A STUDY OF EXTENSIONS OF THE STABLE RULE FOR ROOMMATE PROBLEMS

A Master's Thesis

by

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To my dad,

A STUDY OF EXTENSIONS OF THE STABLE RULE FOR ROOMMATE PROBLEMS

The Graduate School of Economics and Social Sciences
of
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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

A STUDY OF EXTENSIONS OF THE STABLE RULE FOR ROOMMATE PROBLEMS

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Roommate problems might not have a stable solution. But for such problems we are still faced with the problem of matching the agents. One natural approach would be to match the agents in such a way that the resulting matching is "close" to being stable. Such solution concepts should select stable matchings when they exists and select matchings "close" to being stable when the problem does not have any stable matchings. We work with the following solution concepts, *Almost Stability*, *Maximum Irreversibility*, *Maximum Internal Stability*, $\mathcal{P}$*-stability* and Q*-stability*, and define a new solution concept, called *Iterated* $\mathcal{P}$*-stability*. We investigate consistency, population monotonicity, competition sensitivity and resource sensitivity of these solution concepts. We also explore Maskin monotonicity of these solution concepts.

Keywords: Competition Sensitivity, Maskin Monotonicity, Population Monotonicity, Resource Sensitivity, Roommate Problem,

ÖZET

ODA ARKADAŞI PROBLEMİ İÇİN KARARLI KURALLARIN
UZANTILARI ÜZERİNE BİR ÇALIŞMA

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Her oda arkadaşı probleminin kararlı bir çözümü olmayabilir ama kararlı çözüm olmasa bile kişileri eşleştirme problemiyle karşı karşıya kalabiliriz. Bu tarz problemlere doğal bir yaklaşım kişilerin kararlı eşleşemeseler bile kararlıya "yakın" eşleştirilmesidir. Bu tarz çözüm kavramları eğer varsa kararlı eşleşmeleri seçmeli, eğer yoksa kararlıya "yakın" eşleşmeleri seçmelidir. Bu çalışmada "Neredeyse Kararlılık", "Maksimum Geri Dönülmezlik", "Maksimum İçten Kararlılık", "P-kararlılık", "Q-kararlılık" çözüm kavramları çalışılmış ve "Yinelemeli P-kararlılık" adıyla yeni bir çözüm kavramı tanımlanmıştır. Bu çözüm kavramları için tutarlılık, popülasyon monotonluğu, rekabet duyarlılığı, kaynak duyarlılığı ve Maskin monotonluğu özellikleri araştırılmıştır.

Anahtar Kelimeler: Kaynak Duyarlılığı, Maskin Monotonluğu, Oda Arkadaşı Problemi, Popülasyon Monotonluğu, Rekabet Duyarlılığı,

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CHAPTER 1

INTRODUCTION

Matching problems in economic theory are concerned with matching members of a group of agents with one or more members of a disjoint or same group of agents. A *two-sided* matching problem is a problem which matches the members of one group of agents with the members of another group of agents such as men and women, students and schools or patients and kidneys. A *one-sided* matching problem is a problem which matches the members of the same group of agents with each other.

Roommate problems which is introduced by Gale & Shapley (1962) are one-sided matching problems. Some real-life examples of such problems are problems of assigning students as roommates and pairing students for a project. A matching is *stable* if there are no two agents who are not roommates and who prefer each other to their actual roommates and no agent strictly prefers being single to being matched with his current roommate. In a roommate problem, we deal with the following question: "Is there a stable way to assign agents into roommate pairs?".

Gale & Shapley (1962) showed that there always exists a stable matching for marriage problems. On the other hand, they showed that there might not exist stable matchings for roommate problems. The question "Which conditions guarantee the existence of stable matchings for roommate problems?" has arisen after Gale & Shapley (1962)'s paper. The investigation of the conditions which guarantee the existence of stable matchings for a roommate problem, one of the problems we are interested in, is studied with various aspects. Tan (1991) identifies the necessary and sufficient conditions for the existence of a complete stable roommate matching[1]; namely, *the non-existence of any odd party.* Tan works with strict preferences.

Chung (2000) gives a sufficient condition, called "no odd ring" condition[2], for the existence of a stable roommate matching under weak preferences. Marriage problems always satisfy *no-odd ring* condition, this is the reason for the existence of stable matchings for marriage problems. Chung also states several sufficiency conditions for the existence of stable matchings for roommate problem, such as preferences being dichotomous or single-peaked.

On the other hand, Irving (1985) proposes an algorithm which determines if a complete stable matching exists or not. If it exists, it finds such a matching, if it does not exist, then the algorithm reports the non-existence of a complete stable matching. Irving only considers cases where the number of agents is even and he is interested in finding a complete stable matching. Gusfield (1988) extends the Irving Algorithm in order to find the set of all complete stable matchings for roommate problems. In our study, we have no restriction on the number of agents. We study not only complete stable matchings but also stable matchings. We have a condition which can be used to check the non-existence of a stable matching.

Roommate problems might not have stable solutions. But in such problems we are still faced with the problem of matching the agents. One natural ap-

[1] A matching is complete if every agent is paired with another agent.

[2] An odd ring is a cycle between an odd number of agents such that; agent i's best raking is agent j, agent j's best ranking is agent k and agent k's best raking is agent i.

proach would be to match the agents in such a way that the resulting matching is "close" to being stable. Such solution concepts should select stable matchings when they exist and select matchings "close" to being stable when the problem does not have any stable matchings. Several such solution concepts has been introduced in the literature. *An Almost stable matching*, introduced by Abraham et al. (2006), is a matching that has a minimum number of blocking pairs. *A Maximum internally stable matching*, proposed by Tan (1990), is a matching such that there exist a maximal set of agents that are matched to agents within this set and this matching is a complete stable matching among these agents. Tan (1990) introduces the notion of stable partitions and defines a necessary and sufficient condition which based on this notion for the existence of a stable roommate matching. *A $\mathcal{P}$-stable matching* which is based on stable partition notion, is introduced by Iñarra et al. (2008). A $\mathcal{P}$-stable matching is a matching such that there is an ordering of agents in which ith agent matched with the $(i-1)$th or $(i+1)$th agent under a stable partition.

We observe that under $\mathcal{P}$-stable solutions, there might exist unmatched agents even if they are acceptable to each other. In our study, we propose a new solution concept, called *iterated $\mathcal{P}$-stable matchings*, which iteratively construct a matching by applying the $\mathcal{P}$-stability matching concept to the set of unmatched agents at each step.

Biró et al. (2016) propose a core consistent solution concept, called *Maximum irreversible matchings*, which finds the matchings with a maximum number of pairs that are stable within a set of agents. They introduced a new solution concept, called Q-stability, which is the intersection of maximum irreversibility and maximum internal stability.

In roommate problems, we can consider agents as both consumers and resources. When newcomers are considered as additional consumers, the arrival of agents might have a negative effect on the existing agents since the newcomers will be competing for possible roommates. In such situations, we would expect all the agents to be effected in the same direction. This requirement, called *Competition Sensitivity*, is introduced by Thomson (1983). Otherwise, when newcomers are considered as additional resources, the arrival of agents might have a positive effect on the existing agents since the newcomers will be additional possible roommates. In such situations, we would expect all agents to be affected in the same direction. This requirement, called *Resource Sensitivity*, is also introduced by Thomson (1983). Klaus (2011) proves that, on the domain of solvable roommate problems, any selection of the stable solution concepts satisfies competition and resource sensitivity.

Population monotonicity requires the solution concepts to response to changes in populations in such a way that, it is possible for all agents to be affected in the same direction. We investigate population monotonicity, resource and competition sensitivity of the aforementioned solution concepts especially on the unsolvable domain of roommate problems.

A notable property, Maskin monotonicity, is introduced by Maskin (1999), requires that if a matching is a solution for a roommate problem for some preference profile and if the ranking of pairs under this matching improved, then that matching should be a solution for the new preference profile. Klaus (2011) study Maskin monotonicity and unanimity properties for solvable roommate problems. We study Maskin monotonicity on an unrestricted domain of roommate problems. We show that there is no solution concept which satisfies Maskin monotonicity among the solution concepts which we define in this thesis.

Can & Klaus (2013) characterize the core by using population sensitivity, consistency and unanimity properties on various domains of roommate problems.

Consistency of a solution concept requires the situations to be such that whenever a set of matched agents leave the problem, then the remaining agents should still be paired as before. We show that there is no solution concept which satisfies consistency among the aforementioned solution concepts.

CHAPTER 2

PRELIMINARIES

2.1 Roommate Problems

We are interested in matching a set of agents among themselves. Agents have preferences over to whom they want to match. $N = \{1, 2, 3, \ldots, n\}$ denotes a finite set of agents and for each agent $i \in N$. $\succ_i$ is a complete, transitive and asymmetric preference of agent i on N and $(\succ_i)_{i \in N}$ denotes a preference profile. A roommate problem is a pair of a finite set of agents N, and a preference profile of these agents $(\succ_i)_{i \in N}$ and is denoted by $(N, (\succ)_{i \in N})$.

Agent j is *unacceptable* for agent i if $i \succ_i j$. In this case, agent i prefers being unmatched to being matched with agent j.

