]$ *weighted-homogeneous of degree* d if the weighted degree of each monomial in f is d, that is,

$$f = \sum_{i=1}^{m} c_i \left(\prod_{j=0}^{r} x_j^{{d_j}^{(i)}} \right) \text{ where } c_i \in k, m \in \mathbb{N}$$

and

$$\sum_{j=0}^{r} a_j {d_j}^{(i)} = d, \text{ for } 0 \leq i \leq r.$$

An ideal I is weighted-homogeneous if it is generated by weighted-homogeneous elements, see [22, Definition 3.0.9]. The *weighted projective variety* associated to I is the set

$$V(I) = \{ p \in \mathbb{P}(a_0, \ldots, a_r) \mid f(p) = 0 \text{ for all } f \in I \}.$$

Define the ideal associated to V by

$$I(V) = \{ f \in k[x_0, \ldots, x_r] \,|\, f(p) = 0 \text{ for all } p \in V,\ f \text{ is weighted homogeneous}\}.$$

Let g and h be both homogeneous polynomials in $k[x_0, \ldots, x_r]$ of the same degree. Given an irreducible projective variety $V \subset \mathbb{P}_k^r$, *the function field* of V is defined by

$$k(V) := \left\{ \frac{g}{h} \mid h \notin I(V) \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation such that $\dfrac{g}{h}$ is identified with $\dfrac{g'}{h'}$ if and only if $gh' - hg' \in I(V)$. We call the elements of $k(V)$ *rational functions*. The *domain* of a rational function of $f \in k(V)$, denoted $dom(f)$, is the set of all points where f can be represented by $\dfrac{g}{h}$ with $h \neq 0$.

A *rational map* from V to $\mathbb{P}^r_k$ is an equivalence class of $(r+1)$-tuples $[(f_0, \ldots, f_r)]$ where

- $f_0, \ldots, f_r$ are rational functions on V,
- there exists an i such that $f_i \neq 0$, and
- $(f_0, \ldots, f_r)$ is equivalent to $(f'_0, \ldots, f'_r)$ if there exists $h \in k(V)$ such that for all i, $hf_i = f'_i$.

The *homogeneous coordinate ring* $S(V)$ of V is the quotient ring

$$S(V) := k[x_0, \ldots, x_r]/I(V),$$

which is equipped with a graded ring structure. For $f_0, \ldots, f_r \in S(V)$, we use the notation

$$(f_0 : f_1 : \ldots : f_r) = \left[\left(\frac{f_0}{h}, \frac{f_1}{h}, \ldots, \frac{f_r}{h}\right)\right]$$

where $\deg(f_0) = \ldots = \deg(f_r) = \deg(h)$ and $h \in S(V)$. This notation is independent of h since $\dfrac{f_i}{h} = \dfrac{h'}{h}\dfrac{f_i}{h'}$.

2.2 Varieties of square-zero matrices

In this section, we introduce the notation for the affine and projective varieties used to prove the conjectures of Chapter 1.

We suppose that k is an algebraically closed field, n a positive integer, $N = 2n$, and $d = (d_1, d_2, \ldots, d_N)$ an N-tuple of nonincreasing integers. We fix an affine variety U_N, a ring $R(U_N)$, and a subvariety $V_N \subseteq U_N$ as follows:

- U_N is the affine variety of strictly upper triangular $N \times N$ matrices over k, so that $U_N \cong k^{N(N-1)/2}$.
- $R(U_N) = k[\ x_{ij} \mid 1 \leq i < j \leq N\ ]$ is the coordinate ring of U_N.
- V_N is the subvariety of square zero matrices in U_N.

Define an action of the unit group k^* on U_N by $\lambda \cdot (x_{ij}) = (\lambda^{d_i - d_j + 1} x_{ij})$ for $\lambda \in k^*$. Note that the power $d_i - d_j + 1$ is always positive since d is a nonincreasing N-tuple of integers and $i < j$. Using this action we also set:

- $U(d)$ is the weighted projective space given by the quotient of $U_N - \{0\}$ by the action of k^*.
- $R(U(d))$ is the homogeneous coordinate ring of $U(d)$. In other words, $R(U(d))$ is $R(U_N)$ considered as a graded ring with $\deg(x_{ij}) = d_i - d_j + 1$.

If the polynomial $p_{ij} = \displaystyle\sum_{m=i+1}^{j-1} x_{im} x_{mj}$ is in $R(U(d))$, it is homogeneous of degree $d_i - d_j + 2$ whenever $1 \leq i < j \leq N$. Similarly, the $n \times n$-minors of (x_{ij}) are homogeneous polynomials in $R(U(d))$. Hence, we define two subvarieties of $U(d)$ as follows:

- $V(d)$ is the projective variety of square zero matrices in $U(d)$.
- $L(d)$ is the subvariety of matrices of rank less than n in $V(d)$.

Let U be an open subset of $V(d)$. We say $\psi : \mathbb{P}_k^{r-1} \to U$ is a nonconstant morphism if ψ can be represented by a matrix (ψ_{ij}) so that the following conditions are satisfied:

(I) there exists a positive integer m so that each ψ_{ij} is a homogeneous polynomial in the variables $x_1, x_2, \ldots, x_r$ in S of degree $m(d_i - d_j + 1)$ for $1 \leq i < j \leq N$.

(II) for every $\gamma \in \mathbb{P}_k^{r-1}$ there exist i, j such that $\psi_{ij}(\gamma) \neq 0$,

In particular, if $\psi : \mathbb{P}_k^{r-1} \to U$ is a nonconstant morphism, ψ can be considered as a function from $\mathbb{P}_k^{r-1}$ to U represented by a nonconstant polynomial map $\widetilde{\psi}$ from $\mathbb{A}_k^r$ to the cone over U such that $\widetilde{\psi}(\mathbb{A}_k^r - \{0\})$ does not contain the zero matrix in V_N. Each indeterminate x_{ij} can be viewed as homogeneous polynomial in $R(U(d))$. Hence, for $1 \leq \mathcal{R}, \mathcal{C} \leq N$ we define an important subvariety of $V(d)$:

- $V(d)_{\mathcal{RC}}$ is the subvariety of $V(d)$ given by the equations $x_{ij} = 0$ for $i \geq N - \mathcal{R} + 1$ or $j \leq \mathcal{C}$.

In other words, $V(d)_{\mathcal{RC}}$ is the variety of matrices in the following form:

$$
\begin{array}{c}
\begin{array}{cccccccc}
1 & & \mathcal{C} & & & & & \\
\end{array} \\
\left[\begin{array}{cccccccc}
0 & \dots & 0 & p_{1,\mathcal{C}+1} & & \dots & p_{1,N} \\
\vdots & & \vdots & & & & \\
0 & & 0 & & & \ddots & \vdots \\
\vdots & & \vdots & \ddots & & & p_{N-\mathcal{R},N} \\
0 & & 0 & \dots & 0 & \dots & 0 \\
\vdots & & \vdots & & & \ddots & \vdots \\
& & & & & & \\
0 & & 0 & \dots & 0 & \dots & 0
\end{array}\right]
\begin{array}{c}
\\ \\ \\ \\ 1 \\ \\ \\ \mathcal{R}
\end{array}
\end{array}
$$

2.2.1 The action of a Borel subgroup on V_N

Here we introduce the action of a Borel subgroup of $GL_N(k)$ on the varieties discussed in the previous subsection. First we set our notation for the Borel subgroup.

- B_N is the group of invertible upper triangular $N \times N$ matrices with coefficients in k.

The group B_N acts on V_N by conjugation.

- V_N/B_N denotes the set of orbits of the action of B_N on V_N.
- B_X denotes the B_N-orbit that contains $X \in V_N$.

There is a particularly nice representative for each Borel orbit. A *partial permutation matrix* is a matrix having at most one nonzero entry, which is 1, in each row and column. A result of Rothbach (Theorem 1 in [19]) implies that each B_N-orbit of V_N contains a unique partial permutation matrix. Hence we introduce the following notation:

- $\mathbf{PM}(N)$ denotes the set of nonzero $N \times N$ strictly upper triangular square-zero partial permutation matrices.

There is a one-to-one correspondence between $\mathbf{PM}(N)$ and V_N/B_N sending P to B_P.

We can identify these partial permutation matrices with a subset of the symmetric group $Sym(N)$:

- $\mathbf{P}(N)$ is the set of involutions in $Sym(N)$, i.e. , the set of non-identity permutations whose square is the identity.

For $P \in \mathbf{PM}(N)$ and $\sigma \in \mathbf{P}(N)$,

- σ_P denotes the permutation in $\mathbf{P}(N)$ that sends i to j if $P_{ij} = 1$;
- P_σ denotes the partial permutation matrix in $\mathbf{PM}(N)$ that satisfies $(P_\sigma)_{ij} = 1$ if and only if $\sigma(i) = j$ and $i < j$.

Clearly, the assignments $P \mapsto \sigma_P$ and $\sigma \mapsto P_\sigma$ are mutual inverses and so define a one-to-one correspondence between $\mathbf{P}(N)$ and $\mathbf{PM}(N)$.

For instance, one of the Borel orbits of V_4 is represented by both:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow \sigma = (12)(34).$$

2.2.2 A partial order on the set of orbits

There are important partial orders on V_N/B_N, $\mathbf{P}(N)$, $\mathbf{PM}(N)$, which make these isomorphic posets via the one-to-one correspondence mentioned above (cf. [19]). We begin with V_N/B_N. For Borel orbits $B, B' \in V_N/B_N$,

- $B' \leq B$ means the closure of B, considered as a subspace of V_N, contains B'.

Second, we define a partial order on $\mathbf{PM}(N)$. To do this, we consider the ranks of certain minors of partial permutation matrices. In general, for an $N \times N$ matrix X,

- $r_{ij}(X)$ denotes the rank of the lower left $((N-i+1) \times j)$ submatrix of X, where $1 \leq i < j \leq N$.

Example 2.2.1. Consider $\sigma = (1,2)(3,4)(5,6)$. Let P_σ be the partial permutation matrix corresponds to σ in $\mathbf{PM}(6)$. We simply denote $r_{ij}(P_\sigma)$ by r_{ij}:

$$P_\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{so that} \quad \begin{matrix} r_{12} = 1 & , & r_{23} = 0 & , & r_{35} = 1, \\ r_{13} = 1 & , & r_{24} = 1 & , & r_{36} = 2, \\ r_{14} = 2 & , & r_{25} = 1 & , & r_{45} = 0, \\ r_{15} = 2 & , & r_{26} = 2 & , & r_{46} = 1, \\ r_{16} = 3 & , & r_{34} = 1 & , & r_{56} = 1. \end{matrix}$$

In Chapter 4, we will see the fact that $r_{23} = 0$ and $r_{45} = 0$ provide useful information.

For partial permutation matrices $P', P \in \mathbf{PM}(N)$,

- $P' \le P$ means $r_{ij}(P') \le r_{ij}(P)$ for all i, j.

For instance, let $\sigma = (1,2)(3,4)(5,6)$ and $\sigma' = (1,3)(2,4)(5,6)$. We denote the partial permutation matrix corresponds to σ by P, so it is same as in Example 2.2.1, and σ' by P':

$$P' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $P' \le P$ because $r_{12}(P') = 0, r_{34}(P') = 0, r_{35}(P') = 0, r_{36}(P') = 1$, and $r_{ij}(P') = r_{ij}(P)$ otherwise.

Third, we define a partial order on $\mathbf{P}(N)$. For positive integers $i < j$, let $\sigma(i,j)$ denote the product of the permutations σ and the transposition (i,j), and write $\sigma^{(i,j)}$ for the right-conjugate of σ by (i,j). For $\sigma, \sigma' \in \mathbf{P}(N)$,

- $\sigma' \le \sigma$ if σ' can be obtained from σ by a sequence of moves of the following form:
  - Type I replaces σ with $\sigma(i,j)$ if $\sigma(i) = j$ and $i \ne j$.
  - Type II replaces σ with $\sigma^{(i,i')}$ if $\sigma(i) = i < i' < \sigma(i')$.
  - Type III replaces σ with $\sigma^{(j,j')}$ if $\sigma(j) < \sigma(j') < j' < j$.
  - Type IV replaces σ with $\sigma^{(j,j')}$ if $\sigma(j') < j' < j = \sigma(j)$.
  - Type V replaces σ with $\sigma^{(i,j)}$ if $i < \sigma(i) < \sigma(j) < j$.

The idea of describing order via these moves comes from [23]. Although we use different names for moves, the set of possible moves are same.

We represent a permutation $(i_1, j_1)(i_2, j_2)\dots(i_s, j_s)$ in $\mathbf{P}(N)$ by the matrix

$$\begin{pmatrix} i_1 & i_2 & \dots & i_s \\ j_1 & j_2 & \dots & j_s \end{pmatrix}.$$

For example, we draw the Hasse diagram of $\mathbf{P}(4)$ in which each edge is labelled by the type of the move it represents:

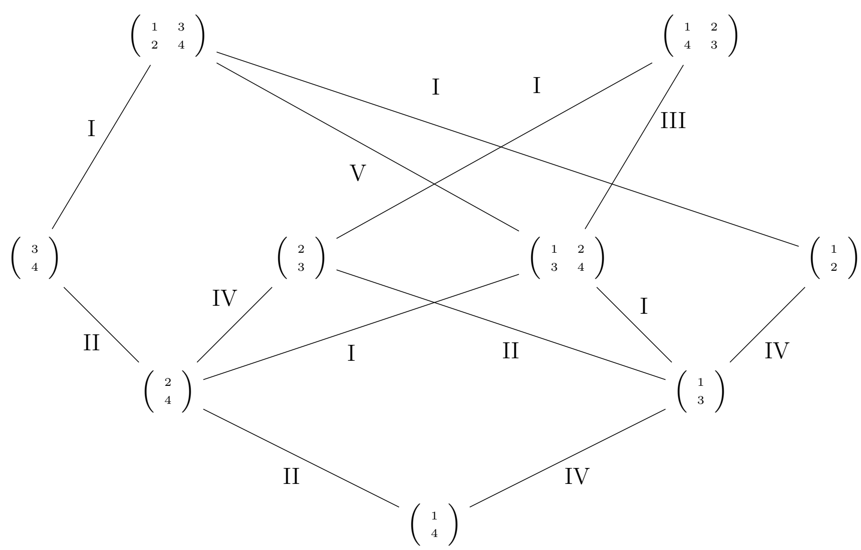


Figure 2.1: Hasse diagram of $\mathbf{P}(4)$

When $N \geq 6$, the Hasse diagram for $\mathbf{P}(N)$ is too large to draw here. We are actually only interested in a small part of this diagram, which we discuss in Section 4.2.

One can consider Figure 2.1 as a stratification of V_4. In the next section, we use the stratification of V_N to stratify $V(d)$.

2.2.3 The stratification of $V(d)$

For $d = (d_1, d_2, \ldots, d_N)$ an N-tuple of nonincreasing integers, $\lambda \in k^*$, and $X = (x_{ij}) \in V_N$, we have

$$\lambda \cdot X = \lambda \cdot (x_{ij}) = (\lambda^{d_i - d_j + 1} x_{ij}) = D_\lambda I_\lambda X D_\lambda^{-1},$$

where D_λ denotes the diagonal matrix with entries $\lambda^{d_1}, \lambda^{d_2}, \ldots, \lambda^{d_N}$ and I_λ is the scalar matrix with all diagonal entries λ. Let $P_X \in \mathbf{PM}(N)$ be the unique partial permutation matrix in the Borel orbit of X. Consider $b \in B_N$ such that

$$P_X = b^{-1} X b.$$

Let $I_{\lambda,X}$ be the diagonal matrix whose j^{th} entry is λ if $(P_X)_{ij} = 1$ for some i and 1 otherwise. Then we have

$$I_\lambda \, P_X \; = \; I_{\lambda,X}^{-1} \, P_X \, I_{\lambda,X}.$$

Hence, we have

$$\lambda \cdot X \; = \; D_\lambda \, b \, I_{\lambda,X}^{-1} \, b^{-1} \, X \, b \, I_{\lambda,X} \, b^{-1} \, D_\lambda^{-1} = Z^{-1} X Z,$$

where $Z = b \, I_{\lambda,X} \, b^{-1} \, D_\lambda^{-1}$ is in B_N. Thus, for any $X \in V(d)$ there exists a well-defined Borel orbit in V_N / B_N that contains a representative of X in V_N. This allows us to set the following notation: For $X \in V(d)$,

- B_X denotes the Borel orbit in V_N / B_N that contains a representative of X in V_N.

