ASSORTMENT PLANNING WITH PREMIUM SERVICES

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By

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Assortment Planning with Premium Services
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ASSORTMENT PLANNING WITH PREMIUM SERVICES

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We consider the assortment planning problem of an online retailer who offers products with multiple service types which differ in delivery time. We assume that customers make their choices according to the multinomial logit model. Our objective is to find the assortment of each service type which maximizes the expected revenue. Service types with faster delivery time are stored in fulfillment centers closer to the customers. The retailer gains less profit from the products offered with faster delivery due to the higher storage and transportation costs; however, faster delivery increases the popularity of the products. Therefore, delivering products faster has a trade off between higher demand and lower revenue. We provide a linear program to solve this problem. We also find structural properties of the optimal assortment and construct a polynomial time algorithm based on the structure of the optimal solution, along with an alternative method developed by modifying an existing algorithm. We also study another version of the problem where the retailer incurs a fixed cost for including a product in the assortment. We analyze the complexity of the problem for two structures of fixed cost. In the first one, the fixed cost is the same for all products. In this case, we show that the problem can be solved in polynomial time. In the second one, the fixed cost is different for all products and the problem is NP-complete. Finally, we conducted a numerical study in which we analyzed the effects of the problem parameters on the optimal assortment.

Keywords: assortment planning, online retail, multinomial logit model.

ÖZET

AYRICALIKLI SERVİS TİPLERİ İLE ÜRÜN SEÇİMİ ENİYİLEMESİ

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Bu çalışmada, ürünlerini tüketicilerine teslim süreleri birbirinden farklı çesitli servis tipleri ile sunan bir elektronik perakendecinin ürün çeşidi en iyilemesi problemi incelenmiştir. Her bir tüketicinin seçimlerini çok sınıflı logit modeline göre yaptığı varsayılmaktadır. Problemin amacı, her servis tipi için beklenen kazancı en çoklayacak ürünleri seçmektir. Hızlı teslim yapan servis tiplerindeki ürünler müşterilere daha yakında yer alan depolarda depolanmaktadır. Perakendeci hızlı teslim edilen ürünlerden depolama ve taşıma maliyetleri nedeniyle daha az kazanç elde eder. Ancak, hızlı teslim ürünlerin popülaritesini arttırır. Bu problemi çözmek için doğrusal programlama modeli geliştirilmiştir. Aynı zamanda, en iyi çözümün yapısal özellikleri saptanıp, kullanılarak polinom zamanda çözülebilen bir algoritma ve başka bir algoritmanın uyarlanmasıyla alternatif bir yöntem daha geliştirilmiştir. Bunlarla birlikte, problemin perakendecinin asortiye aldığı her ürün için sabit bir maliyet ödediği bir versiyonu da incelenmiştir. Bu problemin zorluğu iki farklı maliyet yapısı için analiz edilmiştir. İlkinde, maliyetin her ürün için aynı olduğu varsayılır. Bu problemin polinom zamanda çözülebileceği gösterilmiştir. İkinci problemde ise, maliyetin asortiye katılan her ürün için farklı olduğu durum incelenmiştir ve problemin NP-complete olduğu gösterilmiştir. Son olarak, problemin parametrelerine göre çözümün nasıl değiştiğini inceleyen sayısal bir çalışma yapılmıştır.

Anahtar sözcükler: ürün seçimi en iyilemesi, elektronik perakendecilik, çok sınıflı logit modeli.

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Chapter 1

Introduction

Assortment is defined as the set of products offered to customers in each store. A retailer's assortment affects sales and gross margin significantly; therefore, assortment planning is an active area of research. The goal of assortment planning is to specify the set of products to offer to the customers that maximizes the expected revenue or profit. Deciding on which products to offer is a strategic decision for retailers to satisfy diverse customer needs and increase their profits.

Retailers used to believe that offering large assortments was a way to increase their market share following the assumption that consumers are perfectly knowledgeable about their preferences [1]. This assumption suggested that large product variety increased the chances for consumers to find their ideal product in the assortment; thus, broad assortments decreased the lost sales. However, the downside of larger assortments was increased operational costs. Moreover, most of the time, the retailers have limited shelf space or budget which constrains the number of products they can offer while deciding on the assortments. Therefore, retailers faced a trade-off between increasing the product variety and allocating space or budget for the products.

Another motivator for decreasing the assortment size was that the majority of the consumers decide on which product to buy after visiting the store and observing the assortment, unlike what was assumed [2]. Several empirical studies also questioned whether offering wider assortment better satisfy consumers' needs. Broniarczyk et al. [3] demonstrated that the perception of variety for a consumer might be different than the actual variety. Empirical findings showed that the perceived variety is affected by the following factors: the space allocated to a category [4], whether a favorite product is included in the assortment or not [3, 5], similarity of products [6] and arrangement of the assortment [7]. Consideration of these factors increases the complexity of assortment optimization.

Assortment planning applies to both online and offline retail. In recent years, online retail sales have been growing rapidly, outpacing the growth of sales within physical stores. Fulgoni [8] states that $1 in every $5 of consumers' discretionary spending is attributed to digital commerce and that number is increasing at 20% every year, which is about four times more than the growth of overall retail sales. US retail sales provide more evidence for the rise of the online retail. Forrester Research Inc forecasts that 17% of US retail sales by 2022 will be attributed to e-commerce, compared to the 12.9% in 2017. Forrester's Wu states, "Online retail sales growth started to accelerate back in 2015 and is now currently outpacing offline market growth rate by a factor of five." [9]. With its fast growth, e-commerce is seen as the future of retail.

Online retailers threaten the physical stores even more by offering their customers faster delivery. According to a McKinsey and Company report, faster delivery "integrates the convenience of online retail with the immediacy of physical stores" [10]. A survey conducted by Mckinsey and Company shows that more than half of the respondents would use online retail more frequently if faster delivery was offered. It is also reported that Amazon's move to the same-day delivery in 2009 increased the purchase conversion during the checkout process by 20 to 30 percent [10]. Currently, Amazon offers its customers the options of free 2-hour delivery and same-day delivery for their prime products.

Although evidences show that online retail has significant advantages over traditional stores, the operation of online channels is more complex. In traditional retail, customers visiting a physical store choose an available product from the store's assortment and have immediate access to the purchased product. However, in an online channel, the products are stored in fulfillment centers and the retailer is responsible for supplying the product to the customers. Faster delivery requires the adoption of even more sophisticated models. Amazon has invested in decentralized hubs for prime products to deliver them to the customers within 2-hours or in the same-day. In the network of Amazon, the products offered with faster delivery are stored in fulfillment centers that are closer to the customers. Considering the impact of faster delivery on sales, the assortment of each distribution center gains even more importance in online retail.

In this thesis, we consider the assortment planning problem for an online retailer which offers products to the customers with different service types. We assume that the products offered with the same service type are served from the same fulfillment center and the products stored in fulfillment centers which are closer to the customers have faster delivery time. The faster delivery time increases the popularity of the products for the customers; however, storing the products closer to the customer is more expensive. Thus, the revenue obtained from a product decreases if it is offered with faster delivery. We also assume that a product can be offered with at most one service type. The objective of the problem is to find the assortment of each service type, i.e., the set of products stored in each fulfillment center that maximizes the expected profit.

The remainder of the thesis is organized as follows: In Chapter 2, we provide a literature review on assortment optimization. In Chapter 3, we define the problem in detail and derive the expected profit. In Chapter 4, we analyze the problem described above and present two polynomial-time algorithms. In Chapter 5, we discuss a version of the problem where the retailer incurs a fixed cost for including a product in the assortment of a service type. Finally, in Chapter 6, we provide a numerical study which analyzes how the optimal assortment changes with respect to the parameters of the problem.

Chapter 2

Literature Review

To understand the modeling approaches in the literature, consider the behavior of consumers before shopping. If a consumer finds her favorite product in the assortment of a store and if it is not stocked out, then she buys it. On the other hand, she has several options if she cannot find her favorite product. First, she can settle for an available one, which is called substitution. Second, she can decide to search for her favorite product in other stores and come back later if she cannot find it. Third, she leaves the store without purchasing any other product and the consumer is lost. Moreover, we can elaborate on why the consumer cannot find her favorite product in a store. The product is either carried in the store's assortment; however, it is stocked out in the time of consumer's visit or it is not included in the assortment of the store. The substitution type is classified as stockout-based and assortment-based in these two cases, respectively. The papers have varying approaches on how they model these cases.

Most of the papers in the literature study the assortment optimization problem of a single store and there are very few papers which consider assortments of multiple channels. The research in assortment planning has two streams. While, some papers construct stylized models and focus on characterizing the structure of optimal assortments, the others attempt to find the optimal assortment with considerations on inventory planning, pricing decisions and shelf allocation in more realistic settings. Kök et al. [11] and Karampatsa et al. [1] give broad reviews on assortment planning literature.

The papers have different assumptions for costumer substitution. If the inventory levels of products do not have impact on the choice of consumers, the model is called static. In this setting, the consumers are uninformed about the products before visiting the store and they make their choice after inspecting the assortment. If the product the consumer chooses is out of stock, the consumer does not substitute and the sale is lost. Therefore, only assortment-based substitution can occur and it is called static substitution. In dynamic substitution, stockout-based substitution is also taken into consideration and the retailer can update the assortment throughout the season.

The classification of papers is done according to the demand model representing the consumer choice. The most frequently used demand models in the assortment planning literature are multinomial logit model, exogenouos demand model and locational choice model. In the rest of the chapter, we briefly describe these models and review the literature under each demand model.

2.1 Assortment Planning Under Multinomial Logit Model

Multinomial logit model (MNL) is the most commonly used utility-based discrete choice demand model in the literature. Let $N = \{1, ..., n\}$ be the set of products and S be the subset of products carried by the retailer. Let r_j and v_j denote the revenue and preference weight of product j, respectively. We create product 0 to represent the no-purchase option of the customer. According to MNL, every customer associates a utility $U_j = u_j + \epsilon_j$ for each $j \in S \cup \{0\}$ where u_j is the mean utility customer assigns to product j and ϵ_j is the random component of the utility. The random components are independent and identically distributed random variables which have Gumbel distribution. Gumbel distribution is characterized by the cumulative distribution function, $P(\epsilon_j \leq x) = e^{-e^{\frac{x}{\mu}+\gamma}}$, where γ is the Euler's constant. The mean is zero and variance is $\frac{\mu^2\pi^2}{6}$. According to MNL, the probability that a customer chooses product j from the assortment S can be written in closed form expression, $P_j(S) = \frac{e^{u_j/\mu}}{\sum_{j\in S} e^{u_j/\mu}}$, which makes MNL a convenient option to model consumer choice.

van Ryzin and Mahajan [12] is one of the earliest works which formulates the assortment planning problem using multinomial logit model with static substitution. They study a stylized model where the products have identical prices and costs. The cost is defined as in the newsvendor model. The authors prove the optimal assortment consists of some number of the most popular products.

There are several follow-up papers which use the problem structure studied in van Ryzin and Mahajan [12]. Mahajan and van Ryzin [13] extend their analysis to dynamic substitution. Maddah and Bish [14] extend the van Ryzin and Mahajan model by considering the pricing decisions as well. Another extension is by Li et al. [15], where they analyze the van Ryzin and Mahajan model under continuous store traffic and demonstrate that the optimal assortment is comprised of some number of products which have highest profit rates.

Cachon et al. [16] incorporate consumer search to the van Ryzin and Mahajan model. In the previous studies, the no-purchase option also represents the cases where the consumer chooses to search the other stores. Cachon et al. [16] build a model which considers the consumer search explicitly and in relation with the products in the assortment. The authors compare the search-incorporated models with the no-search model and conclude that no-search model can result in narrower assortments which have an expected revenue considerably less than the optimal. Cachon et al. [16] also prove that the result found by van Ryzin and Mahajan [12] applies to any concave increasing cost function.

Talluri and van Ryzin [17] study an assortment planning problem under MNL. They assume that the parameters of MNL are deterministic and known. They prove that some number of products with the highest revenues are carried in the optimal assortment.

Davis et al. [18] consider constrained assortment planning problems which aim to find the optimal assortment that maximizes the expected revenue restricted by a set of constraints which have totally unimodular structure. They show the following problems can be formulated as a linear program due to their totally unimodular constraint structure: problems with cardinality constraints, problems which consider the locations to display the chosen products, problems with pricing decisions when there is a finite set of feasible prices, problems with precedence constraints, and quality consistent pricing problems.

Kunnumkal et al. [19] study a problem where a fixed cost is associated with each product and it is incurred for every product included in the assortment. The goal of the problem is to maximize the expected profit. They prove that the problem is NP-complete and they construct a polynomial time approximation scheme and a 2-approximation algorithm.

Rusmevichientong et al. [20] work on a problem with a capacity constraint in static and dynamic settings. In static setting, the parameters of MNL are assumed to be known and structural properties of the optimal solution are derived. Based on the structural properties of the optimal solution, they develop a polynomial time algorithm. In the dynamic setting, the parameters of the MNL are unknown. For the dynamic problem, an adaptive policy which learns the parameters from the past data is developed using the structural properties of the static problem.

The papers discussed until now study assortment planning problem in a single channel. For assortment optimization in multi-channel settings under MNL, we review the following two papers.

Singh et al. [21] work on an assortment planning problem which considers inventory decisions in alternative supply chain structures with a wholesaler and multiple retailers extending the van Ryzin and Mahajan model. The authors first consider a traditional channel and drop-shipping channel. Each retailer keeps inventory in the traditional channel. Likewise, the wholesaler keeps inventory in the drop-shipping channel and ships products to the consumers at each retailer's request, while charging an additional drop-shipping fee. The authors show the optimal solution in traditional and drop-shipping channels can be characterized by a single threshold policy. They conclude that when the drop-shipping fee is low and there are large number of retailers, the optimal assortment of drop-shipping channel is larger than the traditional channel. In addition, they study a single firm which fulfills orders from multiple retailer locations and a central warehouse. They find that the popular products should be stored at the retailer, whereas the less popular products should be drop-shipped from the central warehouse.

Finally, Dzyabura and Jagabathula [22] study an assortment optimization problem where an online retailer offers a subset of the assortment in an offline store. The goal of the problem is to find the assortment of the offline channel which maximizes the profit across both channels. The decision problem under this setting is proved to be NP-hard. The authors find the optimal results for some special cases and develop near-optimal approximations for the general case.

2.1.1 Nested Logit Model

The major flaw of the MNL is that it fails to capture how the demand for products change when we add a product similar to the ones in the assortment in comparison to a different one. This is caused by the Independent of Irrelevant Alternatives (IIA) property of MNL which holds only if the choice probabilities of the products are independent. An example which illustrates this deficiency is the "blue bus/red bus paradox": Consider a person who has two options to take while going to work: using her car or taking the bus. Also, let both options have the same probability, i.e., $P(car) = P(bus) = 1/2$. Now, assume that we give three options to the person which are using her car, taking the blue bus or taking the red bus. If the person is indifferent between the colors, we would expect the choice probabilities to be $P(car) = 1/2$, $P(red\ bus) = P(blue\ bus) = 1/4$; however, MNL gives $P(car) = P(red\ bus) = P(blue\ bus) = 1/3$. To alleviate IIA property in MNL, the Nested Logit Model (NL) is introduced by Ben-Akiva and Lerman [23]. In NL, the choice set is partitioned into subsets, which are called nests. The consumers choose the nest first and then make a selection within the chosen nest.

Kök and Xu [24] work on an assortment optimization and pricing problem using NL with two different nest formations. They extend the characterization found in van Ryzin and Mahajan [12] for the optimal solution to these two NL models with different nest structures.

Davis et al. [25] study an assortment optimization problem under NL, where they consider four cases which are characterized by varying assumptions on nest dissimilarity parameters and no-purchase option. They show that if the nest dissimilarity parameter is less than one and consumers always select a product within the chosen nest, the optimal solution can be found in polynomial time; however, if we relax any of these assumptions, the problem becomes NP-hard.

Gallego and Topaloglu [26] consider a constrained problem under NL. with constraints on the set of products offered in each nest. They demonstrate that if there is a cardinality constraint for the set of products offered in each nest, the problem can be formulated as a linear program. On the other hand, the problem with space constraints is proved to be NP-hard.

Rodriguez and Aydın [27] also consider a problem with multiple channels where a manufacturer offers products via a direct channel and through a retailer which offers a subset of the products. The consumer can purchase products either from the manufacturer or the retailer. The customer choice is assumed to be governed by NL. The paper provides insights for the pricing decisions taking the competition between the manufacturer and the retailer into consideration. They also analyze the manufacturer's assortment decisions under different scenarios; however, they do not characterize the retailer's optimal assortment.

2.1.2 Mixed Multinomial Logit Model

Another variation of MNL model is called the Mixed MNL model (MMNL). MMNL is introduced by Boyd and Mellman [28] and Cardell and Dunbar [29] and it does not have the IIA property of MNL. It also provides flexibility as McFadden and Train [30] state, "any discrete choice model derived from random utility maximization has choice probabilities that can be approximated as closely as one pleases by a MMNL model." However, Bront et al. [31] and Rusmevichientong et al. [32] prove that assortment planning problem under MMNL is NP-complete and provide a mixed integer programming formulation. In assortment planning literature, MMNL is also referred as Mixtures of MNL [33], MNL with random choice parameters [32], and latent-class MNL [34].

Rusmevichientong et al. [32] study a problem where the parameters of MNL are random. They assume that there are multiple customer types and each one's preference for the products is different from each other. The firm does not know the type of arriving customers which accounts for the randomness in the parameters. The goal of the problem is to find the assortment which maximizes the expected revenue over all customer types. It is proved that even the problem with two customer segments is NP-complete. They identify special cases where the revenue-ordered assortments are optimal and derive approximation guarantees for cases where they are not optimal.

Feldman and Topaloglu [33] work on an assortment planning problem with multiple customer types. Their goal is to find the assortment that maximizes the expected revenue over all customer types. The authors develop strong upper bounds on the optimal solution.

Şen et al. [44] consider the constrained assortment optimization problem with MMNL. They provide a conic quadratic mixed-integer program which enables the solution of large instances.

2.2 Assortment Planning Under Locational Choice Model

Locational Choice Model (LC) is introduced by Hotelling and extended by Lancaster. In this model, products are characterized by bundles of attributes and a consumer's preference is defined as a combination of the characteristics [35, 36]. The space of all possible combinations of attributes constitutes the preference space. A consumer's most preferred combination of attributes corresponds to her ideal point in the preference space. The utility of a product to a consumer is found in the following way. Let a product have m attributes. Suppose that z_j denotes the location of product j in $\mathbb{R}^m$. To a consumer whose ideal product is defined by $y \in \mathbb{R}^m$, the utility of product j is $U_j = k - r_j - g(y, z_j)$, where k is a positive constant, r_j is the price of product j and $g : \mathbb{R}^m \to \mathbb{R}$ a distance function representing the disutility associated with the product's distance from the ideal point. The consumer selects the product with the her the maximum utility. The major difference of LC from MNL is that substitution can occur between any two products in MNL, whereas in LC, substitution between products is limited to products with locations that are close to each other in the attribute space.

Gaur and Honhon [37] study a single-period assortment optimization and inventory management problem under LC model. They consider both static and dynamic substitution. They show that the optimal assortment under static substitution consists of products that are spaced out so that there is no substitution between them. The paper states that the optimal assortment may not include the most popular product, which contrasts with the result of the same problem under MNL. The authors compare the static model with the dynamic model and conclude that the retailer offers higher variety with dynamic substitution and the optimal assortment includes products closer to each other in the attribute space. The optimal expected profit found under the static substitution serves as a lower bound to the model. An upper bound is also found using retailer-controlled substitution. The paper proposes two heuristics for the dynamic problem and evaluate their performances.

2.3 Assortment Planning Under Exogenous Demand Model

Exogenous demand models (ED) are often used in the inventory management literature; thus, the papers reviewed under this section study the assortment optimization and inventory management problems jointly. In ED, unlike MNL and LC, the consumer behavior is not defined based on a utility model. The demand for each product and what a consumer does when her selected product is not available is directly specified. Let N be the set of products. For all $j \in N$, p_j denotes the probability that a consumer selects product j. Suppose that the consumer chooses product $i \in N$, which is not available, either because it is stocked-out or it is not included in the assortment. Then, the consumer substitutes product j with probability μ_{ij}.

Smith and Agrawal [38] are the first authors who use exogenous demand model in assortment optimization. They build a probabilistic demand model which capture the effects of static substitution and propose a methodology to determine inventory levels for products to maximize the expected profit. They use the negative binomial demand distribution and Newsboy model for the supply process. They obtain the following conclusions by solving illustrative examples. Substitution effects reduce the optimal assortment size in the presence of fixed cost. Even without fixed costs, substitution effects can decrease the assortment size. Finally, the paper states that stocking the most popular items might not be the optimal solution in this setting in contrast to the main result of van Ryzin and Mahajan [12].

Rajaram and Tang [39] analyze the impact of product substitution on the order quantities and expected profits using an exogenous demand model. They employ the Newsboy model and develop a service rate heuristic. They evaluate the performance of the heuristic by obtaining an upper bound from the Lagrangian dual problem and conclude that the heuristic is tractable and accurate to find the order quantities and expected profit under substitution.

Fisher and Kök [40] consider a problem with dynamic substitution when there are constraints on the shelf space, maximum inventory levels and order lead times. They propose a procedure for estimating the substitution rates and develop a heuristic which gives solutions within 0.05% of the optimal solution on average in the computational studies.

Yücel et al. [41] analyze a problem under static substitution and provide a mathematical model. Issues such as supplier selection, shelf space constraint and poor quality procurement are also considered. They evaluate the performance of three modified models, each neglecting one of the issues mentioned, and conclude that neglecting any of them might result in significantly inefficient assortments.

Fadıloğlu et al. [42] consider a static assortment optimization problem and present an optimization model which finds the optimal assortment that maximizes the expected profit. The distinction of their model from the other papers in the literature is it does not require extensive data and easy to implement. They test their model on the shampoo lines of two supermarkets. The results show that the model is computationally applicable and significant boost in the expected profit.

Gao and Su [43] study the impact of offering the option to buy online and pick up in store (BOPS). The offline store modeled as a newsvendor problem, whereas the online channel is exogenous and always in stock. They find that BOPS option is not suitable for all products. They also analyze the impact of BOPS option on attracting customers to the offline store. Finally, they consider a decentralized retail system and conclude that retailers can maximize profits by sharing BOPS revenue between the online and store channels.

In the next chapter, we define our problem and explain its relations with literature.

Chapter 3

Problem Statement

We consider an online retailer which offers its customers multiple services that differ in delivery time (such as delivery within two hours, the same day or the next day), assuming that a product can be offered with at most one service type. Since the delivery times and assortments of service types are different, we can think each service type as a different channel. Thus, we study the assortment optimization problem in multi-channel setting, unlike the majority of papers in the literature.

Customers derive higher utility when the products are shipped faster; therefore, the popularity of a product increases if it is offered with faster delivery. Products designated for faster delivery need to be positioned closer to the customer (e.g., Amazon's Prime Now hubs in metropolitan areas) whereas products on regular delivery can be stored in regular fulfillment centers. Faster fulfillment typically increases the outbound costs; thus, we assume that the profit the retailer obtains from a product depends on the service type it is offered with. We make no distinction between the customers which means that each customer can purchase products offered with any service type. This assumption can be justified in a setting where the prices of products do not change depending on the service type they are offered with; thus, one service type is always more favorable for a product from the perspective of the customers. In that setting, even if a product is offered with multiple service types, the customers would buy the one offered with the most favorable service type. We also handle the cases where the price of the products change depending on the service type they are offered with.

We also consider a scenario, where the retailer incurs a fixed cost for every product included in the assortment. A retailer has a limited space for the products that are offered in the assortment. The fixed cost can be interpreted as the opportunity cost of space. The space restriction can be limited storage space or a page limit for an online retailer. The fixed cost is a penalty for adding a product to the assortment. The fixed cost may be the same for all products or it may vary.

In this thesis, we consider the problem explained above under the assumption that customers make a choice within the offered assortment according to the multinomial logit model (MNL). As in van Ryzin and Mahajan [12], we assume that the costumers are uninformed about the assortment and they decide on what to buy after observing the assortment. Therefore, we study the problem under static substitution. However, unlike their stylized model, the revenue and preference weight of products are different and depend on the service type the products are offered with. The parameters of the MNL are assumed to be known. The problem for the online retailer is to determine the assortments of each service type such that the total profit from the sales of all service types is maximized.