A preference profile satisfies *mutual acceptability* if and only if for all $i, j \in N$, i is acceptable for j iff j is acceptable for i.

A *matching* is a function that assigns each agent to at most one agent such that if an agent i, is assigned to an agent j, then the agent j is assigned to the agent i. In this case, i and j become roommates. If an agent is assigned to herself then we will say that she remains *single*. We give the formal definitions below.

Definition 1. A *matching* $\mu : N \to N$ is a bijection such that for all $\{i, j\} \subseteq$

N $\mu(i) = j$ if and only if $\mu(j) = i$. If $\mu(i) = i$, then agent i is said to be *single* or *unmatched* under matching μ.

Definition 2. A matching satisfies *individual rationality (IR)* if no agent is assigned an agent who is unacceptable to him, i.e., for all $i \in N$, $\mu(i) \succ_i i$

Definition 3. A matching μ is *blocked* by a pair $\{i, j\} \subseteq N$, if each prefers the other to their roommates under μ, i.e, $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$.

Definition 4. A matching is *stable* if it isn't blocked by any pair of agents and no agent is assigned an unacceptable roommate.

Definition 5. A roommate problem is *solvable*, if the set of stable matchings is non-empty.

Definition 6. A matching is *complete* if every agent is paired.

A *complete stable matching* is a stable matching under which there is no unmatched agent.

CHAPTER 3

IRVING'S ALGORITHM

The Roommate Problem is proposed by Gale & Shapley (1962). They show that each marriage problem[1] has at least one stable matching then they give the *roommate problem* as an example of a situation in which there might not exist stable matching. Irving proposes an algorithm which determines whether a complete stable matching exists or not. If it exists, it finds such a matching, if it does not exist, it reports the non-existence of a complete stable matching.

In Irving's algorithm, the preferences of agents are inputs and a complete stable matching or a report of the nonexistence of a complete stable matching is the output.

We assume that the preference profile satisfies mutual acceptability. For each agent $i \in N$, E_i is the set of agents which are *eliminated* from agent i's preference.

The Irving Algorithm is as follows;

Phase 1

1. Take an ordering of N, let $i = 1$. For each agent $i \in N$, $E_i = \emptyset$

[1]The marriage problem consists of two disjoint set of agents, a set of men, denoted by M and a set of women, denoted by W and a preference relation for each agent. A *matching* $\mu : M \cup W \to M \cup W$ is a mapping which assigns each men at most one women.

2. WLOG, agent i proposes to her most preferred agent, let say agent j, from $N \setminus E_i$

   2.1. if agent j does not have an agent at hold[2], then agent j holds agent i's proposal,

      2.1.1. if agent j is the last agent of the ordering, then the algorithm terminates.

      2.1.2. otherwise, agent $i+1$ becomes new proposer, we set i as $i+1$. The algorithm goes to step 2.

   2.2. if agent j holds a proposal,

      2.2.1. if agent j holds a better proposal than agent i, then agent j rejects agent i's proposal. We set E_j as $E_j \cup \{i\}$ and E_i as $E_i \cup \{j\}$. The algorithm goes to step 2.

      2.2.2. if agent j does not hold a better proposal than agent i, then agent j holds agent i's proposal and rejects the proposal that she currently holds, let l be the agent who is rejected by agent j. We set E_l as $E_l \cup \{j\}$ and E_j as $E_j \cup \{l\}$,

         2.2.2.1. agent l becomes proposer and agent l proposes her most preferred agent, let say agent k, from $N \setminus E_l$.

            2.2.2.1.1. if agent k does not have an agent at hold, then agent k holds agent l's proposal. We set i as $i+1$ and the algorithm goes to step 2.

            2.2.2.1.2. if agent k holds a proposal,

               2.2.2.1.2.1. if agent k holds a better proposal than agent l, then agent k rejects agent l's proposal. We set E_k as $E_k \cup \{l\}$ and E_l as $E_l \cup \{k\}$. The algorithm goes to step 2.2.2.1

[2] We are using the word hold as a meaning of temporarily acceptance

2.2.2.1.2.2. if agent k does not hold a better proposal than agent l, then agent k holds agent l's proposal and rejects the proposal that she currently holds, let m be the agent who is rejected by agent k. We set E_m as $E_m \cup \{k\}$ and E_k as $E_k \cup \{m\}$, agent m becomes new proposer, the algorithm goes to step 2.2.2.1.

3. This phase will terminate when

    - each agent holds a proposal, or
    - an agent is rejected by every other agent

4. If an agent is rejected by every other agent, then there is no complete stable matching and the algorithm terminates. Otherwise,

5. If each agent holds a proposal, then we proceed to the elimination step. Assume that agent j keeps a proposal from agent i, we eliminate from E_j;

    - all those to whom agent j prefers agent i, and
    - j from all person's preference list who is deleting from i's preference.

6. After elimination;

    - if there is an agent $i \in N$, such that $E_i = N \setminus \{i\}$ (whose preference list is completely eliminated), then there is no complete stable matching and the algorithm terminates.
    - if every agent's preference contains only one agent, then these preferences yield a complete stable matching.
    - if there is an agents who has more than one person in her preference, then the algorithm continues with *Phase 2*.

Phase 2

1. Find a rotation $R = (a_1, \ldots, a_r) \mid (b_1, \ldots, b_r)$ which is a pair of sequence of agents such that; b_i is the first person on a_i's preference and the first person b_i is the second person on a_{i-1} (mod r).

2. Then eliminate this rotation in such a way;

   - all agents to whom b_i prefers a_{i-1}
   - b_i from all agent's preference who is deleted from b_i's preference in i

3. After eliminating the rotation;

   - if there exists an empty preference then there is no complete stable matching and the algorithm terminates.
   - if each agent has just one person then these preferences yield a complete stable matching
   - if an agent has more than one person in her preference after elimination, then find a new rotation exposed from reduced preference and the algorithm goes to step 1 of *phase 2*.

We study an extension of the Irving Algorithm with *stable matchings* for both even and odd number of agents and agents can be paired or remain single under the matchings.

In Irving Algorithm, if an agent is rejected by everyone then the algorithm terminates. In order to extend the algorithm for finding stable matchings, we continue the execution of the algorithm until everybody holds a proposal or rejected by everyone.

Proposition 1. *For any problem $(N, \succ)$ with an even number of agents, in* phase 1 *of Irving Algorithm, if an odd number of agents are rejected by all agents who are acceptable for each of them, then there does not exist a stable matching.*

Proof. We prove this statement using *proof by contradiction.*

Let $S = \{i_1, i_2, \ldots, i_k\}$ be the set of agents in which each of the agents is rejected by all agents who are acceptable for her and let $\mu : N \to N$ be a stable matching for $(N, \succ)$ and for all $i \in N$ A_i be the set of acceptable agents for agent i. Now our claim is;

Claim 1. For all $j \in A_i$, if agent j rejects agent i, then $\{i, j\}$ cannot be a pair under a stable matching μ.

Proof. If agent j rejects agent i, we know that agent j holds an agent who is preferred to agent i. Assume that agent j has agent k at hold. Then agent k cannot be paired with any agent that she prefers to agent j.

Now, suppose $\{i, j\}$ be paired under matching μ. Then the pair $\{j, k\}$ blocks the matching μ. Therefore, $\{i, j\}$ cannot be paired. $\square$

We have that each agent from the set S is rejected by everyone who is acceptable to her. Therefore, all agents from the set S are single under μ. We have even number of agents there must be at least one more agent who is single and she is not rejected by everyone. Let agent r be single and suppose agent k holds agent r. This means that agent k cannot be paired with any agent that she prefers to agent r. Then $\{k, r\}$ blocks the matching μ, this gives a contradiction to stability of matching μ. $\square$

CHAPTER 4

SOLUTION CONCEPTS

4.1 Solution Concepts

A *solution concept* is a systematic way of finding matchings for roommate problems i.e., a correspondence that maps each roommate problem to a set of matchings.

It is well-known that the roommate problems might not have stable solutions. But in such problems we are still faced with the problem of matching the agents. One natural approach would be to match the agents in such a way that the resulting matching is "close" to being stable. Such solution concepts should select stable matchings when they exist and select matchings "close" to being stable when the problem does not have stable matchings. Several such solution concepts have been introduced in the literature. In this section we define several of them.

4.1.1 Maximum Irreversibility

Maximum irreversibility is introduced by Biró et al. (2016). They define maximum-irreversibility as a suitable way for searching a matching that is as stable as possible.

A matching is T-irreversible if there exist a set of agents and a complete stable matching between these agents and no agent would prefer an agent outside of this set to her pair. A maximum irreversible matching is a matching which is T-irreversible with a largest such a set.

Definition 7. Let $T \subseteq N$. A matching μ is *T-irreversible* if and only if $\mu|_T$ is a complete stable matching for $(T, (\succ_i)_{i \in T})$ and for all $i \in T$, for all $j \in N \setminus T$, $\mu|_T(i) \succ_i j$.