Let $\psi : \mathbb{P}_k^{r-1} \to V(d) - L(d)$ be a nonconstant morphism. There is a lift of this morphism to a morphism from $\mathbb{A}_k^r - \{0\}$ to the cone over $V(d) - L(d)$ that can be extended to a morphism $\widetilde{\psi} : \mathbb{A}_k^r \to V_N$. Since $\mathbb{A}_k^r$ is an irreducible affine variety, there exists a unique maximal Borel orbit among the Borel orbits that intersects the image of $\widetilde{\psi}$ nontrivially. Note that this maximal Borel orbit is independent of the lift and extension we selected because it is also maximal in the set $\{ B_X \mid X \in V(d)\}$. Hence we may associate a permutation to the nonconstant morphism ψ:

- σ_ψ is the permutation that corresponds to the unique maximal Borel orbit which intersects with the image of ψ nontrivially.

Note that every point in the image of a morphism ψ as above must have rank n. Therefore σ_ψ must be a product of n distinct transpositions. In Section 4.2, we will restrict our attention to such permutations.

Chapter 3

New conjectures

This chapter gives the background on Carlsson's conjecture that is necessary to understand the motivation of this thesis. We also introduce our new conjectures. The main objects of study are differential graded modules over a differential graded algebra.

A *differential graded* algebra A_* (or DG-algebra for short) over the field k is a graded k-algebra

$$A_* = \bigoplus_{n \in \mathbb{Z}} A_n,$$

endowed with

- a differential $\partial : A_* \to A_{*-1}$, i.e., a degree -1 homomorphism of graded k-vector spaces, satisfying $\partial^2 = 0$,
- a multiplication $A_n \otimes A_m \to A_{n+m}$ satisfying $ab = (-1)^{nm} ba$,

which are related to each other by

- $\partial(ab) = (\partial a)b + (-1)^n a \partial b$.

We primarily consider polynomial rings in r variables with coefficients in k. The grading is determined by the degree of variables.

Let A be a differential graded algebra. A *differential graded module* M over A (or *DG*-A-module) is a left graded A-module M such that $A_nM_m \subset M_{n+m}$, endowed with a differential $\partial : M_* \to M_{*-1}$, i.e., a degree -1 homomorphism of graded k-vector spaces, satisfying $\partial^2 = 0$ and

$$\partial(am) = \partial(a)m + (-1)^n a\partial(m),$$

for $a \in A_n$ and $m \in M$.

We are concerned with free *DG*-A-modules, i.e., ones where the underlying A-module of M is free.

Let R be a ring. A *chain complex* C_* is a differential graded R-module which has the differential ∂ of degree -1. Let M and N be *DG*-R-modules and $f, g : M \to N$ degree zero *DG*-R-module morphisms. We say f is chain equivalent to g, denoted $f \simeq g$, if there exists a homomorphism $H : M \to N$ of degree 1 such that $f - g = \partial H + H\partial$. Now suppose that $f : M \to N$ and $g : N \to M$ are *DG*-R-module morphisms. Then M is *chain equivalent* to N, written $M \simeq N$, if $f \circ g \simeq \mathrm{id}_M$ and $g \circ f \simeq \mathrm{id}_N$.

3.1 The algebraic background of Carlsson's work

In this section, we examine Carlsson's work in the case $p = 2$.

Let k be an algebraically closed field of characteristic 2, $A = k[y_1, \ldots, y_r]$ the polynomial ring over k in the variables $y_1, \ldots, y_r$ of degree -1, and G an elementary abelian 2-group of rank r, that is,

$$G = (\mathbb{Z}/2\mathbb{Z})^r = \big\langle\, g_1, \ldots, g_r \,|\, g_1^2 = g_2^2 = \ldots = g_r^2 = 1, [\, g_i, g_j \,] = 1 \big\rangle.$$

Note that as an algebra kG is isomorphic to the exterior algebra:

$$kG \cong \Lambda(t_1, \ldots, t_r) \text{ where } t_i = 1 + g_i, \text{ so that } t_i^2 = 0.$$

Conjecture 4. *[15] If C_* is a finite chain complex of free kG-modules with nonzero homology then $\dim_k H_*(C_*) \geq 2^r$.*

Note that the condition $H_*(C_*) \neq 0$ is necessary; otherwise, the trivial chain complex or $C_* : 0 \to kG \to kG \to 0$ would give us trivial homology, so the conjecture fails.

Let $\Lambda := kG$ and (C_*, δ) be a free bounded above Λ-chain complex. In [14], there is a construction of a free differential module βC_* as the graded module $C_* \widetilde{\otimes}_k A$ with the differential

$$\partial(c \otimes f) = \delta c \otimes f + \sum_i^r t_i c \otimes y_i f \text{ for } c \in C_* \text{ and } f \in A.$$

Note that the differential δ is not the standard one on tensor products.

Proposition 3.1.1. *([14, Proposition II.1]) There are natural isomorphisms $\mathrm{H}_*(C_*) \cong \mathrm{H}_*(\beta C_*; k)$ and $\mathrm{H}_*(C_* \otimes_\Lambda k) \cong \mathrm{H}_*(\beta C_*)$.*

The first natural isomorphism of Proposition 3.1.1 comes from the isomorphisms $A \otimes_A k \cong k$ and $C_* \cong C_* \widetilde{\otimes}_k k$, which give us

$$\mathrm{H}_*(C_*) \cong \mathrm{H}_*(C_* \widetilde{\otimes}_k A \otimes_A k) \cong \mathrm{H}_*(\beta C_* \otimes_A k).$$

The second one comes from the fact that $\Lambda \widetilde{\otimes}_k A$ is a resolution of k and so

$$\mathrm{H}_*(\beta C_*) \cong \mathrm{H}_*(C_* \widetilde{\otimes}_k A) \cong \mathrm{H}_*(C_* \widetilde{\otimes}_\Lambda \Lambda \widetilde{\otimes}_k A) \cong \mathrm{H}_*(C_* \widetilde{\otimes}_\Lambda k).$$

Moreover, if C_* is finitely generated as a Λ-module then $\mathrm{H}_*(\beta C_*)$ is finite dimensional as a k-vector space.

Let M be a free DG-A-module. We also construct a map $\tilde{\beta}$ as follows: Given a DG-A module (M, ∂), we define $\tilde{\beta}(M)$ to be the graded module $M \widetilde{\otimes}_k \Lambda$ with differential

$$\delta(m \otimes \alpha) = \partial m \otimes \alpha + \sum_i^r y_i m \otimes t_i \alpha \text{ for } m \in M \text{ and } \alpha \in \Lambda.$$

Proposition 3.1.2. $\mathrm{H}_*(M) \cong \mathrm{H}_*(\tilde{\beta}M; k)$ *and* $\mathrm{H}_*(\tilde{\beta}M) \cong \mathrm{H}_*(M; k)$.

Similarly, the first isomorphism of Proposition 3.1.2 comes from observing

$$\mathrm{H}_*(M) \cong \mathrm{H}_*(M \widetilde{\otimes}_k \Lambda \otimes_\Lambda k) \cong \mathrm{H}_*(\tilde{\beta}M \otimes_\Lambda k).$$

The second one is again obtained by using a resolution of k;

$$\mathrm{H}_*(\tilde{\beta}M) \cong \mathrm{H}_*(M \widetilde{\otimes}_k \Lambda) \cong \mathrm{H}_*(M \widetilde{\otimes}_A A \widetilde{\otimes}_k \Lambda) \cong \mathrm{H}_*(M \otimes_A k).$$

Moreover, $\beta\tilde{\beta}(M) = M \widetilde{\otimes}_k \Lambda \widetilde{\otimes}_k A$ which is chain equivalent to M.

Definition 3.1.3. We say that the *DG*-*A*-module (M, ∂) is *minimal* if

$$\partial \otimes \mathrm{id} : M \otimes_A k \to M \otimes_A k$$

is the zero map. Equivalently, (M, ∂) is minimal if the image of ∂ is contained in $(y_1, \ldots, y_r)M$.

Proposition 3.1.4. *([9, §1 Proposition 7]) Given free DG-A-module M, there exists a minimal DG-A-module $\widetilde{M}$ and a chain equivalence $f : M \to \widetilde{M}$. Moreover* $\dim_k H_*(\widetilde{M}; k) = \dim_A \widetilde{M}$.

Using the results mentioned above, one can show that Conjecture 1 implies Conjecture 4: Given a finite chain complex of free kG-modules C_*, form the *DG*-*A*-module $\beta(C_*)$. If Conjecture 1 holds and $M = \beta(C_*)$, by Proposition 3.1.4, we may consider $\widetilde{\beta(C_*)}$, which satisfies

$$\mathrm{rank}_A(\widetilde{\beta(C_*)}) = \mathrm{rank}_k(\widetilde{\beta(C_*)} \otimes_A k) = \mathrm{rank}_k H_*(\widetilde{\beta(C_*)} \otimes_A k) = \mathrm{rank}_k H_*(\beta(C_*) \otimes_A k).$$

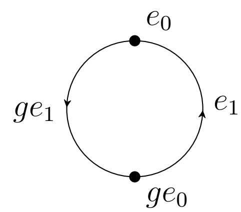
By Proposition 3.1.1, we have

$$\mathrm{rank}_k H_*(\beta(C_*) \otimes_A k) = \mathrm{rank}_k H_*(C).$$

Since $\mathrm{rank}_A(M) = \mathrm{rank}_A(\beta(C_*)) = \mathrm{rank}_A(\widetilde{\beta(C_*)}) \geq 2^r$ by Conjecture 1, we have $\mathrm{rank}_k H_*(C) \geq 2^r$, and thus Conjecture 4 is verified.

Moreover, Carlsson proved that Conjecture 4 implies Conjecture 1 [15, Proposition II.9].

Example 3.1.5. Let $G = \mathbb{Z}/2\mathbb{Z} =< g \mid g^2 = 1 >$ and $X = \mathbb{S}^1$. G acts on X with antipodal action. Consider the following G-CW-complex and induced cellular chain complex of X:



$$\begin{array}{rcl} 0 \longrightarrow C_1(X) & \xrightarrow{\delta=t=(1-g)} & C_0(X) \longrightarrow 0 \\ e_1 & \longrightarrow & e_0 - ge_0 = (1-g)e_0 = te_0 \\ ge_1 & \longrightarrow & ge_0 - e_0 = (g-1)e_0 \end{array}$$

If we apply β, we have the DG-A-module $C_* \widetilde{\otimes}_k A$ and the following map

$$C_* \widetilde{\otimes}_k A \xrightarrow{\partial} C_* \widetilde{\otimes}_k A,$$

where $kG = k[t]/t^2 = k \oplus kt$. We can write

$$C_* \widetilde{\otimes}_k A = (1 \otimes A) \oplus (t \otimes A) \oplus (1 \otimes A) \oplus (t \otimes A),$$

and compute

$$\begin{array}{cccc} \partial : & C_* \widetilde{\otimes}_k A & \longrightarrow & C_* \widetilde{\otimes}_k A \\ & (1 \otimes p(y), 0, 0, 0) & \longmapsto & (0, t \otimes yp(y), 0, t \otimes p(y)) \\ & (0, t \otimes p(y), 0, 0) & \longmapsto & (0, 0, 0, 0) \\ & (0, 0, 1 \otimes p(y), 0) & \longmapsto & (0, 0, 0, t \otimes p(y)) \\ & (0, 0, 0, t \otimes p(y)) & \longmapsto & (0, 0, 0, 0). \end{array}$$

We may express ∂ as a matrix with respect to the given decomposition as

$$\partial = \begin{bmatrix} 0 & y & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $\partial \otimes \mathrm{id}$ is not the zero map since ∂ has the constant entry 1 in the upper right corner. Thus (M, ∂) is not a minimal *DG*-*A*-module. By using the idea in the proof of [9, §1 Proposition 7], we can construct the minimal module $\widetilde{M}$ as follows: By the Primary Decomposition Theorem for modules over a principal ideal domain, M is of the form $M^t \oplus (\bigoplus_\alpha K(b_\alpha, \partial b_\alpha))$, where M^t is a *DG*-*k*-module with trivial differential and $K(b_\alpha, \partial b_\alpha)$ is a *DG*-*k*-module generated by b_α and ∂b_α. Let $\widetilde{b_\alpha}$ be a lifting to M of $b_\alpha \in M \otimes_A k$ and $E \leq M$ the submodule spanned by the elements $\{\widetilde{b_\alpha}, \partial \widetilde{b_\alpha}\}_\alpha$. More precisely,

$$\widetilde{b_\alpha} = (1 \otimes 1, 0, 0, 0) \text{ and } \partial \widetilde{b_\alpha} = (0, t \otimes y, 0, t \otimes 1).$$

Then $\widetilde{M} = M/E =< [(0, t\otimes 1, 0, 0)], [(0, 0, 1\otimes 1, 0)] >$. If we set $v_1 := (0, t\otimes 1, 0, 0)$ and $v_2 := (0, 0, 1 \otimes 1, 0)$, then $\widetilde{\partial}(v_2) = (0, 0, 0, t \otimes y) = (0, t \otimes y^2, 0, 0) = y^2 v_1$. In matrix form,

$$\widetilde{\partial} = \begin{bmatrix} 0 & y^2 \\ 0 & 0 \end{bmatrix}.$$

Consequentially, $(\widetilde{M}, \widetilde{\partial})$ is a minimal *DG*-*A*-module with $\dim_k H_*(\widetilde{M}; k) = \dim_A \widetilde{M}$.

Proposition 3.1.6. *([9, §1 Proposition 8]) Let (M, ∂) be a free, finitely generated DG-A-module with $\dim_k H_*(M) < \infty$. If $\mathbf{m}$ is any maximal ideal of A other than $(y_1, \ldots, y_r)$, then $H_*(M \otimes_A A/\mathbf{m}) = 0$.*

Proposition 3.1.7. *([9, §1 Proposition 9]) Suppose that (M, ∂) is finite dimensional differential module over k. Then*

$$H_*(M) = 0 \Leftrightarrow \dim_k M = 2 \,\mathrm{rank}_k\, \partial.$$

Using these propositions, Carlsson proved Conjecture 1 for $p = 2$, $r \leq 3$.

3.2 Some notes on Conjectures 1, 2, and 3

Let k be an algebraically closed field without any assumption on the characteristic. In this section we verify that Conjecture 3 is the strongest conjecture in Section 1.

Lemma 3.2.1. *Given a DG-A-module (M, ∂) with M free of rank N, then (M, ∂) admits a quotient DG-module $(\widetilde{M}, \widetilde{\partial})$ such that*

- *$\widetilde{M}$ is also A-free,*
- *$(\widetilde{M}, \widetilde{\partial})$ is minimal, and*
- *the map $(M, \partial) \to (\widetilde{M}, \widetilde{\partial})$ is a quasi-isomorphism.*

Proof. If (M, ∂) is already minimal, we are done. Otherwise, there exists a basis $b_1, \ldots, b_N$ of M and there are some i and j such that $\partial(b_i) = cb_j + \sum_{l \neq j} g_l b_l$ for some non-zero $c \in k$ and $g_l \in A$. Replacing b_j with $\partial(b_i)$ gives a new basis $b'_1, \ldots, b'_N$ such that $\partial(b'_i) = b'_j$. Now form the acyclic sub-*DG-A*-module (E, ∂) of (M, ∂) spanned by $\{b'_i, b'_j\}$. We have a short exact sequence $0 \to E \to M \to M/E \to 0$, which gives rise to a long exact sequence

$$\cdots \to H(E) \to H(M) \to H(M/E) \to 0.$$

Since E is acyclic, $H(M) \cong H(M/E)$. Hence the map $(M, \partial) \to (M, \partial)/(E, \partial)$ is a surjective quasi-isomorphism and M/E is free of rank $N - 2$.