For the problem without fixed cost, we derive structural properties of the optimal assortment and extend the result of Talluri and van Ryzin [17] to the assortments of multiple service types. Using the structural properties of the optimal assortment, we develop an algorithm to find the optimal solution.

The static problem studied by Rusmevichientong et al. [20] is similar to our problem in the aspects that MNL is used to represent the consumer choice and only assortment-based substitution is used. The difference of our work is that they study the assortment of a single store and assume there is a capacity constraint on the number of products that can be included in the assortment. However, both problems have a similar geometrical structure which can be exploited to find the optimal solution. Using this similarity, we provide an alternative method by modifying the algorithm of Rusmevichientong et al. [20].

Davis et al. [18] also study the assortment optimization problem under MNL without fixed cost. They show that the problem can be formulated as a linear program if the constraints have a totally unimodular structure. Although we study the problem with multiple service types, their result holds for our problem as well when there is no fixed cost. Therefore, we formulate the problem without fixed cost as a linear program using the approach in Davis et al. [18].

In the second version with fixed cost, we analyze the complexity of the problem. Kunnumkal et al. [19] study the assortment optimization problem with fixed cost under MNL for a single store and they prove the problem is NP-complete if the fixed cost depends on the product. We extend their result to multiple assortments showing that if the fixed cost depends on the product and the service type that the product is offered with, the problem is NP-complete. We also analyze a fixed cost structure where the fixed cost is the same for all products and service types and show that the problem can be solved in polynomial time.

We also provide mathematical formulations for assortment optimization problem with fixed cost. Şen et al. [44] study the constrained assortment optimization problem for a single store under MMNL considering multiple customer classes and formulate it as a conic quadratic mixed-integer program. Although we study the problem with multiple service under MNL and do not consider multiple customer classes, we use their approach to formulate the problem as a conic quadratic mixed-integer program.

The main difference of this thesis from the papers in the litareture is the consideration of the assortment planning problem with multiple service types. Thus, we study the assortment optimization problem in multichannel setting. Moreover, we decide on the assortments of each channel, unlike the majority of papers which study assortment planning in multichannel setting. Although Singh et al. [21] also consider an assortment planning problem under MNL with multiple channels, they study the relationship between assortment size and the structure of the supply chain. Another paper which study an assortment optimization problem under MNL, Dzyabura and Jagabathula [22], focus on the interactions between online and offline channels and analyze the impact of having an offline channel. They only decide on the assortment of the offline channel, whereas we do not consider the interactions between different service types and decide on the assortments of all service types. Rodriguez and Aydın [27] consider a multichannel assortment planning problem under NL. They provide insights on pricing decisions while taking the competition between the two channels into account. In our problem, the channels do not compete with each other and we characterize the optimal assortment rather than studying pricing decisions. Finally, Gao and Su [43] consider a problem with offline store and online channel. However, they analyze the impact of offering online channel and do not focus on assortment decisions, unlike our study. Therefore, our main contribution is characterizing the optimal assortments of each service type.

Before describing how the expected revenue is derived under the multinomial logit model, we define the notation that will be used to model the problem. Let $M = \{1, 2, \ldots, m\}$ be the set of service types and $N = \{1, 2, \ldots, n\}$ be the set of products. Product 0 is also created to represent the no purchase option. Let r_{ik} and v_{ik} denote the profit and preference weight of product k if it is offered with service type i, respectively. v_0 denotes the preference weight of the no purchase option. We use S_i to represent the assortment of products offered with service type i and the total assortment is denoted by $S = (S_1, \ldots, S_m)$. The notation that will be used in this chapter is given in Table 3.1.

Under MNL, customers associate a utility $U_{ik} = u_{ik} + \epsilon_{ik}$ with every product $k \in S_i$ for all $i \in M$. We define $v_{ik} = e^{u_{ik}/\mu}$ for all $k \in S_i$, $i \in M$. Under the multinomial logit model, given an assortment $S = \{S_1, \ldots, S_i, \ldots, S_m\}$, the probability that a customer chooses product k is given by,

$$P_{ik}(S) = \frac{v_{ik}}{v_0 + \sum_{i \in M} \sum_{k \in S_i} v_{ik}}$$

Table 3.1: **Notation**

Sets		
M	:	Set of service types
N	:	Set of products
S	:	Total assortment of the retailer
S_i	:	Assorment of service type i
Parameters		
v_{ik}	:	Preference weight of product k if it is offered with service type i
v_0	:	Preference weight of no purchase option
r_{ik}	:	Profit of product k if it is offered with service type i
K_i	:	Capacity of service type i
c_{ik}	:	Cost of including product k in the assortment of service type i
$P_{ik}(S)$	:	The probability that a customer chooses product k from service type i from the assortment $S = (S_1, \ldots, S_m)$.
$\pi(S)$	:	Expected profit of assortment $S = (S_1, \ldots, S_m)$.

If we offer assortment $S = (S_1, \ldots, S_m)$, we obtain an expected profit of

$$\pi(S) = \sum_{i \in M} \sum_{k \in S_i} r_{ik} P_{ik}(S) = \frac{\sum_{i \in M} \sum_{k \in S_i} r_{ik} v_{ik}}{v_0 + \sum_{i \in M} \sum_{k \in S_i} v_{ik}}.$$

Our goal is to choose an assortment $S = (S_1, \ldots, S_m)$ such that the expected profit is maximized; therefore, the optimization problem under the general setting is

$$\max_{S: S_i \cap S_j = \emptyset \forall i,j \in M: i \neq j} \pi(S) = \frac{\sum_{i \in M} \sum_{k \in S_i} r_{ik} v_{ik}}{v_0 + \sum_{i \in M} \sum_{k \in S_i} v_{ik}}. \tag{3.1}$$

Observe that, we can consider the no purchase option as a service type which has a preference weight of v_0 and a revenue of 0, i.e, $r_{0k} = 0$ and $v_{0k} = v_0$ for all $k \in N$.

For the problem with fixed cost, let c_{ik} be the fixed cost of including product k in the assortment of sevice type i. Then, we need to solve the following optimization problem,

$$\max_{S: S_i \cap S_j = \emptyset \forall i,j \in M: i \neq j} \pi(S) = \frac{\sum_{i \in M} \sum_{k \in S_i} r_{ik} v_{ik}}{v_0 + \sum_{i \in M} \sum_{k \in S_i} v_{ik}} - \sum_{i \in M} \sum_{k \in S_i} c_{ik}. \tag{3.2}$$

We formulate and analyze the problems described above. In Chapter 3, we focus on Problem (3.1) and provide a linear programming formulation. Moreover, we characterize the structure of the optimal solution and construct algorithms based on properties of the optimal solution. In Chapter 4, we study the Problem (3.2) and provide mathematical formulations. We also analyze several cases of the fixed cost problem. Finally, we provide numerical experiments in Chapter 6.

Chapter 4

Assortment Optimization with Premium Services without Fixed Cost

4.1 Mathematical Formulation

In this section, we formulate Problem (3.1) as a linear program. Davis et al. [18] demonstrate that the assortment optimization problem with a set of totally unimodular constraints can be solved as an equivalent linear program. We use the same approach and formulate Problem (3.1) as a linear program using the totally unimodular structure of the constraints.

First, we define a decision variable as follows:

$$x_{ik} = \begin{cases} 1 & \text{if product } k \text{ is offered with service type i} \\ 0 & \text{otherwise} \end{cases}$$

Then, Problem (3.1) is equivalent to the following model:

$$\begin{aligned} \max \quad & \frac{\sum_{i\in M}\sum_{k\in N} r_{ik}v_{ik}x_{ik}}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}} \\ \text{s.t.} \quad & \sum_{i\in M} x_{ik} \le 1, \quad \forall k \in N, \\ & x_{ik} \in \{0,1\}, \quad \forall i \in M, \quad \forall k \in N. \end{aligned} \tag{4.1}$$

Problem (4.1) has a nonlinear objective function and binary decision variables. This problem has an interval constraint matrix, since each row of the constraint matrix consists of consecutive ones. Interval matrices are proved to be totally unimodular by Nemhauser and Wolsey [45]. Using this property, we show the equivalence of (4.1) to the following linear program In Theorem 1.

$$\begin{aligned} \max \quad & \sum_{i\in M}\sum_{k\in N} r_{ik}v_{ik}y_{ik} \\ \text{s.t.} \quad & \sum_{i\in M} y_{ik} \le y_0, \quad \forall k \in N, \\ & \sum_{i\in M}\sum_{k\in N} v_{ik}y_{ik} + v_0y_0 = 1, \\ & y_{ik} \le y_0, \quad \forall i \in M, \quad \forall k \in N, \\ & y_{ik} \ge 0, \quad \forall i \in M, \quad \forall k \in N. \end{aligned} \tag{4.2}$$

Theorem 1. *Problems (4.1) and (4.2) have the same optimal objective value and we can construct an optimal solution to one of these problems by using an optimal solution to the other.*

Proof. Let x^* and z^* be the optimal solution and optimal value of problem (4.1), respectively. Using the decision variable $w = \{w_{ik} : i \in M, k \in N\} \in [0,1]^{m\times n}$, we claim that problem (4.1) is equivalent to the problem

$$\begin{aligned} \max \quad & \sum_{i\in M}\sum_{k\in N} v_{ik}r_{ik}w_{ik} - z^*\left(v_0 + \sum_{i\in M}\sum_{k\in N} v_{ik}w_{ik} - 1\right) \\ \text{s.t.} \quad & \sum_{i\in M} w_{ik} \le 1, \quad \forall k \in N, \\ & 0 \le w_{ik} \le 1, \quad \forall i \in M, \quad \forall k \in N. \end{aligned} \tag{4.3}$$

which is a linear program.

Since

$$z^* = \frac{\sum_{i \in M} \sum_{k \in N} v_{ik} r_{ik} x_{ik}^*}{v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}^*},$$

we get

$$z^* \left(v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}^* \right) = \sum_{i \in M} \sum_{k \in N} v_{ik} r_{ik} x_{ik}^*.$$

Evaluating the objective function of problem (4.3) at feasible solution x^*, we obtain

$$\sum_{i \in M} \sum_{k \in N} v_{ik} r_{ik} x_{ik}^* - z^*(v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}^* - 1) = z^*$$

which implies that the optimal objective value of problem (4.3) is at least as large as the optimal objective value of problem (4.1).

Let, w^* be the optimal solution to problem (4.3). Problem (4.3) has a linear objective function and its constraints have a totally unimodular structure; thus, $w^* \in \{0, 1\}^{m \times n}$. Evaluating the objective function value of problem (4.1) at the feasible solution w^*, we have

$$z^* \geq \frac{\sum_{i \in M} \sum_{k \in N} v_{ik} r_{ik} w_{ik}^*}{v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}^* w_{ik}^*},$$

which implies that

$$z^* \left(v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}^* w_{ik}^* \right) \geq \sum_{i \in M} \sum_{k \in N} v_{ik} r_{ik} w_{ik}^*.$$

Arranging the terms, we obtain

$$z^* \geq \sum_{i \in M} \sum_{k \in N} v_{ik} r_{ik} w_{ik}^* - z^* \left(v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}^* w_{ik}^* - 1 \right).$$

Therefore, the optimal objective value of problem (4.3) is at most as large as the optimal objective value of problem (4.1). Thus, problems (4.1) and (4.3) are equivalent to each other, sharing the same optimal objective value.

Now, we will show that problems (4.2) and (4.3) are equivalent to each other. Let y_0^* and $y^* = \{y_{ik}^* : i \in M, k \in N\}$ be an optimal solution to problem (4.2) with objective value ζ^* and w^* be an optimal solution to problem (4.3). We construct the solution $\hat{y}_0$ and $\hat{y} = \{\hat{y}_{ik} : i \in M, k \in N\}$ to problem (4.2) as

$$\hat{y}_{ik} = \frac{w^*_{ik}}{v_0 + \sum_{i\in M}\sum_{k\in N} v_{ik} w^*_{ik}} \text{ for all } i \in M \text{ and } k \in N \text{ and}$$
$$\hat{y}_0 = \frac{1}{v_0 + \sum_{i\in M}\sum_{k\in N} v_{ik} w^*_{ik}}.$$

It is easy to check that $\hat{y}$ is feasible to problem (4.2). The objective value provided by the feasible solution $\hat{y}$ to problem (4.2) satisfies $\zeta^* \geq z^*$. We construct solution to problem (4.3) as $\hat{w}_{ik} = y^*_{ik}/y^*_0$ for all $i \in M$ and $k \in N$. The objective value provided by the feasible solution $\hat{w}$ to problem (4.3) satisfies

$$\begin{aligned}
z^* &\geq \sum_{i\in M}\sum_{k\in N} v_{ik} r_{ik} \frac{y^*_{ik}}{y^*_0} - z^* \left(v_0 + \sum_{i\in M}\sum_{k\in N} \frac{y^*_{ik}}{y^*_0} - 1 \right) \\
&\geq \sum_{i\in M}\sum_{k\in N} v_{ik} r_{ik} \frac{y^*_{ik}}{y^*_0} - \zeta^* \left(v_0 + \sum_{i\in M}\sum_{k\in N} \frac{y^*_{ik}}{y^*_0} - 1 \right) \\
&\geq \frac{\zeta^*}{y^*_0} - \zeta^* \left(\frac{1}{y^*_0} - 1 \right) \\
&\geq \zeta^*.
\end{aligned}$$

The second inequality comes from $\zeta^* \geq z^*$ and the third inequality is satisfied because of the second constraint in problem (4.2). Therefore, we obtain $z^* \geq \zeta^*$. □

4.2 Analysis of the Problem

In this section, we characterize the structure of the optimal solution. Based on the characteristics of the optimal solution, we construct an algorithm to solve the problem. We also present an alternative approach by modifying an existing algorithm proposed in Rusmevichientong et al. [20].

We assume that, the service types are labeled in increasing order of net utility and their order is the same for all products, i.e., $v_{1j} < v_{2j} < \cdots < v_{mj}$ for all $j \in N$. Let $V(S) = v_0 + \sum_{i\in M}\sum_{k\in S_i} v_{ik}$. We make use of Lemma 2 and Lemma 3 to characterize the structure of the optimal solution in Theorem 4.

Lemma 2. *Let $S^* = (S_1^*, \ldots, S_m^*)$ be the optimal assortment. For all $p \in M$ and $l \in S_p^*$, $\max_{i \in M: i > p} \left\{ \dfrac{v_{il} r_{il} - v_{pl} r_{pl}}{v_{il} - v_{pl}} \right\} \leq \pi(S^*)$.*

Proof. Suppose that there exists $l \in S_p^*$ such that $\frac{v_{jl} r_{jl} - v_{pl} r_{pl}}{v_{jl} - v_{pl}} > \frac{\sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik}}{v_0 + \sum_{i \in M} \sum_{k \in S_i^*} v_{ik}} = \pi(S^*)$ for some $j > p$. Then,

$$\frac{v_{jl} r_{jl} - v_{pl} r_{pl}}{v_{jl} - v_{pl}} > \frac{\sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik}}{V(S^*)} = \pi(S^*)$$

$$(v_{jl} r_{jl} - v_{pl} r_{pl}) V(S^*) > (v_{jl} - v_{pl}) \left(\sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik} \right)$$

$$\left(v_{jl} r_{jl} - v_{pl} r_{pl} + \sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik} \right) V(S^*) > (v_{jl} - v_{pl} + V(S^*)) \left(\sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik} \right)$$

$$\frac{v_{jl} r_{jl} - v_{pl} r_{pl} + \sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik}}{v_{jl} - v_{pl} + V(S^*)} > \frac{\sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik}}{V(S^*)}$$

$$\frac{v_{jl} r_{jl} - v_{pl} r_{pl} + \sum_{i \in M} \sum_{k \in S_i^*} r_{ik} v_{ik}}{v_{jl} - v_{pl} + V(S^*)} > \pi(S^*)$$

which contradicts with S^* being the optimal assortment. Therefore,
$\max_{i \in M: i > p} \left\{ \dfrac{v_{il} r_{il} - v_{pl} r_{pl}}{v_{il} - v_{pl}} \right\} \leq \pi(S^*)$ for all $l \in S_p^*$. □

Lemma 2 provides the following implication: Let assortment $S = (S_1 \ldots, S_m)$ be given and assume that there exists $k \in N$ and $p \in M$ such that $k \in S_p$ and $\frac{v_{il} r_{il} - v_{pl} r_{pl}}{v_{il} - v_{pl}} > \pi(S)$ for some $i > p$. Lemma 2 implies that we can obtain a greater expected profit by including product k in the assortment of service type i instead of service type p, i.e., we can increase the expected profit by offering product k with a faster service type. Moreover, if there exists a product $k \in N$ such that $k \notin \cup_{i \in M} S_i$ and $\max_{i \in M} r_{ik} > \pi(S)$, including product k in the assortment yields a greater expected profit than $\pi(S)$. Similarly, Lemma 3 identifies when we can obtain a greater revenue by offering a product with a slower service type.

Lemma 3. *Let $S^* = (S_1^*, \ldots, S_m^*)$ be the optimal assortment. For all $i \in M$ and $l \in S_i^*$, $\frac{v_{il} r_{il} - v_{pl} r_{pl}}{v_{il} - v_{pl}} \geq \pi(S^*)$ for all $p = 1, \ldots, i-1$.*

Proof. Let us assume there exists $l \in S_j^*$ such that $\pi(S^*) > \frac{v_{jl}r_{jl}-v_{pl}r_{pl}}{v_{jl}-v_{pl}}$ for some $j \in M$ and $p \in M$ such that $p < j$.

$$\frac{\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik}}{V(S^*)} > \frac{v_{jl}r_{jl} - v_{pl}r_{pl}}{v_{jl} - v_{pl}}$$

$$(v_{jl} - v_{pl})\left(\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik}\right) > (v_{jl}r_{jl} - v_{pl}r_{pl})V(S^*)$$

$$-(v_{jl} - v_{pl})\left(\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik}\right) < -(v_{jl}r_{jl} - v_{pl}r_{pl})V(S^*)$$

$$(V(S^*) - v_{jl} + v_{pl})\left(\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik}\right) < \left(\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik} - v_{jl}r_{jl} + v_{pl}r_{pl}\right)V(S^*)$$

$$\frac{\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik}}{V(S^*)} < \frac{\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik} - v_{jl}r_{jl} + v_{pl}r_{pl}}{V(S^*) - v_{jl} + v_{pl}}$$

$$\pi(S^*) < \frac{\sum_{i\in M}\sum_{k\in S_i^*} r_{ik}v_{ik} - v_{jl}r_{jl} + v_{pl}r_{pl}}{V(S^*) - v_{jl} + v_{pl}}$$

which contradicts with S^* being the optimal assortment. Therefore, for all $i \in M$ such that $i > p$ and $l \in S_i^*$ $\frac{v_{il}r_{il}-v_{pl}r_{pl}}{v_{il}-v_{pl}} \geq \pi(S^*)$. $\square$

Let assortment $S = (S_1 \ldots, S_m)$ be given and assume that there exists $k \in N$ and $i \in M$ such that $k \in S_i$ and $\frac{v_{ik}r_{ik}-v_{pk}r_{pk}}{v_{ik}-v_{pk}} < \pi(S)$ for some $p < i$. Lemma 3 implies that we can increase the expected profit by including product k in the assortment of service type p, instead of service type i, i.e., by offering it with a slower service type.

For notational convenience, we define service type 0 to denote the products that are not included in the assortment. Let $v_{0k} = 0$ and $r_{0k} = 0$ be the preference weight and profit of product k when it is not included in the assortment, respectively. Let $R_k(i, j)$ denote the ratio we use to identify the structure of the optimal assortment for product k between service types i and j, i.e, $R_k(i, j) = \frac{v_{ik}r_{ik}-v_{jk}r_{jk}}{v_{ik}-v_{jk}}$. Since, $v_{0k} = 0$ and $r_{0k} = 0$, we have $R_k(i, 0) = \frac{v_{ik}r_{ik}-0}{v_{ik}-0} = r_{ik}$ for all $k \in N$. We also define $\bar{R}_k(p)$ as the maximum ratio of $R_k(i, p)$ between a service type p and service types faster than p for product $k \in N$, i.e, $\bar{R}_k(p) = \max_{i>p}\{R_k(i, p)\}$. For service type 0, we define $\bar{R}_k(0) = \max_{i \in M}\{R_k(i,0)\} = \max_{i \in M}\{r_{ik}\}$. Using the results of Lemma 2 and Lemma 3, we characterize the structure of the optimal solution in Theorem 4.

Theorem 4. *For a given $p \in M \cup \{0\}$, let the products be sorted such that, $\bar{R}_1(p) \geq \cdots \geq \bar{R}_n(p)$. Then, there exists an optimal assortment $S^* = (S_1^*, \ldots, S_m^*)$ where $S_{p+1}^* \cup \cdots \cup S_m^* = \{1, 2, \ldots, k\}$ for some $k \in N$.*

Proof. Let $S^* = (S_1^*, \ldots, S_m^*)$ be the optimal assortment such that, $\cup_{i \in M: i > p} S_i^*$ contains product l but not product k and $k \in S_p^*$ for some $p \in M$ with $k < l$. Let $\pi(S^*)$ be the expected profit obtained from the assortment S^*. Then, by Lemma 2, we obtain $\max_{i \in M: i>p}\{\frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}\} \leq \pi(S^*)$. On the other hand, by Lemma 3, we have $\frac{v_{il}r_{il} - v_{pl}r_{pl}}{v_{il} - v_{pl}} \geq \pi(S^*)$ for $l \in S_i^*$ where $i > p$. Then, we have $\max_{i \in M: i>p}\{\frac{v_{il}r_{il} - v_{pl}r_{pl}}{v_{il} - v_{pl}}\} \geq \frac{v_{il}r_{il} - v_{pl}r_{pl}}{v_{il} - v_{pl}} \geq \pi(S^*) \geq \max_{i \in M: i>p}\{\frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}\}$, which contradicts with $\max_{i \in M: i>p}\{\frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}\} > \max_{i \in M: i>p}\{\frac{v_{il}r_{il} - v_{pl}r_{pl}}{v_{il} - v_{pl}}\}$. Therefore, S^* cannot be the optimal assortment. $\square$

Since service type 0 is defined to represent the products that are not included in the assortment, Theorem 4 also identifies the structure of the total assortment, i.e., $\cup_{i \in M} S_i^*$. The structure of the total assortment is stated in Corollary 4.1

Corollary 4.1. *There exists an optimal assortment $S^* = (S_1^*, \ldots, S_m^*)$ such that, the union of the assortment is of the form $S_1^* \cup \cdots \cup S_m^* = \{1, 2, \ldots, k\}$ for some $k \in N$, assuming the products are ordered such that,* $\max_{i \in M}\{r_{i1}\} \geq \max_{i \in M}\{r_{i2}\} \geq \cdots \geq \max_{i \in M}\{r_{in}\}$.

Next, we provide two algorithms developed based on the structural properties of the optimal solution.

4.2.1 Algorithm 1

We use the results found in the previous section to construct an algorithm to find the optimal assortment. According to Theorem 4, the optimal assortment

$S^* = (S_1^*, \ldots, S_m^*)$ has the following structure: Assortment of the fastest service type, S_m^*, consists of a number of products with the highest $\bar{R}_k(m-1)$ ratio, where k denotes the product. The assortment $S_{m-1}^* \cup S_m^*$ includes a number of products with the highest $\bar{R}_k(m-2)$ ratio. If we continue this way, the total assortment $S^* = \cup_{i \in M} S_i$ consists of a number of products with highest $\bar{R}_k(0)$. Using this result, the algorithm constructed to find the optimal solution is described below.