Definition 8. A matching μ is a *maximum irreversible matching* if and only if there exist $T \subseteq N$ such that μ is a T-irreversible matching and for any $T' \subseteq N$, such that $T \subsetneq T'$, μ is not a T'-irreversible matching.

The maximum irreversible solution concept, denoted MI, selects the set of all maximum irreversible solutions for each matching problem.

4.1.2 Maximum Internal Stability

Tan (1990) proposes a solution concept which is called maximum stability. This solution concept assigns matchings with a maximum number of stable pairs.

Biró et al. (2016) modify Tan's maximum stable matching concept. A matching is T-internally stable if there exist a set of agents that are matched to agents within this set and a complete stable matching among these agents. A maximum internally stable matching is a matching which is T-internally stable with a largest such a set. We give formal definitions below.

Definition 9. Let $T \subseteq N$. A matching μ is *T-internally stable* if and only if $\mu|_T$ is a complete stable matching for $(T, (\succ_i)_{i \in T})$.

Definition 10. A matching μ is a *maximum internally stable* matching if and only if there exist $T \subseteq N$ such that μ is a T-internally stable matching and for any $T' \subseteq N$, such that $T \subseteq T'$, μ is not a T'-internally stable matching.

The maximum internally stable solution concept, denoted MIS, selects the set of all maximum irreversible solutions for each matching problem.

4.1.3 Almost Stability

An almost stable matching, introduced by Abraham et al. (2006), is a matching that has a minimum number of blocking pairs. We give the formal definition below.

We will denote the set of blocking pairs of the matching μ by $bp(\mu)$, i.e., $bp(\mu) = \{\{i, j\} \subseteq N : \{i, j\} \text{ block } \mu\}$.

Definition 11. An almost stable matching is a matching with minimum number of blocking pairs, i.e., a matching μ is almost stable if and only if for any μ', $|bp(\mu)| \leq |bp(\mu')|$.

The almost stability solution concept, denoted AS, selects the set of all almost stable solutions for each matching problem.

4.1.4 $\mathcal{P}$-stability

Tan (1991) proposes a notion of stable partition to introduce a necessary and sufficient condition for the existence of a stable roommate problem. Iñarra et al. (2008) defines a solution concept by using stable partitions. We give formal definitions below.

Definition 12. Let $A = \{a_1, \ldots, a_k\}$ be an ordered set of agents (subscript modulo k);

- A is a *ring*, if $k \geq 3$ and for any $l \in \{1, \ldots, k\}$ $\quad a_{l+1} \succ_{a_l} a_{l-1} \succ_{a_l} a_l$,
- A is *mutually acceptable* agents, if $k = 2$ and for any $l \in \{1, 2\}$, $\quad a_{l+1} \succ_{a_l} a_l$

Given a roommate problem $(N, \succ)$, a *partition* $\mathcal{P}$ of N is a set of nonempty subsets of N such that, $\bigcup_{A \in \mathcal{P}} A = N$ and for any $A, B \in \mathcal{P}$, $(A \neq B)$, $A \cap B = \emptyset$.

Definition 13. A partition $\mathcal{P}$ of N is a stable partition if

- for all $A \in \mathcal{P}$, A is a ring, a pair of mutually acceptable agents or a singleton and
- for all $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ in $\mathcal{P}$ (possibly $A = B$), we have;

$$\forall i \in \{1, \ldots, k\}, \quad \forall j \in \{1, \ldots, l\} \quad b_j \succ_{a_i} a_{i-1} \implies b_{j-1} \succ_{b_j} a_i. \quad (b_j \neq a_{i+1})$$

Iñarra et al. (2008) introduce a solution concept, called $\mathcal{P}$-stability which is characterized by stable partitions.

Definition 14. Let $\mathcal{P}$ be a stable partition for $(N, \succ)$, a matching μ is a $\mathcal{P}$-*stable matching* if for each $A = \{a_1, \ldots, a_k\} \in \mathcal{P}$,

- for any $i \in \{1, \ldots, k\}$, $\quad \mu(i) \in \{i + 1, i - 1\}$
- if k is odd, there exists $j \in \{1, \ldots, k\}$ such that $\mu(j) = j$

The $\mathcal{P}$-stability solution concept, denoted P, selects the set of all $\mathcal{P}$-stable solutions for each matching problem.

4.1.5 Q-stability

Biró et al. (2016) study the relations between solution concepts; maximum irreversibility, almost stability and maximum internal stability. They define a new solution concept, called *Q-stability*, as the intersection of maximum irreversibility and maximum internal stability. They also introduce an efficient algorithm to find Q-stable matchings.

Definition 15. A matching is Q-stable if it is maximum internally stable and maximum irreversible.

The Q-stability solution concept, denoted Q, selects the set of all Q-stable solutions for each matching problem.

4.1.6 Iterated $\mathcal{P}$-stability

We observe that there might exist unmatched agents even if they are acceptable to each other under $\mathcal{P}$-stable solutions. We propose iterated $\mathcal{P}$-stability solution concept which gives solutions that match iteratively to agents who remain single and acceptable to each other.

The algorithm first finds $\mathcal{P}$-stable matchings then the algorithm holds $\mathcal{P}$-stable matching's pairs. Then for each $\mathcal{P}$-stable matching, the algorithm finds new $\mathcal{P}$-stable matchings with unmatched agents in each step. It terminates when there are no agents who are unmatched and acceptable to each other.

We will denote the set of agents in N who are unmatched under the matchig μ with $s(\mu, N)$.

Given any roommate problem $(N, \succ)$ construct a matching μ, which we will call a *iterated $\mathcal{P}$-matching*, for the problem $(N, \succ)$ as follows. Let $N^1 = N$.

1) Choose $\mu^1 \in P(N^1, \succ|_{N^1})$ and define μ on $N^1 \setminus s(\mu^1, N^1)$ as for all $i \in N^1 \setminus s(\mu^1, N^1)$, $\mu(i) = \mu^1(i)$

$\vdots$

$k \geq 2$) Given N^{k-1} and μ^{k-1}, let $N^k = s(\mu^{k-1}, N^{k-1})$

- If $N^k = \emptyset$, then μ has been defined on N.
- If $N^k \neq \emptyset$, then choose $\mu^k \in P(N^k, \succ|_{N^k})$
  * if $s(\mu^k, N^k) = N^k$, then define μ on N^k as for all $i \in N^k$ $\mu(i) = i$. With this μ would be defined on N.
  * if $s(\mu^k, N^k) \neq N^k$, then repeate step k with N^{k-1} replaced with N^k and μ^{k-1} replaced with μ^k.

Note that since N is finite, this algorithm will terminate after a finite number of steps.

The Iterated $\mathcal{P}$-stability solution concept, denoted IP, selects the set of all iterated $\mathcal{P}$-stable solutions for each matching problem.

4.2 Properties

In roommate problems, agents are the commodities therefore, we can consider agents as both consumers and resources. This allows us to study aforementioned solution concepts' behaviors when the set of agents changes. We explain the effects of changes to the population by population monotonicity, resource and competition sensitivity and consistency.

Furthermore, we also study Maskin monotonicity which is the necessary condition for a solution concept to be implementable, for unsolvable roommate problem domains.

4.2.1 Population Monotonicity

Population monotonicity states that, if there is a change in the population, then each agent who exists before and after should be affected in the same direction. Population monotonicity may seem to be a solidarity property, each agent should be affected in the same way as a result of newcomers. We give formal definitions below.

A roommate problem $(N', \succ')$ is an *extension* of $(N, \succ)$ if and only if $N \subset N'$ and the preference profile $\succ'$ over N' such that; for all $i \in N$ and for all $k, l \in N$, $k \succ_i l$ if and only if $k \succ'_i l$.

Definition 16. (Population Monotonicity) A solution concept φ is *population monotonic* if and only if for any problem $(N, \succ)$, for any extension $(N', \succ')$ of $(N, \succ)$, and for any $\mu \in \varphi(N, \succ)$, there exists $\mu' \in \varphi(N', \succ')$ such that, for any $i \in N$, $\mu(i) \succeq_i \mu'(i)$ or for any $i \in N$, $\mu'(i) \succeq_i \mu(i)$.

In this set up, a solution concept is a correspondence, we might not have single-valued solution sets. We need to compare two different sets. There might be various techniques to compare these sets. For example; any solutions from one set with any solutions from the other set or any solution from one set and one solution from the other one. We use the definition which is stated above, since the literature has used this one.

Proposition 2. *Maximum irreversibility solution concept does not satisfy population monotonocity.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $\succ$ be the preference profile illustrated below; [1] [2]

1 : 2	3	4	6	5	7	8	9	10	1
2 : 3	1	10	5	6	7	8	4	9	2
3 : 1	2	4	5	6	7	8	9	10	3
4 : 8	9	3	6	7	10	1	2	5	4
5 : 9	1	6	8	7	4	2	3	10	5
6 : 4	5	7	8	9	2	3	1	10	6
7 : 6	8	9	5	1	2	3	4	10	7
8 : 5	7	4	6	1	2	9	10	3	8
9 : 7	4	5	1	2	3	6	8	10	9
10 : 1	2	3	4	5	6	7	8	9	10

The matchings;

$$\mu_1 = \{\{1,2\},\{3\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$
$$\mu_2 = \{\{1,3\},\{2\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$
$$\mu_3 = \{\{2,3\},\{1\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

are maximum irreversible matchings for $(N, \succ)$, since for $T = \{4,5,6,7,8,9\}$,

[1] Agents are enumerated in the first column and in the other columns, there is an ordering of agents to whom the given agent may be matched with. The ordering is a ranking of agents from most preferred to the least preferred.