We can repeat this process until one arrives at a minimal module $(\widetilde{M}, \widetilde{\partial})$, because each iteration decreases the rank by at least 2.

□

Lemma 3.2.2. *If (M, ∂) is minimal, then there is a homogeneous A-basis $b_1, \ldots, b_N$ of M such that $\partial(b_i) \in \bigoplus_{j=i+1}^{N} Ab_j$ for all $i \in \{1, \ldots N\}$.*

Proof. Pick any basis $c_1, \ldots, c_N$ of M such that $\deg(c_1) \leq \cdots \leq \deg(c_N)$. Let m be such that $\deg(c_{N-m+1}) = \cdots = \deg(c_N)$ and $\deg(c_{N-m}) < \deg(c_{N-m+1})$. For each i, we have $\partial(c_i) = \sum_j g_{ij} c_j$, for some homogeneous polynomials $g_{ij} \in A$. Since the image of ∂ is contained in $(y_1, \ldots, y_r)M$, no g_{ij} is a non-zero constant. Thus, whenever g_{ij} is non-zero, we have $\deg(g_{ij}) \leq -1$ and hence

$\deg(c_j) = \deg(c_i) - 1 - \deg(g_{ij}) \geq \deg(c_i)$. It follows that the differential on M restricts to one on the submodule

$$L = Ac_{N-m+1} \oplus \cdots \oplus Ac_N.$$

More precisely, for all $i \in \{N-m+1, \ldots, N\}$ we have $\partial(c_i) = \sum_{j=N-m+1}^{N} g_{ij}c_j$, where each nonzero g_{ij} is a linear polynomial. Hence, relative to the basis $c_{N-m+1}, \ldots, c_N$, the differential ∂ on L is given by a matrix of the form $y_1X_1 + \cdots + y_rX_r$, where each X_i is an $m \times m$ matrix with entries in k. Since $\partial^2 = 0$, we have $X_i^2 = 0$ and $X_iX_j + X_jX_i = 0$ for all i, j.

If the characteristic of the field k is 2, by a classical result about a commuting set of matrices (see e.g. [24, 6.5, Theorem 7]), there exists an invertible $m \times m$ matrix T with coefficients in k such that $T^{-1}X_iT$ is upper triangular for all $i \in \{1, \ldots, r\}$.

If the characteristic of the field k is not 2, for every polynomial Q in noncommuting r variables, the square of the matrix $Q(X_1, \ldots X_r)(X_iX_j - X_jX_i)$ is zero. Therefore, by a theorem of McCoy as stated in [25] (see also [26], [27]), again there exists a matrix T as above which simultaneously conjugates all X_i's to upper triangular matrices. In other words, there is a k-linear change of basis in which each X_i is upper triangular. It follows that, relative to this new basis $c'_{N-m+1}, \ldots, c'_N$ of L, one has $\partial(c'_i) \in \bigoplus_{j=i+1}^{N} Ac'_j$ for all $i \in \{N-m+1, \ldots N\}$. Note that M/L is a free *DG*-*A*-module whose differential has an image in $(y_1, \ldots, y_r)(M/L)$ and so, by induction on rank, we may assume that M/L admits a basis that makes its differential upper triangular. The union of any lift of this basis to M with the basis $c'_{N-m+1}, \ldots, c'_N$ gives a basis B for M where ∂ is represented by an upper triangular matrix Ψ'. □

To show that Conjecture 3 is the strongest conjecture in Section 1, it is enough to prove the following:

Theorem 3.2.3. *Conjecture 2 implies Conjecture 1.*

Proof. Let k, r, and A be as in Conjecture 1. Let (M, ∂) be a free DG-A-module of rank N that satisfies the hypothesis in Conjecture 1. Without loss of generality, we may assume that N is the smallest rank of all such DG-A-modules.

By Lemma 3.2.1, the image of the differential ∂ is contained in $(y_1, \ldots, y_r)M$, which means that M is minimal. By Lemma 3.2.2, there exists a homogeneous A-basis $b_1, \ldots, b_N$ of M such that $\partial(b_i) \in \bigoplus_{j=i+1}^{N} Ab_j$ for all $i \in \{1, \ldots N\}$. Hence the differential ∂ of M is represented by an upper triangular matrix. We denote the matrix by Ψ'. Moreover, Propositions 3.1.6 and 3.1.7 in [9] work in any characteristic. Hence N is divisible by 2 and for any γ in $k^r - \{0\}$ the evaluation of Ψ' at γ gives a matrix of rank $N/2$.

Let $S = k[x_1, \ldots, x_r]$ be the polynomial algebra with $\deg(x_i) = 1$. For $1 \leq i \leq r$, replace y_i with x_i in Ψ' to obtain Ψ. Note that Ψ can be considered as a a nonconstant morphism from the projective variety $\mathbb{P}_k^{r-1}$ to the weighted quasi-projective variety of rank $N/2$ square-zero upper triangular $N \times N$ matrices $\{(x_{ij})\}$ with $\deg(x_{ij}) = d_i - d_j + 1$, where $d_i = -$(degree of the ith element in B). $\square$

3.2.1 Restatements of the conjectures

We can use the terminology of Section 2.2 to restate Conjecture 2:

Conjecture 5. *Let k be an algebraically closed field, r a positive integer, and d an N-tuple of nonincreasing integers. If there exists a nonconstant morphism ψ from the projective variety $\mathbb{P}_k^{r-1}$ to the quasi-projective variety $V(d) - L(d)$, then $N \geq 2^r$.*

Now we restate the Conjecture 3 as follows:

Conjecture 6. *Let k be an algebraically closed field, r a positive integer, and d an N-tuple of nonincreasing integers. If there exists a nonconstant morphism ψ from the projective variety $\mathbb{P}_k^{r-1}$ to the quasi-projective variety $V(d)_{\mathcal{RC}} - L(d)$, then $N \geq 2^{r-1}(\mathcal{R} + \mathcal{C})$.*

Conjecture 6 implies Conjecture 5, and thus Conjecture 1 as well by Theorem 3.2.3.

We prove Conjecture 6 for $N < 8$ in Theorem 4.2.1. The main idea is the stratification of certain varieties obtained by considering the action of a Borel group. Thus, these varieties are the main interest of this thesis.

Note that Iyengar-Walker [13] disprove Conjecture 1 when the characteristic of k is not 2 and $r \geq 8$. Their work led us to find a counterexample for Conjecture 6 if the characteristic of the field is not 2 and $r = 8$. It is not possible to determine the number $\mathcal{R}$ and $\mathcal{C}$ in the proof of Theorem 3.2.3 since the proof depends on noncanonical choices of basis. Hence we give another proof for Theorem 3.2.3 in which Allday-Puppe's work [28] plays an essential role. For consistency between their work and ours, we maintain their notation such as for differentials and tensor products. Briefly, let k be an algebraically closed field and $A = k[y_1, y_2, \ldots, y_r]$ the polynomial algebra in r variables with coefficients in k, where $deg(y_i) = -1$. Let $\Lambda = \Lambda_k(t_1, t_2, \ldots t_r)$ be the exterior algebra on the r dimensional k-vector space spanned by t_1, t_2, $\ldots$, t_r, where $deg(t_i) = 0$. Given a DG-A-module (M, ∂) let $\Lambda\widetilde{\otimes}M$ denote the DG-Λ-module whose underlying graded Λ-module is $\Lambda \otimes_k M$ and whose differential δ is given by:

$$\delta(\alpha \otimes m) = \alpha \otimes \partial(m) + (-1)^d \sum_{i=1}^{r} \alpha t_i \otimes y_i m,$$

where $\alpha \in \Lambda$ and $m \in M$ with $\deg(m) = d$. Similarly, given a DG-Λ-module (C, δ) let $A\widetilde{\otimes}C$ denote the DG-A-module whose underlying graded A-module is $A \otimes_k C$ and whose differential ∂ is given by

$$\partial(a \otimes \gamma) = a \otimes \delta(\gamma) + (-1)^d \sum_{i=1}^{r} a y_i \otimes t_i \gamma,$$

where $a \in A$ and $\gamma \in C$ with $\deg(\gamma) = d$. One can use the ideas in [28], [15], and [9] to prove the following:

Alternative proof of Theorem 3.2.3. Assume (M, ∂) is a free, finite dimensional DG-A-module whose homology is nonzero and finite dimensional as a k-vector space. Then M is chain homotopy equivalent to $A\widetilde{\otimes}(\Lambda\widetilde{\otimes}M)$, as the idea in the proof of Theorem II.7 in [15] works for any characteristic. By Proposition $B.2.3$ in [28], there exists a differential structure on $A\otimes H(\Lambda\widetilde{\otimes}M)$ that is chain homotopy equivalent to $A\widetilde{\otimes}(\Lambda\widetilde{\otimes}M)$ as a DG-A-module.

Here we discuss the differential structure on $A\otimes H(\Lambda\widetilde{\otimes}M)$ more explicitly by considering the proof of Proposition $B.2.3$ in [28]. Let δ denote the differential of the DG-Λ-module $\Lambda\widetilde{\otimes}M$. Let T be the DG-k-module $\mathrm{im}(\delta)\oplus\mathrm{coker}(\delta)$, where the differential is given by the natural isomorphism from $\mathrm{coker}(\delta)$ to $\mathrm{im}(\delta)$. Note that R, K, L, N', and N'' in the proof of $B.2.3$ in [28] correspond to A, $\Lambda\widetilde{\otimes}M$, $H(\Lambda\widetilde{\otimes}M)$, 0, and T in our proof, respectively. Consider $A\otimes T$ as a DG-A-module with the usual differential on such a tensor product. We define an injective DG-A-module morphism $f : A\otimes T\to A\widetilde{\otimes}(\Lambda\widetilde{\otimes}M)$ by:

$$f(\alpha\otimes(\iota,\kappa)) = \alpha\otimes(0,\iota,\kappa) + (-1)^{d+1}\sum_{i=1}^{r}\alpha y_i\otimes t_i(0,0,\bar{\iota}),$$

where $\iota\in\mathrm{im}(\delta)$, $\kappa\in\mathrm{coker}(\delta)$, $d=\deg(\iota)=\deg(\kappa)$, $\bar{\iota}\in\mathrm{coker}(\delta)$, $\delta(\bar{\iota})=\iota$, and $(0,\iota,\kappa),(0,0,\bar{\iota})$ are in $\Lambda\otimes M$ by considering the isomorphisms:

$$\Lambda\otimes M\cong H(\Lambda\widetilde{\otimes}M)\oplus\mathrm{im}(\delta)\oplus\mathrm{coker}(\delta)\cong H(\Lambda\widetilde{\otimes}M)\oplus T.$$

Note that $\mathrm{coker}(f)$ has a natural DG-A-module structure and, as a graded A-module, it is isomorphic to $A\otimes H(\Lambda\widetilde{\otimes}M)$. This differential structure on $A\otimes H(\Lambda\widetilde{\otimes}M)$ is the same as the differential structure mentioned above. By abuse of notation, let us denote this new DG-A-module by $A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M)$.

Considering page 449 in [28], $k\otimes_A M$ is chain homotopy equivalent to $k\otimes_A A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M)\cong H(\Lambda\widetilde{\otimes}M)$. Hence we have:

$$\begin{aligned}\dim_A(M) &= \dim_k(k\otimes_A M)\\ &\geq \dim_k(H(k\otimes_A M))\\ &= \dim_k(H(k\otimes_A(A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M))))\\ &= \dim_k(H(\Lambda\widetilde{\otimes}M))\\ &= \dim_A(A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M)).\end{aligned}$$

Say $N = \dim_A(A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M))$. Let $I = (t_1, t_2, \ldots t_r)$ be the augmentation ideal of Λ. Then take a basis $B'_{p,q}$ of the k-vector space $I^pH_q(M)/I^{p+1}H_q(M)$. Lift each $B'_{p,q}$ to $B_{p,q}$ a subset of $A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M)$ by lifting $b + I^{p+1}H_q(M)$ in $B'_{p,q}$ to $1 \otimes b$ in $B_{p,q}$. Consider the basis of $A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M)$ given in the following order

$$B = \ldots B_{0,q} \cup B_{1,q} \cup \cdots \cup B_{r,q} \cup B_{0,q+1} \cup \ldots.$$

In this union only finitely many of the $B_{p,q}$ are nonempty; in fact there are N many elements in this basis. Let Ψ' be the matrix that represents the differential on $A\widetilde{\otimes}H(\Lambda\widetilde{\otimes}M)$ with respect to the basis B. Then Ψ' is an upper triangular matrix, all of whose entries are homogeneous polynomials in y_1, y_2, ..., y_r. If the ith element in B belongs to $B_{*,d}$ and the jth element in B belongs to $B_{*,\overline{d}}$ then the degree of the ijth entry in the matrix is $-d + \overline{d} - 1$. Moreover, Propositions 3.1.6 and 3.1.7 in [9] hold in any characteristic. Hence N is divisible by 2 and for any $\gamma \in k^r - \{0\}$ the evaluation of Ψ' at γ gives a matrix of rank $N/2$.

As in the last part of the proof of Theorem 3.2.3, $S = k[x_1, \ldots, x_r]$ denotes the polynomial algebra with $\deg(x_i) = 1$. For $1 \leq i \leq r$, we replace y_i with x_i in Ψ' to obtain Ψ. We consider Ψ as a a nonconstant morphism from the projective variety $\mathbb{P}_k^{r-1}$ to the weighted quasi-projective variety of rank $N/2$ square-zero upper triangular $N \times N$ matrices (x_{ij}) with $\deg(x_{ij}) = d_i - d_j + 1$ where $d_i = -$(degree of ith element in B). □

By using the alternative proof of Theorem 3.2.3, we give an example which shows that Conjectures 1, 5, and 6 do not hold when $r = 8$ and the characteristic of k is not 2. We use the ideas of Iyengar-Walker in [13]. Let Λ be the exterior algebra of a k-vector space with a basis $s_1, \ldots, s_m, t_1, \ldots, t_m$ as in [13, Proposition 2.1]. Define $w := \sum_{i=1}^{m} s_i t_i \in \Lambda_2$, where Λ_2 is the degree 2 part of Λ and the morphism $\lambda_w : \Lambda \to \Sigma^{-2}\Lambda$ with $\lambda_w(1) = w$. Then we have $\operatorname{rank}_k \operatorname{Ker}(\lambda_w) + \operatorname{rank}_k \operatorname{Coker}(\lambda_w) = \binom{2m+2}{m+1}$.

Example 3.2.4. Assume k is an algebraically closed field whose characteristic is not 2. Take w and Λ as above, where Λ is the exterior algebra of a k-vector space with a basis $s_1, \ldots, s_4, t_1, \ldots, t_4$. Then forget the grading of the exterior algebra Λ and consider it as a graded algebra concentrated in degree 0. Define $C_n = \Lambda$

if $n = 0$ or 1, and $C_n = 0$ otherwise. Assume $\partial_1 : C_1 \to C_0$ is multiplication by w. Then C is a DG-Λ module. Hence $A\widetilde{\otimes}C$ is DG-A-module. As in the proof of Theorem 1, there exists a DG-A-module $A\widetilde{\otimes}H(C)$ such that $A\widetilde{\otimes}C$ is chain homotopy equivalent to $A\widetilde{\otimes}H(C)$. This DG-A-module comes with a basis with respect to which the differential is represented by a strictly upper triangular matrix Ψ'. As in the proof of Theorem 1, we can obtain a morphism Ψ that satisfies the hypothesis in Conjecture 2. Note that $A\widetilde{\otimes}H(C)$ is a counterexample to Conjecture 1 because we have $r = 8$ and $N = 252$. The morphism Ψ is a counterexample to Conjecture 2 and 3 since $\mathcal{R} = 1$ and $\mathcal{C} = 1$ for Ψ but $252 \ngeq 256 = 128(1 + 1)$.