For $p = 0, \ldots, m-1$, let $\sigma^p = (\sigma_1^p, \ldots, \sigma_n^p)$ denote the ordering of products according to $\bar{R}_k(p)$, that is,

$$\bar{R}_{\sigma_1^p}(p) \geq \cdots \geq \bar{R}_{\sigma_n^p}(p).$$

For $p = 0, \ldots, m-1$, we construct the aggregate assortments of service types greater than p, $\cup_{i>p} S_i$, as a subset of products included in $\cup_{i>p-1} S_i$ which consists of a certain number of products from σ^p. Let S_{p+1}^k denote the aggregate assortment of service types greater than p that contains the first k products from the ordering σ^p, that is, $S_{p+1}^k = \cup_{i>p} S_i = \{\sigma_1^p, \ldots, \sigma_k^p\}$. We construct a candidate assortment for the optimal solution in each step of the algorithm. Among the candidate assortments obtained at the end of the Algorithm 1, the one with the highest expected revenue is the optimal solution. Let S^i denote the candidate assortment found in step i. How the candidate assortments are constructed in each step is explained in the pseudocode given below.

Algorithm 4.2.1: $<$ALGORITHM 1$>$$(<\sigma^p>)$

Inputs: The orderings σ^p for $p = 0, \dots, m-1$.

Outputs: Candidate assortments $S^i = (S_1^i, \dots, S_m^i)$.

$$
\begin{array}{l}
\textbf{for } p \leftarrow 1 \textbf{ to } m \\
\quad \textbf{do } \left\{ \begin{array}{l}
S_p^0 = \emptyset \\
\textbf{for } k \leftarrow 2 \textbf{ to } n \\
\quad \textbf{do } \left\{ S_p^k \leftarrow S_p^{k-1} \cup \{\sigma_k^{p-1}\} \right.
\end{array} \right. \\
i \leftarrow 1 \\
\textbf{for } k_1 \leftarrow 1 \textbf{ to } n \\
\quad \textbf{do } \left\{ \begin{array}{l}
\textbf{for } k_2 \leftarrow 1 \textbf{ to } k_1 \\
\quad \textbf{do } \left\{ \begin{array}{l}
\bar{S}_1 = S_1^{k_1} \setminus S_2^{k_2} \\
\dots \\
\quad \textbf{for } k_m \leftarrow 1 \textbf{ to } k_{m-1} \\
\quad\quad \textbf{do } \left\{ \begin{array}{l}
\bar{S}_{m-1} \leftarrow S_{m-1}^{k_{m-1}} \setminus S_m^{k_m} \\
\bar{S}_m \leftarrow S_m^{k_m} \\
\text{Construct assortment } S^i = (\bar{S}_1, \dots, \bar{S}_m) \\
\text{Calculate } \pi(S^i) \\
i \leftarrow i + 1
\end{array} \right.
\end{array} \right.
\end{array} \right.
\end{array}
$$

Algorithm 1 enumerates all assortments that satisfy Theorem 4, that is, the aggregate assortment $\cup_{i>p} S_i$ for a service type $p \in M$ in a candidate assortment includes products with highest $\bar{R}(p)$. Since we enumerate all candidates where the assortments of m service types are ordered, Algorithm 1 has complexity of $O(n^m)$ which means it is polynomial time for a fixed m.

How the algorithm works is demonstrated in the following example.

Example 1. The revenue and preference weights of products are given in Tables 4.1 and 4.2. Let $v_0 = 1$.

Table 4.3 shows how the assortments are constructed according to Algorithm 1. The optimal assortment is $S_1 = \{3\}$, $S_2 = \emptyset$ and $S_3 = \{1, 2\}$ with an expected revenue of 0.778.

Table 4.1: **Revenues associated with products**

	1	**2**	**3**
S_1	1.61	1.24	1.09
S_2	1.45	1.12	0.98
S_3	1.37	1.06	0.93

Table 4.2: **Preference weights associated with products**

	1	**2**	**3**
S_1	0.24	0.70	0.53
S_2	0.29	0.84	0.64
S_3	0.43	1.27	0.95

Table 4.3: **Algorithm 1**

$S_1^{k_1}$	$S_2^{k_2}$	$S_3^{k_3}$	$\bar{S}_1$	$\bar{S}_2$	$\bar{S}_3$	$\pi(S^i)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0
	$\emptyset$	$\emptyset$	$\{1\}$	$\emptyset$	$\emptyset$	0.312
$\{1\}$	$\{1\}$	$\emptyset$	$\emptyset$	$\{1\}$	$\emptyset$	0.326
	$\{1\}$	$\{1\}$	$\emptyset$	$\emptyset$	$\{1\}$	0.412
	$\emptyset$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\emptyset$	0.647
	$\{1\}$	$\emptyset$	$\{2\}$	$\{1\}$	$\emptyset$	0.647
$\{1,2\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\emptyset$	$\{1\}$	0.684
	$\{1,2\}$	$\emptyset$	$\emptyset$	$\{1,2\}$	$\emptyset$	0.639
	$\{1,2\}$	$\{1\}$	$\emptyset$	$\{2\}$	$\{1\}$	0.674
	$\{1,2\}$	$\{1,2\}$	$\emptyset$	$\emptyset$	$\{1,2\}$	0.717
	$\emptyset$	$\emptyset$	$\{1,2,3\}$	$\emptyset$	$\emptyset$	0.742
	$\{1\}$	$\emptyset$	$\{2,3\}$	$\{1\}$	$\emptyset$	0.741
	$\{1\}$	$\{1\}$	$\{2,3\}$	$\emptyset$	$\{1\}$	0.765
	$\{1,2\}$	$\emptyset$	$\{3\}$	$\{1,2\}$	$\emptyset$	0.729
$\{1,2,3\}$	$\{1,2\}$	$\{1\}$	$\{3\}$	$\{2\}$	$\{1\}$	0.753
	$\{1,2\}$	$\{1,2\}$	$\mathbf{\{3\}}$	$\emptyset$	$\mathbf{\{1,2\}}$	**0.778***
	$\{1,2,3\}$	$\emptyset$	$\emptyset$	$\{1,2,3\}$	$\emptyset$	0.718
	$\{1,2,3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$	$\{1\}$	0.741
	$\{1,2,3\}$	$\{1,2\}$	$\emptyset$	$\{3\}$	$\{1,2\}$	0.767
	$\{1,2,3\}$	$\{1,2,3\}$	$\emptyset$	$\emptyset$	$\{1,2,3\}$	0.772

4.2.2 Algorithm 2

In this section, we provide an alternative approach by modifying the algorithm proposed by Rusmevichientong et al. [20]. This algorithm is originally developed to solve the capacitated assortment optimization problem under MNL; however, we show a similar approach can be used to solve the assortment optimization problem with multiple service types. Rusmevichientong et al. [20] exploit the geometry of lines in the plane to develop a geometric algorithm which illustrates how the optimal assortment changes with parameter v. The geometry associated with their algorithm gives insight on the sequence the products should be added to the assortment without exceeding the capacity to find the optimal assortment; thus, the problem can be solved in polynomial time. To solve our problem, we use a similar geometry which shows the service type a product should be offered with to maximize the expected profit. However, since there is no capacity constraint in our problem, the sequence of the products do not matter and we add the products until the expected profit decrease.

To construct the algorithm, we express the optimal expected profit Z^* of Problem (3.1) as follows:

$$\begin{aligned} Z^* &= \max\{\lambda \in \mathbb{R} : \exists S = (S_1, \ldots, S_m), \cup_{i\in M} S_i \subset N, \cap_{i\in M} S_i = \emptyset \text{ and } \pi(S) \geq \lambda\} \\ &= \max\{\lambda \in \mathbb{R} : \exists S = (S_1, \ldots, S_m), \cup_{i\in M} S_i \subset N, \cap_{i\in M} S_i = \emptyset \text{ and} \\ &\sum_{i\in M}\sum_{k\in S_i} v_{ik}(r_{ik} - \lambda) \geq v_0\lambda\}, \end{aligned}$$

The second equality follows from the definition of $\pi(S)$ in (1). We define functions $A : \mathbb{R} \to \{S = (S_1, \ldots, S_m) : \cup_{i\in M} S_i \subset N, \cap_{i\in M} S_i = \emptyset\}$ and $g : \mathbb{R} \to \mathbb{R}$ where $A(\lambda) = \arg\max_{\cap_{i\in M} S_i = \emptyset} \sum_{i\in M}\sum_{k\in S_i} v_{ik}(r_{ik} - \lambda)$ and $g(\lambda) = \sum_{i\in M}\sum_{k\in A_i(\lambda)} v_{ik}(r_{ik} - \lambda)$. To find the optimal assortment, we need to enumerate $A(\lambda)$ for all $\lambda \in \mathbb{R}$. For $i = 1, \ldots, m$ and $k = 1, \ldots, n$, we define linear functions $h_{ik} : \mathbb{R} \to \mathbb{R}$ by $h_{ik}(\lambda) = v_{ik}(r_{ik} - \lambda)$ and $h_0 : \mathbb{R} \to \mathbb{R}$ by $h_0(\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Note that, i denotes the service type and k denotes the product in this representation. Enumeration of intersection points among the $h_{ik}(.)$ of product k for all $k \in N$ and the intersection points among h_{ik} and h_0 for all $i \in M$ and $k \in N$ corresponds to the enumeration of $A(\lambda)$ for all $\lambda \in \mathbb{R}$. The number of the intersection points is at most $N \cdot \left(\binom{M}{2} + M\right)$. Therefore, the collection of assortments $\{A(\lambda) : \lambda \in \mathbb{R}\}$ has at most $O(N)$ sets and the algorithm has complexity of $O(N)$.

For $0 < k \leq n$, observe that $R_k(i,j)$ corresponds to the x-coordinate of the intersection point of the lines $h_{ik}(.)$ and $h_{jk}(.)$, that is, $h_{ik}(R_k(i,j)) = h_{jk}(R_k(i,j)) \iff R_k(i,j) = \dfrac{v_{ik}r_{ik} - v_{jk}r_{jk}}{v_{ik} - v_{jk}}$. Let A^p denote the assortment in the interval $(R_p(i,j), R_{p+1}(i,j))$. Let A_i^p be the assortment of service type i, that is, $A^p = (A_1^p, \ldots, A_m^p)$.

For $p = 1, \ldots, P$, we construct A_i^p in the following way: for $k = 1, \ldots, n$, if for some i, $h_{ik}(\lambda) > h_{jk}(\lambda)$ for all $j \in M \setminus \{i\}$, we add product k to A_i^p. If $h_{ik}(\lambda) \leq 0$ for all $i \in M$, we do not include product k in the assortment. The sequence of assortments $\mathcal{A} = \{A^p : p = 0, \ldots, P\}$ found that way corresponds to $A(\lambda)$ for all $\lambda \in R$ and the one with the highest expected profit is the optimal solution. The formal description of the algorithm is given below.

Algorithm 4.2.2: <ALGORITHM 2>($< h_{ik}, R_k(i,j) >$)

Inputs: h_{ik} and the intersection points $R_k(i,j)$ $\forall i \in M$, $\forall k \in N$
Outputs: The sequence of assortments $\mathcal{A} = \{A^p : p = 0, \ldots, P\}$
for $p \leftarrow 0$ **to** P

$$\textbf{do} \begin{cases} \textbf{for } k \leftarrow 1 \textbf{ to } n \\ \quad \textbf{do} \begin{cases} \textbf{if } \exists\, i \ni h_{ik}(\lambda) > h_{jk}(\lambda) \geq 0 \quad \forall j \in M \setminus \{i\}\ \forall\, \lambda \in (R_p(i,j), R_{p+1}(i,j)) \\ \quad \textbf{then } \left\{ A_i^p \leftarrow A_i^p \cup \{k\}. \right. \end{cases} \end{cases}$$

We give the following example to illustrate how the algorithm works.

Example 2. We use the example given for Algorithm 1. The revenue and preference weights of products are given in Tables 4.1 and 4.2. Let $v_0 = 1$. The corresponding lines h_{ik} are shown in Figure 4.1 and the sequence of assortments are given in Table 4.4. Enumerating the assortments we obtained from Figure 4.1, we find that the optimal solution of the problem is $S_1 = \{3\}$,$S_2 = \emptyset$ and $S_2 = \{1,2\}$ with an expected revenue of 0.778.

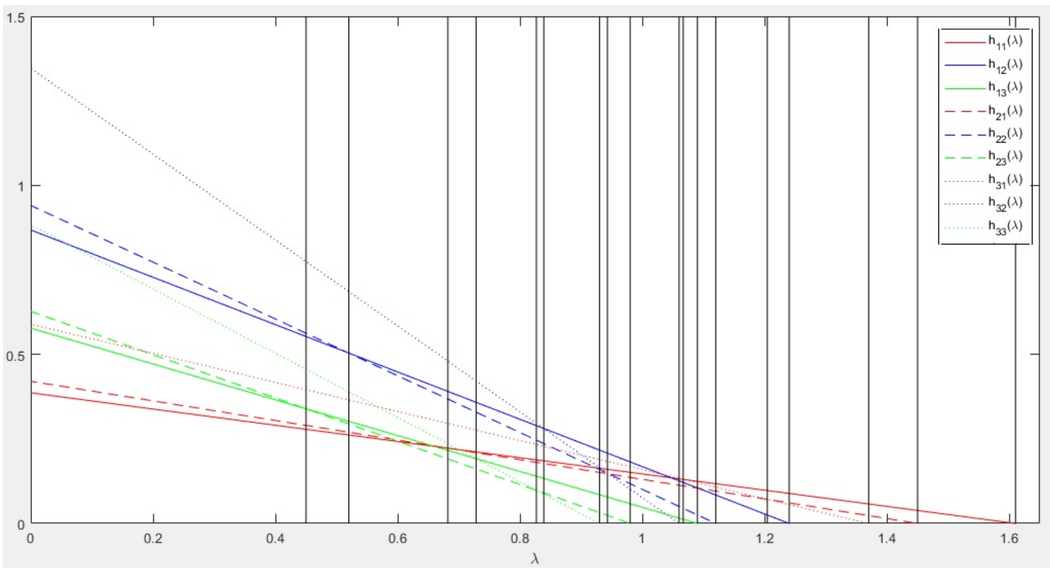

Figure 4.1: h_{ik} lines

Table 4.4: **Sequence of assortments**

S_1	S_2	S_3	**Expected Revenue**
$\emptyset$	$\emptyset$	{1,2,3}	0.772
{3}	$\emptyset$	{1,2}	**0.778**
{1}	$\emptyset$	{2,3}	0.753
{2}	$\emptyset$	{1}	0.684
{2}	{1}	$\emptyset$	0.647
{1,2}	$\emptyset$	$\emptyset$	0.646
{1}	$\emptyset$	$\emptyset$	0.312

4.2.3 Further Insights on the Optimal Assortment

In this section, we find conditions for a service type not to be used. The conditions are derived for the following structures of revenue and preference weight:

1. $v_{ik} = v_{1k} \times \alpha_i$, $r_{ik} = r_{1k} \times \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 1$ and $1 > \beta_1 > \cdots > \beta_m > 0$.

2. $v_{ik} = v_{1k} \times \alpha_i$, $r_{ik} = r_{1k} - \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 1$ and $\beta_m > \cdots > \beta_1 > 0$.

3. $v_{ik} = v_{1k} + \alpha_i$, $r_{ik} = r_{1k} \times \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 0$ and $1 > \beta_1 > \cdots > \beta_m > 0$.

4. $v_{ik} = v_{1k} + \alpha_i$, $r_{ik} = r_{1k} - \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 0$ and $\beta_m > \cdots > \beta_1 > 0$.

In Proposition 9, we derive the condition for the second case. The conditions for the other cases can be found in Appendix A.

Proposition 5. *We consider the assortment of service type $p \in M$. If there exists service types $i, j \in M$ such that, $i > p > j$ which satisfy $\frac{\alpha_j\beta_j - \alpha_p\beta_p}{\alpha_p - \alpha_j} > \frac{\alpha_p\beta_p - \alpha_i\beta_i}{\alpha_i - \alpha_p}$, the optimal assortment for service type p is empty set, i.e., the retailer should not offer service type p.*

Proof. According to Lemma 2 and Lemma 3, for $k \in S_p^*$, we must have $\max_{i\in M: i>p}\{\frac{v_{il}r_{il} - v_{pl}r_{pl}}{v_{il} - v_{pl}}\} \leq \pi(S^*)$ and $\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} \geq \pi(S^*)$ for all $j = 1, \ldots, p-1$. Therefore, $\{\frac{v_{il}r_{il} - v_{pl}r_{pl}}{v_{il} - v_{pl}}\} > \frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}}$ for some $i > p > j \in M$ implies that

$S_p^* = \emptyset$. Arranging the terms, we obtain,

$$\begin{aligned}
\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} &> \frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}} && \implies S_p^* = \emptyset \\
\frac{v_{1k}\alpha_p(r_{1k} - \beta_p) - v_{1k}\alpha_j(r_{1k} - \beta_j)}{v_{1k}(\alpha_p - \alpha_j)} &> \frac{v_{1k}\alpha_i(r_{1k} - \beta_i) - v_{1k}\alpha_p(r_{1k} - \beta_p)}{v_{1k}(\alpha_i - \alpha_p)} && \implies S_p^* = \emptyset \\
\frac{v_{1k}(\alpha_p r_{1k} - \alpha_p\beta_p - \alpha_j r_{1k} + \alpha_j\beta_j)}{v_{1k}(\alpha_p - \alpha_j)} &> \frac{v_{1k}(\alpha_i r_{1k} - \alpha_i\beta_i - \alpha_p r_{1k} + \alpha_p\beta_p)}{v_{1k}(\alpha_i - \alpha_p)} && \implies S_p^* = \emptyset \\
\frac{\alpha_j\beta_j - \alpha_p\beta_p}{\alpha_p - \alpha_j} &> \frac{\alpha_p\beta_p - \alpha_i\beta_i}{\alpha_i - \alpha_p} && \implies S_p^* = \emptyset.
\end{aligned}$$

□

Note that, since $v_{1k} = v_{1k} \times \alpha_1$ and $r_{1k} = r_{1k} - \beta_1$ in Case 2, we have $\alpha_1 = 1$ and $\beta_1 = 0$. Using that, we give the condition for the second service type to be empty in the problem with three service types in Corollary 5.1.

Corollary 5.1. *If $\alpha_2, \alpha_3 > 1$ and $\beta_2, \beta_3 > 0$ satisfy $\frac{-\alpha_2\beta_2}{\alpha_2 - 1} > \frac{\alpha_2\beta_2 - \alpha_3\beta_3}{\alpha_3 - \alpha_2}$, then the assortment of Service Type 2 is empty set, i.e., the retailer should not offer any products with Service Type 2.*

In the next chapter, we consider the same problem with the inclusion of fixed cost. We provide mathematical formulations and analyze the complexity of several cases.

Chapter 5

Assortment Optimization with Premium Services with Fixed Cost

5.1 Mathematical Formulations

In this section, we provide a mixed integer linear programming and a conic formulation to solve assortment optimization problem with premium services and fixed cost.

5.1.1 Mixed Integer Linear Programming Formulation

We show that we can formulate Problem (3.2) as a MILP. First, we define the following decision variables:

$$x_{ik} = \begin{cases} 1, & \text{if product } k \text{ is offered with service type i,} \\ 0, & \text{otherwise.} \end{cases}$$

As defined in Chapter 2, let c_{ik} denote the fixed cost of including product k in the assortment of service type i. Then, the problem can be formulated as:

$$\begin{aligned} \max \quad & \frac{\sum_{i\in M}\sum_{k\in N} r_{ik}v_{ik}x_{ik}}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}} - \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik} \\ \text{s.t.} \quad & \sum_{i\in M} x_{ik} \le 1, \quad \forall k \in N, \\ & x_{ik} \in \{0,1\}, \quad \forall i \in M, \quad \forall k \in N. \end{aligned} \tag{5.1}$$

To formulate this problem as a MILP, we define the following variable:

$$y = \frac{1}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}},$$

Substituting y in Problem (5.1), we obtain the following nonlinear program,

$$\begin{aligned} \max \quad & \sum_{i\in M}\sum_{k\in N} r_{ik}v_{ik}x_{ik}y - \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik} \\ \text{s.t.} \quad & \sum_{i\in M} x_{ik} \le 1, \quad \forall k \in N, \\ & v_0 y + \sum_{i\in M}\sum_{k\in N} r_{ik}v_{ik}x_{ik}y = 1, \\ & x_{ik} \in \{0,1\}, \quad \forall i \in M, \quad \forall k \in N, \\ & y \ge 0. \end{aligned} \tag{5.2}$$

Problem (5.2) has a nonlinear objective function. To linearize its objective function, we define bilinear variables $z_{ik} = x_{ik}y$ for all $i \in M$, $k \in N$ and we obtain the following MILP which is equivalent to Problem (5.2):

$$
\begin{aligned}
\max \quad & \sum_{i \in M} \sum_{k \in N} r_{ik} v_{ik} z_{ik} - \sum_{i \in M} \sum_{k \in N} c_{ik} x_{ik} && (5.3)\\
\text{s.t.} \quad & \sum_{i \in M} x_{ik} \leq 1, \quad \forall k \in N, \\
& v_0 y + \sum_{i \in M} \sum_{k \in N} r_{ik} v_{ik} z_{ik} = 1, \\
& y - z_{ik} \leq M(1 - x_{ik}), \quad \forall i \in M, \quad \forall k \in N, \\
& 0 \leq z_{ik} \leq y, \quad \forall i \in M, \quad \forall k \in N, \\
& z_{ik} \leq M x_{ik}, \quad \forall i \in M, \quad \forall k \in N, \\
& x_{ik} \in \{0, 1\}, \quad \forall i \in M, \quad \forall k \in N, \\
& y \geq 0.
\end{aligned}
$$

Although we can formulate the assortment optimization problem with premium services as a LP when there is no cost, we need to solve a MILP to find the optimal solution of the problem with fixed cost. Therefore, assuming the firm incurs a fixed cost by including products in the assortment complicates the problem considerably.

5.1.2 Conic Formulation

Şen et al. [44] introduce a conic quadratic mixed-integer formulation for the assortment optimization problem under MMNL. Using a similar approach, we provide a conic formulation for the assortment optimization problem with premium services under MNL. Similar to our problem, their objective is to choose the assortment that maximizes the expected profit; however, they do not consider assortments of multiple service types. To formulate the problem as a second-order cone problem, they restate the objective function as minimization. In the same way, we transform our model into a minimization problem. For that purpose, we define $\bar{r} = \sum_{i \in M} \sum_{k \in N} r_{ik}$. Observe that, we can restate the objective function

(5.1) as follows:

$$\max \quad \frac{\sum_{i\in M}\sum_{k\in N} r_{ik}v_{ik}x_{ik}}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}} + \bar{r} - \bar{r} - \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik}.$$

$$\max \quad \bar{r} - \left(\frac{\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}(\bar{r}-r_{ik}) + v_0\bar{r}}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}} + \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik} \right). \tag{5.4}$$

Since the first component in (5.4) is a constant, the following expression is equivalent to (5.4),

$$\min \quad \frac{\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}(\bar{r}-r_{ik}) + v_0\bar{r}}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}} + \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik}. \tag{5.5}$$

Then, we obtain the following nonlinear minimization problem,

$$\min \quad \frac{\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}(\bar{r}-r_{ik}) + v_0\bar{r}}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}} + \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik} \tag{5.6}$$

$$\text{s.t.} \quad \sum_{i\in M} x_{ik} \le 1, \quad \forall k \in N, \tag{5.7}$$

$$x_{ik} \in \{0,1\}, \quad \forall i \in M, \quad \forall k \in N. \tag{5.8}$$

To linearize the objective function, we define the decision variables y and z_{ik} as we did for the MILP formulation and obtain the following minimization problem,

$$\min \quad \sum_{i\in M}\sum_{k\in N} v_{ik}z_{ik}(\bar{r}-r_{ik}) + v_0\bar{r}y + \sum_{i\in M}\sum_{k\in N} c_{ik}x_{ik} \tag{5.9}$$

$$\text{s.t.} \quad \sum_{i\in M} x_{ik} \le 1, \quad \forall k \in N, \tag{5.10}$$

$$y \ge \frac{1}{v_0+\sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik}}, \tag{5.11}$$

$$z_{ik} \ge x_{ik}y, \quad \forall i \in M, \quad \forall k \in N, \tag{5.12}$$

$$x_{ik} \in \{0,1\}, \quad \forall i \in M, \quad \forall k \in N, \tag{5.13}$$

$$y \ge 0, \tag{5.14}$$

$$z_{ik} \ge 0, \quad \forall i \in M, \quad \forall k \in N. \tag{5.15}$$

Since the objective function is minimization, constraints (5.11) and (5.12) are satisfied at equality in an optimal solution. We define,

$$w = v_0 + \sum_{i\in M}\sum_{k\in N} v_{ik}x_{ik} \tag{5.16}$$

and write (5.11) as a hyperbolic constraint:

$$yw \geq 1. \tag{5.17}$$

Observe that, $w, y \geq 0$ and constraint (5.17) is satisfied at equality in an optimal solution. Since x_{ik} are binary variables, we also have $x_{ik} = x_{ik}^2$. Then, we can write (5.17) as hyperbolic constraints:

$$z_{ik} w \geq x_{ik}^2 \quad \forall i \in M, \quad \forall k \in N. \tag{5.18}$$

We also add the constraint,

$$v_0 y + \sum_{i \in M} \sum_{k \in N} v_{ik} z_{ik} \geq 1. \tag{5.19}$$

Finally, we obtain the following conic quadratic 0-1 mixed formulation:

$$\begin{aligned}
\min \quad & \sum_{i \in M} \sum_{k \in N} v_{ik} z_{ik} (\bar{r} - r_{ik}) + v_0 \bar{r} y + \sum_{i \in M} \sum_{k \in N} c_{ik} x_{ik} && (5.20) \\
\text{s.t.} \quad & \sum_{i \in M} x_{ik} \leq 1, \quad \forall k \in N, && (5.21) \\
& w = v_0 + \sum_{i \in M} \sum_{k \in N} v_{ik} x_{ik}, && (5.22) \\
& yw \geq 1, && (5.23) \\
& z_{ik} w \geq x_{ik}^2, \quad \forall i \in M, \quad \forall k \in N, && (5.24) \\
& v_0 y + \sum_{i \in M} \sum_{k \in N} v_{ik} z_{ik} \geq 1, && (5.25) \\
& x_{ik} \in \{0, 1\}, \quad \forall i \in M, \quad \forall k \in N, && (5.26) \\
& y \geq 0, && (5.27) \\
& z_{ik} \geq 0, \quad \forall i \in M, \quad \forall k \in N. && (5.28)
\end{aligned}$$

5.2 Analysis of the Problem

In this section, we analyze two cases of the fixed cost problem. In the first case, we assume the fixed cost is product specific, that is, each product has a fixed cost associated with it which also depends on the service type the product is offered with. In the second case, the service type is the same for all products and service types.