[2] This preference profile is given by Biró et al. (2016)

the matchings;

$$\mu_1|_T = \mu_2|_T = \mu_3|_T = \{\{4,8\},\{5,9\},\{7,6\}\}$$

are complete stable matchings for $(T, (\succ_i)_{i \in T})$. For T, μ_1, μ_2, μ_3 are T-irreversible and for any T', such that $T \subseteq T'$, μ_1, μ_2, μ_3 are not T'-irreversible; therefore, these matchings are maximum irreversible.

Let $N' = \{11\}$ and let $\hat{N} = N \cup N'$ and $\hat{\succ}$ be the preference profile illustrated below;

$$
\begin{array}{l}
1: 2 \quad 3 \quad 4 \quad 6 \quad 5 \quad 7 \quad 8 \quad 9 \quad 10 \quad 1 \\
2: 11 \quad 3 \quad 1 \quad 10 \quad 5 \quad 6 \quad 7 \quad 8 \quad 4 \quad 9 \quad 2 \\
3: 1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 3 \\
4: 8 \quad 9 \quad 3 \quad 6 \quad 7 \quad 10 \quad 1 \quad 2 \quad 5 \quad 4 \\
5: 9 \quad 1 \quad 6 \quad 8 \quad 7 \quad 4 \quad 2 \quad 3 \quad 10 \quad 5 \\
6: 4 \quad 5 \quad 7 \quad 8 \quad 9 \quad 2 \quad 3 \quad 1 \quad 10 \quad 6 \\
7: 6 \quad 8 \quad 9 \quad 5 \quad 1 \quad 2 \quad 3 \quad 4 \quad 10 \quad 7 \\
8: 5 \quad 7 \quad 4 \quad 6 \quad 1 \quad 2 \quad 9 \quad 10 \quad 3 \quad 8 \\
9: 7 \quad 4 \quad 5 \quad 1 \quad 2 \quad 3 \quad 6 \quad 8 \quad 10 \quad 9 \\
10: 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 11 \quad 9 \quad 10 \\
11: 2 \quad 10
\end{array}
$$

The matching $\mu' = \{\{1,3\},\{2,11\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$ is unique maximum irreversible matching since for $\hat{T} = \{2,4,5,6,7,8,9,11\}$, $\mu|_{T'} = \{\{2,11\},\{4,8\},\{5,9\},\{7,6\}\}$ is a complete stable matching for the roommate problem $(\hat{T}, (\hat{\succ}_i)_{i \in \hat{T}})$. For $\hat{T}$, μ is $\hat{T}$-irreversible and for any T'', $\hat{T} \subseteq T''$, μ is not T''-irreversible.

A change from μ_1 to μ' makes agent 1 worse off and agent 3 better off. Therefore maximum irreversible stable solution concept does not satisfy population monotonicity. $\square$

Proposition 3. *Maximum internal stability solution concept does not satisfy population monotonicity.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\succ$ be the preference profile illustrated below;

$$\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:1 & 2 & 5 & 4 & 6 & 3 \\
4:5 & 3 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$\mathcal{P} = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ is the unique stable partition for $(N, \succ)$. The maximum internal stable matchings are

$$\mu_1 = \{\{1, 2\}, \{4, 5\}, \{3, 6\}\}$$
$$\mu_2 = \{\{1, 2\}, \{4, 5\}, \{3\}, \{6\}\}$$
$$\mu_3 = \{\{1, 3\}, \{4, 5\}, \{2, 6\}\}$$
$$\mu_4 = \{\{1, 3\}, \{4, 5\}, \{3\}, \{6\}\}$$
$$\mu_5 = \{\{2, 3\}, \{4, 5\}, \{1, 6\}\}$$
$$\mu_6 = \{\{2, 3\}, \{4, 5\}, \{1\}, \{6\}\}$$

Let $N^{'} = \{7, 8\}$ and $\hat{N} = N \cup N^{'}$, $\hat{N} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and $\hat{\succ}$ be the preference profile illustrated below;

$$
\begin{array}{llllllll}
1:2 & 8 & 3 & 5 & 4 & 6 & 1 & \\
2:3 & 1 & 4 & 5 & 6 & 2 & & \\
3:1 & 2 & 5 & 4 & 6 & 3 & & \\
4:7 & 5 & 3 & 1 & 2 & 6 & 4 & \\
5:4 & 2 & 1 & 3 & 6 & 5 & & \\
6:1 & 8 & 2 & 3 & 7 & 4 & 5 & 6 \\
7:4 & 8 & 6 & & & & & \\
8:6 & 1 & 7 & & & & &
\end{array}
$$

$\mathcal{P} = \{\{1, 2, 3\}, \{4, 7\}, \{6, 8\}, \{5\}\}$ is a stable partition for $(\hat{N}, \hat{\succ})$. This stable partition yields the maximum internally stable matching μ' for $(N^{'}, \succ')$ where;

$$\mu' = \{\{1, 2\}, \{3, 5\}, \{7, 4\}, \{6, 8\}\}$$

A change from μ_1 to μ' makes agent 3 better off and agent 5 worse off. Therefore, this solution concept does not satisfy population monotonicity. $\square$

Proposition 4. *Almost stability solution concept does not satisfy population monotonicity.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\succ$ be the preference profile[3] illustrated

[3] This preference profile is given by Biró et al. (2016)

below;

$$
\begin{array}{l}
1:2\quad 3\quad 8\quad 5\quad 4\quad 7\quad 6\quad 1\\
2:3\quad 1\quad 4\quad 5\quad 6\quad 7\quad 8\quad 2\\
3:4\quad 1\quad 2\quad 5\quad 6\quad 7\quad 8\quad 3\\
4:5\quad 3\quad 1\quad 6\quad 7\quad 8\quad 2\quad 4\\
5:4\quad 6\quad 3\quad 2\quad 7\quad 8\quad 1\quad 5\\
6:5\quad 8\quad 7\quad 4\quad 1\quad 2\quad 3\quad 6\\
7:8\quad 6\quad 4\quad 5\quad 1\quad 2\quad 3\quad 7\\
8:6\quad 7\quad 1\quad 5\quad 4\quad 2\quad 3\quad 8
\end{array}
$$

For the problem $(N, \succ)$, the matching $\mu = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ is the unique almost stable matching since $\{\{4,5\}\}$ is the only blocking pair under the matching μ and there is no other matching which has one blocking pair.

Let $N' = \{9,10\}$ and let $\hat{N} = N' \cup N$, $\hat{N} = \{1,2,3,4,5,6,7,8,9,10\}$ and $\hat{\succ}$ be the preference profile illustrated below;

$$
\begin{array}{l}
1:2 \quad 3 \quad 8 \quad 5 \quad 4 \quad 7 \quad 6 \quad 1 \\
2:3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 2 \\
3:4 \quad 1 \quad 2 \quad 5 \quad 6 \quad 7 \quad 8 \quad 3 \\
4:5 \quad 3 \quad 1 \quad 6 \quad 7 \quad 8 \quad 2 \quad 4 \\
5:4 \quad 10 \quad 6 \quad 3 \quad 2 \quad 7 \quad 8 \quad 1 \quad 5 \\
6:5 \quad 8 \quad 7 \quad 4 \quad 1 \quad 2 \quad 3 \quad 6 \\
7:8 \quad 9 \quad 6 \quad 4 \quad 5 \quad 1 \quad 2 \quad 3 \quad 7 \\
8:6 \quad 10 \quad 7 \quad 1 \quad 5 \quad 4 \quad 2 \quad 3 \quad 8 \\
9:7 \\
10:8 \quad 5
\end{array}
$$

For problem $(\hat{N}, \hat{\succ})$, the matching $\mu' = \{\{1,2\},\{3,4\},\{5,10\},\{6,8\},\{7,9\}\}$ is the unique almost stable matching since $\{\{4,5\}\}$ is the only blocking pair under the matching μ' and there is no other matching which has one blocking pair.

A change from μ to μ' makes agent 8 better off and agent 6 worse off. Therefore, almost stability solution concept does not satisfy population monotonocity.