When $r = 2$ and w is defined in the same manner of [13, Proposition 2.1] the conclusion of Conjecture 6 still holds.

Let k be a field, Λ the exterior algebra of a k-vector space with basis s, t. Set $w := st$, and let λ_w be a morphism such that $\lambda_w : \Lambda \to \Lambda$, where $1 \mapsto st$, $s \mapsto 0$, $t \mapsto 0$ and $st \mapsto 0$. Then we have a DG-Λ-module (C, δ) such that $C = \Lambda \oplus \Lambda$ and

$$\delta = \left[\begin{array}{c|cccc} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline & & & & \\ 0 & & & 0 & \\ & & & & \end{array}\right].$$

Note that $\{s, t, st\}$ is a basis of $\operatorname{Ker}(\lambda_w)$ and $\{1+\operatorname{Im}(\lambda_w), s+\operatorname{Im}(\lambda_w), t+\operatorname{Im}(\lambda_w)\}$ of $\operatorname{Coker}(\lambda_w)$, so $\operatorname{rank}_k \operatorname{Ker}(\lambda_w) + \operatorname{rank}_k \operatorname{Coker}(\lambda_w) = 6 \nless 4$.

Now we associate this calculation with the alternative proof above. First we obtain DG-A-module M by tensoring the chain complex C with A. Then the differential of $A\widetilde{\otimes}C$ forms of the following matrix denoted by ∂

$$\partial = \begin{bmatrix} 0 & -x & y & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \begin{aligned} \partial(b_1) = &\ b_8 - xb_2 - yb_3, \\ \partial(b_2) = &\ -yb_4, \\ \partial(b_3) = &\ xb_4, \\ \partial(b_4) = &\ \partial(b_8) = 0, \\ \partial(b_5) = &\ yb_7 + xb_6, \\ \partial(b_6) = &\ yb_8, \\ \partial(b_7) = &\ -xb_8. \end{aligned}$$

The module $A\widetilde{\otimes}C$ is not minimal since the differential ∂ includes the constant entry. We obtain the minimal module $(\widetilde{M}, \widetilde{\partial})$ by taking the quotient of this module with the submodule spanned by $\{b_1, \partial(b_1)\}$. Then we have $\widetilde{\partial}(b_7) = -x^2b_2 - xyb_3$ and $\widetilde{\partial}(b_6) = xyb_2 + y^2b_3$. As given in the alternative proof we get a new basis $(c_1, \ldots, c_6) = (b_5, b_6, b_7, b_2, b_3, b_4)$, which gives rise to the following example

$$\psi_{\sigma_w}(x, y) = \begin{bmatrix} 0 & x & -y & 0 & 0 & 0 \\ 0 & 0 & 0 & -xy & y^2 & 0 \\ 0 & 0 & 0 & -x^2 & xy & 0 \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\sigma_w = (1, 2)(3, 4)(5, 6)$ with $d = (0, 0, 0, -1, -1, -1)$. Since $r = 2$, $\mathcal{R}+\mathcal{C} = 2$, and $N = 6$ Conjecture 6 holds for ψ_{σ_w}.

Note that the smallest r solution to the inequality

$$\binom{2r+2}{r+1} < 2^{2r}$$

is 8, see [13, Remark 2.5].

Chapter 4

The first main result

In this chapter, we introduce more notation and contain operations on certain matrices in order to obtain our main results. We will use those to prove Theorem A here.

4.1 Operations on polynomial maps from $\mathbb{A}_k^r$ to V_N

Another way to see that B_X is well-defined for $X \in V(d)$ is to consider the fact that a minor of a representative of X is zero if and only if the corresponding minor of another representative is zero. We use this fact several times to prove our main result. Hence we introduce the following notation. For $X \in V_N$,

- $m^{i_1\, i_2\, \dots\, i_k}_{j_1\, j_2\, \dots\, j_k}(X)$ denotes the determinant of the $k \times k$ submatrix obtained by taking the ${i_1}^{\text{th}}$, ${i_2}^{\text{th}}, \dots, {i_k}^{\text{th}}$ rows and ${j_1}^{\text{th}}, {j_2}^{\text{th}}, \dots, {j_k}^{\text{th}}$ columns of X.

Note that $m^{i_1\, i_2\, \dots\, i_k}_{j_1\, j_2\, \dots\, j_k}$ can be considered as a morphism from V_N to k, and hence can be composed with the morphism $\widetilde{\psi}$ mentioned in the previous subsection. We introduce here several other similar morphisms that we can compose with such morphisms. For $u \in k$,

- $R_{i,j}(u)$ is the function that takes a square matrix M and multiplies the i^{th} row of M by u and adds it to the j^{th} row of M while multiplying the j^{th} column of M by u and adding it to the i^{th} column of M.

Note that $R_{i,j}(u)(M)$ is a conjugate of M. In fact, they are in the same Borel orbit when $M \in V_N$ and $i > j$. Hence, for $i > j$, we can consider $R_{i,j}(u)$ as an operation that takes a morphism from $\mathbb{A}_k^s$ to V_N and transforms it to a morphism from $\mathbb{A}_k^{s+1}$ to V_N by considering u as a new indeterminate and applying $R_{i,j}(u)$ to the morphism. For $v \in k^*$,

- $D_i(v)$ denotes the function that takes a square matrix M and multiplies the i^{th} row of M by v and the i^{th} column of M by $1/v$.

Let q be a polynomial in s indeterminates. We define $D_i(q)$ as an operation that takes a rational map from the quasi-affine variety $\mathbb{A}_k^s - Z$ to V_N and transforms it into a rational map from $\mathbb{A}_k^s - Z \cup V(q)$ to V_N by applying $D_i(q)$, using the following notation:

- $V(q_1, q_2, \ldots, q_l)$ is the variety determined by the equations $q_1 = \cdots = q_l = 0$.

We use the above notation also for varieties in projective spaces determined by the homogeneous polynomials $q_1, q_2, \ldots, q_l$.

4.2 The rank of orbits and proof of the first main result

Each $\sigma \in \mathbf{P}(N)$ is a product of disjoint transpositions, so we define the *rank* of such a σ to be the number of transpositions in σ. Note that under the one-to-one corespondence between $\mathbf{P}(N)$ and $\mathbf{PM}(N)$, the rank of a permutation is equal to the rank of the corresponding partial permutation matrix.

- $\mathbf{RP}(N)$ denotes the permutations in $\mathbf{P}(N)$ of rank n.

Note that all moves other than those of type I preserve the rank of σ. Indeed, the only way to obtain a permutation of smaller rank by applying our moves is by deleting a transposition, which is the effect of a move of type I. Also note that it is impossible to have a move of type II or a move of type IV between two permutations in $\mathbf{RP}(N)$.

As an example, we draw the Hasse diagram for $\mathbf{RP}(6)$ where each dotted line denotes a move of type III and solid line denotes a move of type V:

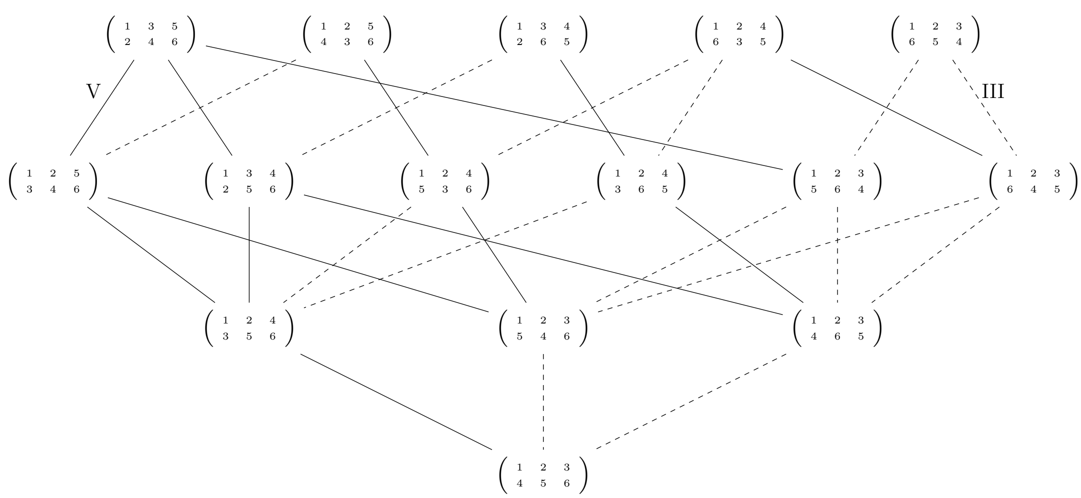


Figure 4.1: Hasse diagram of $\mathbf{RP}(6)$

Such Hasse diagrams, with particular attention paid to the maximal elements, will lead to the proof of our first main result.

Theorem 4.2.1. *[29, Theorem 2] Conjecture 6 holds for $N < 8$.*

Proof. Take $N < 8$, $d = (d_1, d_2, \ldots, d_N)$ an N-tuple of nonincreasing integers, and $\psi : \mathbb{P}_k^{r-1} \to V(d) - L(d)$ a nonconstant morphism. Then $\sigma = \sigma_\psi$ is in $\mathbf{RP}(N)$.

By considering Figures 2.1 and 4.1, we note that there exists a unique maximal $\sigma' \in \mathbf{RP}(N)$ such that σ can be obtained from σ' by a sequence of moves of type III. Since moves of type III do not change the number of leading zero rows and ending zero columns of the corresponding partial permutation matrices, the Borel orbit corresponding to σ is contained in $V(d)_{\mathcal{RC}}$ if and only if the Borel orbit corresponding to σ' is contained in $V(d)_{\mathcal{RC}}$ for all $\mathcal{R}, \mathcal{C}$. Hence it is enough to consider the cases where σ is less than or equal to a maximal element in $\mathbf{RP}(N)$ for $N = 2, 4, 6$. We cover these cases by proving in the following eight statements:

(i) If $\sigma = (1, 2)$ then $r < 2$.

Assume to the contrary that $\sigma = (1, 2)$ and $r \geq 2$. If we also write ψ for its restriction to $\mathbb{P}^1_k \subseteq \mathbb{P}^{r-1}_k$, we get a map of the form

$$\psi(x : y) = \begin{bmatrix} 0 & p_{12} \\ 0 & 0 \end{bmatrix},$$

where p_{12} is a homogeneous polynomial in $k[x, y]$. Since k is algebraically closed and $r \geq 2$, by Theorem 2.1.3 there exists γ in $\mathbb{P}^1_k$ such that $p_{12}(\gamma) = 0$. This means $\psi(\gamma)$ is in $L(d)$, which is a contradiction.

(ii) If $\sigma \leq (1, 2)(3, 4)$ then $r < 3$.

Suppose to the contrary that $\sigma \leq (1, 2)(3, 4)$ and $r \geq 3$. When we restrict ψ to $\mathbb{P}^2_k$, we get a map of the form

$$\psi(x : y : z) = \begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} \\ 0 & 0 & 0 & p_{24} \\ 0 & 0 & 0 & p_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 2.1.3, there exists γ in $\mathbb{P}^2_k$ such that

$$p_{12}(\gamma) = 0 \text{ and } p_{13}(\gamma) = 0.$$

Again this means $\psi(\gamma) \in L(d)$. Hence this case is proved by contradiction as well.

(iii) If $\sigma \leq (1, 4)(2, 3)$ then $r < 2$.

Suppose we have

$$\psi(x:y) = \begin{bmatrix} 0 & 0 & p_{13} & p_{14} \\ 0 & 0 & p_{23} & p_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $m^{i_1\, i_2\, \dots\, i_k}_{j_1\, j_2\, \dots\, j_k}$ be as in Section 4.1 and use the same notation to denote its composition with ψ. Then there exists γ in $\mathbb{P}^1_k$ such that

$$m^{12}_{34}(\gamma) = (p_{13}p_{24} - p_{23}p_{14})(\gamma) = 0.$$

This again gives a contradiction.
(iv) If $\sigma \leq (1,2)(3,4)(5,6)$ then $r < 3$.
Suppose otherwise. We have

$$\psi(x:y:z) = \begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & 0 & 0 & p_{24} & p_{25} & p_{26} \\ 0 & 0 & 0 & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & 0 & p_{46} \\ 0 & 0 & 0 & 0 & 0 & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If p_{12} and p_{13} are not relatively prime homogeneous polynomials then there exists $\gamma \in \mathbb{P}^2_k$ such that

$$p_{12}(\gamma) = 0,\ p_{13}(\gamma) = 0, \text{ and } m^{123}_{456}(\gamma) = 0.$$

Moreover, if $p_{46}(\gamma) = 0$ and $p_{56}(\gamma) = 0$, then the rank of $\psi(\gamma)$ is at most 2, which leads to a contradiction. Hence we have $p_{46}(\gamma) \neq 0$ or $p_{56}(\gamma) \neq 0$. Let

$$c_4(\gamma) := \begin{bmatrix} p_{14}(\gamma) \\ p_{24}(\gamma) \\ p_{34}(\gamma) \end{bmatrix} \text{ and } c_5(\gamma) := \begin{bmatrix} p_{15}(\gamma) \\ p_{25}(\gamma) \\ p_{35}(\gamma) \end{bmatrix}.$$

Since $\psi^2 = 0$,

$$\begin{aligned} p_{12}p_{26} + p_{13}p_{36} + p_{14}p_{46} + p_{15}p_{56} &= 0, \\ p_{24}p_{46} + p_{25}p_{56} &= 0, \\ p_{34}p_{46} + p_{35}p_{56} &= 0. \end{aligned}$$

Thus $c_4(\gamma)p_{46}(\gamma) + c_5(\gamma)p_{56}(\gamma) = 0$. By the fact that $p_{46}(\gamma) \neq 0$ or $p_{56}(\gamma) \neq 0$, $c_4(\gamma)$ and $c_5(\gamma)$ are linearly dependent. Thus the rank of $\psi(\gamma)$ is at most 2, which is a contradiction.

Therefore we may assume p_{12} and p_{13} are relatively prime. Using $\psi^2 = 0$, we have $p_{12}p_{24} + p_{13}p_{34} = 0$ and $p_{12}p_{25} + p_{13}p_{35} = 0$. This implies that p_{12} divides p_{34} and p_{35}, and similarly p_{13} divides p_{24} and p_{25}. Then there exits γ in $\mathbb{P}^2_k$ such that

$$p_{12}(\gamma) = 0 \text{ and } p_{13}(\gamma) = 0.$$

This means p_{12}, p_{13}, p_{24}, p_{25}, p_{34}, and p_{35} all vanish at γ. Hence $\psi(\gamma) \in L(d)$, which is a contradiction.

(v) If $\sigma \leq (1,2)(3,6)(4,5)$ then $r < 3$, and **(vi)** If $\sigma \leq (1,4)(2,3)(5,6)$ then $r < 3$.

These cases are symmetric, so it is enough to prove **(v)**.