5.2.1 Product Specific Fixed Cost

Kunnumkal et al. [19] study the assortment optimization problem under MNL with product costs and show that it is NP-complete. Their model does not consider multiple service types; however, we can make a reduction from their model to our problem with service types. Let the traditional assortment optimization problem with fixed cost be defined as follows.

Let the set of products be $N = \{1, \ldots, n\}$ and $\bar{r}_k$ and $\bar{v}_k$ be the profit and preference weight associated with product k for all $k \in N$, respectively. $\bar{v}_0$ denotes the preference weight of the no purchase option. The objective is to find the assortment S that maximizes the expected revenue. We construct the problem with multiple services in the following way. Let $M = \{1, \ldots, m\}$ be the set of service types and $N = \{1, \ldots, n\}$ be the set of products. We set $v_{ik} = \bar{v}_k$ and $r_{ik} = \bar{r}_k$ for all $i \in M$, $k \in N$. Since a product can be included in the assortment of at most one service type and we have no differences between the service types, the problem becomes equivalent to the problem with a single service type. Therefore, the problem with multiple service types with different product costs is NP-complete.

In the next section, we analyze the case where the product costs are the same for all products and service types.

5.2.2 Cardinality Based Fixed Cost

In this section, we assume the fixed cost of including a product in the assortment of a service type is the same for all products and service types, that is, $c_{ik} = c$ for all $i \in M$ and $k \in N$. Thus, the incurred cost only depends on the cardinality of the offered assortment. We show that although the problem with product specific fixed costs is NP-complete, we can solve the problem with cardinality based fixed cost in polynomial time. Since the cost is dependent upon the number of products in the total assortment, Proposition 6 establishes a connection between the cardinality constrained assortment optimization problem and assortment optimization problem with cardinality based fixed cost.

In the cardinality constrained assortment optimization problem, the objective is to find the assortment that maximizes the expected profit subject to a cardinality constraint on the number of products in the assortment. Let π^K denote the maximum expected profit of the cardinality constrained assortment optimization problem that enforces K products are included in the assortment. The problem is as follows

$$\pi^K = \max_{S:|S|\leq K, S_i\cap S_j=\emptyset \forall i,j\in M: i\neq j} \pi(S) = \max_{S:|S|\leq K, S_i\cap S_j=\emptyset \forall i,j\in M: i\neq j} \frac{\sum_{i\in M}\sum_{k\in S_i} r_{ik}v_{ik}}{v_0+\sum_{i\in M}\sum_{k\in S_i} v_{ik}},$$

where $|S| = |S_1| + \cdots + |S_m|$ for assortment $S = (S_1, \ldots, S_m)$. Let π^F be the maximum expected profit of the problem where fixed cost depends on the cardinality of assortment.

Proposition 6. *Let S^K denote the optimal assortment of problem that enforces K products are included in the assortment.* $\pi^F = \max_{K\leq |N|}\{\pi^K - |S^K|c\}$.

Proof. Let S' be any other feasible solution to the problem which has a cardinality constraint that enforces K products are included in the assortment and $\pi^K(S')$ denote its expected revenue. Since assortments S^K and S' have the same fixed cost and $\pi^K > \pi^K(S')$, $\pi^K(S') - |S'|c$ cannot be the optimal solution. $\square$

π^K can be formulated by adding a cardinality constraint to Problem 4.1. Observe that, cardinality constraint also yields an interval matrix, since the constraint matrix has consecutive ones in each row. As stated by Davis et al. [18], if we combine two interval matrices, total unimodularity is preserved. Thus, total unimodularity in the constraint matrix is maintained in Problem 4.1 when we add the cardinality constraint and Problem 4.1 with the cardinality constraint can be formulated as a linear program as well. Therefore, Proposition 6 implies that π^F can be found by solving $|N|$ linear programs in polynomial time.

Chapter 6

Numeric Results

In this chapter, we make numeric experiments with the problem with premium services without fixed cost, which is discussed in Chapter 4. We analyze how the optimal assortment changes with respect to the amount of increase in preference weight and decrease in revenue if products are offered with higher service types.

We make numeric experiments with problems which have two service types and three service types. In the problem with two service types, let Service Type 1 and Service Type 2 be the products offered with regular and premium delivery, respectively. Similarly, in the problem with three service types, let Service Type 1 be the regular delivery and Service Types 2 and 3 be premium options, 3 being faster than 2. We generate the data for revenue and preference weight in the following way: The revenue that the products have in the regular service type is generated from a uniform distribution between 5 and 15. Similarly, the preference weight of products in the regular service type is generated from a uniform distribution between 0 and 1. To generate the revenue and preference weight of the products when they are offered with the premium service types, we use the following structure given in Section 4.2.3. The parameters we used in this section can be found in https://github.com/iremakcakus/data.git.

$$v_{ik} = v_{1k} \times \alpha_i, \quad i \in M, \quad \forall k \in N,$$
$$r_{ik} = r_{1k} - \beta_i, \quad i \in M, \quad \forall k \in N.$$

In this structure, the changes in the revenue and preference weight are the same for all products. This can be explained by all product being similar to each other; thus, the increase in popularity and decrease in revenue is the same for all of them. In other words, we have a homogeneous products set which consists of products that appeal to customers the same way when they are offered with faster delivery. An example for this kind product set is immediate need products.

Numeric experiments are conducted with two sets of problems. The sets consist of 100 randomly generated problem instances with 10 and 30 products, respectively. We test these sets of problems with two different preference weight for the no-purchase option, 1 and 3. We also solve the problem with two service types for 8 different α and β values; thus, we provide results for 64 (α, β) pairs. We analyze how the optimal solution changes with respect to parameters, using the following measures:

1. Assortment size: For each set of parameters, we report what percent of the products are included in the assortment.

2. Distribution of products: We analyze how the products in the assortment are distributed between different service types.

3. Fill Rate: Fill rate shows what percent of the demand is satisfied. We calculate the fill rate using the following formula,

$$\frac{\sum_{i \in M} \sum_{k \in S_i} v_{ik}}{v_0 + \sum_{i \in M} \sum_{k \in S_i} v_{ik}}.$$

4. Difference in Fill Rate: We calculate the fill rate for the problem which only has the regular service type and find what percent fill rate changes when we offer the premium service type.

5. Expected Revenue: We report the expected revenue for each problem.

6. Difference in Expected Revenue: The change in the expected revenue when the premium service type is offered compared to the problem which only has the regular service type.

The graphs given in Figures B.1-B.9 in Appendix show how these measures change with respect to the parameters in the problem with two service types. In the color maps, the lighter colors indicate higher values. Also, Tables B.1-B.24 in Appendix give the minimum, maximum and average value of each measure in the 100 samples solved for each (α, β) pair. Using these results, we examine the impacts of no-purchase option, popularity and cost.

The behavior of the optimal solution in terms of the mentioned measures is similar for the problem with three service types. Thus, in our experiments with three service types, we only analyze which service types are used in the optimal assortment in 100 runs for each (α, β) pair. The problem is solved for five α and two β values. We also report the average number of products in each service type for each (α, β) pair in Tables B.25 and B.26 in Appendix.

Numerical experiments are done by solving the LP formulation given in Chapter 4.1. Since the problem instances are small, LP is solved in a few seconds. To solve the LP, IBM CPLEX Optimization Studio V12.8 is used. The problems are solved in a computer with an Intel(R) Core(TM) i7-4510U CPU @2.00GHz 2.60 GHz processor, 8 GB RAM and Windows 10 64-bit Operating System.

6.1 Impact of No-Purchase Option

No-purchase preference represents the option that the consumer do not purchase anything from the assortment. In our model, the no-purchase option is assumed to be independent of the products in the assortment. However, it includes the cases where a consumer decides not to purchase anything to look for a better product in other stores. Therefore, a higher no-purchase preference indicates a higher preference for the products offered by other retailers.

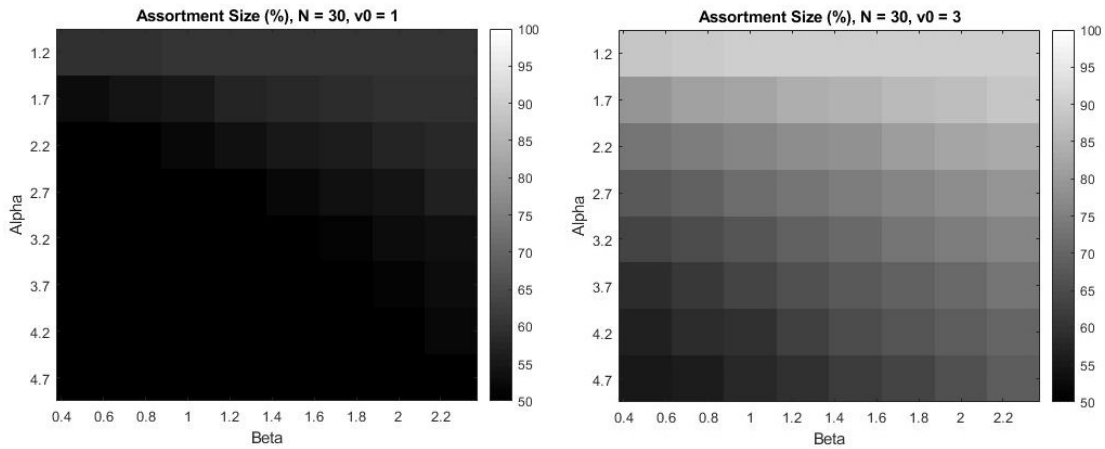

Figure 6.1: Change in assortment size with respect to v_0

Figure 6.1 shows the change in the assortment size when v_0 increases. In the graphs, lighter colors indicate a higher percentage. When v_0 is small, there is a greater chance that a customer will make a purchase from the assortment. Thus, the retailer offers a narrower assortment which includes products with higher revenue. However, as v_0 gets larger, the appeal of the products in other stores increases. In this case, the retailer offers a wider assortment with more popular products to attract the customers. Therefore, large v_0 values result in assortments with greater product variety.

As v_0 increases, the distribution of products in the assortment between service types changes in favor of the premium service type. This alteration is also the result of the need to offer more popular products in the assortment to appeal to the customers. Figure 6.2 shows the change in the percent of the products offered with the premium service. It can be observed that the percent of products offered with the premium service type at higher cost values (higher β values) increases. This indicates that the retailer becomes more willing to offer products with the premium service to attract customers, although it brings lower profit.

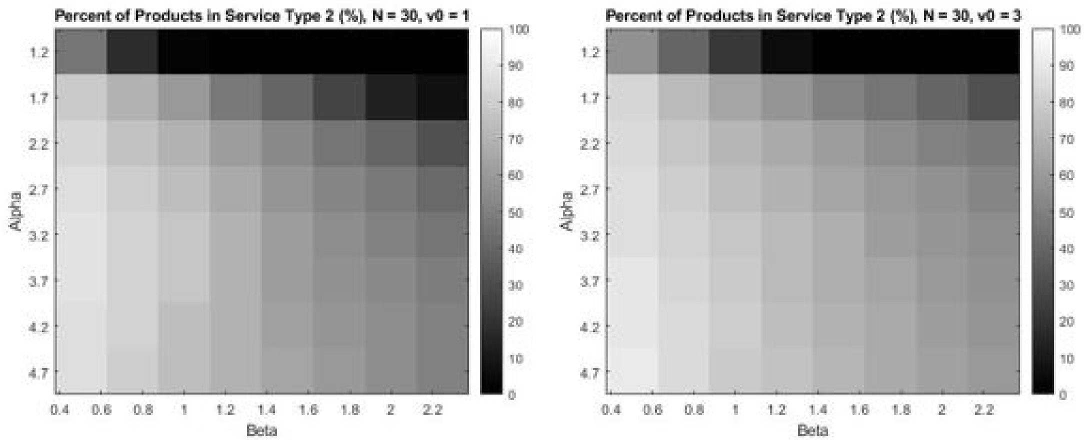

Figure 6.2: Percent of products offered with premium service with respect to v_0

6.2 Impact of Popularity

The increase in the preference weight, i.e., popularity, of the products in the premium service type has the opposite effect from v_0. As the preference weight of the products with premium service type increases, the customers are more likely to purchase. As a result, the retailer can keep a smaller assortment. We examine the impact of the premium service type's popularity by increasing the α value. Table 6.1 shows how the assortment size changes as we increase α at $\beta = 1.5$. Figure 6.1 also demonstrates the change in the assortment size with respect to α.

In Table 6.1, we can also observe the trade-off between α and β values, which shows the complexity of the problem. At $\alpha = 1.2$, although the products gain popularity if they are offered with Service 2, the optimal solution does not use Service 2 in any of the 100 samples. This shows that the gain in the popularity does not balance the incurred cost, which is represented by β. However, as α gets larger, offering majority of the products with Service Type 2 becomes more profitable.

In some instances, the decrease in the assortment size affects the fill rate negatively. Table 6.2 displays the change in the fill rate, using the example in Table

Table 6.1: **Change in assortment size and distribution of products with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
1.2	1.5	30.0	90.0	60.2	100.0	100.0	100.0	0.0	0.0	0.0
1.7	1.5	30.0	90.0	58.5	0.0	100.0	60.9	0.0	100.0	39.1
2.2	1.5	30.0	80.0	54.8	0.0	85.7	46.5	14.3	100.0	53.5
2.7	1.5	20.0	80.0	52.2	0.0	85.7	40.9	14.3	100.0	59.1
3.2	1.5	20.0	80.0	50.2	0.0	85.7	38.6	14.3	100.0	61.4
3.7	1.5	20.0	80.0	47.9	0.0	83.3	37.7	16.7	100.0	62.3
4.2	1.5	20.0	80.0	46.2	0.0	83.3	37.0	16.7	100.0	63.0
4.7	1.5	20.0	80.0	45.2	0.0	83.3	35.4	16.7	100.0	64.6

6.1. For $\alpha = 1.7, 2.2, 2.7$ and 3.2, difference in the fill rate is negative in some instances, which means that the fill rate is lower compared to the problem with only regular service type. This shows that, in some instances offering products with premium service type brings larger expected revenue, even though less of the demand is satisfied. Still, such cases do not occur frequently as the average change in the fill rate compared to the problem with single service type follows an increasing trend as α increases.

Table 6.2: **Difference in fill rate with respect to α, $N = 30$, $v_0 = 1$**

α	β	Fill Rate (%)			Difference in Fill Rate (%)		
		min	max	avg	min	max	avg
1.2	1.5	59.8	83.2	74.9	0.0	0.0	0.0
1.7	1.5	66.8	84.7	78.4	-10.5	17.1	4.97
2.2	1.5	70.2	86.8	81.1	-5.03	25.1	8.59
2.7	1.5	69.3	88.3	83.0	-7.05	30.2	11.2
3.2	1.5	72.9	89.6	84.5	-2.26	33.9	13.2
3.7	1.5	75.6	90.7	85.5	1.37	36.8	14.6
4.2	1.5	77.9	91.5	86.5	3.87	39.0	15.9
4.7	1.5	78.1	92.3	87.4	4.34	40.9	17.2

6.3 Impact of Cost

In our structure of the data, the cost of offering a product with the premium service is represented by β. We observe how the assortment size changes in Table 6.3. At $\alpha = 2.2$, as β decreases, the assortment size gets smaller. This is because as β decreases, the decline in revenue for service type 2 becomes trivial compared to the increase in popularity. Thus, the retailer is better off by offering a narrow assortment which includes products with higher revenue. Since the cost of offering products with service type 2 has less effect as β decreases, the percent of the products offered with service type 2 increases.

Table 6.3: **Change in assortment size and distribution of products with respect to β, $N = 30$, $v_0 = 1$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
2.2	2.25	30.0	90.0	58.1	20.0	100.0	68.3	0.0	80.0	31.7
	2.0	30.0	90.0	57.1	0.0	100.0	60.1	0.0	100.0	39.9
	1.75	30.0	90.0	56.1	0.0	100.0	53.9	0.0	100.0	46.1
	1.5	30.0	80.0	54.8	0.0	85.7	46.5	14.3	100.0	53.5
	1.25	20.0	80.0	53.2	0.0	85.7	37.8	14.3	100.0	62.2
	1.0	20.0	80.0	51.7	0.0	75.0	29.7	25.0	100.0	70.3
	0.75	20.0	80.0	50.6	0.0	71.4	23.7	28.6	100.0	76.3
	0.5	20.0	80.0	48.8	0.0	60.0	16.4	40.0	100.0	83.6

Table 6.4 shows the change in fill rate for a fixed α and varying β values. Similar to the analysis with varying α and fixed β values, fill rate declines in some instances when premium service is offered compared to the problem with a single service type. Yet, on average, fill rate increases when we offer the premium service and as β decreases.

Figure B.8 also reflects the change in the fill rate with respect to α and β values. The graph shows that the fill rate slightly increases as β decreases. The changes in fill rate with respect to β seem smooth, whereas, we can observe drastic changes as we vary α values at a fixed β. This shows that fill rate is not as sensitive to changes in β as it is to the changes in α.

Table 6.4: **Difference in fill rate with respect to β, $N = 30$, $v_0 = 1$**

α	β	Fill Rate (%)			Difference in Fill Rate (%)		
		min	max	avg	min	max	avg
	2.25	70.7	85.8	79.3	-5.03	24.3	6.19
	2.0	70.2	85.8	80.2	-5.03	24.3	7.39
	1.75	70.2	87.4	80.7	-5.03	24.3	7.98
2.2	1.5	70.2	86.8	81.1	-5.03	25.1	8.59
	1.25	68.0	87.6	81.4	-5.03	25.1	9.0
	1.0	64.8	88.4	81.8	-13.1	25.1	9.49
	0.75	64.8	88.4	82.1	-13.1	25.1	9.91
	0.5	64.8	89.6	82.2	-13.1	25.1	10.0

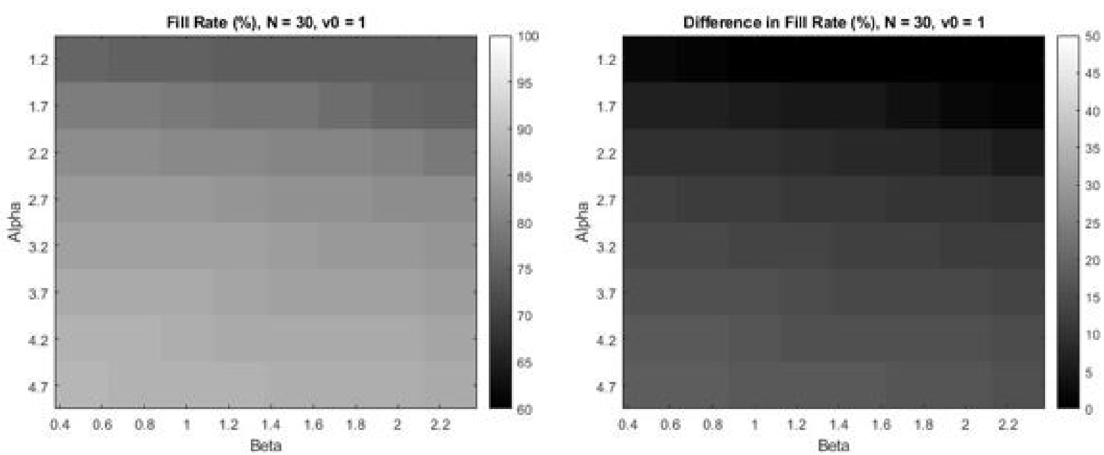

Figure 6.3: Fill Rate with respect to α and β

6.4 The Problem with Three Service Types

To generate the parameters of the problem with three service types, we take $\alpha_2 = \alpha_3 = \alpha$ and $\beta_2 = \beta_3 = \beta$. Tables 6.5 and 6.6 show the change in the service types used and average number of products in each service type in optimal solution with respect to α and β for $v_0 = 1$ and $v_0 = 3$.

In tables, it can be seen that when $\alpha = 1.2$ and $\beta = 1$, only the assortment of Service Type 1 is used in the majority of runs. This is because the increase in the popularity is very low compared to the decrease in revenue even if we deliver the products with the fastest option. In this case, the retailer does not benefit from offering products with faster delivery.

As α increases at $\beta = 1$, the retailer becomes more inclined to use faster delivery options, as the increase in popularity compensates the decrease in revenue. However, when we increase α from 2.4 to 2.8 when $v_0 = 1$, we observe a drop in the average number of products offered with Service Type 3 at both β values. This is caused by the decrease in the assortment size. Although the retailer is better off by offering more products with faster delivery when we increase α, the size of the total assortment decreases as explained in Section 6.1. Since the products become significantly more popular when we offer them with faster delivery, the retailer can capture the demand by offering less products.

When the no-puchase preference increases from 1 to 3, the probability that a customer will leave without any purchase increases. To attract more demand, the retailer offers more products with faster delivery, even when α is low compared to β. The number of instances that more than one service type is used increases. Therefore, when the preference weight of no-purchase option is high, the faster delivery options become more important for the retailer.