$\square$

Proposition 5. *$\mathcal{P}$-stability solution concept does not satisfy population monotonocity.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile illustrated be-

low;

$$\begin{array}{llllll} 1:2 & 3 & 5 & 4 & 6 & 1 \\ 2:3 & 1 & 4 & 5 & 6 & 2 \\ 3:1 & 2 & 5 & 4 & 6 & 3 \\ 4:3 & 5 & 1 & 2 & 6 & 4 \\ 5:4 & 2 & 1 & 3 & 6 & 5 \\ 6:1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$P = \{\{1,2,3\},\{4,5\},\{6\}\}$ is the unique stable partition for $(N, \succ)$ and the matchings

$$\mu_1 = \{\{1,2\},\{4,5\},\{3\},\{6\}\}$$
$$\mu_2 = \{\{1,3\},\{4,5\},\{2\},\{6\}\}$$
$$\mu_3 = \{\{2,3\},\{4,5\},\{1\},\{6\}\}$$

are $\mathcal{P}$-stable matchings. Let $N' = \{7,8\}$ and let $\hat{N} = N \cup N'$ and $\hat{N} = \{1,2,3,4,5,6,7,8\}$. $\hat{R}$ be the preference profile illustrated below;

$$\begin{array}{llllllll} 1:2 & 8 & 3 & 5 & 4 & 6 & 1 & \\ 2:3 & 1 & 4 & 5 & 8 & 6 & 2 & \\ 3:1 & 2 & 5 & 4 & 8 & 6 & 3 & \\ 4:3 & 5 & 1 & 2 & 8 & 6 & 4 & \\ 5:4 & 2 & 1 & 3 & 8 & 6 & 5 & \\ 6:7 & 1 & 2 & 3 & 4 & 5 & 8 & 6 \\ 7:8 & 6 & & & & & & \\ 8:1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

The matching

$$\mu' = \{\{1,2\},\{3\},\{4,5\},\{7,6\},\{8\}\}$$

is $\mathcal{P}$-stable for the problem $(\hat{N},\hat{\succ})$. A change from μ_3 to μ' makes agent 1 better off and agent 2 worse off. Therefore, $\mathcal{P}$-stability does not satisfy population monotonicity. □

Proposition 6. *Q-stability solution concept does not satisfy population monotonicity.*

Proof. Let $N = \{1,2,3,4,5,6,7,8,9,10\}$ and $\succ$ be the preference profile illustrated below; [4]

$$\begin{array}{lllllllllll}
1: & 2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 & 10 & 1 \\
2: & 3 & 1 & 10 & 5 & 6 & 7 & 8 & 4 & 9 & 2 \\
3: & 1 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 3 \\
4: & 8 & 9 & 3 & 6 & 7 & 10 & 1 & 2 & 5 & 4 \\
5: & 9 & 1 & 6 & 8 & 7 & 4 & 2 & 3 & 10 & 5 \\
6: & 4 & 5 & 7 & 8 & 9 & 2 & 3 & 1 & 10 & 6 \\
7: & 6 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 10 & 7 \\
8: & 5 & 7 & 4 & 6 & 1 & 2 & 9 & 10 & 3 & 8 \\
9: & 7 & 4 & 5 & 1 & 2 & 3 & 6 & 8 & 10 & 9 \\
10: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

[4] This preference profile is given by Biró et al. (2016)

The matching

$$\mu = \{\{1,2\},\{3\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

is Q-stable. Let $N' = \{11\}$ and let $\hat{N} = N' \cup N$, $\hat{N} = \{1,2,3,4,5,6,7,8,9,10,11\}$ and $\succ'$ be the preference profile shown below;

$$\begin{array}{l}
1:2 \quad 3 \quad 4 \quad 6 \quad 5 \quad 7 \quad 8 \quad 9 \quad 10 \quad 1 \\
2:11 \quad 3 \quad 1 \quad 10 \quad 5 \quad 6 \quad 7 \quad 8 \quad 4 \quad 9 \quad 2 \\
3:1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 3 \\
4:8 \quad 9 \quad 3 \quad 6 \quad 7 \quad 10 \quad 1 \quad 2 \quad 5 \quad 4 \\
5:9 \quad 1 \quad 6 \quad 8 \quad 7 \quad 4 \quad 2 \quad 3 \quad 10 \quad 5 \\
6:4 \quad 5 \quad 7 \quad 8 \quad 9 \quad 2 \quad 3 \quad 1 \quad 10 \quad 6 \\
7:6 \quad 8 \quad 9 \quad 5 \quad 1 \quad 2 \quad 3 \quad 4 \quad 10 \quad 7 \\
8:5 \quad 7 \quad 4 \quad 6 \quad 1 \quad 2 \quad 9 \quad 10 \quad 3 \quad 8 \\
9:7 \quad 4 \quad 5 \quad 1 \quad 2 \quad 3 \quad 6 \quad 8 \quad 10 \quad 9 \\
10:1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 11 \quad 10 \\
11:2 \quad 10
\end{array}$$

The matching $\mu' = \{\{1,3\},\{2,11\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$ is Q-stable for the problem $(\hat{N}, \hat{\succ})$. A change from μ to μ' makes agent 1 worse off and agent 3 better off. Therefore, Q-stability solution concept does not satisfy population monotonicity.

$\square$

Proposition 7. *Iterated $\mathcal{P}$-stability solution concept does not satisfy population monotonicity.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\succ$ be the preference profile illustrated below;

$$\begin{array}{llllll} 1:2 & 3 & 5 & 4 & 6 & 1 \\ 2:3 & 1 & 4 & 5 & 6 & 2 \\ 3:1 & 2 & 5 & 4 & 6 & 3 \\ 4:3 & 5 & 1 & 2 & 6 & 4 \\ 5:4 & 2 & 1 & 3 & 6 & 5 \\ 6:1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$P = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ is a stable partition for $(N, \succ)$ and the matchings $\mu_1 = \{\{1, 2\}, \{4, 5\}, \{3\}, \{6\}\}$, $\mu_2 = \{\{1, 3\}, \{4, 5\}, \{2\}, \{6\}\}$ and $\mu_3 = \{\{2, 3\}, \{4, 5\}, \{1\}, \{6\}\}$ are $\mathcal{P}$-stable matchings.

The sets of pairs of $\mathcal{P}$-stable matchings;

$$\nu_{\mu_1} = \{\{1, 2\}, \{4, 5\}\}$$
$$\nu_{\mu_2} = \{\{1, 3\}, \{4, 5\}\}$$
$$\nu_{\mu_3} = \{\{2, 3\}, \{4, 5\}\}$$

The set of single agents of matching μ_1 is $\{\{3\}, \{6\}\}$, let $N' = \{3, 6\}$. $P' = \{\{3, 6\}\}$ is the unique stable partition for $(N', \succ_{N'})$ and $\mu'_{P'} = \{\{3, 6\}\}$ then the iterated $\mathcal{P}$-stable matching is

$$\mu^I_{\mu_1} = \{\{1, 2\}, \{4, 5\}, \{3, 6\}\}$$

By the same process, $\mathcal{P}$-stable matchings which are illustrated above yields the iterated $\mathcal{P}$-stable matchings;

$$\mu^I_{\mu_2} = \{\{1,3\},\{4,5\},\{2,6\}\}$$
$$\mu^I_{\mu_3} = \{\{2,3\},\{4,5\},\{1,6\}\}$$

Let $N' = \{7,8\}$ and let $\hat{N} = N \cup N'$ so $\hat{N} = \{1,2,3,4,5,6,7,8\}$, $\hat{\succ}$ be the preference profile illustrated below;

$$\begin{array}{l}
1:2 \quad 8 \quad 3 \quad 5 \quad 4 \quad 6 \quad 1 \\
2:3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 2 \\
3:1 \quad 2 \quad 5 \quad 4 \quad 6 \quad 3 \\
4:3 \quad 7 \quad 5 \quad 1 \quad 2 \quad 6 \quad 4 \\
5:4 \quad 2 \quad 1 \quad 3 \quad 6 \quad 5 \\
6:1 \quad 8 \quad 7 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
7:4 \quad 8 \quad 6 \\
8:6 \quad 1 \quad 7
\end{array}$$

$P' = \{\{1,2,3\},\{4,7\},\{6,8\},\{5\}\}$ is a stable partition for $(\hat{N}, \succ_{\hat{N}})$ and the iterated $\mathcal{P}$-stable matchings are

$$\mu^I_{\mu_{1'}} = \{\{1,2\},\{4,7\},\{6,8\},\{3,5\}\}$$
$$\mu^I_{\mu_{2'}} = \{\{1,3\},\{4,7\},\{6,8\},\{2,5\}\}$$
$$\mu^I_{\mu_{3'}} = \{\{2,3\},\{4,7\},\{6,8\},\{1,5\}\}$$

A change from $\mu^I_{\mu_1}$ to $\mu^I_{\mu_{1'}}$ makes agent 3 better off and agent 5 worse off. Therefore, iterated P-stability does not satisfy population monotonicity. $\square$

4.2.2 Competition and Resource Sensitivity

If there exist newcomers in a roommate problem, then these newcomers might affect any of the incumbent agents negatively or positively. *Competition sensitivity* means that if two existing agents become newly paired after newcomers arrived, then at least one of them becomes worse off. This property captures the negative effects of newcomers. A formal definition is given below.

Definition 17. (Competition sensitivity) A solution concept φ is *competition sensitive* if and only if for any roommate problem $(N, \succ)$ and for any extension $(N', \succ')$ of $(N, \succ)$, and for any $\mu \in \varphi(N, \succ)$, there exists $\mu' \in \varphi(N', \succ')$ such that for any $i \in N$, $\mu(i) \succeq_i \mu'(i)$.