Consider

$$\psi(x:y:z) = \begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & 0 & 0 & 0 & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & p_{45} & p_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We modify ψ by the operations in Section 4.1. First apply $R_{6,5}(u)$ to ψ for a new variable u. If $p_{46} + up_{45} \neq 0$, apply $D_5(1/(p_{46} + up_{45}))$ and then $R_{5,6}(-p_{45})$ to obtain a matrix of the form

$$\begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} & * & * \\ 0 & 0 & 0 & 0 & m^{24}_{56} & p_{26} + up_{25} \\ 0 & 0 & 0 & 0 & m^{34}_{56} & p_{36} + up_{35} \\ 0 & 0 & 0 & 0 & 0 & p_{46} + up_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $V(m^{24}_{56}, m^{34}_{56}) \nsubseteq V(p_{45})$, then we can find γ in $\mathbb{P}^2_k$ such that $m^{24}_{56}(\gamma) = 0$ and $m^{34}_{56} = 0$ but $p_{45}(\gamma) \neq 0$. By selecting a correct value for u in order to satisfy the condition $p_{46} + up_{45} \neq 0$, we have rank 2 matrix, which is a contradiction. Hence we may assume

$$V(m_{56}^{24}, m_{56}^{34}) \subseteq V(p_{45}).$$

Similarly, we may also assume

$$V(m_{56}^{23}, m_{56}^{34}) \subseteq V(p_{35}) \text{ and } V(m_{56}^{23}, m_{56}^{24}) \subseteq V(p_{25}).$$

Therefore,

$$V(m_{56}^{23}, m_{56}^{34}, m_{56}^{24}) \subseteq V(p_{25}, p_{35}, p_{45}) = \emptyset.$$

Thus, $\{m_{56}^{23}, m_{56}^{34}, m_{56}^{24}\}$ is a regular sequence in $k[x, y, z]$. If p_{45} and p_{46} are not relatively prime, there exists γ such that $m_{56}^{23}(\gamma) = 0$, and $p_{45}(\gamma) = p_{46}(\gamma) = 0$. Hence, we may assume p_{45} and p_{46} are relatively prime.

Clearly,

$$p_{45}m_{56}^{23} + p_{25}m_{56}^{34} - p_{35}m_{56}^{24} = 0. \tag{4.1}$$

Since $\{m_{56}^{23}, m_{56}^{34}, m_{56}^{24}\}$ is a regular sequence, Equation 4.1 gives us

$$p_{45} = z_1 m_{56}^{34} + z_2 m_{56}^{24},$$

where z_1 and z_2 are homogeneous polynomials. Then

$$p_{45}(1 + z_1 p_{36} + z_2 p_{26}) = p_{46}(z_1 p_{35} + z_2 p_{25}).$$

This implies that p_{46} divides $1 + z_1 p_{36} + z_2 p_{26}$, but it leads a contradiction since p_{46} is a homogeneous polynomial.

(vii) If $\sigma \leq (1, 6)(2, 3)(4, 5)$ then $r < 2$.
To prove this case, consider

$$\psi(x : y) = \begin{bmatrix} 0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & 0 & p_{23} & p_{24} & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & p_{45} & p_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose that

$$\begin{aligned} a &:= gcd(p_{13}, p_{14}), & p_{13} &= \overline{p_{13}}a, & p_{14} &= \overline{p_{14}}a, \\ b &:= gcd(p_{23}, p_{24}), & p_{23} &= \overline{p_{23}}b, & p_{24} &= \overline{p_{24}}b, \\ c &:= gcd(p_{35}, p_{45}), & p_{35} &= \overline{p_{35}}c, & p_{45} &= \overline{p_{45}}c, \\ d &:= gdc(p_{36}, p_{46}), & p_{36} &= \overline{p_{36}}d, & p_{46} &= \overline{p_{46}}d. \end{aligned}$$

Since $\psi^2 = 0$, we have $p_{13}p_{35} + p_{14}p_{45} = 0$, which implies that $\overline{p_{13}}\,\overline{p_{35}} = -\overline{p_{14}}\,\overline{p_{45}}$. We may write $u\overline{p_{13}} = \overline{p_{45}}$ and $-u\overline{p_{14}} = \overline{p_{35}}$ where u is an unit. Similarly, for some units u, v, and w we have

$$\psi(x:y) = \begin{bmatrix} 0 & 0 & \overline{p_{13}}a & \overline{p_{14}}a & p_{15} & p_{16} \\ 0 & 0 & v\overline{p_{13}}b & v\overline{p_{14}}b & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & -u\overline{p_{14}}c & -w\overline{p_{14}}d \\ 0 & 0 & 0 & 0 & u\overline{p_{13}}c & w\overline{p_{13}}d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Set

$$A := \begin{bmatrix} a & p_{15} & p_{16} \\ vb & p_{25} & p_{26} \\ 0 & -uc & -wd \end{bmatrix}.$$

There exists nonzero γ such that $\det A(\gamma) = 0$. Then we have

$$\operatorname{rank} \begin{bmatrix} \overline{p_{13}}a(\gamma) & \overline{p_{14}}a(\gamma) & p_{15}(\gamma) & p_{16}(\gamma) \\ v\overline{p_{13}}b(\gamma) & v\overline{p_{14}}b(\gamma) & p_{25}(\gamma) & p_{26}(\gamma) \\ 0 & 0 & -u\overline{p_{14}}c(\gamma) & -w\overline{p_{14}}d(\gamma) \\ 0 & 0 & u\overline{p_{13}}c(\gamma) & w\overline{p_{13}}d(\gamma) \end{bmatrix} \leq 2,$$

which is a contradiction.

(viii) If $\sigma \leq (1,6)(2,5)(3,4)$ then $r < 2$.

As a final case, consider

$$\psi(x:y) = \begin{bmatrix} 0 & 0 & 0 & p_{14} & p_{15} & p_{16} \\ 0 & 0 & 0 & p_{24} & p_{25} & p_{26} \\ 0 & 0 & 0 & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is enough to consider a root of m_{456}^{123} to prove this case. □

Note that in the above proof the last two cases prove Conjecture 6 when $N \leq 6$ and $r \leq 2$. In the rest of the paper we will generalize these ideas to prove the conjecture for $r \leq 2$. To do this, we examine the dimensions of these varieties.

Chapter 5

The second main result

5.1 Orbit dimensions and proof of second main result

We now introduce notation for dimensions of the varieties in Chapter 2 and 4. If $\sigma \in \mathbf{P}(N)$ has rank s, then we obtain two sequences of numbers $i_1, \ldots, i_s$ and $j_1, \ldots, j_s$ such that

$$\sigma = (i_1, j_1)(i_2, j_2) \ldots (i_s, j_s)$$

with $i_1 < i_2 < \cdots < i_s$ and $i_a < j_a$ for all $1 \leq a \leq s$. In [30], Melnikov gives a formula for the dimension of a Borel orbit B_σ for σ in $\mathbf{P}(N)$ as follows:

- $f_t(\sigma) := \#\{j_p \mid p < t, j_p < j_t\} \ + \ \#\{j_p \mid p < t, j_p < i_t\}$ for $2 \leq t \leq s$,
- $\dim(B_\sigma) = Ns + \sum_{t=1}^{s}(i_t - j_t) - \sum_{t=2}^{s} f_t(\sigma)$.

We define a new subset of $\mathbf{RP}(N)$:

- $\mathbf{DP}(N)$ is the set of all σ in $\mathbf{RP}(N)$ such that $\dim(B_{\sigma'}) = \dim(B_\sigma) - 1$ whenever σ' is a permutation obtained by applying a single move of type I to σ.

For instance, the following is the Hasse diagram of $\mathbf{DP}(8)$.

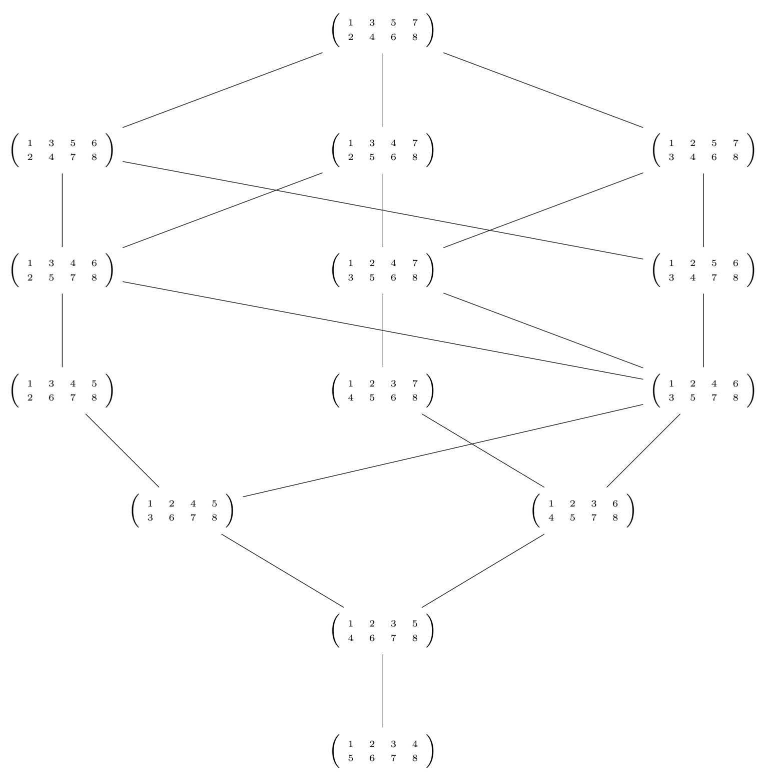


Figure 5.1: Hasse diagram of $\mathbf{DP}(8)$

Note that in the Hasse diagram of $\mathbf{DP}(8)$ all moves are of type V. This is generally the case, which we will prove below. Before we do so, we will prove an easier result that will introduce some notation and our style of argument.

Fix $\sigma \in \mathbf{DP}(N)$. Since $\mathbf{DP}(N)$ has full rank, every number between 1 and N must be moved by σ. In particular in the presentation of σ, 1 must appear so our convention for writing σ forces $i_1 = 1$.

For $q \in \{1, \ldots, n\}$, let σ' be the result of applying the move of type I that deletes the qth transposition of σ, so that

$$\sigma' = (1, j_1) \ldots (i_{q-1}, j_{q-1})(i_{q+1}, j_{q+1}) \ldots (i_n, j_n).$$

As a matrix,

$$\sigma' = \begin{pmatrix} 1 & i_2 & \dots & i_{q-1} & \widehat{i_q} & i_{q+1} & \dots & i_n \\ \\ j_1 & j_2 & \dots & j_{q-1} & \widehat{j_q} & j_{q+1} & \dots & j_n \end{pmatrix}.$$

Then by Melnikov's formula, we have

$$\dim(B_\sigma) = Nn + \sum_{t=1}^{n}(i_t - j_t) - \sum_{t=2}^{n} f_t(\sigma)\,, \text{ and}$$

$$\dim(B_{\sigma'}) = N(n-1) + \sum_{t=1}^{n-1}(i_t - j_t) - \sum_{t=2}^{n-1} f_t(\sigma').$$

To simplify our calculation, we write $f_t(\sigma) = f_t^1(\sigma) + f_t^2(\sigma)$ for $2 \le t \le n$, where

$$f_t^1(\sigma) = \#\{j_p \mid p < t, j_p < j_t\} \text{ and } f_t^2(\sigma) = \#\{j_p \mid p < t, j_p < i_t\},$$

and we use the notation:

$$f_{t,q}^l(\sigma') = \begin{cases} f_t^l(\sigma') & \text{if } t \le q-1 \\ 0 & \text{if } t = q \\ f_{t-1}^l(\sigma') & \text{if } t \ge q+1 \end{cases}$$

for $l = 1, 2$.

Lemma 5.1.1. *If $N \neq 2$ and the transposition $(1, N)$ appears in σ, then $\sigma \notin \mathbf{DP}(N)$.*

Proof. If $\sigma \in \mathbf{DP}(N)$ contains $(1, N)$, let $q = 2$ and σ' be the result of deleting the second transposition from σ. Since the i's are increasing and $i_a < j_a$ for all $1 \le a \le n$, we have $i_2 = 2$, so

$$\sigma' = \begin{pmatrix} 1 & \widehat{2} & i_3 & \dots & i_n \\ \\ N & \widehat{j_2} & j_3 & \dots & j_n \end{pmatrix}.$$

Thus $3 \le j_2 \le N-1$. Let $j_2 = N - b$ for some $1 \le b \le N-3$. Then

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & i_n \\ \\ N & N-b & \dots & j_n \end{pmatrix},$$

and any number between $N-b$ and N has to appear as a j or an i that is bigger than j_2. Therefore,

$$\left(\sum_{t=2}^{n} f_t^1(\sigma) - f_{t,2}^1(\sigma')\right) + \left(\sum_{t=2}^{n} f_t^2(\sigma) - f_{t,2}^2(\sigma')\right) = b-1.$$

Hence, $\dim(B_\sigma) - \dim(B_{\sigma'}) = N + 2 - N + b - (b-1) = 3$, so $\sigma \notin \mathbf{DP}(N)$. $\square$

Now we prove the main proposition of this section.

Proposition 5.1.2. *If $\sigma \in \mathbf{DP}(N)$, then $j_p < j_t$ for all $p < t$, and therefore we cannot apply a move of type* III *to σ. Conversely, if $\sigma \in \mathbf{RP}(N)$ and we cannot apply a move of type of* III *to σ, then $\sigma \in \mathbf{DP}(N)$.*

Proof. Assume that $\sigma \in \mathbf{DP}(N)$. We will prove the following statement by induction on k:

$$j_{n-k} < \ldots < j_{n-1} < j_n \text{ and } \forall\ (p < n-k),\ j_p < j_{n-k}. \tag{$*$}$$

Suppose $k = 0$. To prove $(*)$, we need to show that $\forall\ p < n$, $j_p < j_n$. Let σ' be obtained by deleting nth transposition of σ.

$$\sigma' = \begin{pmatrix} 1 & i_2 & \ldots & i_{n-1} & \widehat{i_n} \\ \\ j_1 & j_2 & \ldots & j_{n-1} & \widehat{j_n} \end{pmatrix}.$$

Since $\sigma \in \mathbf{DP}(N)$, we have

$$1 = \dim(B_\sigma) - \dim(B_{\sigma'}) = N + (i_n - j_n) - \left(f_n^1(\sigma) + f_n^2(\sigma)\right). \tag{5.1}$$

Since the total number of possible j excluding j_n is $n-1$, and any number between i_n and j_n has to appear as some j, we have $f_n^2(\sigma) = n - 1 - (j_n - i_n - 1)$. By the equation (5.1), $f_n^1(\sigma) = n-1$, so that $\forall\ p < n$, $j_p < j_n$ is true. Therefore $j_n = N$.

Now assume the statement $(*)$ is true for k. Then we can visualise σ as follows:

$$\sigma = \begin{pmatrix} 1 & < & i_2 & < & \ldots & < & i_{n-k-1} & < & i_{n-k} & < & i_{n-k+1} & < & \ldots & < i_n \\ \\ j_1 & & j_2 & & \ldots & & j_{n-k-1} & & j_{n-k} & < & j_{n-k+1} & < & \ldots & < j_n \end{pmatrix}$$

We need to prove $(*)$ for $k+1$, that is,

$$j_{n-k-1} < \ldots < j_{n-1} < j_n \text{ and } \forall\ (p < n-k-1),\ j_p < j_{n-k-1}.$$

By the second part of the inductive hypothesis for k, we have $j_{n-k-1} < j_{n-k}$ so the first part of $(*)$ is already true, and we only need to show that the second part holds. In other words, it is enough to show that $f^1_{n-k-1}(\sigma) = n-k-2$. Let σ' be obtained by deleting the $(n-k-1)$th transposition of σ. Then we have

$$\sum_{t=n-k}^{n} f^1_t(\sigma) - f^1_{t,n-k-1}(\sigma') = k+1.$$

Let $w := \#\{i_p \mid i_{n-k-1} < i_p < j_{n-k-1}\}$. Then

$$f^2_{n-k-1}(\sigma) = n-(k+2)-(j_{n-k-1} - i_{n-k-1} - 1 - w),$$

and

$$\sum_{t=n-k}^{n} f^2_t(\sigma) - f^2_{t,n-k-1}(\sigma') = k+1-w.$$

By the fact that $\dim(B_\sigma) - \dim(B_{\sigma'}) = 1$, we have $f^1_{n-k-1}(\sigma) = n-k-2$. Thus the first claim is proved.