Table 6.5: **Change in the distribution of products to service types with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Service Types Used (Out of 100 runs)							Average # of Products		
		S_1	S_2	S_3	S_1, S_2	S_1, S_3	S_2, S_3	S_1, S_2, S_3	$\|S_1\|$	$\|S_2\|$	$\|S_3\|$
1.2	1.0	100.0	0.0	0.0	0.0	0.0	0.0	0.0	11.88	0.0	0.0
	0.5	11.0	0.0	0.0	35.0	19.0	0.0	35.0	9.56	1.18	1.09
1.6	1.0	1.0	0.0	0.0	61.0	2.0	0.0	36.0	8.41	2.55	0.64
	0.5	0.0	0.0	0.0	0.0	18.0	1.0	81.0	4.14	1.69	4.58
2.0	1.0	0.0	0.0	0.0	28.0	2.0	0.0	70.0	6.39	3.21	1.41
	0.5	0.0	0.0	0.0	0.0	19.0	3.0	78.0	3.2	1.71	4.44
2.4	1.0	0.0	0.0	0.0	12.0	3.0	0.0	85.0	5.25	3.3	1.73
	0.5	0.0	0.0	2.0	0.0	16.0	5.0	77.0	2.82	1.71	4.11
2.8	1.0	0.0	0.0	0.0	12.0	2.0	0.0	86.0	4.85	3.34	1.69
	0.5	0.0	0.0	1.0	0.0	16.0	5.0	78.0	2.38	1.8	3.72

Table 6.6: **Change in the distribution of products to service types with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Service Types Used (Out of 100 runs)							Average # of Products		
		S_1	S_2	S_3	S_1, S_2	S_1, S_3	S_2, S_3	S_1, S_2, S_3	$\|S_1\|$	$\|S_2\|$	$\|S_3\|$
1.2	1.0	64.0	0.0	0.0	35.0	0.0	0.0	1.0	17.92	0.56	0.02
	0.5	0.0	0.0	0.0	0.0	14.0	0.0	86.0	9.3	1.7	6.96
1.6	1.0	0.0	0.0	0.0	0.0	2.0	0.0	98.0	8.18	3.31	5.47
	0.5	0.0	0.0	0.0	0.0	24.0	1.0	75.0	4.27	1.43	9.46
2.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	99.0	6.08	3.25	5.94
	0.5	0.0	0.0	0.0	0.0	17.0	3.0	80.0	3.04	1.45	8.59
2.4	1.0	0.0	0.0	0.0	0.0	3.0	0.0	97.0	5.11	3.34	5.55
	0.5	0.0	0.0	2.0	0.0	21.0	5.0	72.0	2.37	1.65	7.55
2.8	1.0	0.0	0.0	0.0	0.0	4.0	0.0	96.0	4.44	3.31	5.2
	0.5	0.0	0.0	1.0	0.0	12.0	8.0	79.0	2.18	1.69	6.78

Chapter 7

Conclusion

Online retailing has been gaining more importance each year and outpacing the growth of sales within the traditional retail. Offering customers faster delivery makes online retail even more advantageous compared to the physical stores, as it provides immediate access to the products with more convenience. However, adopting faster delivery options requires more complex networks which increases the cost of products offered with faster delivery. Therefore, retailers face a trade-off between increased popularity and lower revenue while considering which products to offer with faster option. Moreover, the assortment offered by a retailer has a huge impact on sales and gross margin. Thus, assortment planning is a significant strategic decision for the retailers. The goal of the assortment planning is to find the optimal assortment which maximizes the expected revenue.

In this thesis, we consider the assortment optimization problem of an online retailer which offers multiple services to its customers. We assume that service types differ in delivery time and the products offered with a faster service type are preferred more than the products offered with regular delivery. On the other hand, the revenue gained from the products is higher if they are offered with regular service type rather than a faster service type. We assume that customers purchase products from the assortment of any service type according to the multinomial logit model (MNL); therefore, a product can be included in the assortment of at most one service type. Our objective is to find the optimal assortments for all service types which maximizes the expected revenue.

We consider two versions of the problem. In the first version, we do not consider a fixed cost for including a product in the assortment. We show that this problem can be formulated as a linear program. We also characterize the structure of the optimal assortment and based on the properties of the optimal solution, we construct a polynomial time algorithm. By modifying an existing polynomial time algorithm in the literature, we also propose an alternative solution method.

In the second version, we assume that the retailer incurs a fixed cost for including a product in the assortment. We provide a mixed integer linear and a mixed integer conic formulation to solve this problem. We also analyze the complexity of two special cases of the fixed cost, which we call product specific fixed cost and cardinality based fixed cost. If the cost structure is product specific, the cost of including a product in the assortment is different for each product. If the fixed cost is cardinality based, the retailer incurs the same amount of fixed cost for including each product in the assortment. We show that the problem with product specific fixed cost is NP-complete, whereas, the problem with cardinality based fixed cost can be solved in polynomial time.

We conduct numerical experiments with the problem without fixed cost and analyze how the optimal assortment changes with respect to the parameters of the problem, such as v_0, the popularity of the products if they are offered with faster delivery or the amount of decrease in their revenue. We use the following measures to assess the optimal assortment: assortment size, distribution of products among service types, fill rate and the change in the fill rate when premium service is offered and expected revenue and the increase in the expected revenue when premium service is offered.

According to the numeric experiments, we conclude that as v_0 increases, it is better to offer more product variety to attract the customers. We observe an opposite effect when α increases, as the chance that a customer will make purchase increases. Thus, the retailer offers narrower assortments which includes products with higher revenue as α increases. We also observe that the average fill rate increases as α gets larger; but, in some instances, offering the premium service type decreases the fill rate compared to the problem with a single service type. Decreasing β at a fixed α value shows similar results for assortment size and fill rate. However, the changes in fill rate with respect to β do not result in drastic changes, unlike the changes with respect to α.

The problem with fixed cost is a potential research area. Heuristics can be developed for the problem with product specific fixed cost. A third type of fixed cost structure can also be defined, where the fixed cost depends on the assortment that a product is included in. We attempted to find out the complexity of this case as well; but, failed to do so. Another possible research direction might be studying the assortment planning problem with multiple services with considerations of different customer segments which have different privileges in accessing products from different service types.

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Appendix A

Insights on Optimal Assortment

Different structures of preference weight and revenue:

1. $v_{ik} = v_{1k} \times \alpha_i$, $r_{ik} = r_{1k} \times \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 1$ and $1 > \beta_1 > \cdots > \beta_m > 0$.

2. $v_{ik} = v_{1k} \times \alpha_i$, $r_{ik} = r_{1k} - \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 1$ and $\beta_m > \cdots > \beta_1 > 0$.

3. $v_{ik} = v_{1k} + \alpha_i$, $r_{ik} = r_{1k} \times \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 0$ and $1 > \beta_1 > \cdots > \beta_m > 0$.

4. $v_{ik} = v_{1k} + \alpha_i$, $r_{ik} = r_{1k} - \beta_i$, $\forall i \in M$, $\forall k \in N$, where $\alpha_m > \cdots > \alpha_1 > 0$ and $\beta_m > \cdots > \beta_1 > 0$.

We derive the conditions for a service type not to be used for the first, third and forth cases.

Proposition 7. *Let the structure of preference weight and revenue be in the form of Case 1. We consider the assortment of service type $p \in M$. If there exists service types $i, j \in M$ such that, $i > p > j$ which satisfy $\frac{\alpha_p \beta_p - \alpha_j \beta_j}{\alpha_p - \alpha_j} > \frac{\alpha_i \beta_i - \alpha_p \beta_p}{\alpha_i - \alpha_p}$, the optimal assortment for service type p is empty set, i.e., the retailer should not offer service type p.*

Proof. According to Lemma 2 and Lemma 3, for $k \in S_p^*$, we must have $\max_{i \in M: i>p}\{\frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}\} \le \pi(S^*)$ and $\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} \ge \pi(S^*)$ for all $j = 1, \ldots, p-1$. Therefore, $\frac{\alpha_p\beta_p - \alpha_j\beta_j}{\alpha_p - \alpha_j} > \frac{\alpha_i\beta_i - \alpha_p\beta_p}{\alpha_i - \alpha_p}$ for some $i > p > j \in M$ implies that $S_p^* = \emptyset$. Arranging the terms, we obtain,

$$\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} > \frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}} \qquad \Longrightarrow S_p^* = \emptyset$$
$$\frac{v_{1k}\alpha_p r_{1k}\beta_p - v_{1k}\alpha_j r_{1k}\beta_j}{v_{1k}(\alpha_p - \alpha_j)} > \frac{v_{1k}\alpha_i r_{1k}\beta_i - v_{1k}\alpha_p r_{1k}\beta_p}{v_{1k}(\alpha_i - \alpha_p)} \qquad \Longrightarrow S_p^* = \emptyset$$
$$\frac{v_{1k}r_{1k}(\alpha_p\beta_p - \alpha_j\beta_j)}{v_{1k}(\alpha_p - \alpha_j)} > \frac{v_{1k}r_{1k}(\alpha_i\beta_i - \alpha_p\beta_p)}{v_{1k}(\alpha_i - \alpha_p)} \qquad \Longrightarrow S_p^* = \emptyset$$
$$\frac{\alpha_p\beta_p - \alpha_j\beta_j}{\alpha_p - \alpha_j} > \frac{\alpha_i\beta_i - \alpha_p\beta_p}{\alpha_i - \alpha_p} \qquad \Longrightarrow S_p^* = \emptyset.$$

□

Proposition 8. *Let the structure of preference weight and revenue be in the form of Case 3. We consider the assortment of service type $p \in M$. If there exists service types $i, j \in M$ such that, $i > p > j$ which satisfy $\frac{v_{1k}\beta_p + \alpha_p\beta_p - \alpha_j\beta_j - v_{1k}\beta_j}{\alpha_p - \alpha_j} > \frac{v_{1k}\beta_i + \alpha_i\beta_i - \alpha_p\beta_p - v_{1k}\beta_p}{\alpha_i - \alpha_p}$, the optimal assortment for service type p is empty set, i.e., the retailer should not offer service type p.*

Proof. According to Lemma 2 and Lemma 3, for $k \in S_p^*$, we must have $\max_{i \in M: i>p}\{\frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}\} \le \pi(S^*)$ and $\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} \ge \pi(S^*)$ for all $j = 1, \ldots, p-1$. Therefore, $\{\frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}\} > \frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}}$ for some $i > p > j \in M$ implies that $S_p^* = \emptyset$. Arranging the terms, we obtain,

$$\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} > \frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}} \qquad \Longrightarrow S_p^* = \emptyset$$
$$\frac{(v_{1k} + \alpha_p)r_{1k}\beta_p - (\alpha_j + v_{1k})r_{1k}\beta_j}{\alpha_p - \alpha_j} > \frac{(v_{1k} + \alpha_i)r_{1k}\beta_i - (v_{1k} + \alpha_p)r_{1k}\beta_p}{\alpha_i - \alpha_p} \qquad \Longrightarrow S_p^* = \emptyset$$
$$\frac{r_{1k}(v_{1k}\beta_p + \alpha_p\beta_p - \alpha_j\beta_j - v_{1k}\beta_j)}{\alpha_p - \alpha_j} > \frac{r_{1k}(v_{1k}\beta_i + \alpha_i\beta_i - \alpha_p\beta_p - v_{1k}\beta_p)}{\alpha_i - \alpha_p} \qquad \Longrightarrow S_p^* = \emptyset$$
$$\frac{v_{1k}\beta_p + \alpha_p\beta_p - \alpha_j\beta_j - v_{1k}\beta_j}{\alpha_p - \alpha_j} > \frac{v_{1k}\beta_i + \alpha_i\beta_i - \alpha_p\beta_p - v_{1k}\beta_p}{\alpha_i - \alpha_p} \qquad \Longrightarrow S_p^* = \emptyset.$$

□

Proposition 9. *Let the structure of preference weight and revenue be in the form of Case 4. We consider the assortment of service type $p \in M$. If there exists* *service types $i, j \in M$ such that, $i > p > j$ which satisfy* $\frac{-v_{1k}\beta_p-\alpha_p\beta_p+\alpha_j\beta_j+v_{1k}\beta_j}{\alpha_p-\alpha_j} > \frac{-v_{1k}\beta_i-\alpha_i\beta_i+\alpha_p\beta_p+v_{1k}\beta_p}{\alpha_i-\alpha_p}$*, the optimal assortment for service type p is empty set, i.e., the retailer should not offer service type p.*

Proof. According to Lemma 2 and Lemma 3, for $k \in S_p^*$, we must have $\max_{i\in M:i>p}\{\frac{v_{ik}r_{ik}-v_{pk}r_{pk}}{v_{ik}-v_{pk}}\} \leq \pi(S^*)$ and $\frac{v_{pk}r_{pk}-v_{jk}r_{jk}}{v_{pk}-v_{jk}} \geq \pi(S^*)$ for all $j = 1, \ldots, p-1$. Therefore, $\{\frac{v_{ik}r_{ik}-v_{pk}r_{pk}}{v_{ik}-v_{pk}}\} > \frac{v_{pk}r_{pk}-v_{jk}r_{jk}}{v_{pk}-v_{jk}}$ for some $i > p > j \in M$ implies that $S_p^* = \emptyset$. Arranging the terms, we obtain,

$$\frac{v_{pk}r_{pk} - v_{jk}r_{jk}}{v_{pk} - v_{jk}} > \frac{v_{ik}r_{ik} - v_{pk}r_{pk}}{v_{ik} - v_{pk}}$$

$$\frac{(v_{1k}+\alpha_p)(r_{1k}-\beta_p) - (\alpha_j+v_{1k})(r_{1k}-\beta_j)}{\alpha_p-\alpha_j} > \frac{(v_{1k}+\alpha_i)(r_{1k}-\beta_i) - (v_{1k}+\alpha_p)(r_{1k}-\beta_p)}{\alpha_i-\alpha_p}$$

$$\frac{-v_{1k}\beta_p - \alpha_p\beta_p + \alpha_j\beta_j + v_{1k}\beta_j}{\alpha_p-\alpha_j} > \frac{-v_{1k}\beta_i - \alpha_i\beta_i + \alpha_p\beta_p + v_{1k}\beta_p}{\alpha_i-\alpha_p} \implies S_p^* = \emptyset.$$

□

Appendix B

Numeric Results

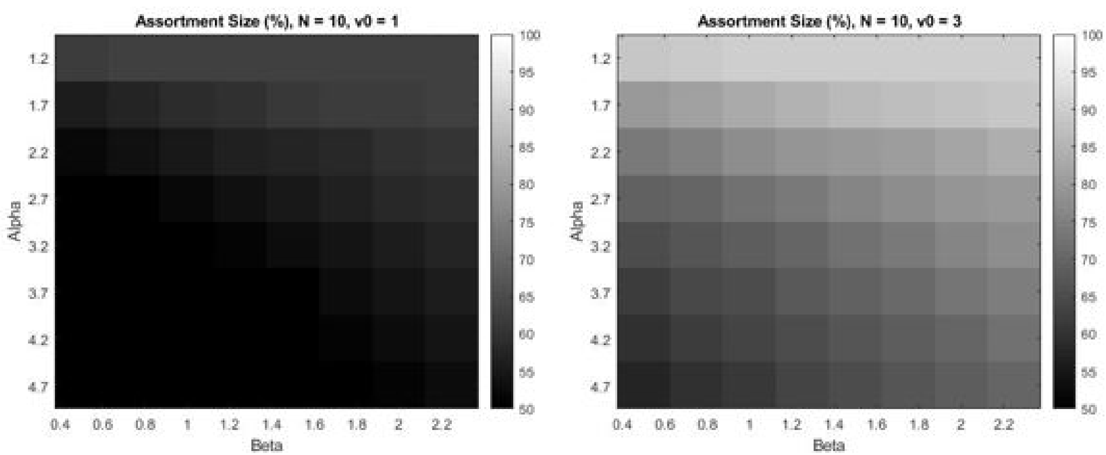

Figure B.1: Assortment size with respect to α and β, $N = 10$

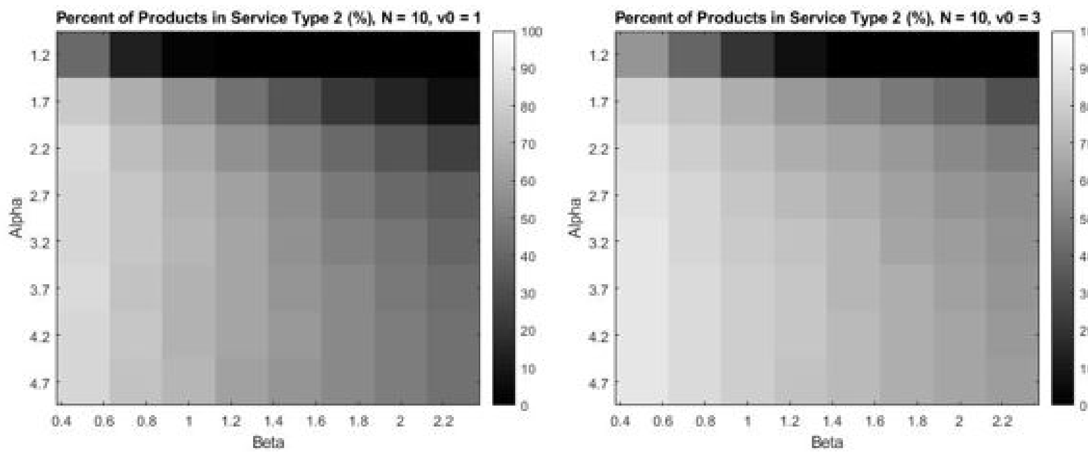

Figure B.2: Percent of products offered with premium service with respect to α and β, $N = 30$