Resource sensitivity captures the possible positive effects of newcomers on existing agents. Resource sensitivity means that if two incumbent agents become newly paired after newcomers arrived, then at least one of them becomes better off.

Definition 18. (Resource sensitivity) A solution concept φ is *resource sensitive* if and only if for any roommate problem $(N, \succ)$ and for any extension $(N', \succ')$ of $(N, \succ)$, and for any $\mu \in \varphi(N, \succ)$, there exists $\mu' \in \varphi(N', \succ')$ such that, for any $i \in N$, $\mu'(i) \succeq_i \mu(i)$.

We have that if a solution concept does not satisfy population monotonicity, then it does not satisfy competition or resource sensitivity. Since there is no aforementioned solution concept which satisfies population monotonicity, there is also no solution concept that satisfies competition or resource sensitivity.

4.2.3 Maskin Monotonicity

Maskin Monotinicity introduced by Maskin (1999). A solution concept is Maskin monotonic if and only if a matching is a solution for a roommate problem under a solution concept, then it is also a solution for a roommate problem which is constructed by a Maskin monotonic transformation. Maskin monotonic transformation basically means that a solution improved in the all agents' preference rankings.

For any agent $i \in N$ and a matching μ, the *lower contour set* of $\succeq_i$ at μ is $L(\succ_i, \mu) = \{\mu' | \mu \succeq_i \mu'\}$. A preference profile $\succ'$ is obtained by a *Maskin monotonic transformation* of $\succ$ at μ if for all $i \in N$, $L(\succ_i, \mu) \subseteq L(\succ'_i, \mu)$. Let $MT(\succ, \mu)$ be the set of preference profiles which are obtained by a *Maskin monotonic* transformation of $\succ$.

Definition 19. (Maskin Monotonicity) A solution concept φ is
Maskin monotonic if and only if for all roommate problems $(N, \succ)$, and all
$\mu \in \varphi(N, \succ)$, if $\succ' \in MT(\succ, \mu)$, then $\mu \in \varphi(N, \succ')$.

Proposition 8. *Maximum irreversible solution concept does not satisfy Maskin Monotonicity.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $\succ$ be the preference profile illustrated below; [5]

[5]This preference profile is given by Biró et al. (2016)

$1 : 2$	3	4	6	5	7	8	9	10	1
$2 : 3$	1	10	5	6	7	8	4	9	2
$3 : 1$	2	4	5	6	7	8	9	10	3
$4 : 8$	9	3	6	7	10	1	2	5	4
$5 : 9$	1	6	8	7	4	2	3	10	5
$6 : 4$	5	7	8	9	2	3	1	10	6
$7 : 6$	8	9	5	1	2	3	4	10	7
$8 : 5$	7	4	6	1	2	9	10	3	8
$9 : 7$	4	5	1	2	3	6	8	10	9
$10 : 1$	2	3	4	5	6	7	8	9	10

The matchings;

$$\mu_1 = \{\{1,2\},\{3\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$
$$\mu_2 = \{\{1,3\},\{2\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$
$$\mu_3 = \{\{2,3\},\{1\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

are maximum irreversible matchings for $(N, \succ)$, since for $T = \{4,5,6,7,8,9\}$, the matchings;

$$\mu_1|_T = \mu_2|_T = \mu_3|_T = \{\{4,8\},\{5,9\},\{7,6\}\}$$

are complete stable matchings for $(T, (\succ_i)_{i \in T})$. For T, μ_1, μ_2, μ_3 are T-irreversible and for any T', such that $T \subseteq T'$, μ_1, μ_2, μ_3 are not T'-irreversible.

Let $\succ'$, a Maskin Monontonic transformation of $\succ$ at μ_3, be the preference profile illustrated below;

$$\begin{array}{llllllllll}
1:3 & 2 & 4 & 6 & 5 & 7 & 8 & 9 & 10 & 1 \\
2:3 & 1 & 10 & 5 & 6 & 7 & 8 & 4 & 9 & 2 \\
3:1 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 3 \\
4:8 & 9 & 3 & 6 & 7 & 10 & 1 & 2 & 5 & 4 \\
5:9 & 1 & 6 & 8 & 7 & 4 & 2 & 3 & 10 & 5 \\
6:4 & 5 & 7 & 8 & 9 & 2 & 3 & 1 & 10 & 6 \\
7:6 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 10 & 7 \\
8:5 & 7 & 4 & 6 & 1 & 2 & 9 & 10 & 3 & 8 \\
9:7 & 4 & 5 & 1 & 2 & 3 & 6 & 8 & 10 & 9 \\
10:1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

The matching

$$\mu' = \{\{1,3\},\{2\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

is unique maximum irreversible matching, since for $T' = \{1,3,4,5,6,7,8,9\}$, $\mu \mid_{T'} = \{\{1,3\},\{4,8\},\{5,9\},\{7,6\}\}$ is a complete stable matching for the roommate problem $(T', (\succ'_i)_{i\in T'})$. For T', μ' is T'-irreversible and for any $\hat{T}$ such that $T' \subseteq \hat{T}$, μ' is not $\hat{T}$-irreversible.

The matching μ_3 is not maximum irreversible for $(N, \succ')$. Therefore, this solution concept does not satisfy Maskin Monotonicity. □

Proposition 9. *Maximum internal stability solution concept does not satisfy Maskin Monotonicity.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile illustrated be-

low;

$$\begin{array}{l} 1: 2 \quad 3 \quad 5 \quad 4 \quad 6 \quad 1 \\ 2: 3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 2 \\ 3: 1 \quad 2 \quad 5 \quad 4 \quad 6 \quad 3 \\ 4: 3 \quad 5 \quad 1 \quad 2 \quad 6 \quad 4 \\ 5: 4 \quad 2 \quad 1 \quad 3 \quad 6 \quad 5 \\ 6: 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array}$$

The matching $\mu = \{\{1,2\},\{4,5\},\{3,6\}\}$ is maximum internally stable for $(N, \succ)$.

Let $\succ'$, a Maskin monotonic transformation of $\succ$ at μ, be the preference profile illustrated below;

$$\begin{array}{l} 1: 2 \quad 3 \quad 5 \quad 4 \quad 6 \quad 1 \\ 2: 3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 2 \\ 3: 2 \quad 1 \quad 5 \quad 4 \quad 6 \quad 3 \\ 4: 3 \quad 5 \quad 1 \quad 2 \quad 6 \quad 4 \\ 5: 4 \quad 2 \quad 1 \quad 3 \quad 6 \quad 5 \\ 6: 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array}$$

The matchings $\mu' = \{\{2,3\},\{4,5\},\{1,6\}\}$ and $\mu'' = \{\{2,3\},\{4,5\},\{1\},\{6\}\}$ are maximum internally stable matchings. The matching μ is not maximum internally stable for the problem $(N, \succ')$. Thefore this solution concept is not Maskin monotonic. $\square$

Proposition 10. *Almost stability solution concept does not satisfy Maskin monotonicity.*

Proof. Let $n = \{1, 2, 3, 4, 5, 6\}$ and $\succ$ be the preference profile illustrated below;

$$\begin{array}{llllll}
1:5 & 2 & 3 & 6 & 4 & 1 \\
2:4 & 1 & 5 & 3 & 6 & 2 \\
3:2 & 4 & 1 & 5 & 6 & 3 \\
4:5 & 1 & 2 & 3 & 6 & 4 \\
5:4 & 2 & 6 & 1 & 3 & 5 \\
6:2 & 4 & 1 & 5 & 3 & 6
\end{array}$$

The matching $\mu = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ is unique almost stable matching for the problem $(N, \succ)$ with only blocking pair $\{\{4, 5\}\}$ as a blocking pair.

Let $\succ'$, a Maskin monotonic transformation of $\succ$ at μ, be the preference profile illustrated below;

$$\begin{array}{llllll}
1:5 & 2 & 3 & 6 & 4 & 1 \\
2:4 & 1 & 5 & 3 & 6 & 2 \\
3:2 & 4 & 1 & 5 & 6 & 3 \\
4:2 & 5 & 1 & 3 & 6 & 4 \\
5:6 & 4 & 2 & 1 & 3 & 5 \\
6:2 & 4 & 1 & 5 & 3 & 6
\end{array}$$

$(N, R^{'})$, $\mu^{'} = \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$ is almost stable for $(N, \succ')$ and μ is not almost stable for $(N, \succ')$. Therefore, almost stability solution concept does not satisfy Maskin Monotonicity. $\square$

Proposition 11. *$\mathcal{P}$-stability solution concept does not satisfy Maskin Monotonicity.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile illustrated below;

$$\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:1 & 2 & 5 & 4 & 6 & 3 \\
4:3 & 5 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$P = \{\{1,2,3\},\{4,5\},\{6\}\}$ is the unique stable partition for $(N,\succ)$ and the matchings

$$\mu_1 = \{\{1,2\},\{4,5\},\{3\},\{6\}\}$$
$$\mu_2 = \{\{1,3\},\{4,5\},\{3\},\{6\}\}$$
$$\mu_3 = \{\{2,3\},\{4,5\},\{1\},\{6\}\}$$

are $\mathcal{P}$-stable matchings.