Conversely, given $\sigma \in \mathbf{RP}(N)$, suppose that σ' is the result of applying the move of type I that deletes the q-th transposition from σ. Note that $f^1_q(\sigma) = q-1$ and $\sum_{t=q+1}^{n} f^1_t(\sigma) - f^1_{t,q}(\sigma') = n-q$. Hence,

$$\sum_{t=2}^{n} f^1_t(\sigma) - f^1_{t,q}(\sigma') = n-1.$$

Then,

$$\dim(B_\sigma) - \dim(B_{\sigma'}) = N + (i_q - j_q) - (n-1) - \left(\sum_{t=2}^{n} f^2_t(\sigma) - f^2_{t,q}(\sigma')\right).$$

We also have the difference $f^2_t(\sigma) - f^2_t(\sigma') = 0$ when $t \in \{1, \ldots, q-1\}$. Therefore,

$$\begin{aligned}\sum_{t=2}^{n} f^2_t(\sigma) - f^2_{t,q}(\sigma') &= \sum_{t=q}^{n} \#\{j_p \mid p < t, j_p < i_t\} \\ &\quad - \sum_{t=q+1}^{n} \#\{j_p \mid p < t, p \neq q, j_p < i_t\} \\ &= \#\{j_p \mid j_p < i_q\} + \#\{i_t \mid j_q < i_t\}.\end{aligned}$$

Let $F = \#\{j_p \mid j_p < i_q\}$ and $G = \#\{i_t \mid j_q < i_t\}$. Note that numbers between i_q and j_q must appear as j_l for $l < q$ or as i_s where $s > q$. Let $a = \#\{j_p \mid i_q < j_p < j_q\}$ and $b = \#\{i_t \mid i_q < i_t < j_q\}$. Then $a + b = j_q - i_q - 1$. Let $A = \#\{j_p \mid i_q < j_p\}$ and $B = \#\{i_t \mid i_t < j_q\}$. We have $A - a + B - b - 1 = n$. Therefore, $A + B = n + j_q - i_q$. The following is the picture we have in mind:

$$\sigma = \begin{pmatrix} \overbrace{i_1,\ldots\ldots,i_q\underbrace{\ldots}_{\text{b}}}^{\text{B}}\overbrace{\ldots,i_n}^{\text{G}} \\ \\ \underbrace{j_1,\ldots}_{\text{F}}\underbrace{\overbrace{\ldots}^{\text{a}}j_q,\ldots\ldots,j_n}_{\text{A}} \end{pmatrix}$$

Since $\sigma \in \mathbf{RP}(N)$, $A+B+F+G = 2n$. Then $F+G = n - j_q + i_q$. Consequently, $\dim(B_\sigma) - \dim(B_{\sigma'}) = 1$, so $\sigma \in \mathbf{DP}(N)$. □

- For $i \le i'$, $j \le j'$ and $\sigma \in \mathbf{DP}(N)$, let $A^{i,i'}_{j,j'}$ denote the submatrix of the partial permutation matrix P_σ obtained by considering the rows from i to i' and columns from j to j'.

Corollary 5.1.3. *If $\sigma \in \mathbf{DP}(N)$, then for any transposition (i, j) in σ, $A^{i+1,N}_{1,j-1}$ is the zero matrix. In other words, for every nonzero entry of a partial permutation matrix P_σ, where $\sigma \in \mathbf{DP}(N)$, there is no nonzero below and to the left of that entry.*

For instance, for $\sigma = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \in \mathbf{DP}(N)$, the corresponding partial permutation matrix is in the following form

$$P_\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 5.1.4. *Suppose that σ satisfies the assumptions in Conjecture 6, that is, k is an algebraically closed field and there exist an N-tuple of nonincreasing* *integers $d = (d_1, d_2, \dots, d_N)$, two positive integers $\mathcal{R}$ and $\mathcal{C}$, and a nonconstant morphism $\psi : \mathbb{P}_k^{r-1} \to V(d)_{\mathcal{RC}} - L(d)$. If $\sigma \in \mathbf{DP}(N)$ and $\mathcal{R} + \mathcal{C} > n$, then $A^{1,\mathcal{C}}_{N-\mathcal{R}+1,N}$ must have $(\mathcal{R} + \mathcal{C} - n)$-many 1's.*

Proof. Write $\sigma_\psi = (i_1, j_1)(i_2, j_2)\dots(i_n, j_n)$ with $1 = i_1 < i_2 < \dots < i_n$ and $i_a < j_a$ for all $1 \le a \le n$. Then we obtain the zero matrix $A^{1,N}_{1,\mathcal{C}}$. Similarly, $A^{N-\mathcal{R}+1,N}_{1,N}$ is the zero matrix. If we also assume that σ is in $\mathbf{DP}(N)$, by Proposition 5.1.2, we have $j_1 < j_2 < \dots < j_n = N$. Therefore, $\mathcal{C} = j_1 - 1$ and $\mathcal{R} = N - i_n$. Moreover, for every a we have $j_a > \mathcal{C}$ and $i_a < N - \mathcal{R} + 1$. Set $I := \{i_1, \dots, i_n\}$. Note that $\{1, \dots, \mathcal{C}\} \subseteq I$ since $\forall a$, $j_a > \mathcal{C}$. So, $i_a = a$ if $a \le \mathcal{C}$. Similarly, $j_{a-n} = a$ if $a \ge N - \mathcal{R} + 1$. Hence, $A^{1,\mathcal{C}}_{1,N}$ has $\mathcal{C}$-many 1's and $A^{1,N}_{N-\mathcal{R}+1,N}$ has $\mathcal{R}$-many 1's. Clearly, $A^{1,\mathcal{C}}_{N-\mathcal{R}+1,N}$ must have $(\mathcal{R} + \mathcal{C} - n)$-many 1's. □

Lemma 5.1.5. *For every X in $V(d) - L(d)$ we have*

$$r_{ij}(X) \ge j - i + 1 - n.$$

Proof. The rank of X is n, so $r_{1N}(X) = n$. The result follows from the inequality

$$r_{ij}(X) + (i-1) + (N-j) \ge r_{1N}(X).$$

□

Theorem 5.1.6. *Let k, d and N be as in Conjecture 6 and $r = 2$. If there exists a nonconstant morphism $\psi : \mathbb{P}_k^1 \to V(d)_{\mathcal{RC}} - L(d)$ and $\sigma_\psi \in \mathbf{DP}(N)$, then $N \ge 2(\mathcal{R} + \mathcal{C})$.*

Proof. Suppose to the contrary that there exists an N-tuple of nonincreasing integers $d = (d_1, d_2, \dots, d_N)$, two positive integers $\mathcal{R}$ and $\mathcal{C}$, and a nonconstant morphism $\psi : \mathbb{P}_k^1 \to V(d)_{\mathcal{RC}} - L(d)$ such that $N < 2(\mathcal{R} + \mathcal{C})$, or equivalently $n < \mathcal{R} + \mathcal{C}$. Write $\sigma_\psi = (i_1, j_1)(i_2, j_2)\dots(i_n, j_n)$ with $1 = i_1 < i_2 < \dots < i_n$ and $i_a < j_a$ for all $1 \le a \le n$.

Since σ is in $\mathbf{DP}(N)$, by Proposition 5.1.2, we have $j_1 < j_2 < \dots < j_n = N$. Therefore, $\mathcal{C} = j_1 - 1$ and $\mathcal{R} = N - i_n$. As in the proof of Lemma 5.1.4, $i_a = a$ if

$a \le \mathcal{C}$ and $j_{a-n} = a$ if $a \ge N - \mathcal{R} + 1$. Set

$$\underline{i} := n - \mathcal{R} + 1 \quad \text{and} \quad \bar{i} := \mathcal{C}$$

and

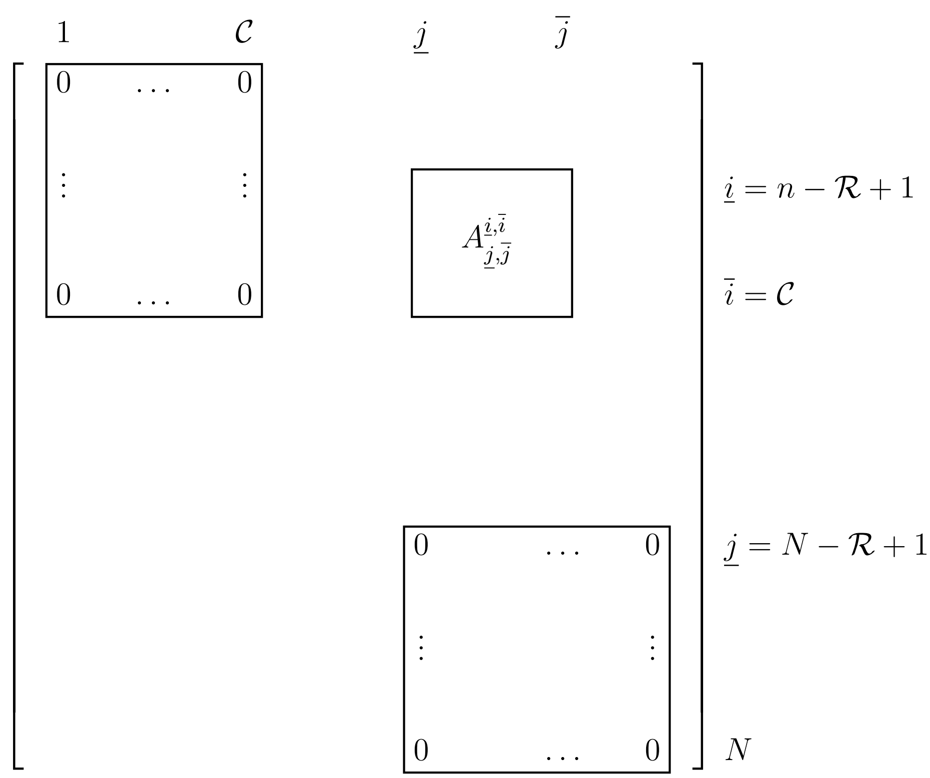
$$\underline{j} := N - \mathcal{R} + 1 \quad \text{and} \quad \bar{j} := \mathcal{C} + n.$$

We have

$$\bar{i} - \underline{i} = \mathcal{R} + \mathcal{C} - n - 1 = \bar{j} - \underline{j}.$$

In particular, $\underline{i} \le \bar{i}$ and $\underline{j} \le \bar{j}$, since we suppose that $n < \mathcal{R} + \mathcal{C}$.

By Lemma 5.1.4, $A^{1,\bar{i}}_{\underline{j},N}$ has $(\mathcal{R} + \mathcal{C} - n)$ many 1's. However, there is no 1 in $A^{1,\underline{i}-1}_{\underline{j},N}$; otherwise there would exist an a such that $i_a < \underline{i} = n - \mathcal{R} + 1$ and $j_a \ge \underline{j}$, which leads to a contradiction by considering the number of 1's in the region determined by the union of $A^{1,i_a-1}_{1,N}$ and $A^{1,N}_{j_a+1,N}$. Similarly, there is no 1 in $A^{1,\bar{i}}_{\bar{j}+1,N}$. Hence the $(\mathcal{R} + \mathcal{C} - n) \times (\mathcal{R} + \mathcal{C} - n)$-submatrix $A^{\underline{i},\bar{i}}_{\underline{j},\bar{j}}$ contains $(\mathcal{R} + \mathcal{C} - n)$-many 1's. Thus, $A^{\underline{i},\bar{i}}_{\underline{j},\bar{j}}$ is the identity matrix of dimension $\mathcal{R} + \mathcal{C} - n$.



This means in particular that for every X in the image of ψ we have

$$r_{\underline{i},\underline{j}-1}(X) = r_{n-\mathcal{R}+1,N-\mathcal{R}}(X) = 0$$

and

$$r_{\bar{i}+1,\bar{j}}(X) = r_{\mathcal{C}+1,\mathcal{C}+n}(X) = 0.$$

Since k is algebraically closed, there exists a root of the minor $m^{\underline{i},\underline{i}+1\dots,\bar{i}}_{\underline{j},\underline{j}+1\dots,\bar{j}}$. Thus, there exists X in the image of ψ such that

$$r_{\underline{i},\bar{j}}(X) \leq \bar{i} - \underline{i},$$

which means

$$r_{n-\mathcal{R}+1,\mathcal{C}+n}(X) \leq \mathcal{R} + \mathcal{C} - n - 1.$$

Lemma 5.1.5 implies that for every X in the image of ψ we have

$$r_{n-\mathcal{R}+1,\mathcal{C}+n}(X) \geq \mathcal{C} + n - (n - \mathcal{R} + 1) + 1 - n = \mathcal{R} + \mathcal{C} - n.$$

This is a contradiction. □

Note that for $r = 2$, Conjecture 6 claims that $n \geq \mathcal{C} + \mathcal{R}$, so σ is of the form

$$\sigma = \begin{pmatrix} 1 & \dots & \mathcal{C} & & & \\ & & & N-\mathcal{R}+1 & \dots & N \end{pmatrix} \text{ or } \sigma = \begin{pmatrix} 1 & \dots & \mathcal{C} & \dots & & & \\ & & & \dots & N-\mathcal{R}+1 & \dots & N \end{pmatrix}$$

We now define our last set of permutations:

- $\mathbf{MP}(r, N)$ is the set of minimal permutations in $\mathbf{P}(N)$ that appear as a permutation in the form σ_ψ for some nonconstant morphism $\psi : \mathbb{P}^{r-1}_k \to V(d) - L(d)$ for some d.

For instance, $\sigma_\psi = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$ is in $\mathbf{MP}(2, 6)$: If it were not, using Figure 4.1, $\sigma^{(1)} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}$ or $\sigma^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \end{pmatrix}$ would be in $\mathbf{MP}(2, 6)$. For $\sigma^{(1)}$, if we restrict the corresponding morphism $\psi^{(1)}$ to $\mathbb{P}^1_k$, also denoted $\psi^{(1)}$, we get

$$\psi^{(1)}(x_1, x_2) = \begin{bmatrix} 0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & 0 & 0 & 0 & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & 0 & p_{36} \\ 0 & 0 & 0 & 0 & 0 & p_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the p_{ij}'s are homogeneous polynomials in $k[x,y]$. But by Theorem 2.1.3, there exists $\gamma \in \mathbb{P}^1_k$ such that $p_{25}(\gamma) = 0$ since k is algebraically closed field. Then the rank of the matrix is less than 3, so that $\psi^{(1)}(\gamma) \in L(d)$, which gives a contradiction. Similarly, for $\sigma^{(2)}$, if we restrict the corresponding morphism $\psi^{(2)}$ to $\mathbb{P}^1_k$, we have

$$\psi^{(2)}(x_1, x_2) = \begin{bmatrix} 0 & 0 & 0 & p_{14} & p_{15} & p_{16} \\ 0 & 0 & 0 & 0 & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

But again by Theorem 2.1.3, there exists $\lambda \in \mathbb{P}^1_k$ such that $m^{123}_{456}(\lambda) = 0$ since k is algebraically closed field. Then $\psi^{(2)}(\lambda) \in L(d)$, which is a contradiction. Therefore, $\sigma_\psi = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$ is in $\mathbf{MP}(2,6)$.