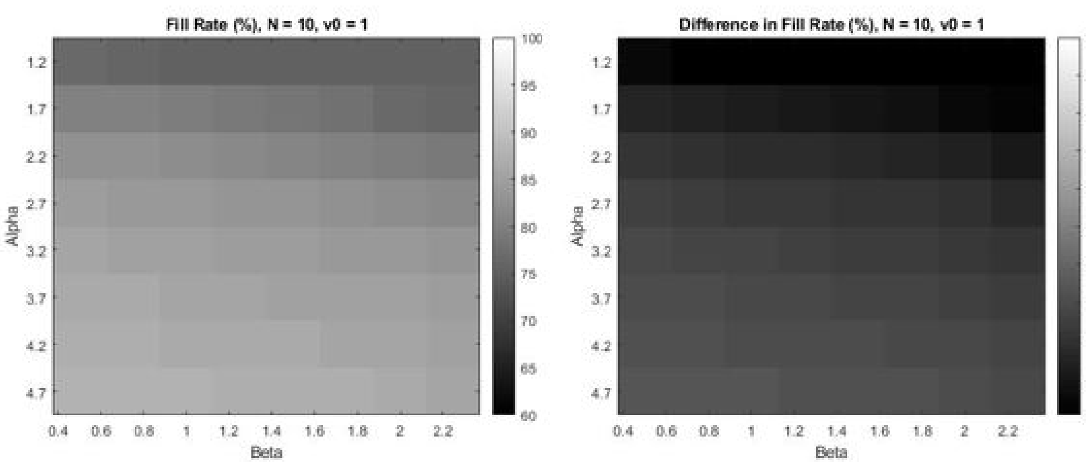

Figure B.3: Fill rate with respect to α and β, $N = 10$, $v_0 = 1$

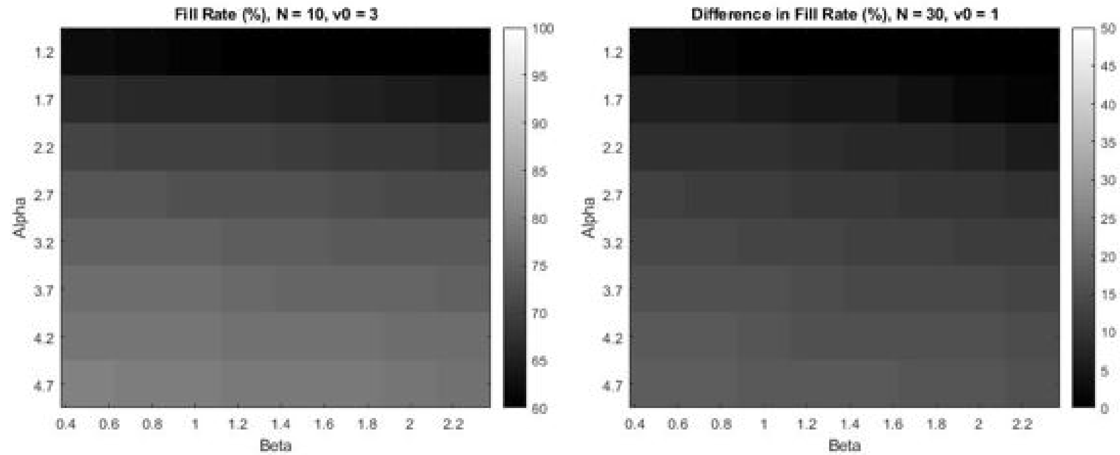

Figure B.4: Fill rate with respect to α and β, $N = 10$, $v_0 = 3$

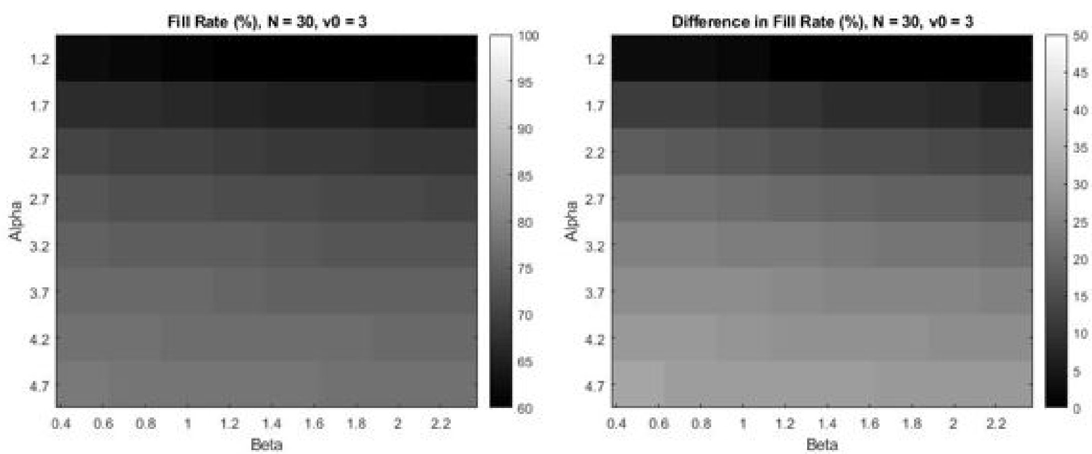

Figure B.5: Fill rate with respect to α and β, $N = 30$, $v_0 = 3$

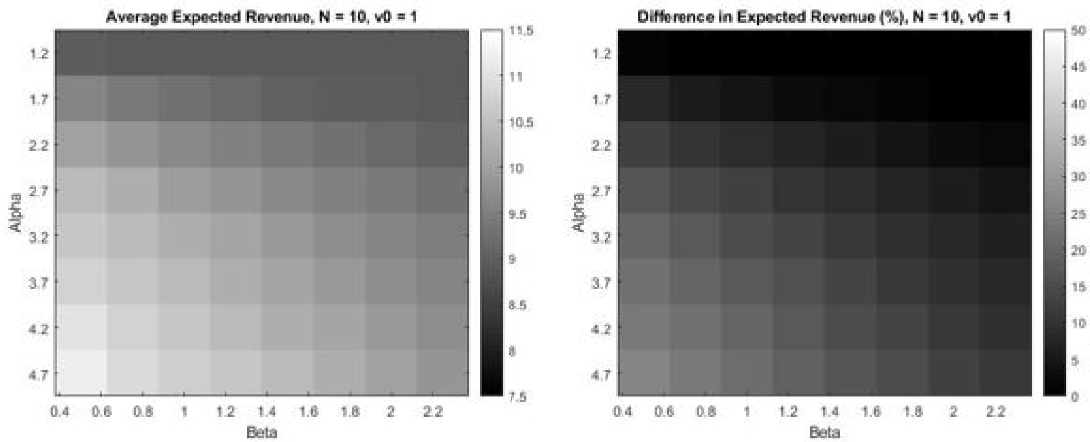

Figure B.6: Expected revenue with respect to α and β, $N = 10$, $v_0 = 1$

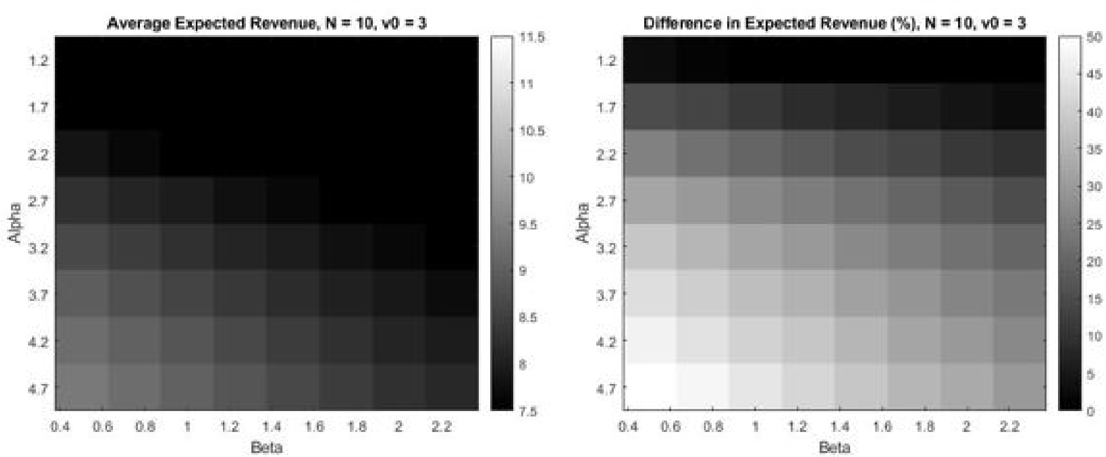

Figure B.7: Expected revenue with respect to α and β, $N = 10$, $v_0 = 3$

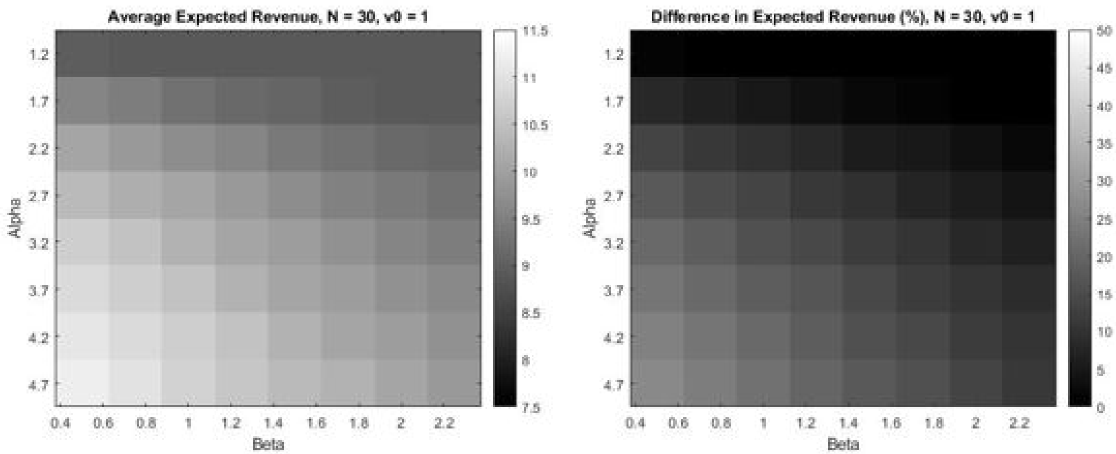

Figure B.8: Expected revenue with respect to α and β, $N = 30$, $v_0 = 1$

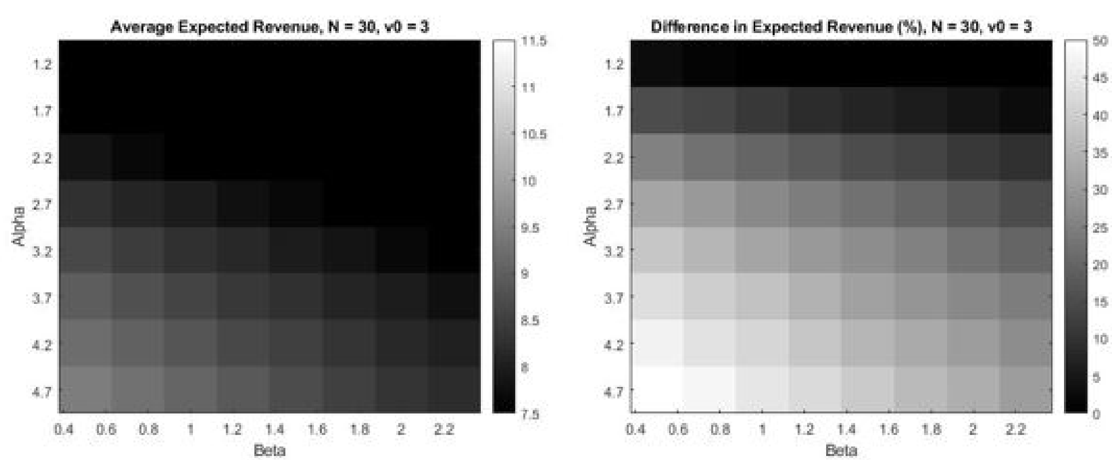

Figure B.9: Expected revenue with respect to α and β, $N = 30$, $v_0 = 3$

Table B.1: **Change in assortment size and distribution of products with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
1.2	2.25	30.0	100.0	62.9	100.0	100.0	100.0	0.0	0.0	0.0
	2.0	30.0	100.0	62.9	100.0	100.0	100.0	0.0	0.0	0.0
	1.75	30.0	100.0	62.9	100.0	100.0	100.0	0.0	0.0	0.0
	1.5	30.0	100.0	62.9	100.0	100.0	100.0	0.0	0.0	0.0
	1.25	30.0	100.0	62.9	100.0	100.0	100.0	0.0	0.0	0.0
	1.0	30.0	100.0	62.9	75.0	100.0	98.1	0.0	25.0	1.91
	0.75	30.0	100.0	62.5	40.0	100.0	86.0	0.0	60.0	14.0
	0.5	30.0	100.0	62.0	0.0	87.5	58.6	12.5	100.0	41.4
1.7	2.25	30.0	100.0	62.5	57.1	100.0	92.2	0.0	42.9	7.8
	2.0	30.0	100.0	62.2	25.0	100.0	85.3	0.0	75.0	14.7
	1.75	30.0	100.0	61.8	25.0	100.0	77.5	0.0	75.0	22.5
	1.5	30.0	100.0	61.1	16.7	100.0	65.8	0.0	83.3	34.2
	1.25	30.0	100.0	59.8	0.0	85.7	55.5	14.3	100.0	44.5
	1.0	30.0	100.0	58.6	0.0	83.3	43.5	16.7	100.0	56.5
	0.75	30.0	100.0	57.6	0.0	66.7	31.6	33.3	100.0	68.4
	0.5	30.0	100.0	55.9	0.0	60.0	20.5	40.0	100.0	79.5
2.2	2.25	30.0	100.0	60.7	25.0	100.0	73.7	0.0	75.0	26.3
	2.0	30.0	100.0	60.0	16.7	100.0	66.9	0.0	83.3	33.1
	1.75	30.0	100.0	58.5	0.0	85.7	58.4	14.3	100.0	41.6
	1.5	30.0	100.0	57.7	0.0	85.7	50.6	14.3	100.0	49.4
	1.25	30.0	100.0	56.6	0.0	85.7	42.7	14.3	100.0	57.3
	1.0	30.0	90.0	55.0	0.0	80.0	34.2	20.0	100.0	65.8
	0.75	30.0	90.0	53.6	0.0	66.7	25.1	33.3	100.0	74.9
	0.5	20.0	80.0	51.7	0.0	50.0	15.3	50.0	100.0	84.7
2.7	2.25	30.0	100.0	58.6	0.0	100.0	64.0	0.0	100.0	36.0
	2.0	30.0	100.0	57.9	0.0	85.7	58.1	14.3	100.0	41.9
	1.75	30.0	90.0	56.7	0.0	85.7	51.7	14.3	100.0	48.3
	1.5	30.0	90.0	55.2	0.0	83.3	45.2	16.7	100.0	54.8
	1.25	20.0	90.0	53.7	0.0	83.3	36.9	16.7	100.0	63.1
	1.0	20.0	90.0	52.1	0.0	80.0	30.4	20.0	100.0	69.6
	0.75	20.0	80.0	50.1	0.0	66.7	22.5	33.3	100.0	77.5
	0.5	20.0	80.0	49.0	0.0	60.0	16.2	40.0	100.0	83.8

Table B.2: **Change in assortment size and distribution of products with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
3.2	2.25	30.0	90.0	57.6	0.0	85.7	60.5	14.3	100.0	39.5
	2.0	20.0	90.0	56.0	0.0	85.7	53.5	14.3	100.0	46.5
	1.75	20.0	90.0	54.4	0.0	83.3	48.5	16.7	100.0	51.5
	1.5	20.0	90.0	52.7	0.0	83.3	42.2	16.7	100.0	57.8
	1.25	20.0	90.0	50.9	0.0	83.3	34.9	16.7	100.0	65.1
	1.0	20.0	80.0	49.5	0.0	80.0	28.6	20.0	100.0	71.4
	0.75	20.0	80.0	48.6	0.0	66.7	23.2	33.3	100.0	76.8
	0.5	20.0	80.0	46.6	0.0	60.0	16.5	40.0	100.0	83.5
3.7	2.25	20.0	90.0	55.8	0.0	85.7	57.8	14.3	100.0	42.2
	2.0	20.0	90.0	54.2	0.0	85.7	51.8	14.3	100.0	48.2
	1.75	20.0	90.0	52.6	0.0	83.3	46.6	16.7	100.0	53.4
	1.5	20.0	80.0	50.7	0.0	83.3	40.9	16.7	100.0	59.1
	1.25	20.0	80.0	49.3	0.0	83.3	35.2	16.7	100.0	64.8
	1.0	20.0	80.0	48.2	0.0	83.3	29.7	16.7	100.0	70.3
	0.75	20.0	80.0	46.3	0.0	75.0	23.5	25.0	100.0	76.5
	0.5	20.0	70.0	44.0	0.0	66.7	15.4	33.3	100.0	84.6
4.2	2.25	20.0	90.0	54.1	0.0	85.7	56.2	14.3	100.0	43.8
	2.0	20.0	90.0	52.4	0.0	85.7	51.1	14.3	100.0	48.9
	1.75	20.0	80.0	50.8	0.0	85.7	45.6	14.3	100.0	54.4
	1.5	20.0	80.0	49.2	0.0	83.3	40.3	16.7	100.0	59.7
	1.25	20.0	80.0	48.6	0.0	83.3	35.9	16.7	100.0	64.1
	1.0	20.0	80.0	46.3	0.0	83.3	30.4	16.7	100.0	69.6
	0.75	20.0	80.0	44.0	0.0	75.0	22.2	25.0	100.0	77.8
	0.5	20.0	70.0	42.6	0.0	66.7	16.9	33.3	100.0	83.1
4.7	2.25	20.0	90.0	52.9	0.0	85.7	54.8	14.3	100.0	45.2
	2.0	20.0	90.0	51.3	0.0	85.7	50.8	14.3	100.0	49.2
	1.75	20.0	80.0	49.6	0.0	85.7	45.5	14.3	100.0	54.5
	1.5	20.0	80.0	48.8	0.0	83.3	40.9	16.7	100.0	59.1
	1.25	20.0	80.0	47.1	0.0	80.0	36.2	20.0	100.0	63.8
	1.0	20.0	80.0	44.5	0.0	80.0	29.5	20.0	100.0	70.5
	0.75	20.0	70.0	42.7	0.0	75.0	23.9	25.0	100.0	76.1
	0.5	20.0	70.0	41.0	0.0	66.7	17.1	33.3	100.0	82.9

Table B.3: **Difference in fill rate with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	Fill Rate (%)			Difference in Fill Rate (%)		
		min	max	avg	min	max	avg
1.2	2.25	62.0	83.7	75.3	0.0	0.0	0.0
	2.0	62.0	83.7	75.3	0.0	0.0	0.0
	1.75	62.0	83.7	75.3	0.0	0.0	0.0
	1.5	62.0	83.7	75.3	0.0	0.0	0.0
	1.25	62.0	83.7	75.3	0.0	0.0	0.0
	1.0	62.0	83.7	75.4	0.0	2.7	0.144
	0.75	65.7	83.7	75.8	-9.31	6.03	0.606
	0.5	65.7	84.5	76.5	-7.71	6.03	1.62
1.7	2.25	66.7	83.8	76.1	-7.76	11.5	1.14
	2.0	66.9	83.8	76.8	-7.76	17.6	2.06
	1.75	69.3	83.7	77.6	-7.76	17.6	3.16
	1.5	70.2	86.2	78.4	-11.2	17.6	4.24
	1.25	70.2	86.2	78.9	-11.1	17.6	4.92
	1.0	70.2	86.2	79.5	-11.1	17.6	5.77
	0.75	72.3	87.0	80.2	-7.74	17.6	6.63
	0.5	68.3	87.8	80.5	-9.55	17.6	7.12
2.2	2.25	70.3	84.6	79.1	-10.9	25.1	5.28
	2.0	72.5	87.6	79.9	-8.72	25.1	6.27
	1.75	72.5	87.6	80.5	-8.72	25.1	7.17
	1.5	72.5	87.9	81.2	-8.72	25.1	8.08
	1.25	72.5	88.1	81.7	-8.72	25.1	8.75
	1.0	72.5	88.5	82.2	-8.72	24.3	9.36
	0.75	72.8	88.5	82.7	-3.95	24.3	9.98
	0.5	64.3	90.0	82.9	-14.5	24.3	10.3
2.7	2.25	71.1	88.7	81.4	-6.5	30.5	8.31
	2.0	74.3	88.7	82.2	-6.5	30.5	9.45
	1.75	74.3	89.4	82.8	-6.5	30.5	10.2
	1.5	74.3	89.6	83.2	-6.5	29.9	10.7
	1.25	68.8	88.7	83.6	-8.38	29.9	11.3
	1.0	68.8	89.9	83.9	-8.38	29.9	11.6
	0.75	68.8	89.9	84.2	-8.38	29.9	12.1
	0.5	68.8	90.6	84.5	-8.38	29.9	12.5

Table B.4: **Change in fill rate with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	Fill Rate(%)			Difference in Fill Rate(%)		
		min	max	avg	min	max	avg
3.2	2.25	72.4	89.6	83.2	-4.69	34.4	10.7
	2.0	72.4	90.5	83.9	-4.69	34.0	11.7
	1.75	72.4	90.5	84.2	-4.69	34.0	12.1
	1.5	72.4	90.0	84.4	-4.69	34.0	12.4
	1.25	72.4	90.0	84.8	-7.37	34.0	12.9
	1.0	72.4	91.0	85.1	-10.3	34.0	13.3
	0.75	72.4	91.0	85.5	-10.3	34.0	13.9
	0.5	72.4	91.6	85.7	-7.68	34.0	14.2
3.7	2.25	69.2	90.4	84.4	-8.36	37.1	12.3
	2.0	74.9	91.4	85.0	-3.01	37.1	13.2
	1.75	74.6	90.6	85.3	-5.33	37.1	13.5
	1.5	74.6	91.0	85.5	-5.33	37.1	13.9
	1.25	71.7	91.1	85.7	-7.15	37.1	14.1
	1.0	71.7	91.2	86.2	-9.57	37.1	14.8
	0.75	71.7	92.2	86.4	-9.57	37.1	15.1
	0.5	73.4	92.2	86.6	-9.57	37.1	15.4
4.2	2.25	70.8	92.2	85.4	-6.21	39.6	13.7
	2.0	76.1	91.2	85.8	-3.67	39.6	14.2
	1.75	76.1	91.6	86.2	-4.52	39.6	14.8
	1.5	73.5	91.8	86.4	-4.52	39.6	15.1
	1.25	73.5	91.9	86.7	-7.36	39.6	15.5
	1.0	73.5	92.0	86.8	-7.36	39.6	15.5
	0.75	73.5	93.0	87.2	-7.36	39.6	16.1
	0.5	73.5	93.0	87.3	-7.36	39.6	16.3
4.7	2.25	72.3	92.8	86.2	-4.17	41.6	14.8
	2.0	75.2	92.0	86.6	-2.37	41.6	15.3
	1.75	75.2	92.4	86.9	-2.37	41.6	15.7
	1.5	75.2	92.5	87.3	-5.38	41.6	16.3
	1.25	75.2	92.5	87.4	-5.38	41.6	16.4
	1.0	75.2	92.4	87.5	-5.38	41.6	16.5
	0.75	72.1	92.8	87.7	-5.38	41.6	16.8
	0.5	72.1	93.7	87.9	-5.38	41.6	17.1

Table B.5: **Change in expected revenue with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
1.2	2.25	5.61	10.7	8.89	0.0	1.78E-14	1.78E-16
	2.0	5.61	10.7	8.89	0.0	1.78E-14	1.78E-16
	1.75	5.61	10.7	8.89	0.0	1.78E-14	1.78E-16
	1.5	5.61	10.7	8.89	0.0	1.78E-14	1.78E-16
	1.25	5.61	10.7	8.89	0.0	1.78E-14	1.78E-16
	1.0	5.61	10.7	8.89	0.0	0.446	0.017
	0.75	5.61	10.7	8.91	0.0	1.44	0.23
	0.5	5.65	10.8	8.97	0.05	3.02	0.92
1.7	2.25	5.63	10.7	8.91	0.0	2.36	0.25
	2.0	5.65	10.7	8.94	0.0	3.6	0.61
	1.75	5.66	10.7	8.98	0.0	4.84	1.17
	1.5	5.69	10.8	9.06	0.0	6.58	1.99
	1.25	5.77	10.9	9.16	0.35	8.54	3.11
	1.0	5.9	11.0	9.28	1.23	10.5	4.48
	0.75	6.04	11.1	9.42	2.2	12.5	6.08
	0.5	6.21	11.3	9.58	3.56	14.4	7.91
2.2	2.25	5.71	10.7	9.06	0.0	7.91	2.12
	2.0	5.75	10.8	9.15	0.0	9.4	3.1
	1.75	5.83	10.9	9.26	0.32	11.5	4.32
	1.5	5.98	11.0	9.38	1.24	13.6	5.74
	1.25	6.14	11.2	9.52	2.27	15.7	7.31
	1.0	6.32	11.3	9.68	3.3	17.8	9.04
	0.75	6.5	11.5	9.84	4.74	20.0	10.9
	0.5	6.68	11.7	10.0	6.33	22.1	13.0
2.7	2.25	5.84	10.8	9.26	0.0	12.4	4.36
	2.0	5.93	10.9	9.38	0.49	14.2	5.72
	1.75	6.1	11.1	9.51	1.54	16.4	7.24
	1.5	6.29	11.3	9.66	2.69	18.6	8.89
	1.25	6.48	11.4	9.82	3.85	20.9	10.7
	1.0	6.66	11.6	9.99	5.11	23.1	12.6
	0.75	6.85	11.8	10.2	6.67	25.4	14.6
	0.5	7.04	12.0	10.4	8.34	27.8	16.8

Table B.6: **Change in expected revenue with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
3.2	2.25	5.97	11.0	9.44	0.45	16.5	6.42
	2.0	6.13	11.1	9.57	1.38	18.5	7.98
	1.75	6.32	11.3	9.72	2.64	21.2	9.68
	1.5	6.52	11.4	9.88	3.89	23.8	11.5
	1.25	6.71	11.6	10.1	5.15	26.4	13.4
	1.0	6.91	11.8	10.2	6.52	29.1	15.4
	0.75	7.1	12.0	10.4	8.21	31.7	17.5
	0.5	7.3	12.2	10.6	9.99	34.4	19.8
3.7	2.25	6.12	11.1	9.59	0.89	20.4	8.19
	2.0	6.3	11.2	9.74	2.24	23.2	9.91
	1.75	6.5	11.4	9.9	3.59	25.9	11.7
	1.5	6.7	11.6	10.1	4.93	28.6	13.7
	1.25	6.9	11.8	10.2	6.28	31.4	15.7
	1.0	7.11	12.0	10.4	7.68	34.1	17.7
	0.75	7.31	12.1	10.6	9.53	36.8	19.9
	0.5	7.51	12.3	10.8	11.4	39.6	22.2
4.2	2.25	6.26	11.2	9.73	1.55	24.1	9.76
	2.0	6.44	11.4	9.89	2.97	26.9	11.6
	1.75	6.65	11.5	10.1	4.4	29.8	13.5
	1.5	6.85	11.7	10.2	5.82	32.6	15.5
	1.25	7.06	11.9	10.4	7.24	35.4	17.6
	1.0	7.26	12.1	10.6	8.74	38.2	19.7
	0.75	7.47	12.3	10.8	10.6	41.1	21.9
	0.5	7.67	12.5	11.0	12.5	43.9	24.2
4.7	2.25	6.39	11.3	9.84	2.12	27.2	11.1
	2.0	6.56	11.5	10.0	3.61	30.1	13.0
	1.75	6.77	11.6	10.2	5.1	33.0	15.0
	1.5	6.98	11.8	10.4	6.58	35.9	17.1
	1.25	7.19	12.0	10.6	8.1	38.7	19.2
	1.0	7.4	12.2	10.7	9.7	41.6	21.4
	0.75	7.61	12.4	10.9	11.5	44.5	23.6
	0.5	7.82	12.6	11.2	13.4	47.4	26.0

Table B.7: **Change in assortment size and distribution of products with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
1.2	2.25	70.0	100.0	90.4	100.0	100.0	100.0	0.0	0.0	0.0
	2.0	70.0	100.0	90.4	100.0	100.0	100.0	0.0	0.0	0.0
	1.75	70.0	100.0	90.4	100.0	100.0	100.0	0.0	0.0	0.0
	1.5	70.0	100.0	90.4	87.5	100.0	99.7	0.0	12.5	0.325
	1.25	60.0	100.0	90.3	57.1	100.0	93.1	0.0	42.9	6.92
	1.0	60.0	100.0	89.9	37.5	100.0	79.0	0.0	62.5	21.0
	0.75	60.0	100.0	89.2	22.2	90.0	59.6	10.0	77.8	40.4
	0.5	60.0	100.0	88.7	0.0	80.0	42.0	20.0	100.0	58.0
1.7	2.25	60.0	100.0	88.5	33.3	90.0	67.3	10.0	66.7	32.7
	2.0	60.0	100.0	87.9	25.0	90.0	58.8	10.0	75.0	41.2
	1.75	60.0	100.0	87.4	16.7	90.0	53.0	10.0	83.3	47.0
	1.5	60.0	100.0	86.6	12.5	80.0	46.2	20.0	87.5	53.8
	1.25	60.0	100.0	84.9	0.0	80.0	39.4	20.0	100.0	60.6
	1.0	60.0	100.0	83.0	0.0	70.0	32.2	30.0	100.0	67.8
	0.75	40.0	100.0	81.3	0.0	57.1	24.7	42.9	100.0	75.3
	0.5	40.0	100.0	80.4	0.0	50.0	17.8	50.0	100.0	82.2
2.2	2.25	50.0	100.0	84.2	12.5	80.0	51.4	20.0	87.5	48.6
	2.0	40.0	100.0	82.2	12.5	80.0	45.5	20.0	87.5	54.5
	1.75	40.0	100.0	81.1	0.0	80.0	40.4	20.0	100.0	59.6
	1.5	40.0	100.0	80.0	0.0	71.4	35.7	28.6	100.0	64.3
	1.25	40.0	100.0	79.5	0.0	62.5	31.4	37.5	100.0	68.6
	1.0	40.0	100.0	77.7	0.0	57.1	25.7	42.9	100.0	74.3
	0.75	40.0	100.0	75.7	0.0	57.1	19.7	42.9	100.0	80.3
	0.5	40.0	100.0	73.5	0.0	42.9	13.6	57.1	100.0	86.4
2.7	2.25	40.0	100.0	80.4	0.0	80.0	45.2	20.0	100.0	54.8
	2.0	40.0	100.0	79.5	0.0	77.8	41.7	22.2	100.0	58.3
	1.75	40.0	100.0	77.9	0.0	77.8	36.5	22.2	100.0	63.5
	1.5	40.0	100.0	75.9	0.0	71.4	31.4	28.6	100.0	68.6
	1.25	40.0	100.0	74.2	0.0	57.1	27.2	42.9	100.0	72.8
	1.0	40.0	100.0	72.0	0.0	57.1	22.2	42.9	100.0	77.8
	0.75	40.0	100.0	70.3	0.0	57.1	16.9	42.9	100.0	83.1
	0.5	40.0	100.0	68.9	0.0	57.1	11.8	42.9	100.0	88.2