Let $\succ'$, a Maskin monotonic transformation of $\succ$ at μ_1, be the preference profile shown below;

$$\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:2 & 1 & 5 & 4 & 6 & 3 \\
4:3 & 5 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$P = \{\{1,6\},\{2,3\},\{4,5\}\}$ is a stable partition for $(N, \succ')$. The matching $\mu' = \{\{2,3\},\{4,5\},\{1,6\}\}$ is the unique $\mathcal{P}$-stable matching for $(N, \succ')$ and the matching μ_1 is not a solution for the problem $(N, \succ')$. Therefore this solution concept does not satisfy Maskin Monotonicity. $\square$

Proposition 12. *Q-stability solution concept does not satisfy Maskin Monotonicity.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile shown below;

$$
\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:1 & 2 & 5 & 4 & 6 & 3 \\
4:3 & 5 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

The matching $\mu = \{\{1,2\},\{4,5\},\{3\},\{6\}\}$ is Q-stable for $(N, \succ)$.

Let $\succ'$, a Maskin monotonic transformation of $\succ$ at μ, be the preference profile illustrated below;

$$
\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:2 & 1 & 5 & 4 & 6 & 3 \\
4:3 & 5 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

The matching $\mu^{'} = \{\{2,3\},\{4,5\},\{1,6\}\}$ is Q-stable. The matching μ is not a Q-stable solution for the problem $(N, \succ')$. Thefore this solution concept is not Maskin Monotonic. □

Proposition 13. *Iterated $\mathcal{P}$-stability solution concept does not satisfy Maskin Monotonicity.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile illustrated below;

$$\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:1 & 2 & 5 & 4 & 6 & 3 \\
4:3 & 5 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$P = \{\{1,2,3\},\{4,5\},\{6\}\}$ is a stable partition for $(N, \succ)$ and the matchings $\mu_1 = \{\{1,2\},\{4,5\},\{3\},\{6\}\}$, $\mu_2 = \{\{1,3\},\{4,5\},\{2\},\{6\}\}$ and $\mu_3 = \{\{2,3\},\{4,5\},\{1\},\{6\}\}$ are $\mathcal{P}$-stable matchings.

The sets of pairs of $\mathcal{P}$-stable matchings;

$$\nu_{\mu_1} = \{\{1,2\},\{4,5\}\}$$
$$\nu_{\mu_2} = \{\{1,3\},\{4,5\}\}$$
$$\nu_{\mu_3} = \{\{2,2\},\{4,5\}\}$$

The set of single agents of matching μ_1 is $\{3,6\}$. If $N' = \{3,6\}$, then $P' = \{\{3,6\}\}$ is unique stable partition of $(N', \succ_{N'})$ and $\mu'_{P'} = \{\{3,6\}\}$ then the iterated $\mathcal{P}$-stable matching is

$$\mu^I_{\mu_1} = \{\{1,2\},\{4,5\},\{3,6\}\}$$

By the same process, $\mathcal{P}$-stable matchings which are illustrated above yields the iterated $\mathcal{P}$-stable matchings;

$$\mu^I_{\mu_2} = \{\{1,3\},\{4,5\},\{2,6\}\}$$
$$\mu^I_{\mu_3} = \{\{2,3\},\{4,5\},\{1,6\}\}$$

Let $\succ'$, a Maskin Monontonic transformation of $\succ$ at $\mu^I_{\mu_1}$, be the preference profile illustrated below;

$$\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1 \\
2:3 & 1 & 4 & 5 & 6 & 2 \\
3:2 & 1 & 5 & 4 & 6 & 3 \\
4:3 & 5 & 1 & 2 & 6 & 4 \\
5:4 & 2 & 1 & 3 & 6 & 5 \\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$P' = \{\{2,3\},\{4,5\},\{1,6\}\}$ is a stable partition for $(N', \succ')$ then the iterated $\mathcal{P}$-stable mathing is

$$\mu^I_{\mu'} = \{\{2,3\},\{4,5\},\{1,6\}\}$$

The matchings $\mu^I_{\mu_1}$ is not a solution for $(N', \succ')$; therefore, iterated $\mathcal{P}$-stability does not satisfy Maskin Monotonicity. $\square$

4.2.4 Consistency

Consistency requires that if there exist a set of matched agents who leave the problem, then the remaining agents should still be paired as before.

A roommate problem $(N', \succ')$ is a reduced problem of $(N, \succ)$ at μ to N' if and only if $N' \subset N$ and the preference profile $\succ'$ over N' such that; for all $i \in N'$, for all $k, l \in N'$, $\quad k \succ_i' l \iff k \succ_i l$ and $\succ_{N'} = (\succ_i')_{i \in N'}$. The restriction of μ to agents in N' is $\mu_{N'}$.

Definition 20. (Consistency) A solution concept φ is *consistent* if and only if for any roommate problem $(N, \succ)$ and for any reduced problem $(N', \succ')$ of $(N, \succ)$, for any $\mu \in \varphi(N, \succ)$ implies $\mu_{N'} \in \varphi(N', \succ_{N'})$.

Proposition 14. *Maximum irreversibility solution concept does not satisfy consistency.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $\succ$ be the preference profile illustrated below; [6]

[6] This preference profile is given by Biró et al. (2016)

$$
\begin{array}{llllllllll}
1:2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 & 10 & 1 \\
2:3 & 1 & 10 & 5 & 6 & 7 & 8 & 4 & 9 & 2 \\
3:1 & 2 & 10 & 5 & 6 & 7 & 8 & 9 & 4 & 3 \\
4:8 & 9 & 3 & 6 & 7 & 10 & 1 & 2 & 5 & 4 \\
5:9 & 1 & 6 & 8 & 7 & 4 & 2 & 3 & 10 & 5 \\
6:4 & 5 & 7 & 8 & 9 & 2 & 3 & 1 & 10 & 6 \\
7:6 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 10 & 7 \\
8:5 & 7 & 4 & 6 & 1 & 2 & 9 & 10 & 3 & 8 \\
9:7 & 4 & 5 & 1 & 2 & 3 & 6 & 8 & 10 & 9 \\
10:1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
$$

The matchings;

$$\mu_1 = \{\{1,2\},\{3\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$
$$\mu_2 = \{\{1,3\},\{2\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$
$$\mu_3 = \{\{2,3\},\{1\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

are maximum irreversible matchings. Let $N' = \{3,4,5,8,9,10\}$ and $(N', \succ_{N'})$ be the reduced problem of $(N, \succ)$. Let $T' = \{3,4,5,8,9,10\}$ and $\mu' = \{\{3,10\},\{4,8\},\{5,9\}\}$ is T'-irreversible. There is no $\hat{T} \subseteq N'$, s.t $T' \subseteq \hat{T}$; therefore, μ' is the unique maximum irreversible matching for $(N', \succ_{N'})$. We have that $\mu_{1N'} = \{\{3\},\{4,8\},\{5,9\},\{10\}\}$ and $\mu_{1N'}$ is not a maximum irreversible matching for $(N', \succ')$. Therefore, maximum irreversibility does not satisfy consistency.

$\square$

Proposition 15. *Maximum internal stability solution concept does not satisfy consistency.*

Proof. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $\succ$ be the preference profile illustrated below;[7]

$$
\begin{array}{rcccccccccc}
1: & 2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 & 10 & 1 \\
2: & 3 & 1 & 10 & 5 & 6 & 7 & 8 & 4 & 9 & 2 \\
3: & 1 & 2 & 10 & 5 & 6 & 7 & 8 & 9 & 4 & 3 \\
4: & 8 & 9 & 3 & 6 & 7 & 10 & 1 & 2 & 5 & 4 \\
5: & 9 & 1 & 6 & 8 & 7 & 4 & 2 & 3 & 10 & 5 \\
6: & 4 & 5 & 7 & 8 & 9 & 2 & 3 & 1 & 10 & 6 \\
7: & 6 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 10 & 7 \\
8: & 5 & 7 & 4 & 6 & 1 & 2 & 9 & 10 & 3 & 8 \\
9: & 7 & 4 & 5 & 1 & 2 & 3 & 6 & 8 & 10 & 9 \\
10: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
$$

[7]This preference profile is given by Biró et al. (2016)

The matchings;

$$\mu_1 = \{\{1,2\},\{3\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

$$\mu_1 = \{\{1,2\},\{3,10\},\{4,8\},\{5,9\},\{7,6\}\}$$

$$\mu_2 = \{\{1,3\},\{2\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

$$\mu_2 = \{\{1,3\},\{2,10\},\{4,8\},\{5,9\},\{7,6\}\}$$

$$\mu_3 = \{\{2,3\},\{1\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

$$\mu_3 = \{\{2,3\},\{1,10\},\{4,8\},\{5,9\},\{7,6\}\}$$

are maximum internally stable. Let $N' = \{3,4,5,8,9,10\}$ and $(N', \succ_{N'})$ be a reduced problem of $(N, \succ)$. We have that $\mu_{1N'} = \{\{3\},\{4,8\},\{5,9\},\{10\}\}$. For the reduced problem, for $T = \{3,4,5,8,9,10\}$, $\mu' = \{\{3,10\},\{4,8\},\{5,9\}\}$ is the unique maximum internally stable matching for $(N', \succ_{N'})$. Since $\mu_{1N'}$ is not a maximum internally stable matching for $(N', \succ_{N'})$, the maximum internal stability does not satisfy consistency.