Similarly, it is easy to see that $\sigma_{\psi'} = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$ is also in $\mathbf{MP}(2,6)$. Otherwise, using Figure 4.1, $\sigma^{(1)}_{\psi'} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}$ and $\sigma^{(2)}_{\psi'} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \end{pmatrix}$ would be in $\mathbf{MP}(2,6)$. We already showed that $\sigma^{(1)}_{\psi'} = \sigma^{(1)} \notin \mathbf{MP}(2,6)$. For $\sigma^{(2)}_{\psi'}$, we have

$$\psi'^{(2)}(x_1, x_2) = \begin{bmatrix} 0 & 0 & 0 & p_{14} & p_{15} & p_{16} \\ 0 & 0 & 0 & p_{24} & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & 0 & p_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\psi'^{(2)}$ is the symmetric case of $\psi^{(2)}$ with respect to anti-diagonal so $\sigma^{(2)}_{\psi'} \notin \mathbf{MP}(2,6)$.

Now we explain the process given in the proof of the statement: Conjecture 6 holds for $r \leq 2$ (see, Theorem 5.1.7) when $r = 2$ and σ is not in $\mathbf{DP}(N)$. Since the proof will be technical, we illustrate special case of it with a numerical example. Our aim is to obtain a new σ, called $\tilde{\sigma}$, such that it is no longer possible to apply a move of type III to $\tilde{\sigma}$ so that $j_p < j_t$ for all $p < t$ in $\tilde{\sigma}$. By Proposition 5.1.2, this implies that $\tilde{\sigma}$ is in $\mathbf{DP}(N)$.

Let $\sigma_\psi = (1,6)(2,3)(4,5)$. Clearly, σ is not in $\mathbf{DP}(6)$.
We have two possible move of type III on σ_ψ. We start with changing the first and second rows by using the operations on polynomial maps from $\mathbb{A}_k^r$ to V_N in Section 4.1. In the proof of Theorem 5.1.7, we determine an order about applying a move of type III to σ not in $\mathbf{DP}(N)$ and we will generalize it.

$$\begin{bmatrix} 0 & 0 & \boxed{\begin{matrix} 0 & 0 & 0 & 1_\downarrow \\ 1^\uparrow & 0 & 0 & 0 \end{matrix}} \\ 0 & 0 & \\ 0 & 0 & 0 \; 0 \; 0 \; 0 \\ 0 & 0 & 0 \; 0 \; 1 \; 0 \\ 0 & 0 & 0 \; 0 \; 0 \; 0 \\ 0 & 0 & 0 \; 0 \; 0 \; 0 \end{bmatrix}.$$

Set $\psi^0 = \psi$ and $Z_0 = \emptyset$. Then we have a rational map $\psi^0 : \mathbb{A}_k^2 - Z_0 \to V_6$. Consider the affine variety Z_1 as a subvariety of $\mathbb{A}_k^4$ by considering $\mathbb{A}_k^2 \subset \mathbb{A}_k^4$. Here we write (x, y, u_1, u_2) to denote a point in $\mathbb{A}_k^4$. Thus, $\mathbb{A}_k^2$ corresponds to the points where $u_1 = u_2 = 0$.

Represent ψ^0 by a matrix (ψ_{ij}^0) whose (i,j)-entry $\psi_{i,j}^0$ is a rational function in $k(x,y)$.

$$\psi^0(x,y) = \begin{bmatrix} 0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & 0 & p_{23} & p_{24} & p_{25} & p_{26} \\ 0 & 0 & 0 & 0 & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & p_{45} & p_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ where } p_{ij} \in k(x,y)$$

Let p_1 denote the entry $\psi_{1,3}^0$ and p_2 denote the entry $\psi_{2,3}^0$. Let $\bar{p}$ denote the greatest monic polynomial in $k[x,y]$ that divides p_1 and p_2. Set $p_1' := p_1/\bar{p}$ and $p_2' = p_2/\bar{p}$. We define $Z_1 = V(u_1, u_1p_1' + u_2p_2')$ and obtain ψ^1 from ψ^0 by first applying $D_1(u_1)$, second applying $R_{2,1}(u_2)$, then $D_2(u_1p_1' + u_2p_2')$. If we let U:=$u_1p_1' + u_2p_2'$, then we have

$$\begin{bmatrix} 0 & 0 & p_1u_1+p_2u_2 & p_{14}u_1+p_{24}u_2 & p_{15}u_1+p_{25}u_2 & p_{16}u_1+p_{26}u_2 \\ 0 & 0 & Up_2 & Up_{24} & Up_{25} & Up_{26} \\ 0 & 0 & 0 & 0 & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & p_{45} & p_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally we apply $R_{1,2}(p_2')$ to above matrix. Since $r_{14}(P) = 1$ where P is the partial permutation matrix of σ_ψ, $m_{34}^{12}(\psi^0) = m_{34}^{12} = 0$. As a result,

$$\psi^1(x,y,u_1,u_2) = \begin{bmatrix} 0 & 0 & p_1u_1+p_2u_2 & p_{14}u_1+p_{24}u_2 & p_{15}u_1+p_{25}u_2 & p_{16}u_1+p_{26}u_2 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \frac{u_1}{\bar{p}}m_{35}^{12} & \frac{u_1}{\bar{p}}m_{36}^{12} \\ 0 & 0 & 0 & 0 & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & p_{45} & p_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We obtain $\bar{\sigma} = (1,3)(2,6)(4,5)$ which is also not in $\mathbf{DP}(6)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {}_{\leftarrow}1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{\rightarrow} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let p_5 denote the entry ψ_{45}^1 and p_6 denote the entry ψ_{46}^1. Let $\bar{q}$ denote the greatest monic polynomial in $k[x,y,u_1,u_2]$ that divides p_5 and p_6. Set $p_5' = p_5/\bar{q}$ and $p_6' = p_6/\bar{q}$. We define $Z_2 = Z_1 \cup V(u_3, u_3p_6' + u_4p_5')$ and obtain ψ^2 from ψ^1 by applying the following operators. First we apply $D_6(1/u_3)$, second $R_{6,5}(u_4)$.

Then we have

$$\begin{bmatrix} 0 & 0 & * & * & * & (p_{16}u_1+p_{26}u_2)u_3+(p_{15}u_1+p_{25}u_2)u_4 \\ 0 & 0 & 0 & 0 & \frac{u_1}{\bar{p}}m^{12}_{35} & (\frac{u_1}{\bar{p}}m^{12}_{36})u_3+(\frac{u_1}{\bar{p}}m^{12}_{35})u_4 \\ 0 & 0 & 0 & 0 & p_{35} & p_{36}u_3+p_{35}u_4 \\ 0 & 0 & 0 & 0 & p_5 & p_6u_3+p_5u_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Set $E := (p_{16}u_1+p_{26}u_2)u_3+(p_{15}u_1+p_{25}u_2)u_4$ and $F := (\frac{u_1}{\bar{p}}m^{12}_{36})u_3+(\frac{u_1}{\bar{p}}m^{12}_{35})u_4$. If we apply the operator $D_5(1/u_3p'_6+u_4p'_5)$, then we have

$$\begin{bmatrix} 0 & 0 & * & * & (p_{15}u_1+p_{25}u_2)(u_3p'_6+u_4p'_5) & E \\ 0 & 0 & 0 & 0 & \frac{u_1}{\bar{p}}m^{12}_{35}(u_3p'_6+u_4p'_5) & F \\ 0 & 0 & 0 & 0 & p_{35}(u_3p'_6+u_4p'_5) & p_{36}u_3+p_{35}u_4 \\ 0 & 0 & 0 & 0 & p_5(u_3p'_6+u_4p'_5) & p_6u_3+p_5u_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and finally apply $R_{5,6}(p'_5)$;

$$\begin{bmatrix} 0 & 0 & * & * & (p_{15}u_1+p_{25}u_2)(u_3p'_6+u_4p'_5)+Ep'_5 & E \\ 0 & 0 & 0 & 0 & \frac{u_1}{\bar{p}}m^{12}_{35}(u_3p'_6+u_4p'_5)+Fp'_5 & F \\ 0 & 0 & 0 & 0 & \frac{u_3}{\bar{q}}m^{34}_{56} & p_{36}u_3+p_{35}u_4 \\ 0 & 0 & 0 & 0 & \mathbf{0} & p_6u_3+p_5u_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $r_{3,6}(\bar{P}) = 1$, where $\bar{P}$ is the partial permutation matrix of $\bar{\sigma}$, $m^{34}_{56} = 0$. Therefore, $\psi^2(x,y,u_1,u_2,u_3,u_4)$ is in the following form

$$\begin{bmatrix} 0 & 0 & p_1u_1+p_2u_2 & p_{14}u_1+p_{24}u_2 & \frac{u_1u_3}{\bar{q}}m^{14}_{56}+\frac{u_2u_3}{\bar{q}}m^{24}_{56} & E \\ 0 & 0 & 0 & 0 & \frac{u_1u_3}{\bar{p}}(p'_6m^{12}_{35}+p'_5m^{12}_{36}) & F \\ 0 & 0 & 0 & 0 & \mathbf{0} & p_{36}u_3+p_{35}u_4 \\ 0 & 0 & 0 & 0 & \mathbf{0} & p_6u_3+p_5u_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently, we obtain $\tilde{\sigma} = (1,3)(2,5)(4,6)$ which is in $\mathbf{DP}(6)$.

We now state and prove our second main result.

Theorem 5.1.7. *[29, Theorem 3] Conjecture 6 holds for $r \leq 2$.*

Proof. We have $\mathbf{MP}(1,N) = \{(1,n+1)(2,n+2)\dots(n,N)\}$ (see Example 6.1.1). This means Conjecture 6 holds for $r = 1$, because $N - n + (n+1) - 1 = N \leq N$. Hence it is enough to prove Conjecture 6 for $r = 2$. Suppose that Conjecture 6 does not hold for $r = 2$. Then there exists an N-tuple of nonincreasing integers $d = (d_1, d_2, \dots, d_N)$, two positive integers $\mathcal{R}$ and $\mathcal{C}$, and a nonconstant morphism $\psi : \mathbb{P}^1_k \to V(d)_{\mathcal{RC}} - L(d)$ such that $n < \mathcal{R} + \mathcal{C}$. Write $\sigma_\psi = (i_1, j_1)(i_2, j_2)\dots(i_n, j_n)$ with $1 = i_1 < i_2 < \dots < i_n$ and $i_a < j_a$ for all $1 \leq a \leq n$.

If σ is in $\mathbf{DP}(N)$, then by Theorem 5.1.6 we are done. Hence assume that $\sigma \notin \mathbf{DP}(N)$. We recursively define perturbations of ψ so that we can again use the square submatrix $A^{\underline{i},\bar{i}}_{\underline{j},\bar{j}}$ to get a contradiction similar to that of the previous case. Set $\psi^0 = \psi$, $n_0 = 0$, $Z_0 = \emptyset$. We have a rational map $\psi^0 : \mathbb{A}^{2+n_0}_k - Z_0 \to V_N$. Now given

$$\psi^s : \mathbb{A}^{2+n_s}_k - Z_s \to V_N$$

we define $\psi^{s+1} : \mathbb{A}^{2+n_{s+1}}_k - Z_{s+1} \to V_N$ when $\sigma_s = \sigma_{\psi^s}$ is not in $\mathbf{DP}(N)$.

Assume $\sigma_s \notin \mathbf{DP}(N)$. By Proposition 5.1.2, there exists a move of type III that we can apply to σ_s. Hence, we may define

$$l'_s := \min\{\, l' \mid l' < l < \sigma_s(l) < \sigma_s(l') \,\}$$

and

$$l_s := \sigma_s \left(\min\{\, \sigma_s(l) \mid l'_s < l < \sigma_s(l) < \sigma_s(l'_s) \,\}\right).$$

In case $l'_s < \underline{i}$, we define

$$n_{s+1} := n_s + l_s - l'_s + 1.$$

Note that $l_s > l'_s$ and so $n_{s+1} \geq n_s + 2$. Hence the affine variety Z_s can be considered as a subvariety of $\mathbb{A}^{2+n_{s+1}}_k$ by considering $\mathbb{A}^{2+n_s}_k \subset \mathbb{A}^{2+n_{s+1}}_k$. Here we write $(x, y, u_1, u_2, \dots u_{n_{s+1}})$ to denote a point in $\mathbb{A}_k^{2+n_{s+1}}$. Hence $\mathbb{A}_k^{2+n_s}$ corresponds to the points where $u_{n_s+1} = \dots = u_{n_{(s+1)}} = 0$

Represent ψ^s by the matrix (ψ^s_{ij}) whose (i,j)-entry is a rational function in $k(x, y, u_1, u_2, \dots u_{n_s})$. Let p_i denote the entry $\psi^s_{i,\sigma(l_s)}$, which is a polynomial. Also let $\bar{p}$ be the greatest common divisor of $p_{l'_s}, p_{l'_s+1}, \dots, p_{l_s}$. Set $p'_i := p_i/\bar{p}$ for $i \in \{l'_s, l'_s+1, \dots, l_s\}$. We define

$$Z_{s+1} = Z_s \cup V\left(u_{n_s+1}\,,\ \sum_{i:=l'_s}^{l_s} u_{n_s+i-l'_s+1}p'_i\right).$$

We obtain ψ^{s+1} from ψ^s by first applying $D_{l'_s}(u_{n_s+1})$, then $R_{i,l'_s}(u_i)$ where $l'_s < i \le l_s$, then $D_i(\sum_{i:=l'_s}^{l_s} u_{n_s+i-l'_s+1}p'_i)$ for $l'_s < i \le l_s$, and finally applying $R_{l'_s,i}(-p'_i)$ for $l'_s < i \le l_s$. Note that p'_i also depends on s, so we write $p'_{s,i}$ instead of p'_i when s is not clear.

We can repeat this process until it is no longer possible to find a move of type III with $l'_s < \underline{i}$. At the end of this part of the process we obtain a rational map ψ^t for some t. Then we can continue with the symmetric (with respect to the diagonal of the matrix running from the lower left entry to the upper right entry) operations assuming ψ^s is defined for $s \ge t$. We define ψ^{s+1} as follows:

$$l'_s := \max\{\, l' \mid l' < l < \sigma_s(l) < \sigma_s(l')\,\}$$

and

$$l_s := \sigma_s\left(\max\{\, \sigma_s(l) \mid l'_s < l < \sigma_s(l) < \sigma_s(l'_s)\,\}\right).$$

We repeat the symmetric operations as long as we have $\sigma_s(l'_s) > \bar{j}$. We define $n_{s+1} := n_s + \sigma_s(l'_s) - \sigma_s(l_s) + 1$. Let p_j denote the entry $\psi^s_{l_s,j}$, which is a polynomial. Also let $\bar{q}$ be the greatest common divisor of $p_{\sigma_s(l_s)}, p_{\sigma(l_s)+1}, \dots, p_{\sigma_s(l'_s)}$. Similarly, we write $p'_{s,j}$ instead of p'_j when s is not clear.