Table B.8: **Change in assortment size and distribution of products with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
3.2	2.25	40.0	100.0	77.5	12.5	77.8	42.5	22.2	87.5	57.5
	2.0	40.0	100.0	76.0	0.0	77.8	38.9	22.2	100.0	61.1
	1.75	40.0	100.0	73.9	0.0	77.8	34.5	22.2	100.0	65.5
	1.5	40.0	100.0	71.9	0.0	71.4	28.4	28.6	100.0	71.6
	1.25	40.0	100.0	69.9	0.0	66.7	24.4	33.3	100.0	75.6
	1.0	40.0	100.0	68.5	0.0	57.1	20.5	42.9	100.0	79.5
	0.75	40.0	100.0	66.7	0.0	50.0	16.0	50.0	100.0	84.0
	0.5	40.0	100.0	64.9	0.0	50.0	10.9	50.0	100.0	89.1
3.7	2.25	40.0	100.0	74.8	12.5	77.8	40.7	22.2	87.5	59.3
	2.0	40.0	100.0	72.7	0.0	77.8	36.3	22.2	100.0	63.7
	1.75	40.0	100.0	70.8	0.0	77.8	32.7	22.2	100.0	67.3
	1.5	40.0	100.0	69.1	0.0	66.7	28.7	33.3	100.0	71.3
	1.25	40.0	100.0	67.4	0.0	62.5	23.3	37.5	100.0	76.7
	1.0	40.0	100.0	65.4	0.0	57.1	19.1	42.9	100.0	80.9
	0.75	30.0	100.0	64.1	0.0	57.1	15.2	42.9	100.0	84.8
	0.5	30.0	100.0	62.4	0.0	50.0	10.5	50.0	100.0	89.5
4.2	2.25	40.0	100.0	72.0	12.5	77.8	39.5	22.2	87.5	60.5
	2.0	40.0	100.0	70.2	0.0	77.8	35.3	22.2	100.0	64.7
	1.75	40.0	100.0	68.4	0.0	75.0	31.5	25.0	100.0	68.5
	1.5	40.0	100.0	66.8	0.0	75.0	28.0	25.0	100.0	72.0
	1.25	30.0	100.0	64.9	0.0	62.5	23.4	37.5	100.0	76.6
	1.0	30.0	100.0	63.6	0.0	62.5	19.4	37.5	100.0	80.6
	0.75	30.0	100.0	61.8	0.0	50.0	14.5	50.0	100.0	85.5
	0.5	30.0	100.0	59.6	0.0	42.9	9.66	57.1	100.0	90.3
4.7	2.25	40.0	100.0	69.8	0.0	75.0	38.8	25.0	100.0	61.2
	2.0	40.0	100.0	68.2	0.0	75.0	34.4	25.0	100.0	65.6
	1.75	40.0	100.0	66.9	0.0	75.0	31.7	25.0	100.0	68.3
	1.5	30.0	100.0	64.9	0.0	75.0	27.5	25.0	100.0	72.5
	1.25	30.0	100.0	63.3	0.0	75.0	23.9	25.0	100.0	76.1
	1.0	30.0	100.0	61.4	0.0	62.5	19.0	37.5	100.0	81.0
	0.75	30.0	100.0	59.4	0.0	62.5	14.7	37.5	100.0	85.3
	0.5	30.0	100.0	57.4	0.0	42.9	9.46	57.1	100.0	90.5

Table B.9: **Difference in fill rate with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	Fill Rate (%)			Difference in Fill Rate (%)		
		min	max	avg	min	max	avg
1.2	2.25	47.2	70.2	59.7	0.0	0.0	0.0
	2.0	47.2	70.2	59.7	0.0	0.0	0.0
	1.75	47.2	70.2	59.7	0.0	0.0	0.0
	1.5	47.2	70.2	59.7	0.0	1.94	0.1
	1.25	48.4	70.6	60.1	-0.48	4.09	0.68
	1.0	49.7	70.6	60.7	-5.54	7.82	1.7
	0.75	50.8	70.6	61.3	-5.54	8.36	2.82
	0.5	50.8	70.6	61.9	-5.03	8.36	3.82
1.7	2.25	52.5	71.8	64.1	-3.05	24.3	7.54
	2.0	54.5	71.8	64.8	-1.49	24.3	8.83
	1.75	54.5	71.9	65.4	-0.86	25.3	9.85
	1.5	56.8	73.5	66.0	1.04	25.4	10.9
	1.25	55.9	75.0	66.3	0.436	25.4	11.3
	1.0	55.8	75.6	66.7	-9.2	26.6	11.9
	0.75	54.3	76.4	66.8	-9.2	26.6	12.2
	0.5	54.3	76.6	67.3	-8.19	26.6	13.0
2.2	2.25	56.2	74.5	68.3	-1.03	37.2	14.8
	2.0	59.2	76.7	68.9	-6.79	37.2	15.8
	1.75	60.4	78.3	69.3	-6.79	37.2	16.4
	1.5	60.4	78.6	69.6	-6.79	37.2	17.0
	1.25	60.4	78.9	70.2	-1.43	37.2	17.9
	1.0	57.6	79.7	70.4	-1.43	38.8	18.3
	0.75	57.6	80.4	70.6	-1.03	38.8	18.6
	0.5	57.6	79.7	70.7	0.137	38.8	18.9
2.7	2.25	60.1	78.8	71.4	-4.56	46.1	20.1
	2.0	60.1	79.6	71.8	-4.56	46.1	20.8
	1.75	60.1	81.1	72.1	-4.56	46.1	21.3
	1.5	60.1	81.6	72.5	-4.56	46.1	21.9
	1.25	60.1	81.6	72.8	2.1	46.1	22.3
	1.0	62.2	82.6	72.9	2.1	46.1	22.6
	0.75	59.3	82.6	73.2	-2.35	47.9	23.0
	0.5	59.3	82.6	73.4	-2.35	47.9	23.5

Table B.10: **Difference in fill rate with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	Fill Rate(%)			Difference in Fill Rate(%)		
		min	max	avg	min	max	avg
3.2	2.25	63.8	81.3	73.8	-2.53	53.0	24.2
	2.0	63.8	83.3	74.0	-2.53	53.0	24.5
	1.75	63.8	83.3	74.2	-2.53	53.0	24.8
	1.5	63.4	83.3	74.7	-2.53	53.0	25.6
	1.25	63.4	83.3	74.9	1.62	53.0	26.0
	1.0	63.4	84.7	75.1	1.62	53.0	26.4
	0.75	63.4	84.7	75.3	-1.3	53.0	26.7
	0.5	63.4	84.1	75.5	4.29	54.8	27.0
3.7	2.25	63.0	83.1	75.5	-0.67	58.5	27.0
	2.0	65.7	85.0	75.9	-0.67	58.5	27.6
	1.75	65.7	85.0	75.9	-5.38	58.5	27.8
	1.5	65.7	85.0	76.2	-5.38	58.5	28.2
	1.25	63.6	85.0	76.6	-9.44	58.5	29.0
	1.0	66.4	85.0	76.9	2.21	58.5	29.4
	0.75	66.7	85.9	77.2	2.21	58.5	29.9
	0.5	66.7	85.8	77.4	2.21	58.5	30.2
4.2	2.25	64.7	84.5	76.9	-3.23	62.9	29.4
	2.0	65.3	84.8	77.4	-6.91	62.9	30.3
	1.75	65.3	86.4	77.5	-6.91	62.9	30.5
	1.5	65.3	86.4	77.6	-6.91	62.9	30.7
	1.25	65.3	86.4	78.0	-6.91	62.9	31.3
	1.0	68.5	86.2	78.3	5.23	62.9	31.8
	0.75	68.9	86.2	78.6	5.23	62.9	32.3
	0.5	66.4	87.3	78.7	2.92	62.9	32.5
4.7	2.25	66.1	85.7	77.9	-4.68	66.6	31.3
	2.0	66.9	86.1	78.6	-4.68	66.6	32.3
	1.75	66.9	87.6	78.8	-4.68	66.6	32.7
	1.5	66.9	87.6	79.0	-4.68	66.6	33.0
	1.25	66.9	87.6	79.1	-4.68	66.6	33.1
	1.0	64.3	87.0	79.4	-8.43	66.6	33.7
	0.75	64.3	87.4	79.5	-8.43	66.6	33.9
	0.5	66.7	87.4	80.0	5.81	66.7	34.6

Table B.11: **Change in expected revenue with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
1.2	2.25	3.76	8.29	6.27	0.0	3.27E-14	9.17E-16
	2.0	3.76	8.29	6.27	0.0	3.27E-14	1.05E-15
	1.75	3.76	8.29	6.27	0.0	3.27E-14	1.05E-15
	1.5	3.76	8.29	6.27	0.0	0.146	0.01
	1.25	3.76	8.29	6.27	0.0	0.82	0.09
	1.0	3.77	8.29	6.3	0.0	1.8	0.54
	0.75	3.79	8.34	6.36	0.402	3.32	1.45
	0.5	3.85	8.43	6.44	1.48	5.11	2.84
1.7	2.25	3.82	8.38	6.45	0.174	8.28	3.0
	2.0	3.88	8.44	6.53	0.964	10.1	4.26
	1.75	3.95	8.51	6.62	1.82	12.0	5.7
	1.5	4.03	8.61	6.72	3.09	14.2	7.36
	1.25	4.1	8.74	6.84	4.93	16.4	9.18
	1.0	4.18	8.89	6.96	6.38	18.6	11.2
	0.75	4.26	9.05	7.1	7.91	20.9	13.3
	0.5	4.38	9.22	7.24	10.2	23.1	15.6
2.2	2.25	4.09	8.67	6.85	3.32	19.0	9.42
	2.0	4.18	8.78	6.97	4.71	21.3	11.3
	1.75	4.27	8.91	7.09	6.56	23.9	13.3
	1.5	4.36	9.06	7.23	8.64	26.5	15.5
	1.25	4.45	9.22	7.37	10.7	29.0	17.7
	1.0	4.54	9.38	7.51	13.0	31.6	20.1
	0.75	4.64	9.56	7.67	15.3	34.2	22.6
	0.5	4.78	9.75	7.83	17.6	36.8	25.2
2.7	2.25	4.35	8.94	7.2	6.47	28.4	15.1
	2.0	4.45	9.09	7.34	8.32	31.0	17.3
	1.75	4.55	9.23	7.48	10.5	33.8	19.6
	1.5	4.66	9.4	7.63	12.7	36.5	22.0
	1.25	4.76	9.57	7.78	15.0	39.3	24.5
	1.0	4.87	9.75	7.95	17.3	42.1	27.1
	0.75	4.97	9.93	8.11	19.8	44.9	29.8
	0.5	5.09	10.1	8.28	22.2	47.7	32.6

Table B.12: **Change in expected revenue with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
3.2	2.25	4.56	9.18	7.49	9.11	35.9	19.9
	2.0	4.67	9.34	7.64	11.2	38.7	22.3
	1.75	4.79	9.49	7.8	13.6	41.4	24.8
	1.5	4.9	9.66	7.96	15.9	44.3	27.3
	1.25	5.02	9.85	8.12	18.2	47.3	30.0
	1.0	5.13	10.0	8.29	20.6	50.2	32.7
	0.75	5.25	10.2	8.47	23.1	53.2	35.6
	0.5	5.36	10.4	8.65	25.7	56.1	38.5
3.7	2.25	4.75	9.38	7.74	11.4	42.1	23.9
	2.0	4.87	9.54	7.9	13.6	45.1	26.5
	1.75	5.0	9.71	8.06	16.0	48.1	29.1
	1.5	5.12	9.91	8.23	18.4	51.1	31.8
	1.25	5.24	10.1	8.4	20.8	54.1	34.5
	1.0	5.37	10.3	8.58	23.2	57.1	37.4
	0.75	5.49	10.5	8.76	25.7	60.1	40.3
	0.5	5.62	10.7	8.95	28.5	63.1	43.3
4.2	2.25	4.91	9.54	7.95	13.2	47.8	27.3
	2.0	5.05	9.73	8.12	15.5	50.9	30.0
	1.75	5.18	9.94	8.29	17.9	54.1	32.8
	1.5	5.31	10.1	8.46	20.4	57.2	35.5
	1.25	5.44	10.4	8.64	22.9	60.4	38.4
	1.0	5.57	10.6	8.82	25.4	63.5	41.3
	0.75	5.71	10.8	9.01	28.0	66.6	44.3
	0.5	5.84	11.0	9.2	30.6	69.8	47.4
4.7	2.25	5.06	9.71	8.14	14.8	52.7	30.3
	2.0	5.2	9.92	8.31	17.1	56.0	33.1
	1.75	5.34	10.1	8.48	19.6	59.2	35.9
	1.5	5.48	10.3	8.66	22.1	62.5	38.8
	1.25	5.62	10.5	8.84	24.7	65.7	41.7
	1.0	5.76	10.8	9.03	27.3	69.0	44.7
	0.75	5.91	11.0	9.22	29.9	72.3	47.7
	0.5	6.07	11.2	9.41	32.6	75.6	50.9

Table B.13: **Change in assortment size and distribution of products with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
1.2	2.25	30.0	90.0	60.2	100.0	100.0	100.0	0.0	0.0	0.0
	2.0	30.0	90.0	60.2	100.0	100.0	100.0	0.0	0.0	0.0
	1.75	30.0	90.0	60.2	100.0	100.0	100.0	0.0	0.0	0.0
	1.5	30.0	90.0	60.2	100.0	100.0	100.0	0.0	0.0	0.0
	1.25	30.0	90.0	60.2	100.0	100.0	100.0	0.0	0.0	0.0
	1.0	30.0	90.0	60.2	66.7	100.0	97.3	0.0	33.3	2.71
	0.75	30.0	90.0	59.9	33.3	100.0	82.7	0.0	66.7	17.3
	0.5	30.0	90.0	59.4	0.0	88.9	53.9	11.1	100.0	46.1
1.7	2.25	30.0	90.0	60.0	50.0	100.0	92.7	0.0	50.0	7.3
	2.0	30.0	90.0	59.5	50.0	100.0	86.6	0.0	50.0	13.4
	1.75	30.0	90.0	58.7	20.0	100.0	72.5	0.0	80.0	27.5
	1.5	30.0	90.0	58.5	0.0	100.0	60.9	0.0	100.0	39.1
	1.25	30.0	90.0	57.4	0.0	88.9	51.7	11.1	100.0	48.3
	1.0	30.0	80.0	55.4	0.0	85.7	39.2	14.3	100.0	60.8
	0.75	30.0	80.0	54.4	0.0	75.0	30.1	25.0	100.0	69.9
	0.5	20.0	80.0	52.9	0.0	60.0	21.3	40.0	100.0	78.7
2.2	2.25	30.0	90.0	58.1	20.0	100.0	68.3	0.0	80.0	31.7
	2.0	30.0	90.0	57.1	0.0	100.0	60.1	0.0	100.0	39.9
	1.75	30.0	90.0	56.1	0.0	100.0	53.9	0.0	100.0	46.1
	1.5	30.0	80.0	54.8	0.0	85.7	46.5	14.3	100.0	53.5
	1.25	20.0	80.0	53.2	0.0	85.7	37.8	14.3	100.0	62.2
	1.0	20.0	80.0	51.7	0.0	75.0	29.7	25.0	100.0	70.3
	0.75	20.0	80.0	50.6	0.0	71.4	23.7	28.6	100.0	76.3
	0.5	20.0	80.0	48.8	0.0	60.0	16.4	40.0	100.0	83.6
2.7	2.25	30.0	90.0	56.3	0.0	100.0	58.7	0.0	100.0	41.3
	2.0	20.0	90.0	54.6	0.0	87.5	52.2	12.5	100.0	47.8
	1.75	20.0	80.0	53.2	0.0	87.5	47.1	12.5	100.0	52.9
	1.5	20.0	80.0	52.2	0.0	85.7	40.9	14.3	100.0	59.1
	1.25	20.0	80.0	50.6	0.0	75.0	33.5	25.0	100.0	66.5
	1.0	20.0	80.0	49.3	0.0	75.0	25.8	25.0	100.0	74.2
	0.75	20.0	80.0	46.7	0.0	75.0	19.7	25.0	100.0	80.3
	0.5	20.0	80.0	45.6	0.0	66.7	13.2	33.3	100.0	86.8

Table B.14: **Change in assortment size and distribution of products with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
3.2	2.25	20.0	90.0	53.9	0.0	100.0	53.4	0.0	100.0	46.6
	2.0	20.0	80.0	52.8	0.0	87.5	48.7	12.5	100.0	51.3
	1.75	20.0	80.0	51.5	0.0	85.7	43.9	14.3	100.0	56.1
	1.5	20.0	80.0	50.2	0.0	85.7	38.6	14.3	100.0	61.4
	1.25	20.0	80.0	48.4	0.0	75.0	30.5	25.0	100.0	69.5
	1.0	20.0	80.0	46.1	0.0	75.0	23.3	25.0	100.0	76.7
	0.75	20.0	80.0	45.1	0.0	66.7	18.4	33.3	100.0	81.6
	0.5	20.0	80.0	43.7	0.0	66.7	12.1	33.3	100.0	87.9
3.7	2.25	20.0	90.0	52.5	0.0	87.5	51.0	12.5	100.0	49.0
	2.0	20.0	80.0	51.4	0.0	87.5	46.5	12.5	100.0	53.5
	1.75	20.0	80.0	49.6	0.0	83.3	42.6	16.7	100.0	57.4
	1.5	20.0	80.0	47.9	0.0	83.3	37.7	16.7	100.0	62.3
	1.25	20.0	80.0	46.1	0.0	83.3	30.7	16.7	100.0	69.3
	1.0	20.0	80.0	44.8	0.0	75.0	22.4	25.0	100.0	77.6
	0.75	20.0	80.0	43.4	0.0	66.7	17.7	33.3	100.0	82.3
	0.5	20.0	80.0	41.7	0.0	66.7	12.3	33.3	100.0	87.7
4.2	2.25	20.0	80.0	51.6	0.0	87.5	50.0	12.5	100.0	50.0
	2.0	20.0	80.0	49.7	0.0	87.5	45.0	12.5	100.0	55.0
	1.75	20.0	80.0	48.3	0.0	83.3	41.9	16.7	100.0	58.1
	1.5	20.0	80.0	46.2	0.0	83.3	37.0	16.7	100.0	63.0
	1.25	20.0	80.0	45.0	0.0	83.3	29.8	16.7	100.0	70.2
	1.0	20.0	80.0	43.6	0.0	71.4	25.4	28.6	100.0	74.6
	0.75	20.0	80.0	42.4	0.0	66.7	18.6	33.3	100.0	81.4
	0.5	20.0	80.0	40.5	0.0	66.7	12.7	33.3	100.0	87.3
4.7	2.25	20.0	80.0	50.2	0.0	87.5	48.8	12.5	100.0	51.2
	2.0	20.0	80.0	48.6	0.0	87.5	44.3	12.5	100.0	55.7
	1.75	20.0	80.0	46.4	0.0	83.3	39.6	16.7	100.0	60.4
	1.5	20.0	80.0	45.2	0.0	83.3	35.4	16.7	100.0	64.6
	1.25	20.0	80.0	44.1	0.0	83.3	31.1	16.7	100.0	68.9
	1.0	20.0	80.0	42.8	0.0	71.4	25.1	28.6	100.0	74.9
	0.75	20.0	80.0	41.1	0.0	71.4	19.8	28.6	100.0	80.2
	0.5	20.0	70.0	39.5	0.0	66.7	13.2	33.3	100.0	86.8

Table B.15: **Difference in fill rate with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Fill Rate (%)			Difference in Fill Rate (%)		
		min	max	avg	min	max	avg
1.2	2.25	59.8	83.2	74.9	0.0	0.0	0.0
	2.0	59.8	83.2	74.9	0.0	0.0	0.0
	1.75	59.8	83.2	74.9	0.0	0.0	0.0
	1.5	59.8	83.2	74.9	0.0	0.0	0.0
	1.25	59.8	83.2	74.9	0.0	0.0	0.0
	1.0	62.1	83.2	75.0	0.0	3.85	0.16
	0.75	62.3	83.2	75.5	-6.37	5.95	0.90
	0.5	62.3	83.7	76.2	-9.88	5.95	1.85
1.7	2.25	63.1	83.2	75.6	0.0	13.3	1.07
	2.0	63.1	83.2	76.1	-4.88	13.3	1.8
	1.75	66.8	84.7	77.3	-10.5	17.1	3.41
	1.5	66.8	84.7	78.4	-10.5	17.1	4.97
	1.25	65.5	85.8	78.7	-10.5	17.7	5.34
	1.0	65.5	85.9	79.1	-10.5	17.7	5.8
	0.75	65.5	86.3	79.6	-10.5	17.7	6.46
	0.5	66.6	87.4	79.8	-10.5	17.7	6.72
2.2	2.25	70.7	85.8	79.3	-5.03	24.3	6.19
	2.0	70.2	85.8	80.2	-5.03	24.3	7.39
	1.75	70.2	87.4	80.7	-5.03	24.3	7.98
	1.5	70.2	86.8	81.1	-5.03	25.1	8.59
	1.25	68.0	87.6	81.4	-5.03	25.1	9.0
	1.0	64.8	88.4	81.8	-13.1	25.1	9.49
	0.75	64.8	88.4	82.1	-13.1	25.1	9.91
	0.5	64.8	89.6	82.2	-13.1	25.1	10.0
2.7	2.25	73.2	87.2	81.9	-0.761	29.3	9.66
	2.0	69.3	88.7	82.3	-7.05	29.3	10.2
	1.75	69.3	88.3	82.6	-7.05	30.2	10.6
	1.5	69.3	88.3	83.0	-7.05	30.2	11.2
	1.25	69.3	90.0	83.4	-7.05	30.2	11.7
	1.0	69.3	90.0	83.9	-7.05	30.2	12.3
	0.75	69.3	90.0	83.8	-7.05	30.2	12.2
	0.5	69.3	89.8	84.1	-7.05	30.2	12.6

Table B.16: **Difference in fill rate with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Fill Rate(%)			Difference in Fill Rate(%)		
		min	max	avg	min	max	avg
3.2	2.25	72.9	88.3	83.5	-2.26	33.1	11.9
	2.0	72.9	89.4	83.9	-2.26	33.1	12.4
	1.75	72.9	89.6	84.3	-2.26	33.9	12.9
	1.5	72.9	89.6	84.5	-2.26	33.9	13.2
	1.25	72.9	91.2	85.0	-2.26	33.9	13.8
	1.0	72.9	91.2	85.1	-2.26	33.9	14.0
	0.75	72.9	90.6	85.4	-2.26	33.9	14.4
	0.5	72.9	91.0	85.6	-2.26	33.9	14.7
3.7	2.25	75.6	90.3	84.8	0.69	36.0	13.7
	2.0	75.6	90.6	85.4	1.37	36.0	14.4
	1.75	75.6	90.6	85.4	1.37	36.8	14.5
	1.5	75.6	90.7	85.5	1.37	36.8	14.6
	1.25	75.6	92.2	85.8	1.37	36.8	15.0
	1.0	75.6	92.2	86.3	1.37	36.8	15.7
	0.75	75.1	91.7	86.5	-0.59	36.8	15.9
	0.5	75.6	91.7	86.7	-0.59	36.8	16.1
4.2	2.25	77.9	91.1	85.9	2.48	38.3	15.2
	2.0	77.9	91.5	86.3	2.51	38.3	15.7
	1.75	77.9	91.5	86.5	3.87	38.3	15.9
	1.5	77.9	91.5	86.5	3.87	39.0	15.9
	1.25	76.7	93.0	86.8	3.52	39.0	16.3
	1.0	76.7	92.6	87.0	2.06	39.0	16.6
	0.75	76.7	92.6	87.5	2.06	39.0	17.2
	0.5	71.1	92.6	87.6	-3.97	39.0	17.3
4.7	2.25	79.8	91.8	86.8	4.09	40.2	16.4
	2.0	79.8	92.3	87.2	3.58	40.2	16.9
	1.75	78.1	92.3	87.3	4.96	40.2	17.0
	1.5	78.1	92.3	87.4	4.34	40.9	17.2
	1.25	78.1	93.0	87.6	4.34	40.9	17.4
	1.0	78.1	93.3	88.0	4.34	40.9	17.9
	0.75	73.3	93.3	88.1	-1.06	40.9	18.1
	0.5	73.3	93.3	88.4	-1.06	40.9	18.5