□

Proposition 16. *Almost stability solution concept does not satisfy consistency.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile illustrated below;

$$1: 2 \quad 3 \quad 5 \quad 4 \quad 6 \quad 1$$

$$2: 3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 2$$

$$3: 4 \quad 1 \quad 2 \quad 5 \quad 6 \quad 3$$

$$4: 5 \quad 3 \quad 1 \quad 6 \quad 2 \quad 4$$

$$5: 4 \quad 6 \quad 3 \quad 2 \quad 1 \quad 5$$

$$6: 5 \quad 4 \quad 1 \quad 2 \quad 3 \quad 6$$

The matching $\mu = \{\{1,2\},\{3,4\},\{5,6\}\}$ is almost stable for the problem $(N,\succ)$. Let $N' = \{3,4,5,6\}$ and $(N',\succ')$ be a reduced market of $(N,\succ)$. $\mu' = \{\{3,6\},\{4,5\}\}$ is the unique almost stable matching for $(N',\succ')$. We have that $\mu_{N'} = \{\{3,4\},\{5,6\}\}$. The matching $\mu_{N'}$ is not almost stable; therefore, this solution concept does not satisfy consistency. □

Proposition 17. *$\mathcal{P}$-stability solution concept does not satisfy consistency.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile shown below;

$$\begin{array}{llllll} 1:2 & 3 & 5 & 4 & 6 & 1 \\ 2:3 & 1 & 4 & 5 & 6 & 2 \\ 3:1 & 2 & 5 & 4 & 6 & 3 \\ 4:3 & 5 & 1 & 2 & 6 & 4 \\ 5:4 & 2 & 1 & 3 & 6 & 5 \\ 6:1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$P = \{\{1,2,3\},\{4,5\},\{6\}\}$ is a stable partition for $(N,\succ)$ and the matchings $\mu_1 = \{\{1,2\},\{4,5\},\{3\},\{6\}\}$, $\mu_2 = \{\{1,3\},\{4,5\},\{2\},\{6\}\}$ and $\mu_3 = \{\{2,3\},\{4,5\},\{1\},\{6\}\}$ are $\mathcal{P}$-stable. Let $N^{'} = \{3,4,5,6\}$ and $(N^{'},\succ_{N'})$ be a reduced problem of $(N,\succ)$. $P = \{\{3,4,5\},\{6\}\}$ is a stable partition for $(N',\succ_{N'})$. The matching $\mu' = \{\{3,4\},\{5\},\{6\}\}$ is $\mathcal{P}$-stable matching for $(N^{'},\succ_{N'})$. The matching $\mu_{1N'}$ is not $\mathcal{P}$-stable matching for $(N^{'},\succ_{N'})$. Therefore, this solution concept does not satisfy consistency. □

Proposition 18. *Q-stability solution concept does not satisfy consistency.*

Proof. Let $N = \{1,2,3,4,5,6,7,8,9,10\}$ and $\succ$ be the preference profile illustrated below;[8]

[8]This preference profile is given by Biró et al. (2016)

1 : 2	3	4	6	5	7	8	9	10	1
2 : 3	1	10	5	6	7	8	4	9	2
3 : 1	2	10	5	6	7	8	9	4	3
4 : 8	9	3	6	7	10	1	2	5	4
5 : 9	1	6	8	7	4	2	3	10	5
6 : 4	5	7	8	9	2	3	1	10	6
7 : 6	8	9	5	1	2	3	4	10	7
8 : 5	7	4	6	1	2	9	10	3	8
9 : 7	4	5	1	2	3	6	8	10	9
10 : 1	2	3	4	5	6	7	8	9	10

The matching

$$\mu = \{\{1,2\},\{3\},\{4,8\},\{5,9\},\{7,6\},\{10\}\}$$

is Q-stable. Let $N' = \{3,4,5,8,9,10\}$ and $(N', \succ_{N'})$ be a reduced problem of $(N, \succ)$. We have that $\mu_{N'} = \{\{3\},\{4,8\},\{5,9\},\{10\}\}$. For the reduced problem, the matching $\mu' = \{\{3,10\},\{4,8\},\{5,9\}\}$ is the unique Q-stable matching for $(N', \succ_{N'})$. Since $\mu_{N'}$ is not a Q-stable matching for $(N', \succ_{N'})$, the Q-stability does not satisfy consistency.

□

Proposition 19. *Iterated $\mathcal{P}$-stability solution concept does not satisfy consistency.*

Proof. Let $N = \{1,2,3,4,5,6\}$ and $\succ$ be the preference profile illustrated be-

low;

$$\begin{array}{llllll}
1:2 & 3 & 5 & 4 & 6 & 1\\
2:3 & 1 & 4 & 5 & 6 & 2\\
3:1 & 2 & 5 & 4 & 6 & 3\\
4:3 & 5 & 1 & 2 & 6 & 4\\
5:4 & 2 & 1 & 3 & 6 & 5\\
6:1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$P = \{\{1,2,3\},\{4,5\},\{6\}\}$ is a stable partition for $(N,\succ)$ and the matchings $\mu_1 = \{\{1,2\},\{4,5\},\{3\},\{6\}\}$, $\mu_2 = \{\{1,3\},\{4,5\},\{2\},\{6\}\}$ and $\mu_3 = \{\{2,3\},\{4,5\},\{1\},\{6\}\}$ are $\mathcal{P}$-stable matchings.

The sets of pairs of $\mathcal{P}$-stable matchings;

$$\nu_{\mu_1} = \{\{1,2\},\{4,5\}\}$$
$$\nu_{\mu_2} = \{\{1,3\},\{4,5\}\}$$
$$\nu_{\mu_3} = \{\{2,2\},\{4,5\}\}$$

The set of single agents of matching μ_1 is $\{3,6\}$. If $N' = \{3,6\}$, then $P' = \{\{3,6\}\}$ is unique stable partition of $(N',\succ_{N'})$ and $\mu'_{P'} = \{\{3,6\}\}$ then the iterated $\mathcal{P}$-stable matching is

$$\mu^I_{\mu_1} = \{\{1,2\},\{4,5\},\{3,6\}\}$$

By same process, $\mathcal{P}$-stable matchings which are illustrated above yields the iterated $\mathcal{P}$-stable matchings;

$$\mu^I_{\mu_2} = \{\{1,3\},\{4,5\},\{2,6\}\}$$
$$\mu^I_{\mu_3} = \{\{2,3\},\{4,5\},\{1,6\}\}$$

Let $N' = \{3,4,5,6\}$ and $(N', \succ_{N'})$ be a reduced problem of $(N, \succ)$. $P = \{\{3,4,5\},\{6\}\}$ is a stable partition for $(N', \succ_{N'})$. The matching $\mu' = \{\{3,4\},\{5,6\}\}$ is iterated $\mathcal{P}$-stable matching for $(N', \succ_{N'})$. The matching $\mu^I_{\mu_1 N'}$ is not an iterated $\mathcal{P}$-stable matching for $(N', \succ_{N'})$; therefore, this solution concept does not satisfy consistency. $\square$

CHAPTER 5

CONCLUSION

It is well-known that roommate problems are one-sided matching problems which might not have any stable solutions. Therefore, the question "Which conditions guarantee the non-existence of stable matchings for roommate problems?" has arisen after Gale & Shapley (1962)'s paper. On the other hand, when there exist stable roommate matchings, "How we can find the stable mathcings for roommate problems?" is another question. We are first interested in Irving Algorithm which finds a complete stable matching if it exists. In our study, we have no restriction on the number of agents. We study not only complete stable matchings but also stable matchings. We have a condition which can be used to check the non-existence of stable matchings.

In this work, we study to match the agents in a such a way that the resulting matching is "close" to be stable. Such solution concepts should select stable matchings when they exist and select matchings "close" to being stable when the problem does not have any stable matchings. Several solution concepts such as Almost Stability, Maximum irreversibility, Maximum Internal stability, $\mathcal{P}$-stability and Q-stability has introduced in the literature. In this study, we introduce a new solution concept, called Iterative $\mathcal{P}$-stability in which we extend $\mathcal{P}$-stability by an iterative algorithm.

Finally, we study population monotonicity, resource and competition sensitivity, consistency and Maskin monotonicity of these solution concepts. We show that there are no aforementioned solution concepts which satisfies population monotonicity, competition sensitivity and resource sensitivity and consistency. There are also no aforementioned solution concepts which satisfies Maskin monotonicity.

Table 1: Summary of Results

Solution Concepts	Population Monotonicity	Competition Sensitivity	Resource Sensitivity	Maskin Monotonicity	Consistency
Almost Stability	No	No	No	No	No
Maximum Internal Stability	No	No	No	No	No
Maximum Irreversibility	No	No	No	No	No
P-stability	No	No	No	No	No
Q-stability	No	No	No	No	No
Iterated P-stability	No	No	No	No	No
Stable Rule	Yes	Yes	Yes	Yes	Yes

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