At the end of this process, we obtain a rational map $\psi^{\bar{t}}$ from the quasi-affine variety $U = \mathbb{A}_k^{2+n_{\bar{t}}} - Z_{\bar{t}}$ to V_N where we have $r_{\underline{i},\underline{j}-1}(X) = r_{n-\mathcal{R}+1,N-\mathcal{R}}(X) = 0$ and $r_{\bar{i}+1,\bar{j}}(X) = r_{\mathcal{C}+1,\mathcal{C}+n}(X) = 0$ for every X in the image of this rational map. Denote the composition of $m^{\underline{i},\dots,\bar{i}}_{\underline{j},\dots\bar{j}}$ with $\psi^{\bar{t}}$ by m. Notice that m is a polynomial in

$k[x, y, u_1, \ldots u_{n_{\bar{t}}}]$. Define another polynomial p in the same polynomial algebra as follows:

$$p = \left(\prod_{s=0}^{t-1} u_{n_s+1} \sum_{i=l'_s}^{l_s} u_{n_s+i-l'_s+1} p'_{s,i} \right) \left(\prod_{s=t}^{\bar{t}-1} u_{n_s+1} \sum_{j=\sigma(l_s)}^{\sigma(l'_s)} u_{n_s+\sigma(l'_s)-j+1} p'_{s,j} \right).$$

Note that $p'_{s,l'_s}, p'_{s,l'_s+1}, \ldots, p'_{s,l_s}$ are relatively prime for $1 \leq s \leq t$ and similarly $p'_{s,\sigma(l_s)}, p'_{s,\sigma(l_s)+1}, \ldots, p'_{s,\sigma(l'_s)}$ are also relatively prime for $t+1 \leq s \leq \bar{t}$. Moreover, the polynomial m has an irreducible factor in $k[x, y]$ or for some s, the polynomial m has an irreducible factor in the form $f u_{n_s} + g$, where f and g are in $k[x, y, u_1, \ldots u_{n_s-1}]$ so that f is neither an associate of p'_{s,l_s} nor $p'_{s,\sigma(l_s)}$. Hence, there exists a solution to the equations $p = 1$ and $m = 0$, which is again a contradiction by Lemma 5.1.5. □

Chapter 6

Examples and open problems

6.1 Examples

Recall Conjecture 6:

If there exists a nonconstant morphism $\psi : \mathbb{P}_k^{r-1} \to V(d)_{\mathcal{R}\mathcal{C}} - L(d)$, then $N \geq 2^{r-1}(\mathcal{R} + \mathcal{C})$.

The last inequality is equivalent to

$$r \leq \left\lfloor \log_2 \left(\frac{N}{\mathcal{R} + \mathcal{C}} \right) \right\rfloor + 1.$$

One might ask how strict this upper bound for r is. In the following examples, we define ψ for different values of r, N, and $\mathcal{R}+\mathcal{C}$ where $r = \left\lfloor \log_2 \left(\frac{N}{\mathcal{R}+\mathcal{C}} \right) \right\rfloor + 1$. It follows that we cannot have a better upper bound for r in these cases. Therefore, the conjecture cannot be stronger in general.

Remember that ψ is a strictly upper-triangular $N \times N$ matrices (x_{ij}) of rank $N/2$ with $\deg(x_{ij}) = d_i - d_j + 1$. Furthermore, we have $\mathcal{C}$-many leading zero columns and $\mathcal{R}$-many ending zero rows.

Example 6.1.1. For $r = 1$, $N = 2n$, $d = (0, \ldots, 0)$ and $\mathcal{R} + \mathcal{C} = N$, define

$$\psi(x) = \left[\begin{array}{c|c} 0 & M \\ \hline 0 & 0 \end{array}\right] \quad \text{where } M = \begin{bmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{bmatrix}.$$

Note that $\sigma_\psi = (1, n+1)(2, n+2)\ldots(n, N)$. This example shows that

$$\mathbf{MP}(1, N) = \{(1, n+1)(2, n+2)\ldots(n, N)\}.$$

Example 6.1.2. For $r = 2$ and $N = 4$, $d = (0, 0, 0, 0)$ and $\mathcal{R} + \mathcal{C} = 2$, define

$$\psi(x, y) = \begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this example, $\sigma_\psi = (1, 2)(3, 4)$. Hence,

$$\mathbf{MP}(2, 4) = \{(1, 2)(3, 4)\}.$$

Example 6.1.3. For $r = 2$, $N = 6$, $d = (0, -1, -1, -1, -1, -1)$, and $\mathcal{R} + \mathcal{C} = 3$, set:

$$\psi(x, y) = \begin{bmatrix} 0 & x^2 & xy & y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, we have $\sigma_\psi = (1, 2)(3, 5)(4, 6)$.

Now we consider the symmetric case of σ_ψ with respect to the anti-diagonal:

For $r = 2$, $N = 6$, $d = (0, 0, 0, 0, 0, -1)$, and $\mathcal{R} + \mathcal{C} = 3$, set:

$$\psi'(x, y) = \begin{bmatrix} 0 & 0 & -x & y & 0 & 0 \\ 0 & 0 & 0 & -x & y & 0 \\ 0 & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & 0 & 0 & xy \\ 0 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This time we have $\sigma_{\psi'} = (1, 3)(2, 4)(5, 6)$.

Considering the Hasse diagram for $\mathbf{RP}(6)$ in Figure 4.1, it is clear that

$$\mathbf{MP}(2, 6) = \{\, (1, 2)(3, 5)(4, 6)\, ,\ (1, 3)(2, 4)(5, 6)\, \}.$$

Using *GAP* we verified that

$$\mathbf{MP}(2, 8) = \{\, (1, 2)(3, 6)(4, 7)(5, 8)\, ,\ (1, 4)(2, 5)(3, 6)(7, 8)\, ,\ (1, 3)(2, 5)(4, 7)(6, 8)\, \}.$$

Our *GAP* calculations also suggest:

$$\begin{aligned} \mathbf{MP}(2, 10) = \{\, & (1, 2)(3, 7)(4, 8)(5, 9)(6, 10)\, ,\ (1, 5)(2, 6)(3, 7)(4, 8)(9, 10) \\ & (1, 4)(2, 6)(3, 7)(5, 9)(8, 10)\, ,\ (1, 3)(2, 6)(4, 8)(5, 9)(7, 10)\, \}. \end{aligned}$$

The above example can be generalized:

Example 6.1.4. For $r = 2$, $N = 2n$, $d = (0, -n+2, \ldots, -n+2)$, and $\mathcal{R}+\mathcal{C} = n$, set:

$$\psi(x, y) = \begin{bmatrix} 0 & x^{n-1} & x^{n-2}y & \ldots & y^{n-1} & 0 & & \ldots & 0 \\ & 0 & 0 & \ldots & 0 & y & & & \vdots \\ & & 0 & & & -x & y & & \\ & & & & & & -x & \ddots & 0 \\ & & & & \ddots & & & \ddots & y \\ & & & & & & & & -x \\ & & & & & & & & 0 \\ & & & & & & & & \vdots \\ & & & & & & & & 0 \end{bmatrix}.$$

The corresponding permutation is $\sigma_\psi = (1,2)(3,n+2)(4,n+3)\dots(n+1,N)$.

Similarly we obtain the symmetric case of σ_ψ with respect to anti-diagonal. For $r=2$, $N=2n$, $d=(0,\dots,0,-n+2)$, and $\mathcal{R}+\mathcal{C}=n$, set:

$$\psi'(x,y) = \begin{bmatrix} 0 & \dots & 0 & -x & y & 0 & & & \dots & 0 \\ & & & & -x & y & & & & \vdots \\ & & & & & -x & \ddots & & & \\ & & & & & & \ddots & y & & \\ & & & & & & & -x & y & 0 \\ & & & & & & & & & y^{n-1} \\ & & & & & & & & & xy^{n-2} \\ & & & & & & & & & \vdots \\ & & & & & & & & & x^{n-1} \\ & & & & & & & & & 0 \end{bmatrix}.$$

Here $\sigma_{\psi'} = (1,n)(2,n+1)\dots(n-1,N-2)(N-1,N)$.

We can use the above examples to obtain new ones by the chess board construction:

Construction 6.1.1. Let $(l_1,l_2,\dots,l_m)$ be an m-tuple of positive integers and $V(d)_{(l_1,l_2,\dots,l_m)}$ the subvariety of $V(d)$ such that $x_{ij}=0$ when $l_1+l_2+\dots+l_{(s-1)}+1 \le i < j \le l_1+l_2+\dots+l_s$ for some $1 \le s \le m$. For example, the following matrix $\psi(x,y)$ is in $V(d)_{(1,3,2)}$ where d is 6-tuple of nonincreasing integers:

$$\psi(x,y) = \begin{bmatrix} \boxed{0} & x^2 & xy & y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -x & y \\ 0 & 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Take $\psi_1 \in V(d_1)_{(l_1^1,l_2^1,\dots,l_m^1)}$ where d_1 is N_1-tuple nonpositive integers and $\psi_2 \in V(d_2)_{(l_1^2,l_2^2,\dots,l_m^2)}$ where d_2 is N_2-tuple of nonincreasing integers. We arrange a $(N_1+N_2)\times(N_1+N_2)$ matrix in a $2m\times 2m$-chessboard as follows: The ij-square contains a $l^{\varepsilon_i}_{\lfloor \frac{i+1}{2} \rfloor} \times l^{\varepsilon_j}_{\lfloor \frac{j+1}{2} \rfloor}$ matrix such that $\varepsilon_k = 1$ if k is odd or $\varepsilon_k = 2$ if k is even integer. Now we color the ij square black if $\varepsilon_i \neq \varepsilon_j$ and white if $\varepsilon_i = \varepsilon_j$. Fill in the ij square with zeros if it is a black square and otherwise fill it in with $(x_{i'j'})$ where $\underline{i} \leq i' \leq \overline{i}$ and $\underline{j} \leq j' \leq \overline{j}$ part of ψ_{ε_i} where $\underline{i}, \overline{i}, \underline{j}, \overline{j}$ are defined by

$$\underline{s} = \sum_{m=1}^{\lfloor \frac{s+1}{2} \rfloor - 1} l^{\varepsilon_s}_m + 1 \text{ and } \overline{s} = \sum_{m=1}^{\lfloor \frac{s+1}{2} \rfloor} l^{\varepsilon_s}_m.$$

For instance, using chessboard construction we can obtain an example:

Example 6.1.5. For $r = 2$, $N = 4{+}6 = 10$, $d = (0, 0, -1, -1, -1, -1, -1, -1, -1, -1)$, and $\mathcal{R} + \mathcal{C} = 3 + 2 = 5$, we obtain an example by applying the chess board construction on the morphisms in Examples 6.1.2 and 6.1.3.

0		x^2	xy	y^2			0	0	
	0				x	y			0
0		0	0	0			y	0	
0		0	0	0			$-x$	y	
0		0	0	0			0	$-x$	
	0				0	0			y
	0				0	0			$-x$
0		0	0	0			0	0	
0		0	0	0			0	0	
	0				0	0			0

We also have other well-known constructions like the Koszul complex construction [16] giving us examples as below.

Example 6.1.6. For $r = 3$, $N = 8$ and $\mathcal{R} + \mathcal{C} = 2$, define

$$\psi(x,y,z) = \begin{bmatrix} 0 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & x & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this example, $\sigma_\psi = (1,2)(3,5)(4,6)(7,8)$.

6.2 Problems

We end with a few questions for future research. Note that all examples discussed above are in $\mathbf{DP}(N)$. Hence one can ask :

Question 6.2.1. Is $\mathbf{MP}(r, N) \subseteq \mathbf{DP}(N)$?

For all these examples $\frac{N}{2^{r-1}}$ is an integer. For instance, we can find an example for $r = 3$, $N = 12$ (see Example 6.2.1) but we do not know the answer of the following question:

Question 6.2.2. Is there any example for $r = 3$ and $N = 10$? More precisely, is $\mathbf{MP}(3, 10)$ is nonempty?

Note that the following example cannot be obtained by the constructions we mentioned above.

Example 6.2.1. For $r = 3$, $N = 12$, $d = (0, 0, -1, \ldots, -1)$ and $\mathcal{R} + \mathcal{C} = 3$, consider

$$\begin{bmatrix} 0 & x & y^2 & yz & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y^2 & yz & 0 & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 0 & -z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & y & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & -y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that we can obtain an example for $r = 3$ and $N = 4s$ for every $s \geq 2$ by using the examples for $r = 3$, $N = 8$ together with the example for $r = 3$, $N = 12$ and applying the chessboard construction as many times as necessary. If the answer to Question 6.2.2 is negative, then one can ask the following question:

Question 6.2.3. Do there exist any periodicity results about nonemptiness of $\mathbf{MP}(r, N)$?

Another observation we make about these examples is that there always exists a sequence of permutations $\sigma_1 < \sigma_2 < \cdots < \sigma_r$ such that the image of the morphism contains a point from each Borel orbit corresponding to the these σ_i's and each pair of consecutive σ_i's consist of distinct transpositions. For example, putting $x = 1$ and $y = 0$ to ψ in Example 6.1.2, we get a point in the Borel orbit corresponding to permutation $\sigma_2 = (1, 2)(3, 4)$. and putting $x = 0$ and $y = 1$, we get $\sigma_1 = (1, 3)(2, 4)$. Hence one could ask:

Question 6.2.4. Given σ in $\mathbf{MP}(r, N)$ does there always exists a morphism $\psi : \mathbb{P}_k^{r-1} \to V(d) - L(d)$ with a sequence permutations $\sigma_1 < \sigma_2 < \cdots < \sigma_r$ and points X_1, X_2, … ,X_r in the image of ψ such that $\sigma_\psi = \sigma$ and X_i is in the Borel orbit of σ_i for all i and σ_i and σ_{i+1} has no common transpositions?

If the answer is affirmative to this question, then one can say that the inequalities

$$\frac{n(n+1)}{2} \leq \dim(B_{\sigma_i}) \leq n^2$$

and

$$\dim(B_{\sigma_i}) + \left\lceil \frac{n}{2} \right\rceil \leq \dim(B_{\sigma_{i+1}})$$

hold and they give the inequality $N \geq 2r$.

Note that Allday and Puppe [28] have related results: If k, A, r, N, and M are as in Conjecture 1, then they prove $N \geq 2r$. Moreover, Avramov, Buchweitz and Iyengar [16] verified that $N \geq 2r$ in a more general case.

Another approach to solve Carlsson's conjecture is relevant to verify the converse of Proposition 3.1.6. Consider the simplest case:

Let k be an algebraically closed field of characteristic 2, $A = k[x]$ and M a free, finitely generated DG-A-module with $H_*(M, A/\mathbf{m}) = 0$ for all maximal ideal $\mathbf{m}$ of A other than (x). Then $m = (x + \alpha)$ where $0 \neq \alpha \in k$. The differential of M is

$$\partial(M) = \begin{bmatrix} 0 & p(x) \\ 0 & 0 \end{bmatrix},$$

where $p(x) = cx^n$ with $n \geq 1$, $c \neq 0$. Otherwise, we can take $m = (x + \beta)$, where β is a root of $p(x)$. Then

$$\partial(M \otimes A/\mathbf{m}) = \begin{bmatrix} 0 & p(x) \otimes A/\mathbf{m} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p(\beta) \\ 0 & 0 \end{bmatrix} \equiv 0.$$

It implies that $H_*(M, A/\mathbf{m}) = M \otimes A/\mathbf{m}$, which is not trivial. This contradicts our assumption. Now, consider $H_*(M)$; we have

$$\mathrm{Ker}(M) = \{(f(x), 0) \mid f(x) \in A\} \text{ and}$$

$$\mathrm{Im}(M) = \{(p(x)g(x), 0) \mid g(x) \in A\}.$$

By the division algorithm, there exist $p(x)$ and $r(x)$ such that

$$f(x) = p(x)g(x) + r(x),$$

where $\deg r(x) < \deg p(x)$. Then $f(x) \equiv r(x)$ in $\operatorname{Ker}(M)/\operatorname{Im}(M)$, but $\deg r(x)$ is finite, so $\dim_k H_*(M)$ is finite.

Note that for the case $A = k[x, y]$, if $\dim_k M = 2$, then there is no example satisfying the condition $H_*(M, A/\mathbf{m}) = 0$ where $\mathbf{m}$ is any maximal ideal of A other than (x, y), but there is such an example when $\dim_k M = 4$. Similarly, we can eliminate the case $A = k[x, y, z]$ and $\dim_k M = 4$. Hence one can ask

Question 6.2.5. Let k be an algebraically closed field of characteristic 2 and M be a free, finitely generated DG-A-module. Suppose that $H_*(M, A/\mathbf{m}) = 0$ for all maximal ideal $\mathbf{m}$ of A other than $(x_1, \ldots, x_r)$. Then under what conditions is $\dim_k H_*(M)$ finite?

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