Table B.17: **Change in expected revenue with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
	2.25	5.96	11.1	8.89	0.0	2.11E-14	2.11E-16
	2.0	5.96	11.1	8.89	0.0	2.11E-14	2.11E-16
	1.75	5.96	11.1	8.89	0.0	2.11E-14	2.11E-16
1.2	1.5	5.96	11.1	8.89	0.0	2.11E-14	2.11E-16
	1.25	5.96	11.1	8.89	0.0	2.11E-14	2.11E-16
	1.0	5.96	11.1	8.89	0.0	0.48	0.03
	0.75	5.96	11.1	8.91	0.0	1.77	0.22
	0.5	5.96	11.1	8.98	0.02	3.16	1.02
	2.25	5.96	11.1	8.91	0.0	3.69	0.25
	2.0	5.96	11.1	8.93	0.0	5.4	0.57
	1.75	5.96	11.1	8.99	0.0	7.1	1.2
1.7	1.5	5.96	11.1	9.07	0.0	8.81	2.17
	1.25	5.97	11.2	9.18	0.28	10.5	3.38
	1.0	6.07	11.3	9.31	1.6	12.2	4.82
	0.75	6.2	11.5	9.46	2.57	14.2	6.5
	0.5	6.36	11.6	9.62	3.75	16.3	8.35
	2.25	5.96	11.1	9.07	0.0	11.1	2.21
	2.0	5.96	11.1	9.17	0.0	13.0	3.34
	1.75	5.96	11.2	9.29	0.0	15.1	4.69
2.2	1.5	6.06	11.3	9.42	1.35	17.1	6.19
	1.25	6.18	11.5	9.57	2.93	19.2	7.85
	1.0	6.31	11.6	9.73	4.06	21.4	9.68
	0.75	6.48	11.8	9.91	5.17	23.7	11.6
	0.5	6.66	12.0	10.1	6.79	26.2	13.7
	2.25	5.96	11.1	9.29	0.0	17.2	4.66
	2.0	5.96	11.2	9.42	0.08	19.5	6.15
	1.75	6.08	11.4	9.56	1.62	21.8	7.78
2.7	1.5	6.22	11.5	9.71	3.41	24.2	9.53
	1.25	6.35	11.7	9.88	4.78	26.5	11.4
	1.0	6.51	11.9	10.1	6.04	28.9	13.4
	0.75	6.69	12.1	10.2	7.36	31.2	15.5
	0.5	6.88	12.3	10.4	9.04	33.6	17.7

Table B.18: **Change in expected revenue with respect to α and β, $N = 30$, $v_0 = 1$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
3.2	2.25	5.96	11.2	9.48	0.0	22.4	6.84
	2.0	6.05	11.4	9.62	1.38	24.9	8.53
	1.75	6.2	11.5	9.78	3.23	27.4	10.3
	1.5	6.34	11.7	9.95	4.82	29.8	12.2
	1.25	6.49	11.9	10.1	6.29	32.4	14.2
	1.0	6.66	12.1	10.3	7.76	35.1	16.3
	0.75	6.84	12.3	10.5	9.23	37.8	18.5
	0.5	7.03	12.5	10.7	10.8	40.5	20.8
3.7	2.25	6.0	11.3	9.64	0.65	26.5	8.7
	2.0	6.14	11.5	9.8	2.48	29.1	10.5
	1.75	6.3	11.7	9.97	4.42	31.7	12.4
	1.5	6.45	11.8	10.1	5.97	34.5	14.4
	1.25	6.61	12.0	10.3	7.52	37.3	16.5
	1.0	6.78	12.3	10.5	9.07	40.1	18.7
	0.75	6.97	12.5	10.7	10.6	42.9	20.9
	0.5	7.16	12.7	10.9	12.2	45.7	23.3
4.2	2.25	6.06	11.4	9.77	1.6	29.9	10.3
	2.0	6.22	11.6	9.94	3.42	32.8	12.2
	1.75	6.38	11.8	10.1	5.33	35.7	14.2
	1.5	6.55	11.9	10.3	6.95	38.6	16.3
	1.25	6.71	12.2	10.5	8.57	41.4	18.4
	1.0	6.88	12.4	10.7	10.2	44.3	20.6
	0.75	7.08	12.6	10.9	11.8	47.2	22.9
	0.5	7.27	12.9	11.1	13.4	50.1	25.3
4.7	2.25	6.12	11.5	9.89	2.18	33.0	11.6
	2.0	6.29	11.7	10.1	4.23	36.0	13.6
	1.75	6.45	11.8	10.3	6.09	38.9	15.7
	1.5	6.62	12.1	10.4	7.77	41.9	17.9
	1.25	6.79	12.3	10.6	9.45	44.8	20.0
	1.0	6.97	12.5	10.8	11.1	47.8	22.3
	0.75	7.16	12.7	11.0	12.8	50.8	24.6
	0.5	7.36	13.0	11.2	14.5	53.7	27.0

Table B.19: **Change in assortment size and distribution of products with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Assortment Size (%)			Proportion of Products (%)					
		min	max	avg	S_1			S_2		
					min	max	avg	min	max	avg
1.2	2.25	60.0	100.0	90.6	100.0	100.0	100.0	0.0	0.0	0.0
	2.0	60.0	100.0	90.6	100.0	100.0	100.0	0.0	0.0	0.0
	1.75	60.0	100.0	90.6	100.0	100.0	100.0	0.0	0.0	0.0
	1.5	60.0	100.0	90.6	88.9	100.0	99.5	0.0	11.1	0.511
	1.25	60.0	100.0	90.6	77.8	100.0	94.1	0.0	22.2	5.91
	1.0	60.0	100.0	90.2	50.0	100.0	77.0	0.0	50.0	23.0
	0.75	60.0	100.0	89.7	25.0	90.0	59.5	10.0	75.0	40.5
	0.5	60.0	100.0	88.3	11.1	80.0	43.3	20.0	88.9	56.7
1.7	2.25	60.0	100.0	88.4	33.3	100.0	67.7	0.0	66.7	32.3
	2.0	60.0	100.0	87.3	25.0	90.0	59.4	10.0	75.0	40.6
	1.75	60.0	100.0	86.2	22.2	80.0	54.6	20.0	77.8	45.4
	1.5	60.0	100.0	85.0	12.5	80.0	49.0	20.0	87.5	51.0
	1.25	60.0	100.0	83.9	0.0	80.0	41.5	20.0	100.0	58.5
	1.0	60.0	100.0	82.5	0.0	70.0	35.4	30.0	100.0	64.6
	0.75	50.0	100.0	81.3	0.0	70.0	27.6	30.0	100.0	72.4
	0.5	50.0	100.0	79.2	0.0	50.0	17.0	50.0	100.0	83.0
2.2	2.25	60.0	100.0	83.5	12.5	80.0	53.1	20.0	87.5	46.9
	2.0	50.0	100.0	82.1	12.5	80.0	49.0	20.0	87.5	51.0
	1.75	50.0	100.0	80.7	0.0	80.0	44.5	20.0	100.0	55.5
	1.5	50.0	100.0	78.8	0.0	77.8	38.7	22.2	100.0	61.3
	1.25	50.0	100.0	77.4	0.0	77.8	33.5	22.2	100.0	66.5
	1.0	50.0	100.0	76.0	0.0	66.7	28.4	33.3	100.0	71.6
	0.75	40.0	100.0	74.3	0.0	66.7	21.9	33.3	100.0	78.1
	0.5	40.0	100.0	72.7	0.0	55.6	15.2	44.4	100.0	84.8
2.7	2.25	50.0	100.0	79.2	12.5	80.0	47.7	20.0	87.5	52.3
	2.0	50.0	100.0	77.4	12.5	77.8	43.6	22.2	87.5	56.4
	1.75	50.0	100.0	76.5	0.0	77.8	40.6	22.2	100.0	59.4
	1.5	40.0	100.0	74.6	0.0	77.8	34.4	22.2	100.0	65.6
	1.25	40.0	100.0	73.3	0.0	71.4	30.2	28.6	100.0	69.8
	1.0	40.0	100.0	71.7	0.0	62.5	25.6	37.5	100.0	74.4
	0.75	40.0	100.0	69.5	0.0	62.5	20.3	37.5	100.0	79.7
	0.5	40.0	100.0	67.5	0.0	62.5	13.6	37.5	100.0	86.4

Table B.20: **Change in assortment size and distribution of products with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Assortment Size (%)			Proportion of Products (%)					
					S_1			S_2		
		min	max	avg	min	max	avg	min	max	avg
3.2	2.25	40.0	100.0	76.0	12.5	80.0	44.8	20.0	87.5	55.2
	2.0	40.0	100.0	74.4	0.0	75.0	41.1	25.0	100.0	58.9
	1.75	40.0	100.0	72.7	0.0	75.0	37.8	25.0	100.0	62.2
	1.5	40.0	100.0	70.9	0.0	71.4	32.6	28.6	100.0	67.4
	1.25	40.0	100.0	69.0	0.0	71.4	27.3	28.6	100.0	72.7
	1.0	30.0	100.0	66.8	0.0	71.4	22.5	28.6	100.0	77.5
	0.75	30.0	100.0	65.3	0.0	57.1	18.0	42.9	100.0	82.0
	0.5	30.0	100.0	63.8	0.0	50.0	13.5	50.0	100.0	86.5
3.7	2.25	40.0	100.0	73.3	0.0	75.0	43.3	25.0	100.0	56.7
	2.0	40.0	100.0	71.0	0.0	75.0	39.2	25.0	100.0	60.8
	1.75	40.0	100.0	69.4	0.0	75.0	35.4	25.0	100.0	64.6
	1.5	30.0	100.0	67.4	0.0	71.4	32.0	28.6	100.0	68.0
	1.25	30.0	100.0	65.7	0.0	66.7	26.8	33.3	100.0	73.2
	1.0	30.0	100.0	64.0	0.0	66.7	21.4	33.3	100.0	78.6
	0.75	30.0	90.0	61.5	0.0	60.0	16.6	40.0	100.0	83.4
	0.5	30.0	90.0	59.3	0.0	42.9	10.7	57.1	100.0	89.3
4.2	2.25	40.0	100.0	70.3	0.0	75.0	41.3	25.0	100.0	58.7
	2.0	30.0	100.0	68.1	0.0	75.0	37.8	25.0	100.0	62.2
	1.75	30.0	100.0	66.7	0.0	71.4	33.9	28.6	100.0	66.1
	1.5	30.0	100.0	64.9	0.0	66.7	31.0	33.3	100.0	69.0
	1.25	30.0	100.0	62.8	0.0	66.7	26.4	33.3	100.0	73.6
	1.0	30.0	90.0	60.1	0.0	60.0	20.0	40.0	100.0	80.0
	0.75	30.0	90.0	58.7	0.0	50.0	15.0	50.0	100.0	85.0
	0.5	30.0	90.0	56.8	0.0	50.0	10.2	50.0	100.0	89.8
4.7	2.25	30.0	100.0	68.1	0.0	75.0	40.4	25.0	100.0	59.6
	2.0	30.0	100.0	65.9	0.0	75.0	36.4	25.0	100.0	63.6
	1.75	30.0	100.0	64.0	0.0	71.4	33.5	28.6	100.0	66.5
	1.5	30.0	100.0	62.0	0.0	66.7	28.3	33.3	100.0	71.7
	1.25	30.0	100.0	59.6	0.0	60.0	24.1	40.0	100.0	75.9
	1.0	30.0	90.0	58.3	0.0	60.0	20.7	40.0	100.0	79.3
	0.75	30.0	90.0	56.1	0.0	50.0	15.0	50.0	100.0	85.0
	0.5	30.0	80.0	54.7	0.0	40.0	8.91	60.0	100.0	91.1

Table B.21: **Change in fill rate with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Fill Rate (%)			Difference in Fill Rate (%)		
		min	max	avg	min	max	avg
1.2	2.25	50.9	69.2	60.0	0.0	0.0	0.0
	2.0	50.9	69.2	60.0	0.0	0.0	0.0
	1.75	50.9	69.2	60.0	0.0	0.0	0.0
	1.5	51.3	69.2	60.1	0.0	1.32	0.05
	1.25	51.6	69.7	60.4	0.0	2.87	0.53
	1.0	51.8	70.3	61.1	-5.15	4.34	1.72
	0.75	51.0	69.2	61.6	-5.15	6.44	2.7
	0.5	51.0	70.2	61.9	-7.22	7.73	3.12
1.7	2.25	51.8	72.7	64.1	-0.79	16.0	6.89
	2.0	55.0	72.7	64.9	-0.79	16.3	8.29
	1.75	55.0	71.3	65.2	-2.33	19.7	8.81
	1.5	55.0	73.3	65.5	-2.33	19.1	9.26
	1.25	55.0	73.8	66.1	-2.33	22.4	10.2
	1.0	55.0	74.7	66.6	0.32	22.4	11.0
	0.75	54.5	73.7	67.0	-3.04	22.4	11.8
	0.5	55.0	73.7	67.4	-4.21	22.5	12.4
2.2	2.25	58.3	74.2	68.3	2.0	28.0	13.9
	2.0	58.3	76.3	68.7	1.55	28.0	14.6
	1.75	58.3	77.6	68.9	1.55	28.1	14.9
	1.5	58.3	77.7	69.3	1.66	29.4	15.6
	1.25	59.3	77.7	69.7	1.66	32.2	16.4
	1.0	59.3	77.7	70.1	2.36	32.2	17.0
	0.75	60.6	77.8	70.4	2.36	32.2	17.5
	0.5	58.6	78.4	70.9	2.12	32.2	18.3
2.7	2.25	61.2	78.9	71.1	2.89	34.6	18.6
	2.0	61.2	80.8	71.3	1.86	34.3	19.0
	1.75	61.2	80.8	71.6	1.86	34.4	19.5
	1.5	61.2	80.9	72.1	1.86	36.9	20.3
	1.25	62.9	80.9	72.4	1.86	39.3	20.9
	1.0	62.9	80.9	72.8	1.86	39.3	21.5
	0.75	62.0	81.1	73.0	5.36	39.3	21.9
	0.5	61.9	81.1	73.4	5.27	39.3	22.5

Table B.22: **Change in fill rate with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Fill Rate(%)			Difference in Fill Rate(%)		
		min	max	avg	min	max	avg
3.2	2.25	63.7	81.3	73.3	5.24	39.8	22.4
	2.0	63.7	83.1	73.6	5.24	40.2	22.9
	1.75	62.9	83.1	73.7	3.37	40.2	23.0
	1.5	61.5	83.3	74.1	1.03	42.8	23.7
	1.25	61.5	83.3	74.4	1.03	44.5	24.3
	1.0	62.1	83.3	74.6	2.23	44.5	24.6
	0.75	63.9	83.3	74.9	2.23	44.5	25.1
	0.5	63.9	83.5	75.2	9.58	44.5	25.6
3.7	2.25	63.4	83.1	75.1	3.51	44.8	25.5
	2.0	63.4	83.1	75.3	4.2	45.5	25.8
	1.75	62.3	84.5	75.5	4.2	45.5	26.1
	1.5	62.3	84.9	75.5	3.39	47.6	26.1
	1.25	62.3	85.2	75.9	4.2	47.6	26.8
	1.0	62.3	85.2	76.3	4.0	48.6	27.4
	0.75	62.5	85.2	76.3	4.0	48.6	27.5
	0.5	60.8	85.2	76.6	3.07	48.6	27.9
4.2	2.25	64.7	84.7	76.6	5.94	49.0	28.0
	2.0	64.7	84.2	76.7	5.94	49.0	28.1
	1.75	64.7	86.1	77.0	5.95	49.7	28.6
	1.5	62.1	86.1	77.0	1.29	51.6	28.6
	1.25	62.1	86.7	77.1	1.29	51.6	28.7
	1.0	62.1	86.7	77.3	6.24	50.6	29.1
	0.75	61.8	86.7	77.7	1.61	50.6	29.8
	0.5	63.7	86.7	77.8	3.72	51.9	30.0
4.7	2.25	66.7	86.0	77.8	8.14	52.6	30.1
	2.0	64.5	85.6	78.0	4.16	52.6	30.4
	1.75	64.5	85.7	78.1	4.16	52.6	30.4
	1.5	64.5	87.3	78.3	4.16	53.2	30.8
	1.25	60.2	87.6	78.2	-1.12	54.9	30.7
	1.0	60.2	87.9	78.4	-1.12	54.0	30.9
	0.75	60.2	87.9	78.5	-1.12	53.2	31.2
	0.5	60.2	87.7	79.1	-1.12	53.2	32.1

Table B.23: **Change in expected revenue with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
1.2	2.25	4.03	8.34	6.28	0.0	4.66E-14	7.92E-16
	2.0	4.03	8.34	6.28	0.0	4.66E-14	7.92E-16
	1.75	4.03	8.34	6.28	0.0	4.66E-14	1.06E-15
	1.5	4.03	8.34	6.28	0.0	0.07	0.01
	1.25	4.03	8.34	6.28	0.0	0.67	0.08
	1.0	4.03	8.34	6.31	0.0	1.89	0.56
	0.75	4.04	8.41	6.37	0.242	3.61	1.49
	0.5	4.11	8.53	6.45	1.53	5.34	2.84
1.7	2.25	4.03	8.45	6.47	0.0	9.06	3.12
	2.0	4.05	8.54	6.55	0.471	11.3	4.37
	1.75	4.11	8.65	6.64	1.7	13.6	5.83
	1.5	4.19	8.78	6.74	3.18	15.9	7.44
	1.25	4.3	8.93	6.85	4.99	18.1	9.2
	1.0	4.43	9.09	6.97	7.05	20.4	11.2
	0.75	4.56	9.26	7.1	9.37	22.7	13.3
	0.5	4.7	9.43	7.25	11.9	25.0	15.6
2.2	2.25	4.19	8.86	6.87	3.16	21.4	9.68
	2.0	4.29	8.99	6.99	4.96	23.9	11.5
	1.75	4.4	9.15	7.11	6.8	26.5	13.5
	1.5	4.53	9.31	7.24	9.01	29.1	15.5
	1.25	4.68	9.49	7.38	11.3	31.7	17.7
	1.0	4.83	9.67	7.52	14.1	34.2	20.1
	0.75	4.97	9.86	7.68	16.8	36.8	22.5
	0.5	5.12	10.1	7.84	19.6	39.4	25.1
2.7	2.25	4.4	9.2	7.23	6.84	31.0	15.4
	2.0	4.52	9.36	7.36	8.85	33.8	17.6
	1.75	4.66	9.53	7.5	11.3	36.6	19.8
	1.5	4.81	9.7	7.65	13.7	39.4	22.1
	1.25	4.97	9.9	7.8	16.1	42.2	24.5
	1.0	5.13	10.1	7.96	19.0	45.0	27.1
	0.75	5.29	10.3	8.12	22.0	47.8	29.7
	0.5	5.45	10.5	8.29	25.1	50.6	32.4

Table B.24: **Change in expected revenue with respect to α and β, $N = 30$, $v_0 = 3$**

α	β	Expected Revenue			Increase in Expected Revenue (%)		
		min	max	avg	min	max	avg
3.2	2.25	4.57	9.48	7.53	9.94	38.4	20.3
	2.0	4.72	9.65	7.68	12.3	41.4	22.6
	1.75	4.87	9.84	7.83	14.8	44.3	25.0
	1.5	5.03	10.0	7.98	17.4	47.3	27.5
	1.25	5.2	10.2	8.14	19.9	50.3	30.1
	1.0	5.37	10.4	8.31	22.8	53.3	32.8
	0.75	5.53	10.6	8.48	25.8	56.3	35.5
	0.5	5.7	10.8	8.66	28.9	59.3	38.3
3.7	2.25	4.73	9.71	7.78	12.5	44.6	24.4
	2.0	4.88	9.89	7.94	15.0	47.7	26.9
	1.75	5.04	10.1	8.1	17.6	50.8	29.4
	1.5	5.21	10.3	8.26	20.3	54.0	32.0
	1.25	5.38	10.5	8.43	23.0	57.1	34.7
	1.0	5.56	10.7	8.6	25.7	60.2	37.5
	0.75	5.74	10.9	8.78	28.7	63.3	40.4
	0.5	5.92	11.1	8.96	31.8	66.4	43.3
4.2	2.25	4.86	9.91	8.0	14.7	49.7	27.9
	2.0	5.02	10.1	8.16	17.2	52.9	30.5
	1.75	5.19	10.3	8.33	20.0	56.1	33.2
	1.5	5.36	10.5	8.5	22.7	59.4	35.9
	1.25	5.54	10.7	8.67	25.5	62.6	38.7
	1.0	5.72	10.9	8.85	28.3	65.8	41.6
	0.75	5.91	11.1	9.03	31.1	69.1	44.5
	0.5	6.1	11.3	9.22	34.3	72.3	47.5
4.7	2.25	4.97	10.1	8.19	16.7	53.9	30.9
	2.0	5.14	10.3	8.36	19.1	57.2	33.7
	1.75	5.31	10.4	8.53	22.0	60.6	36.4
	1.5	5.48	10.6	8.71	24.8	63.9	39.3
	1.25	5.67	10.8	8.89	27.6	67.2	42.2
	1.0	5.86	11.0	9.07	30.5	70.5	45.1
	0.75	6.05	11.2	9.25	33.3	73.9	48.1
	0.5	6.25	11.4	9.44	36.3	77.2	51.2

Table B.25: **Change in the distribution of products to service types with respect to α and β, $N = 10$, $v_0 = 1$**

α	β	**Service Types Used (Out of 100 runs)**							**Average # of Products**		
		S_1	S_2	S_3	S_1, S_2	S_1, S_3	S_2, S_3	S_1, S_2, S_3	$\|S_1\|$	$\|S_2\|$	$\|S_3\|$
1.2	1.0	82.0	0.0	0.0	11.0	5.0	0.0	2.0	6.02	0.13	0.07
	0.5	0.0	0.0	1.0	2.0	56.0	1.0	40.0	3.48	0.54	2.09
1.6	1.0	0.0	1.0	0.0	7.0	31.0	4.0	57.0	2.95	1.21	1.66
	0.5	0.0	0.0	18.0	0.0	45.0	8.0	29.0	1.32	0.5	3.31
2.0	1.0	0.0	0.0	5.0	1.0	20.0	11.0	63.0	1.94	1.29	1.96
	0.5	0.0	0.0	23.0	0.0	29.0	17.0	31.0	0.93	0.63	2.99
2.4	1.0	0.0	0.0	5.0	1.0	24.0	15.0	55.0	1.67	1.25	1.89
	0.5	0.0	0.0	24.0	0.0	17.0	30.0	29.0	0.67	0.79	2.58
2.8	1.0	0.0	0.0	5.0	1.0	22.0	21.0	51.0	1.52	1.3	1.74
	0.5	0.0	0.0	23.0	0.0	23.0	25.0	29.0	0.74	0.72	2.36

Table B.26: **Change in the distribution of products to service types with respect to α and β, $N = 10$, $v_0 = 3$**

α	β	**Service Types Used (Out of 100 runs)**							**Average # of Products**		
		S_1	S_2	S_3	S_1, S_2	S_1, S_3	S_2, S_3	S_1, S_2, S_3	$\|S_1\|$	$\|S_2\|$	$\|S_3\|$
1.2	1.0	1.0	0.0	0.0	21.0	37.0	0.0	41.0	6.61	0.98	1.34
	0.5	0.0	0.0	0.0	0.0	55.0	0.0	45.0	3.22	0.57	4.78
1.6	1.0	0.0	0.0	1.0	0.0	36.0	2.0	61.0	2.76	1.08	4.03
	0.5	0.0	0.0	14.0	0.0	47.0	8.0	31.0	1.39	0.53	5.36
2.0	1.0	0.0	0.0	5.0	0.0	30.0	6.0	59.0	2.08	1.04	3.94
	0.5	0.0	0.0	23.0	0.0	34.0	14.0	29.0	1.01	0.51	4.82
2.4	1.0	0.0	0.0	4.0	0.0	37.0	13.0	46.0	1.81	0.94	3.7
	0.5	0.0	0.0	25.0	0.0	33.0	12.0	30.0	0.96	0.57	4.26
2.8	1.0	0.0	0.0	7.0	0.0	34.0	16.0	43.0	1.55	0.95	3.42
	0.5	0.0	0.0	28.0	0.0	30.0	22.0	20.0	0.74	0.55	3.87